

Time scales of the turbulent heat-flux equation and their near-wall behavior

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Abstract. The dissipation rate $\varepsilon_{i\theta}$ of the turbulent heat flux $\overline{u_i\theta}$ is split by the two-point correlation method, exactly and at any Prandtl number, into a homogeneous part $\varepsilon_{i\theta,h}$, the curvature of the velocity–temperature correlation at coincidence, and an inhomogeneous part $\frac{1}{4}(\nu + \kappa)\Delta_x \overline{u_i\theta}$ absorbed by the molecular diffusion, the $(\nu - \kappa)$ terms being confined to a cross-diffusion in divergence form. The homogeneous part vanishes under local isotropy, measures small-scale anisotropy and is not sign-definite; the heat-flux time scale defined from the flux equation itself, $\tau_{i\theta,h} = \overline{u_i\theta}/\varepsilon_{i\theta,h}$, is therefore signed, and its ratios to the mechanical and thermal time scales have as quotient the thermal-to-mechanical ratio of the temperature-variance analysis. The near-wall behavior follows analytically for the four thermal wall conditions. At an isothermal wall the two parts of the streamwise dissipation are equal, the wall-normal slopes are in the ratio 1:3, and both components share the anchors $2Pr/(1 + Pr)$ and $2/(1 + Pr)$: the heat-flux time scale is the harmonic mean of the thermal and mechanical time scales, thermally controlled at low Prandtl number. At walls with fluctuating temperature the wall-normal dissipation vanishes through the cancellation of finite parts, the wall-normal ratio attains the universal anchor $-2Pr/(1 + Pr)$, the homogeneous dissipation acting as a source of the wall-normal flux over a layer whose thickness is a wall-parallel microscale of the wall-temperature field, and the streamwise ratio diverges as $(y^+)^{-1}$ with a parameter-free amplitude. Channel- and pipe-flow direct numerical simulations from four codes at $Pr = 0.71, 0.025$ and 0.01 reproduce the splits and anchors to within 0.4–7% and locate the source layer at 2–9 viscous units; at liquid-metal Prandtl numbers the cross-diffusion reaches 40% of the peak dissipation and the heat-flux time scale remains of the order of the thermal one across the channel. The exact equation and the wall identities are parameter-free constraints for heat-flux closures and data-driven representations of the scalar flux, and acceptance tests for their training data.

1 Introduction

The transport equation for the turbulent heat flux $\overline{u_i\theta}$ forms the basis of second-moment analysis and modeling of the thermal field [1–4]. The equation contains its own dissipation rate $\varepsilon_{i\theta}$, and with it a time scale intrinsic to the flux, $\overline{u_i\theta}/\varepsilon_{i\theta}$. This time scale has not been used as the basis of a closure, nor has its behavior been established analytically. Two circumstances account for this. At high Reynolds number $\varepsilon_{i\theta}$ is neglected on the basis of local isotropy: an isotropic small-scale field admits no invariant vector, so the correlation between velocity and scalar gradients must vanish [3]. However, $\varepsilon_{i\theta}$ is of leading order in the near-wall budgets [5, 6] and it is not a dissipation in the conventional sense: it is not sign-definite and can act locally as transport or as a source, so that the intrinsic time scale is signed, vanishing or diverging where the flux or its dissipation crosses zero. This property is made exact in this paper.

The time scale of the heat flux has consequently been prescribed from outside the heat-flux equation. Models based on the generalized gradient-diffusion hypothesis, which improve on the Reynolds analogy by admitting separate time scales for the velocity and temperature

fields, organize the closure around a single time scale for the flux, and its choice has remained empirical: the mechanical scale $\tau_u = k/\varepsilon$ in the original formulation, the thermal scale $\tau_\theta = \overline{\theta^2}/(2\varepsilon_\theta)$, argued more appropriate for buoyancy-driven flows at low Prandtl number, or a hybrid, commonly the geometric mean $(\tau_u\tau_\theta)^{1/2}$ [3, 7–12]. The geometric mean is an interpolation without a physical mechanism, the dimensionally consistent scale-symmetric combination of the two available scales, and its a priori investigation on Rayleigh–Bénard direct numerical simulation data [11, 13] found the thermal time scale, not the hybrid formulation, appropriate for the buoyancy-driven low-Prandtl-number case. Each of these formulations takes its time scale from another transport equation, that of k or of $\overline{\theta^2}$; none is derived from the equation for $\overline{u_i\theta}$, whose own $\varepsilon_{i\theta}$ term sets the time scale of the flux. A heat-flux time scale derived from the heat-flux equation itself, the natural formulation, has not been available, because of the two circumstances above: $\varepsilon_{i\theta}$ has been treated as negligible or damped empirically, see e.g. [14, 15], and its signed character has not been established analytically. The present paper resolves both.

The near-wall behavior of the heat-flux budget itself is well documented. The leading powers of the wall distance of $\overline{u_i\theta}$ and of the budget terms at an isothermal wall were established from DNS by Kasagi et al. [5], and near-wall closures reproduce them by construction: the damped forms of Lai and So [14], the two-equation model of Sommer et al. [16] with an asymptotically consistent $\varepsilon_{i\theta}$, and the elliptic-blending differential flux models of Dehoux et al. [17], which Mangeon et al. [18] extended to imposed-flux and conjugate walls by Taylor expansion of the budget and calibration on the DNS of Flageul et al. [6], with a thermal-to-mechanical time-scale ratio that depends on the thermal wall condition. Those results concern the asymptotics of the total budget terms and their representation in a model. The present analysis concerns the structure of $\varepsilon_{i\theta}$ itself, obtained by the two-point decomposition: the exact split into a closed molecular part and a signed homogeneous part, the wall values of that split, the time scale of the flux defined from its own equation and the composition of that time scale from the mechanical and thermal ones. None of these appear in the works cited, which are used in §4 as the source of the tabulated budgets and in §5 as the models to which the constraints apply.

For the kinetic-energy dissipation, Kolovandin and Vatutin [19] and Jovanović et al. [20] showed, by the two-point correlation technique introduced by Chou [21] and Burgers [22], that ε separates into a homogeneous part ε_h and an inhomogeneous part confined to the wall vicinity. Otić et al. [13] used the technique to model the turbulent diffusion of the temperature variance and its dissipation rate, and Jakirlić and Hanjalić [23] to model the stress dissipation with the correct near-wall asymptotics. In Otić [24] the technique was applied to the temperature-variance dissipation rate ε_θ : the inhomogeneous part is absorbed exactly by the molecular diffusion of the variance, the remaining homogeneous part defines the homogeneous thermal-to-mechanical time-scale ratio $\mathcal{R}_h = \overline{\theta^2} \varepsilon_h / (2k \varepsilon_{\theta,h})$, and the near-wall behavior of $\mathcal{R}_h$ follows analytically for the ideal and conjugate thermal wall conditions, anchored at $\mathcal{R}_h|_w = Pr$ for an isothermal wall and singular as $\overline{\theta^2}_w(y^+)^{-2}$ at walls with fluctuating temperature.

Here the technique is applied to the heat-flux dissipation $\varepsilon_{i\theta}$. The decomposition yields an exact transport equation in which the inhomogeneous part of $\varepsilon_{i\theta}$ is closed and the destruction is concentrated in the homogeneous part $\varepsilon_{i\theta,h}$. The heat-flux time scale is then defined from the equation itself, $\tau_{i\theta,h} = \overline{u_i\theta}/\varepsilon_{i\theta,h}$, a signed quantity per flux component, and its comparison with the homogeneous mechanical and thermal time scales of the variance analysis defines two homogeneous time-scale ratios, $\mathcal{R}_{i\theta,h}^u = \tau_{i\theta,h}/\tau_{u,h}$ and $\mathcal{R}_{i\theta,h}^\theta = \tau_{i\theta,h}/\tau_{\theta,h}$, whose quotient equals $\mathcal{R}_h$ identically. The construction is thereby consistent with the variance analysis, and any two of the three ratios determine the third. The near-wall behavior of the time scale and of both ratios follows analytically. Because the turbulent heat flux is a vector and its dissipation a cross-correlation, the extension is structurally different from the scalar variance case, and three differences govern the analysis. First, the two-point identity for $\varepsilon_{i\theta}$ itself contains no differential-diffusivity term at any Prandtl number. The $(\nu - \kappa)$ structure of the molecular terms appears only in the molecular diffusion, which separates exactly into a symmetric part that

absorbs the inhomogeneous dissipation and a cross-diffusion in divergence form (§2). Second, the cross-correlation between velocity and temperature has no extremal property at coincidence, so $\varepsilon_{i\theta,h}$, and with it the time scale and the ratios, are not sign-definite; under local isotropy $\varepsilon_{i\theta,h}$ vanishes identically, and it therefore measures the small-scale anisotropy of the velocity–scalar correlation. Third, the near-wall behavior is component-dependent and more differentiated than that of the variance (§3): at an isothermal wall both flux components share the anchors $\mathcal{R}_{i\theta,h}^u|_w = 2Pr/(1+Pr)$ and $\mathcal{R}_{i\theta,h}^\theta|_w = 2/(1+Pr)$, the harmonic mean of the thermal and mechanical time scales, while at walls with fluctuating temperature the wall-normal dissipation vanishes through an exact cancellation of finite parts, the wall-normal ratio attains the universal negative anchor $\mathcal{R}_{3\theta,h}^u|_w = -2Pr/(1+Pr)$, and the streamwise ratio diverges as $(y^+)^{-1}$; the source has a kinematic origin, and the thickness of the source layer is derived as a wall-parallel microscale of the wall-temperature field weighted by the shear. A signed curvature length of the cross-correlation, replacing the Taylor microscale, provides a geometric formulation of these results (§2.4): linear in the wall distance at an isothermal wall for both components, of negative square at the fluctuating wall, where the correlation of wall-normal velocity and temperature is locally minimal at coincidence.

The analytical results are assessed in §4 against four direct numerical simulation databases from independent codes: the channel-flow simulations of Abe et al. [25] at $Pr = 0.71$ and 0.025 , of Flageul et al. [6] at $Pr = 0.71$ under Dirichlet, Neumann, Robin and conjugate conditions, of Tiselj and Cizelj [26] at $Pr = 0.01$, and the pipe-flow DNS of Straub et al. [27] at $Pr = 0.71$ and 0.025 under isothermal, mixed-type and isoflux conditions, which adds an independent geometry and code family. The wall identities of §3 serve as acceptance tests for the data and fix the conventions, as in Otić [24]. The wall-resolved databases reproduce the splits and the anchors of both components to within 1–6% at $Pr \leq 0.025$ and the streamwise anchor to 0.4% at $Pr = 0.71$; the negative anchor $-2Pr/(1+Pr)$ is found at every wall with fluctuating temperature, in the channel and in the pipe, at all three Prandtl numbers. Beyond the wall values, the data provide the quantities that the expansions leave undetermined: the thickness of the layer in which the homogeneous dissipation acts as a source, the range of validity of the streamwise asymptote and its Prandtl-number dependence, the magnitude of the cross-diffusion, and the interior composition of the heat-flux time scale, which at $Pr = 0.01$ remains thermally controlled across the channel. In §5 the exact equation and the wall identities are formulated as constraints on heat-flux closures and on data-driven representations of the scalar flux, and on the data from which such representations are constructed; §6 concludes.

2 Two-point decomposition of the heat-flux dissipation

2.1 Transport equation

With the Reynolds decomposition $U_i = \overline{U}_i + u_i$, $T = \overline{T} + \theta$, where $\overline{(\cdot)}$ is an ensemble average, the turbulent heat flux $\overline{u_i\theta}$ of an incompressible constant-property flow obeys the exact transport equation [2, 3, 28, 29]

$$\frac{D\overline{u_i\theta}}{Dt} = \underbrace{-\overline{u_i u_j} \frac{\partial \overline{T}}{\partial x_j} - \overline{u_j \theta} \frac{\partial \overline{U}_i}{\partial x_j}}_{P_{i\theta}} + \underbrace{\overline{f_i \theta}}_{G_{i\theta}} + \underbrace{\frac{\overline{p}}{\rho} \frac{\partial \overline{\theta}}{\partial x_i}}_{\Pi_{i\theta}} - \underbrace{\frac{\partial}{\partial x_j} \left(\overline{u_i u_j \theta} + \frac{\overline{p\theta}}{\rho} \delta_{ij} \right)}_{D_{i\theta}^t} + \underbrace{\nu \overline{\theta \Delta_x u_i} + \kappa \overline{u_i \Delta_x \theta}}_{M_{i\theta}}, \quad (1)$$

with $D/Dt = \partial_t + \overline{U}_j \partial/\partial x_j$, $\Delta_x \equiv \partial^2/\partial x_j \partial x_j$, $\partial_j \equiv \partial/\partial x_j$, p the fluctuating pressure, ν the kinematic viscosity, κ the thermal diffusivity and f_i a fluctuating body force per unit mass. The terms are the production by the mean gradients $P_{i\theta}$, the body-force correlation $G_{i\theta}$, the pressure–temperature-gradient correlation $\Pi_{i\theta}$, the turbulent and pressure diffusion $D_{i\theta}^t$ and the molecular contribution $M_{i\theta}$. For Boussinesq buoyancy the fluctuating body force

is $f_i = -g_i\beta\theta$, with g_i the gravitational acceleration and β the thermal expansion coefficient, so that $G_{i\theta} = -g_i\beta\bar{\theta}^2$; the general form is kept.

The molecular contribution $M_{i\theta}$ is exact, and it is commonly separated into diffusion and destruction. The identities $\theta\Delta_x u_i = \partial_j\theta\partial_j u_i - \partial_j\theta\partial_j u_i$ and $u_i\Delta_x\theta = \partial_j u_i\partial_j\theta - \partial_j u_i\partial_j\theta$, with the one-point gradient correlations

$$S_j \equiv \overline{\theta\partial_j u_i}, \quad T_j \equiv \overline{u_i\partial_j\theta}, \quad S_j + T_j = \partial_j\overline{u_i\theta}, \quad A_j \equiv S_j - T_j, \quad (2)$$

give

$$M_{i\theta} = \underbrace{\partial_j(\nu S_j + \kappa T_j)}_{D_{i\theta}^m} - \underbrace{(\nu + \kappa)\overline{\partial_j u_i\partial_j\theta}}_{\varepsilon_{i\theta}}, \quad (3)$$

which defines the molecular diffusion $D_{i\theta}^m$ and the dissipation $\varepsilon_{i\theta} = (\nu + \kappa)\overline{\partial_j u_i\partial_j\theta}$ exactly for any Pr .

In Launder [3] the viscous stress is carried in its deformation form, $\nu\partial_j(\partial_j u_i + \partial_i u_j)$, and the corresponding split reads

$$M_{i\theta} = \partial_j \left[\nu(\overline{\theta\partial_j u_i} + \overline{\theta\partial_i u_j}) + \kappa\overline{u_i\partial_j\theta} \right] - \left[\nu\overline{\partial_j\theta(\partial_j u_i + \partial_i u_j)} + \kappa\overline{\partial_j u_i\partial_j\theta} \right]. \quad (4)$$

The two conventions describe the same equation: by continuity $\partial_j(\theta\partial_i u_j) = \partial_j\theta\partial_i u_j$, so the additional term in the destruction group of (4) is the exact divergence $\nu\overline{\partial_j\theta\partial_i u_j} = \partial_j(\nu\overline{\theta\partial_i u_j})$, and the same divergence is added to the diffusion group: the difference is an exact transfer between the two groups. The contracted correlation $\overline{\partial_j u_i\partial_j\theta}$, common to both conventions and tabulated as the dissipation in the DNS budget databases used in §4 [5, 6], is the object to which the two-point method applies; the deformation-form contribution, being the divergence of a one-point correlation, vanishes in homogeneous fields and is grouped with the diffusion in (3). The diffusion separates by the elementary identity $\nu S_j + \kappa T_j = \frac{1}{2}(\nu + \kappa)(S_j + T_j) + \frac{1}{2}(\nu - \kappa)(S_j - T_j)$ into

$$D_{i\theta}^m = \frac{1}{2}(\nu + \kappa)\Delta_x\overline{u_i\theta} + \frac{1}{2}(\nu - \kappa)\partial_j A_j, \quad (5)$$

a symmetric diffusion of the flux itself and a cross-diffusion carried by the antisymmetric gradient correlation A_j . At $Pr = 1$, and in a homogeneous field, where A_j is uniform, the cross-diffusion vanishes. Only at $\nu = \kappa$ does $D_{i\theta}^m$ reduce to $\frac{1}{2}(\nu + \kappa)\Delta_x\overline{u_i\theta}$; the frequently written form $(\nu + \kappa)\partial^2\overline{u_i\theta}/\partial x_j\partial x_j$ is not exact at $Pr \neq 1$.

With (3) and (5), the transport equation (1) takes the form

$$\frac{D\overline{u_i\theta}}{Dt} = P_{i\theta} + G_{i\theta} + \Pi_{i\theta} + D_{i\theta}^t + \frac{1}{2}(\nu + \kappa)\Delta_x\overline{u_i\theta} + \frac{1}{2}(\nu - \kappa)\partial_j A_j - \varepsilon_{i\theta}, \quad (6)$$

exact for any Prandtl number, in which the molecular contribution appears as a symmetric diffusion, a cross-diffusion in divergence form and the dissipation $\varepsilon_{i\theta}$.

2.2 Decomposition of the dissipation rate

Following Chou [21], Burgers [22], see also Hinze [2], the fluctuating fields are referred to two points A, B through the relative and centroid coordinates

$$\xi_k = (x_k)_B - (x_k)_A, \quad (x_k)_{AB} = \frac{1}{2}[(x_k)_A + (x_k)_B], \quad (7)$$

so that, with $\partial_{\xi_k} \equiv \partial/\partial\xi_k$ and $(\partial_k)_{AB} \equiv \partial/\partial(x_k)_{AB}$, the single-point derivatives are $(\partial_k)_A = \frac{1}{2}(\partial_k)_{AB} - \partial_{\xi_k}$ and $(\partial_k)_B = \frac{1}{2}(\partial_k)_{AB} + \partial_{\xi_k}$. With summation over the repeated index k and $(\partial_k^2)_{AB} \equiv (\partial_k)_{AB}(\partial_k)_{AB}$, the mixed second derivatives commute and cancel in the product, leaving

$$(\partial_k)_A(\partial_k)_B = \frac{1}{4}(\partial_k^2)_{AB} - \partial_{\xi_k}^2. \quad (8)$$

Kinematic relations of this multi-point type, and their role in the nonlocality of turbulence, are analyzed by Kholmyansky and Tsinober [30] and Kholmyansky et al. [31]. Define the velocity–temperature two-point correlation

$$Q_i(x, \xi) \equiv \overline{(u_i)_A (\theta)_B}, \quad (9)$$

a vector function of the centroid position x and the separation ξ , with $Q_i(x, 0) = \overline{u_i \theta}$. The free index i is unaffected: the operator (8) contracts over the derivative index k only, so the identity below holds per component. Applying (8) to Q_i and taking $\xi \rightarrow 0$, in which $(\partial_k^2)_{AB} \rightarrow \Delta_x$, gives, exactly and for any Pr ,

$$\frac{\varepsilon_{i\theta}}{\nu + \kappa} = \overline{\partial_k u_i \partial_k \theta} = \frac{1}{4} \Delta_x \overline{u_i \theta} - (\Delta_\xi Q_i)_0, \quad (10)$$

with $\Delta_\xi \equiv \partial^2 / \partial \xi_k \partial \xi_k$ and the subscript 0 the limit $\xi \rightarrow 0$. No differential-diffusivity term appears: the $(\nu - \kappa)$ structure of (5) is a property of the molecular diffusion, not of the dissipation. Therefore

$$\varepsilon_{i\theta} = \varepsilon_{i\theta,h} + \varepsilon_{i\theta,ih}, \quad \text{where} \quad \varepsilon_{i\theta,h} \equiv -(\nu + \kappa)(\Delta_\xi Q_i)_0 \quad \text{and} \quad \varepsilon_{i\theta,ih} \equiv \frac{1}{4}(\nu + \kappa) \Delta_x \overline{u_i \theta}, \quad (11)$$

the homogeneous part being the curvature of the two-point correlation at coincidence and the inhomogeneous part a function of the mean flux field. In a homogeneous field $\Delta_x \overline{u_i \theta} = 0$ and $\varepsilon_{i\theta} = \varepsilon_{i\theta,h}$. The inhomogeneous part is proportional to the Laplacian of the flux and changes sign at inflection points of the $\overline{u_i \theta}$ profile.

Two properties distinguish the present case from the kinetic energy and the temperature variance. First, for the autocorrelations the two-point correlation is maximal at coincidence, so the curvature term is non-negative in a homogeneous field; the cross-correlation Q_i has no extremal property at $\xi = 0$ (the Cauchy–Schwarz inequality bounds it by the autocorrelations but does not locate an extremum there), so neither part of (11) is sign-definite even in a homogeneous field. Consequently $\varepsilon_{i\theta}$ is not, strictly, a sink of $\overline{u_i \theta}$; the near-wall analysis of §3 quantifies this property exactly. Second, under local isotropy of the small-scale field the vector $(\Delta_\xi Q_i)_0$, a first-rank tensor formed from the small scales, must vanish: an isotropic field admits no invariant vector. This is the two-point form of the classical argument for neglecting $\varepsilon_{i\theta}$ at high Reynolds number [3]. The homogeneous part $\varepsilon_{i\theta,h}$ therefore measures the small-scale anisotropy of the velocity–scalar correlation; it is finite where the small scales are anisotropic, in particular in the wall layer, and decays away from the wall as a kinematic consequence rather than by a modeling assumption. The name dissipation rate is retained for $\varepsilon_{i\theta}$ by convention, following the budget databases and the literature; the conventional interpretation of a positive-definite destruction is not, and the decomposition (11) supplies the consistent replacement, a closed molecular-transport part and a signed anisotropy measure.

The antisymmetric correlation A_j of (2) has a two-point interpretation that identifies the cross-diffusion as an intrinsically inhomogeneous quantity. Differentiating (9) at one point, $S_j = [(\partial_j)_A Q_i]_0 = \frac{1}{2} \partial_j \overline{u_i \theta} - (\partial_{\xi_j} Q_i)_0$ and $T_j = [(\partial_j)_B Q_i]_0 = \frac{1}{2} \partial_j \overline{u_i \theta} + (\partial_{\xi_j} Q_i)_0$, so that

$$A_j = -2 (\partial_{\xi_j} Q_i)_0, \quad \frac{1}{2}(\nu - \kappa) \partial_j A_j = -(\nu - \kappa) \partial_{x_j} (\partial_{\xi_j} Q_i)_0 : \quad (12)$$

the cross-diffusion is the mixed centroid–separation derivative of the two-point correlation, which has no counterpart for the autocorrelations (it is antisymmetric under $A \leftrightarrow B$) and vanishes identically in a homogeneous field.

2.3 Exact transport equation and time-scale ratios

Substituting (3), (5) and (11) into (1), the inhomogeneous part of the dissipation is absorbed exactly by the symmetric molecular diffusion,

$$\frac{\partial \overline{u_i \theta}}{\partial t} + \overline{U_j} \frac{\partial \overline{u_i \theta}}{\partial x_j} = P_{i\theta} + G_{i\theta} + D_{i\theta}^t + \Pi_{i\theta} + \frac{1}{4}(\nu + \kappa) \Delta_x \overline{u_i \theta} + \frac{1}{2}(\nu - \kappa) \partial_j A_j - \varepsilon_{i\theta,h}, \quad (13)$$

in exact parallel to the variance equation of Otić [24], where the absorbed fraction is one half of $\kappa\Delta_x\overline{\theta^2}$; here it is one half of the symmetric diffusion, and the cross-diffusion $\frac{1}{2}(\nu - \kappa)\partial_j A_j$ remains as an exact term in divergence form. It vanishes at $Pr = 1$ and in homogeneous fields, is evaluable from DNS budgets as the difference between the tabulated molecular diffusion and $\frac{1}{2}(\nu + \kappa)\Delta_x\overline{u_i\theta}$, and its near-wall expansion is derived in §3. Being a divergence, it redistributes the flux without net production or destruction; as a one-point correlation not expressible through $\overline{u_i\theta}$, it requires closure in a model, and its wall asymptotics below provide the corresponding constraint.

With the homogeneous time scales of the velocity and variance fields, $\tau_{u,h} = k/\varepsilon_h$ and $\tau_{\theta,h} = \overline{\theta^2}/(2\varepsilon_{\theta,h})$ [24], and the signed heat-flux time scale

$$\tau_{i\theta,h} \equiv \frac{\overline{u_i\theta}}{\varepsilon_{i\theta,h}}, \quad (14)$$

two homogeneous time-scale ratios are defined,

$$\mathcal{R}_{i\theta,h}^u \equiv \frac{\tau_{i\theta,h}}{\tau_{u,h}} = \frac{\overline{u_i\theta}\varepsilon_h}{k\varepsilon_{i\theta,h}}, \quad \mathcal{R}_{i\theta,h}^\theta \equiv \frac{\tau_{i\theta,h}}{\tau_{\theta,h}} = \frac{2\overline{u_i\theta}\varepsilon_{\theta,h}}{\overline{\theta^2}\varepsilon_{i\theta,h}}, \quad (15)$$

which satisfy, at every point and by the definitions alone, the identity

$$\frac{\mathcal{R}_{i\theta,h}^u}{\mathcal{R}_{i\theta,h}^\theta} = \frac{\overline{\theta^2}\varepsilon_h}{2k\varepsilon_{\theta,h}} = \mathcal{R}_h. \quad (16)$$

The construction thus connects to the homogeneous thermal-to-mechanical ratio $\mathcal{R}_h$ of the variance analysis [24]: any two of the three ratios determine the third.

The quantity (14) is dimensionally a time but is not positive: neither $\overline{u_i\theta}$ nor $\varepsilon_{i\theta,h}$ is sign-definite, so $\tau_{i\theta,h}$ and the ratios (15) are signed. This property is not a deficiency of the definition but carries its physical content, which is read from the exact equation (13), in which the homogeneous destruction appears as $-\varepsilon_{i\theta,h} = -\overline{u_i\theta}/\tau_{i\theta,h}$, equivalently $-\omega_{i\theta,h}\overline{u_i\theta}$ with the signed rate $\omega_{i\theta,h} \equiv 1/\tau_{i\theta,h} = \varepsilon_{i\theta,h}/\overline{u_i\theta}$: where $\tau_{i\theta,h} > 0$ the homogeneous dissipation relaxes the flux component on the time $|\tau_{i\theta,h}|$, and where $\tau_{i\theta,h} < 0$ amplifies it, $|\tau_{i\theta,h}|$ being the growth time, in accordance with the usual interpretation of a negative relaxation time as exponential growth. The mechanical and thermal time scales are positive in the flows considered, the homogeneous dissipations ε_h and $\varepsilon_{\theta,h}$ being positive at every point of every DNS case analyzed in Otić [24], so the ratios carry the sign of $\tau_{i\theta,h}$, that of the product $\overline{u_i\theta}\varepsilon_{i\theta,h}$ per component without summation: a negative ratio states that the homogeneous dissipation acts as a source of the flux component, with the magnitude comparing the growth time to the mechanical or thermal scale. The zeros and poles of the ratios correspond to physical crossings rather than to artifacts of the definition: $\mathcal{R}_{i\theta,h}^u$ passes through zero where $\varepsilon_{i\theta,h}$ changes sign at finite flux, the transition between destruction and amplification, and diverges where the flux component crosses zero at finite $\varepsilon_{i\theta,h}$. All statements below are therefore component-wise and local, and the sign structure is itself determined exactly by the near-wall analysis: at walls with fluctuating temperature the wall-normal ratio attains a negative anchor (§3.5), the wall expression of the fact that $\varepsilon_{i\theta}$ does not act as a sink. If a single scalar time is required, the frame-invariant projection

$$\tau_{\theta,h}^\parallel \equiv \frac{\overline{u_k\theta}\overline{u_k\theta}}{\overline{u_k\theta}\varepsilon_{k\theta,h}}, \quad \text{summation over } k, \quad (17)$$

measures the signed time scale of the flux vector along itself; near a wall it is dominated by the component of lowest order in the wall distance and reduces to the streamwise limits of §3. A positive-definite alternative to (14) is not introduced, since any construction enforcing positivity removes the source region established by the near-wall analysis.

2.4 Signed cross-correlation microscale

The homogeneous dissipations of the kinetic energy and the variance admit Taylor-microscale forms, $\varepsilon_h = 10\nu k/\lambda^2$ [32–34] and $\varepsilon_{\theta,h} = C_\theta \kappa \overline{\theta^2}/\lambda_\theta^2$ with $C_\theta = 3$ from the homogeneous isotropic normalization [24, 35]. The corresponding quantity for the heat flux is necessarily signed. Define, per component, the signed curvature length

$$\lambda_{i\theta}^2 \equiv \frac{C_{u\theta} (\nu + \kappa) \overline{u_i \theta}}{\varepsilon_{i\theta,h}} = C_{u\theta} (\nu + \kappa) \tau_{i\theta,h}, \quad (18)$$

real of either sign; the second equality shows that $\lambda_{i\theta}^2$ inherits the sign and locality of the signed time scale (14). Its geometric content follows from the two-point form: with $\varepsilon_{i\theta,h} = -(\nu + \kappa)(\Delta_\xi Q_i)_0$, the definition (18) reads $(\Delta_\xi Q_i)_0 = -C_{u\theta} Q_i(0)/\lambda_{i\theta}^2$, and the Taylor expansion of the normalized correlation $\rho_i(\xi) = Q_i(x, \xi)/Q_i(x, 0)$, averaged over the directions of the separation ($\langle \xi_k \xi_l \rangle = r^2 \delta_{kl}/3$ on the sphere of radius r), gives

$$\langle \rho_i \rangle(r) = 1 - \frac{C_{u\theta} r^2}{6 \lambda_{i\theta}^2} + O(r^4) : \quad (19)$$

for $\lambda_{i\theta}^2 > 0$ the correlation decreases away from coincidence, and for $\lambda_{i\theta}^2 < 0$ coincidence is a local minimum of the direction-averaged correlation. No isotropic normalization of $C_{u\theta}$ exists, since both $u_i \theta$ and $\varepsilon_{i\theta,h}$ vanish in isotropic turbulence and their quotient has no canonical value; the value of $C_{u\theta}$ is therefore conventional, and it cancels in every wall limit of §3. In terms of the microscales, the ratios (15) take the form

$$\mathcal{R}_{i\theta,h}^u = \frac{10}{C_{u\theta}} \frac{Pr}{1 + Pr} \left(\frac{\lambda_{i\theta}}{\lambda} \right)^2, \quad \mathcal{R}_{i\theta,h}^\theta = \frac{2C_\theta}{C_{u\theta}} \frac{1}{1 + Pr} \left(\frac{\lambda_{i\theta}}{\lambda_\theta} \right)^2, \quad (20)$$

by $\nu/(\nu + \kappa) = Pr/(1 + Pr)$ and $\kappa/(\nu + \kappa) = 1/(1 + Pr)$: the entire dependence on the molecular diffusivities is contained in the prefactors, while the microscale ratio carries the remaining variation. The quotient of the two forms is $(5/C_\theta) Pr (\lambda_\theta/\lambda)^2$, the microscale form of $\mathcal{R}_h$ in Otić [24], so that the identity (16) is preserved at the microscale level, including the normalization constants. The prefactor $(\nu + \kappa)$ in (18) admits a direct physical interpretation: the cross-correlation is attenuated by both molecular diffusivities acting on a single curvature length, so that the rate $\tau_{i\theta,h}^{-1}$ is additive, $\tau_{i\theta,h}^{-1} = C_{u\theta} \nu/\lambda_{i\theta}^2 + C_{u\theta} \kappa/\lambda_{i\theta}^2$. The additivity of the rates at a common length corresponds to a harmonic combination of the time scales, and at the wall, where all three microscales become linear with their normalization constants cancelling (§3.4), this reduces exactly to the harmonic mean of (47).

3 Near-wall behavior

3.1 Wall expansions and coefficient identities

Let $y \equiv x_3$ be the wall-normal coordinate, the subscript w denote evaluation at the wall ($y \rightarrow 0$), and the fluctuations be expanded as

$$u_1 = b_1 y + c_1 y^2 + d_1 y^3 + \dots, \quad u_3 = c_3 y^2 + d_3 y^3 + \dots, \quad \theta = a_\theta + b_\theta y + \gamma_\theta y^2 + \delta_\theta y^3 + \dots, \quad (21)$$

the coefficients being fluctuating fields on (x_1, x_2, t) . No slip gives $u_i|_w = 0$, and continuity $\partial_i u_i = 0$ evaluated at the wall forces $\partial_3 u_3|_w = 0$, so $u_3 = O(y^2)$; at order y , continuity further gives

$$2c_3 = -(\partial_1 b_1 + \partial_2 b_2) \implies c_3 = -\frac{1}{2} \partial_\alpha b_\alpha, \quad (22)$$

Greek indices running over the wall-parallel directions, with $\nabla_{\parallel} \equiv (\partial_1, \partial_2)$. The fluctuating wall-parallel momentum balance at the wall, where all convective terms and the time derivative vanish with u_i , reduces to $0 = -(1/\rho) \partial_{\alpha} p|_w + \nu \Delta_x u_{\alpha}|_w = -(1/\rho) \partial_{\alpha} p|_w + 2\nu c_{\alpha}$, whence

$$c_{\alpha} = \frac{1}{2\rho\nu} \partial_{\alpha} p|_w : \quad (23)$$

c_{α} is the fluctuating wall pressure gradient, c_1 its streamwise component. A further identity used below combines (22) with the wall-parallel homogeneity of the statistics: in channel and pipe flow the wall-parallel derivative of any averaged quantity vanishes, so the divergence term produced by integrating by parts drops out,

$$\overline{a_{\theta} c_3} = -\frac{1}{2} \overline{a_{\theta} \partial_{\alpha} b_{\alpha}} = -\frac{1}{2} \partial_{\alpha} \overline{a_{\theta} b_{\alpha}} + \frac{1}{2} \overline{b_{\alpha} \partial_{\alpha} a_{\theta}} = \frac{1}{2} \overline{b_{\alpha} \partial_{\alpha} a_{\theta}}, \quad (24)$$

the correlation of the fluctuating wall shear with the wall-parallel gradient of the fluctuating wall temperature. The thermal wall condition enters only through (a_{θ}, b_{θ}) :

$$\text{Dirichlet: } a_{\theta} = 0; \quad \text{Neumann: } b_{\theta} = 0; \quad \text{Robin: } a_{\theta} = -L b_{\theta}; \quad \text{conjugate: } a_{\theta}, b_{\theta} \text{ free}, \quad (25)$$

with L the Robin length and the conjugate correlations set by the thermal activity ratio K and the solid thickness d^+ [6, 26]. By spanwise reflection symmetry $\overline{u_2 \theta} = 0$, and only the components $i = 1, 3$ are considered. For the mechanical field the expansions give, independently of the thermal condition [24],

$$k = \frac{1}{2} \overline{(b_1^2 + b_2^2)} y^2 + O(y^3), \quad \varepsilon_h|_w = \frac{1}{2} \varepsilon|_w = \frac{1}{2} \nu \overline{(b_1^2 + b_2^2)}, \quad \frac{\varepsilon_h}{k} \xrightarrow{y \rightarrow 0} \frac{\nu}{y^2}, \quad (26)$$

and for the thermal field at a Dirichlet wall $\varepsilon_{\theta,h}|_w = \frac{1}{2} \varepsilon_{\theta}|_w = \frac{1}{2} \kappa \overline{b_{\theta}^2}$ with $\overline{\theta^2} = \overline{b_{\theta}^2} y^2$, so $\tau_{\theta,h} \rightarrow y^2/\kappa$, while at walls with fluctuating temperature $\overline{\theta^2}|_w = \overline{a_{\theta}^2}$ is finite and $\mathcal{R}_h \sim \overline{\theta^2}_w (y^+)^{-2}$ [24].

3.2 Streamwise component

All expansions are given first for general (a_{θ}, b_{θ}) and specialized afterwards. The flux follows by direct multiplication of (21) after averaging,

$$\overline{u_1 \theta} = \overline{b_1 a_{\theta}} y + (\overline{b_1 b_{\theta}} + \overline{c_1 a_{\theta}}) y^2 + (\overline{b_1 \gamma_{\theta}} + \overline{c_1 b_{\theta}} + \overline{d_1 a_{\theta}}) y^3 + O(y^4), \quad (27)$$

so that

$$\Delta_x \overline{u_1 \theta} = \frac{d^2 \overline{u_1 \theta}}{dy^2} = 2(\overline{b_1 b_{\theta}} + \overline{c_1 a_{\theta}}) + 6(\overline{b_1 \gamma_{\theta}} + \overline{c_1 b_{\theta}} + \overline{d_1 a_{\theta}}) y + O(y^2). \quad (28)$$

For the dissipation, the wall-normal gradient product is

$$\begin{aligned} \partial_3 u_1 \partial_3 \theta &= (b_1 + 2c_1 y + 3d_1 y^2)(b_{\theta} + 2\gamma_{\theta} y + 3\delta_{\theta} y^2) \\ &= b_1 b_{\theta} + 2(b_1 \gamma_{\theta} + c_1 b_{\theta}) y + (3b_1 \delta_{\theta} + 4c_1 \gamma_{\theta} + 3d_1 b_{\theta}) y^2 + \dots, \end{aligned} \quad (29)$$

while the wall-parallel product starts at order y because $\nabla_{\parallel} u_1 = \nabla_{\parallel} b_1 y + O(y^2)$ whereas $\nabla_{\parallel} \theta = \nabla_{\parallel} a_{\theta} + O(y)$:

$$\nabla_{\parallel} u_1 \cdot \nabla_{\parallel} \theta = \nabla_{\parallel} b_1 \cdot \nabla_{\parallel} a_{\theta} y + O(y^2). \quad (30)$$

Averaging,

$$\frac{\varepsilon_{1\theta}}{\nu + \kappa} = \overline{b_1 b_{\theta}} + [2 \overline{b_1 \gamma_{\theta}} + 2 \overline{c_1 b_{\theta}} + \overline{\nabla_{\parallel} b_1 \cdot \nabla_{\parallel} a_{\theta}}] y + O(y^2), \quad (31)$$

with the wall value $\varepsilon_{1\theta}|_w = (\nu + \kappa) \overline{b_1 b_{\theta}}$ for every thermal condition. The antisymmetric correlation requires the two products

$$\begin{aligned} \theta \partial_3 u_1 &= (a_{\theta} + b_{\theta} y + \gamma_{\theta} y^2)(b_1 + 2c_1 y + 3d_1 y^2) \\ &= a_{\theta} b_1 + (b_{\theta} b_1 + 2a_{\theta} c_1) y + (\gamma_{\theta} b_1 + 2b_{\theta} c_1 + 3a_{\theta} d_1) y^2 + \dots, \end{aligned} \quad (32)$$

$$u_1 \partial_3 \theta = (b_1 y + c_1 y^2)(b_\theta + 2\gamma_\theta y) = b_1 b_\theta y + (2b_1 \gamma_\theta + c_1 b_\theta) y^2 + \dots, \quad (33)$$

whose averaged difference is

$$A_3^{(1)} = \overline{a_\theta b_1} + 2 \overline{a_\theta c_1} y + (\overline{b_\theta c_1} - \overline{b_1 \gamma_\theta} + 3 \overline{a_\theta d_1}) y^2 + O(y^3), \quad (34)$$

so the cross-diffusion of (13) is $\frac{1}{2}(\nu - \kappa) dA_3^{(1)}/dy = (\nu - \kappa) \overline{a_\theta c_1} + O(y)$: it enters at the wall only where the wall temperature fluctuates and is carried by the correlation of the fluctuating wall temperature with the fluctuating wall pressure gradient, by (23). From (11) and (28),

$$\varepsilon_{1\theta,ih}|_w = \frac{1}{2}(\nu + \kappa)(\overline{b_1 b_\theta} + \overline{c_1 a_\theta}), \quad \varepsilon_{1\theta,h}|_w = \varepsilon_{1\theta}|_w - \varepsilon_{1\theta,ih}|_w = \frac{1}{2}(\nu + \kappa)(\overline{b_1 b_\theta} - \overline{c_1 a_\theta}). \quad (35)$$

3.3 Wall-normal component

The same steps give

$$\overline{u_3 \theta} = \overline{c_3 a_\theta} y^2 + (\overline{c_3 b_\theta} + \overline{d_3 a_\theta}) y^3 + O(y^4), \quad \Delta_x \overline{u_3 \theta} = 2 \overline{c_3 a_\theta} + 6(\overline{c_3 b_\theta} + \overline{d_3 a_\theta}) y + O(y^2). \quad (36)$$

The gradient products are

$$\begin{aligned} \partial_3 u_3 \partial_3 \theta &= (2c_3 y + 3d_3 y^2)(b_\theta + 2\gamma_\theta y) = 2c_3 b_\theta y + (4c_3 \gamma_\theta + 3d_3 b_\theta) y^2 + \dots, \\ \nabla_{\parallel} u_3 \cdot \nabla_{\parallel} \theta &= \nabla_{\parallel} c_3 \cdot \nabla_{\parallel} a_\theta y^2 + O(y^3), \end{aligned} \quad (37)$$

both vanishing at the wall: $\partial_3 u_3|_w = 0$ by continuity and $\nabla_{\parallel} u_3 = O(y^2)$. Hence

$$\frac{\varepsilon_{3\theta}}{\nu + \kappa} = 2 \overline{c_3 b_\theta} y + [4 \overline{c_3 \gamma_\theta} + 3 \overline{d_3 b_\theta} + \overline{\nabla_{\parallel} c_3 \cdot \nabla_{\parallel} a_\theta}] y^2 + O(y^3), \quad (38)$$

and consequently

$$\varepsilon_{3\theta}|_w = 0 \quad \text{for every thermal condition.} \quad (39)$$

The antisymmetric correlation follows from $\theta \partial_3 u_3 = (a_\theta + b_\theta y + \gamma_\theta y^2)(2c_3 y + 3d_3 y^2) = 2a_\theta c_3 y + (2b_\theta c_3 + 3a_\theta d_3) y^2 + \dots$ and $u_3 \partial_3 \theta = (c_3 y^2 + d_3 y^3)(b_\theta + 2\gamma_\theta y) = c_3 b_\theta y^2 + \dots$:

$$A_3^{(3)} = 2 \overline{a_\theta c_3} y + (\overline{b_\theta c_3} + 3 \overline{a_\theta d_3}) y^2 + O(y^3), \quad (40)$$

so the cross-diffusion is $(\nu - \kappa) \overline{a_\theta c_3} + O(y)$ at the wall. The two parts of the dissipation are

$$\begin{aligned} \varepsilon_{3\theta,ih} &= \frac{1}{2}(\nu + \kappa) \overline{c_3 a_\theta} + \frac{3}{2}(\nu + \kappa)(\overline{c_3 b_\theta} + \overline{d_3 a_\theta}) y + O(y^2), \\ \varepsilon_{3\theta,h} &= -\frac{1}{2}(\nu + \kappa) \overline{c_3 a_\theta} + (\nu + \kappa) \left[\frac{1}{2} \overline{c_3 b_\theta} - \frac{3}{2} \overline{d_3 a_\theta} \right] y + O(y^2), \end{aligned} \quad (41)$$

the second obtained as $\varepsilon_{3\theta} - \varepsilon_{3\theta,ih}$ from (38). At the wall,

$$\varepsilon_{3\theta,h}|_w = -\varepsilon_{3\theta,ih}|_w = -\frac{1}{2}(\nu + \kappa) \overline{c_3 a_\theta} : \quad (42)$$

whenever the wall temperature fluctuates, the total wall-normal dissipation vanishes at the wall through the exact cancellation of two finite parts. By (24) their common magnitude is set by the correlation of the fluctuating wall shear with the wall-parallel advection of the fluctuating wall temperature, the mechanism that generates wall-normal flux at a fluctuating wall.

3.4 Dirichlet wall

With $a_\theta = 0$ every correlation containing a_θ drops. The streamwise component inherits the exact equipartition identity of the variance and kinetic-energy dissipations,

$$\varepsilon_{1\theta,h}|_w = \varepsilon_{1\theta,ih}|_w = \frac{1}{2} \varepsilon_{1\theta}|_w = \frac{1}{2}(\nu + \kappa) \overline{b_1 b_\theta}, \quad (43)$$

while for the wall-normal component both parts vanish with the total and the leading slopes split, from (41) with $a_\theta = 0$, in the Pr -independent ratio 1:3,

$$\varepsilon_{3\theta,h} = \frac{1}{2}(\nu + \kappa) \overline{c_3 b_\theta} y + O(y^2), \quad \varepsilon_{3\theta,ih} = \frac{3}{2}(\nu + \kappa) \overline{c_3 b_\theta} y + O(y^2), \quad (44)$$

i.e. homogeneous and inhomogeneous fractions $\frac{1}{4}$ and $\frac{3}{4}$ of $\varepsilon_{3\theta}$ at the wall. The cross-diffusion is $O(y)$ for both components and does not affect the wall values. In the ratios (15) the wall statistics cancel. For the streamwise component, with (27), (43) and (26),

$$\mathcal{R}_{1\theta,h}^u|_w = \lim_{y \rightarrow 0} \frac{\overline{u_1 \theta} (\varepsilon_h/k)}{\varepsilon_{1\theta,h}} = \frac{\overline{b_1 b_\theta} y^2 \cdot \nu/y^2}{\frac{1}{2}(\nu + \kappa) \overline{b_1 b_\theta}} = \frac{2\nu}{\nu + \kappa}, \quad (45)$$

and for the wall-normal component, with $\overline{u_3 \theta} = \overline{c_3 b_\theta} y^3$ and (44),

$$\mathcal{R}_{3\theta,h}^u|_w = \frac{\overline{c_3 b_\theta} y^3 \cdot \nu/y^2}{\frac{1}{2}(\nu + \kappa) \overline{c_3 b_\theta} y} = \frac{2\nu}{\nu + \kappa}. \quad (46)$$

By the identity $(\nu + \kappa) = \nu(1 + 1/Pr) = \kappa(1 + Pr)$, and with $\mathcal{R}_{i\theta,h}^\theta = \mathcal{R}_{i\theta,h}^u/\mathcal{R}_h$ and $\mathcal{R}_h|_w = Pr$ [24],

$$\mathcal{R}_{1\theta,h}^u|_w = \mathcal{R}_{3\theta,h}^u|_w = \frac{2Pr}{1 + Pr}, \quad \mathcal{R}_{1\theta,h}^\theta|_w = \mathcal{R}_{3\theta,h}^\theta|_w = \frac{2}{1 + Pr}, \quad (47)$$

while the thermal-side anchors also follow directly, e.g. $\mathcal{R}_{1\theta,h}^\theta|_w = 2 \overline{b_1 b_\theta} y^2 \cdot \frac{1}{2} \kappa \overline{b_\theta^2} / [\overline{b_\theta^2} y^2 \cdot \frac{1}{2}(\nu + \kappa) \overline{b_1 b_\theta}] = 2\kappa/(\nu + \kappa)$. The anchors are component-independent, free of the wall statistics, and $2Pr/(1 + Pr) = 2/(1 + 1/Pr)$ is the harmonic mean of Pr and unity: at an isothermal wall the heat-flux time scale is the harmonic mean of the thermal and mechanical time scales. At $Pr = 1$ all four anchors equal one; at $Pr \ll 1$, $\mathcal{R}_{\theta,h}^u \rightarrow 2Pr$ and $\mathcal{R}_{\theta,h}^\theta \rightarrow 2$. The identity $\mathcal{R}_{\theta,h}^u/\mathcal{R}_{\theta,h}^\theta = Pr = \mathcal{R}_h|_w$ is consistent with the wall limit of the variance analysis [24].

The microscale (18) is linear at the Dirichlet wall for both components. Inserting (27) with $a_\theta = 0$ and (43),

$$\lambda_{1\theta}^2 = \frac{C_{u\theta}(\nu + \kappa) \overline{b_1 b_\theta} y^2}{\frac{1}{2}(\nu + \kappa) \overline{b_1 b_\theta}} = 2C_{u\theta} y^2, \quad (48)$$

and inserting (36) with $a_\theta = 0$ and (44),

$$\lambda_{3\theta}^2 = \frac{C_{u\theta}(\nu + \kappa) \overline{c_3 b_\theta} y^3}{\frac{1}{2}(\nu + \kappa) \overline{c_3 b_\theta} y} = 2C_{u\theta} y^2 : \quad (49)$$

for each component the leading order of the flux exceeds that of the homogeneous dissipation by two, y^2 against y^0 for $i = 1$ and y^3 against y^1 for $i = 3$, so that the quotient is of order y^2 in both cases and

$$\lambda_{i\theta} = \sqrt{2C_{u\theta}} y, \quad i = 1, 3, \quad (50)$$

the exact analog of $\lambda_\theta = \sqrt{2C_\theta} y$ [24]. With $\lambda = \sqrt{10} y$, the anchors (47) are recovered independently from the microscale form (20) with both constants cancelling, $(\lambda_{i\theta}/\lambda)^2 = C_{u\theta}/5$ and $(\lambda_{i\theta}/\lambda_\theta)^2 = C_{u\theta}/C_\theta$, in parallel with the cancellation of C_θ in the wall limit of $\mathcal{R}_h$. At the wall, where the three microscales are linear, the additive rate of §2.4 reduces to the exact harmonic identity: with $\tau_{u,h} \rightarrow y^2/\nu$, $\tau_{\theta,h} \rightarrow y^2/\kappa$ and $\tau_{i\theta,h} \rightarrow 2y^2/(\nu + \kappa)$,

$$\frac{1}{\tau_{i\theta,h}} \Big|_w = \frac{1}{2} \left(\frac{1}{\tau_{u,h}} + \frac{1}{\tau_{\theta,h}} \right) \Big|_w. \quad (51)$$

3.5 Walls with fluctuating temperature

Every condition with $a_\theta \neq 0$, Neumann, Robin and conjugate, shares the following structure.

Wall-normal component: universal negative anchor. By (42) the two parts are finite and opposite at the wall while the total vanishes; the flux $\overline{u_3\theta} \simeq \overline{c_3 a_\theta} y^2$ carries the same correlation, which cancels in the ratio:

$$\mathcal{R}_{3\theta,h}^u|_w = \lim_{y \rightarrow 0} \frac{\overline{c_3 a_\theta} y^2 \cdot \nu / y^2}{-\frac{1}{2}(\nu + \kappa) \overline{c_3 a_\theta}} = -\frac{2Pr}{1 + Pr}, \quad (52)$$

the negative of the Dirichlet anchor, independent of the wall statistics and of the particular fluctuating condition. The homogeneous dissipation opposes the wall-normal flux at the wall: it acts there as a source of $\overline{u_3\theta}$, the exact expression of the fact that $\varepsilon_{i\theta}$ does not act as a sink. Consistently with the chain (16) and $\mathcal{R}_h \sim \overline{\theta^2}_w (y^+)^{-2}$ at these walls, $\mathcal{R}_{3\theta,h}^\theta = \mathcal{R}_{3\theta,h}^u / \mathcal{R}_h \rightarrow -[2Pr/(1 + Pr)] (\varepsilon_{\theta,h}|_w / \overline{\theta^2}_w)^+ (y^+)^2 \rightarrow 0$. In microscale form, inserting (36) and (42) into (18),

$$\lambda_{3\theta}^2 = \frac{C_{u\theta}(\nu + \kappa) \overline{c_3 a_\theta} y^2}{-\frac{1}{2}(\nu + \kappa) \overline{c_3 a_\theta}} = -2C_{u\theta} y^2, \quad (53)$$

the Dirichlet magnitude (49) with reversed sign, the geometric expression of (52). By (19) the constant $C_{u\theta}$ cancels and the direction-averaged normalized correlation of u_3 and θ is locally minimal at coincidence, increasing on a scale set by the wall distance, $\langle \rho_3 \rangle(r) = 1 + r^2/(12y^2) + O(r^4)$.

The origin of the source is kinematic. Split the coincidence Laplacian of Q_3 into its wall-normal and wall-parallel separation curvatures, $(\Delta_\xi Q_3)_0 = \mathcal{C}_\perp + \mathcal{C}_\parallel$. Applying (8) direction by direction, with wall-parallel homogeneity removing $\nabla_\parallel^2 \overline{u_3\theta}$,

$$\mathcal{C}_\perp = \frac{1}{4} \frac{d^2 \overline{u_3\theta}}{dy^2} - \overline{\partial_3 u_3 \partial_3 \theta}, \quad \mathcal{C}_\parallel = -\overline{\nabla_\parallel u_3 \cdot \nabla_\parallel \theta}, \quad \varepsilon_{3\theta,h} = -(\nu + \kappa)(\mathcal{C}_\perp + \mathcal{C}_\parallel), \quad (54)$$

exactly. The wall-parallel curvature is the gradient product of the two fields: where u_3 and θ are correlated on similar wall-parallel scales it has the sign of the flux, so that it is of destruction type, and it is $O(y^2)$ because $\nabla_\parallel u_3 = O(y^2)$. The wall-normal curvature is finite at the wall, $\mathcal{C}_\perp|_w = \frac{1}{2} \overline{c_3 a_\theta}$, and carries (42). Its origin is visible in the two-point function itself. At centroid height y and wall-normal separation ξ_3 the velocity point lies at $y - \xi_3/2$ and the temperature point at $y + \xi_3/2$, so that by (21)

$$Q_3(y; \xi_3) = \overline{c_3 a_\theta} (y - \frac{1}{2}\xi_3)^2 + O(y^3), \quad (55)$$

a parabola in ξ_3 with its vertex at $\xi_3 = 2y$, where the velocity point is on the wall. Coincidence lies on the rising flank: the correlation increases when the velocity point moves away from the wall and the temperature point towards it, because u_3 grows as y^2 while the wall temperature is flat in y . The curvature $\frac{1}{2} \overline{c_3 a_\theta}$ therefore has the sign of the flux and $-(\nu + \kappa)\mathcal{C}_\perp$ opposes it; averaging (55) over directions removes the odd term and returns $\langle \rho_3 \rangle = 1 + r^2/(12y^2)$. Only no slip, continuity and a finite wall temperature enter, which is why the anchor is independent of the wall statistics and of the type of fluctuating condition; (24) supplies the amplitude, the advection of the wall-temperature field by the wall shear. The thickness of the layer over which the source persists is derived at the end of this section.

Streamwise component: first-power divergence. With $\overline{u_1\theta} \simeq \overline{b_1 a_\theta} y$ against $k \sim y^2$, the ratio diverges with one power of the wall distance less than $\mathcal{R}_h$:

$$\mathcal{R}_{1\theta,h}^u \rightarrow \frac{\nu \overline{b_1 a_\theta}}{\frac{1}{2}(\nu + \kappa)(\overline{b_1 \theta} - \overline{c_1 a_\theta})} \frac{1}{y} = \frac{(\partial_y \overline{u_1\theta})_w^+}{\varepsilon_{1\theta,h}^+|_w} (y^+)^{-1}, \quad \mathcal{R}_{1\theta,h}^\theta \sim y^+ \rightarrow 0, \quad (56)$$

the second form in wall units, with the amplitude fixed by the wall slope of the flux and the wall value (35) of the homogeneous dissipation: the asymptote is parameter-free and directly comparable with DNS, in analogy with the singular asymptote of $\mathcal{R}_h$ in Otić [24]. The corresponding microscale is linear rather than quadratic in the wall distance,

$$\lambda_{1\theta}^2 = \frac{2C_{u\theta} \overline{b_1 a_\theta}}{\overline{b_1 b_\theta} - \overline{c_1 a_\theta}} y, \quad (57)$$

of either sign, so that $|\lambda_{1\theta}| \propto y^{1/2}$ is non-analytic at the wall; inserting (57) into the microscale form (20) with $\lambda^2 = 10y^2$ reproduces (56) with the same amplitude, which confirms the internal consistency of the microscale formulation.

Neumann wall. With $b_\theta = 0$ the total streamwise dissipation vanishes at the wall exactly, $\varepsilon_{1\theta}|_w = (\nu + \kappa) \overline{b_1 b_\theta} = 0$: the wall-normal gradient product requires b_θ , and the wall-parallel velocity gradients vanish at the wall. This contrasts with the variance dissipation, which remains finite at a Neumann wall through $\kappa |\overline{\nabla_\parallel a_\theta}|^2$ [24]: at an isoflux wall the fluctuating temperature field dissipates, the heat flux does not. The parts remain finite and opposite, $\varepsilon_{1\theta,h}|_w = -\varepsilon_{1\theta,ih}|_w = -\frac{1}{2}(\nu + \kappa) \overline{c_1 a_\theta}$, carried, by (23), by the correlation of the fluctuating wall temperature with the fluctuating wall pressure gradient; the streamwise divergence (56) has amplitude $-2\nu \overline{b_1 a_\theta} / [(\nu + \kappa) \overline{c_1 a_\theta}]$, and the wall-normal total is $O(y^2)$ rather than $O(y)$, from (38) with $b_\theta = 0$.

Robin wall. With $a_\theta = -L b_\theta$, all leading correlations reduce to Dirichlet-type correlations times powers of L ; in particular

$$\overline{u_1 \theta} = -L \overline{b_1 b_\theta} y + (\overline{b_1 b_\theta} - L \overline{c_1 b_\theta}) y^2 + O(y^3), \quad (58)$$

so the linear term, of the fluctuating class, dominates for $y \lesssim L$ and the quadratic term, of the Dirichlet class, beyond. The wall value is that of the fluctuating class, with the anchors and asymptotes above, and the Dirichlet-class behavior is approached beyond the Robin length. The wall-normal ratio therefore connects the anchor $-2Pr/(1+Pr)$ at the wall to the positive Dirichlet-class values through a sign change of $\varepsilon_{3\theta,h}$, in analogy with the anchored-singular interpolation of $\mathcal{R}_h$; the location of that sign change is set by L together with a length of the wall velocity field, (64) below, and is determined from the data in §4.3.

Conjugate wall. The general formulas apply with the amplitudes carried by the $\overline{a_\theta(\cdot)}$ correlations, set by the thermal activity ratio K and the wall thickness d^+ : as $K \rightarrow 0$ the Dirichlet limit is approached and for $K \gg 1$ the Neumann limit, as for the wall variance [24, 26]. Unlike ε_θ , which is discontinuous at a conjugate interface [36], the heat-flux dissipation has no solid-side counterpart, $u_i \equiv 0$ in the solid, so no interface-continuity question arises.

Thickness of the source layer. The layer in which $\varepsilon_{3\theta,h}$ opposes $\overline{u_3 \theta}$ ends where the wall-parallel curvature overtakes the wall-normal one, $\mathcal{C}_\perp + \mathcal{C}_\parallel = 0$. Its thickness δ_s follows from the expansion of (54) to $O(y^2)$, which requires, beyond (22) and (23), the next orders of continuity and of the momentum and energy balances at the wall. With $u_3 = c_3 y^2 + d_3 y^3 + e_3 y^4 + \dots$ and $p = p_w + p_1 y + \dots$,

$$d_3 = -\frac{1}{3} \partial_\alpha c_\alpha, \quad e_3 = -\frac{1}{4} \partial_\alpha d_\alpha = \frac{1}{12\nu} \partial_t c_3 - \frac{1}{6} \nabla_\parallel^2 c_3, \quad \gamma_\theta = \frac{\partial_t a_\theta}{2\kappa} - \frac{1}{2} \nabla_\parallel^2 a_\theta, \quad (59)$$

where the expression for e_3 follows from the wall-parallel momentum balance at $O(y)$, $\partial_t b_\alpha = -\rho^{-1} \partial_\alpha p_1 + \nu (\nabla_\parallel^2 b_\alpha + 6d_\alpha)$, with the wall-normal balance at the wall, $p_1 = 2\rho\nu c_3$, and (22); the

last identity is the energy equation at the wall, where $u_i = 0$. With (59), wall-parallel homogeneity and stationarity, every coefficient of the expansion reduces to a wall-parallel advection correlation of the wall fields,

$$\mathcal{A} \equiv \overline{b_\alpha \partial_\alpha a_\theta}, \quad \mathcal{A}_b \equiv \overline{b_\alpha \partial_\alpha b_\theta}, \quad \mathcal{P} \equiv \overline{c_\alpha \partial_\alpha a_\theta} = \frac{\overline{\nabla_\parallel p_w \cdot \nabla_\parallel a_\theta}}{2\rho\nu}, \quad \mathcal{B} \equiv \overline{\partial_\alpha b_\alpha \nabla_\parallel^2 a_\theta}, \quad \mathcal{T} \equiv \overline{c_3 \partial_t a_\theta} : \quad (60)$$

the wall temperature advected by the shear, $\mathcal{A} = 2\overline{c_3 a_\theta}$ by (24); the wall heat-flux fluctuation advected by the shear, $\mathcal{A}_b = 2\overline{c_3 b_\theta}$, the Dirichlet-class correlation; the wall temperature advected by the pressure-gradient-driven curvature of the velocity profile, $\mathcal{P} = 3\overline{d_3 a_\theta}$; the shear convergence correlated with the wall-parallel Laplacian of the wall temperature, $\mathcal{B} = 2\overline{\nabla_\parallel c_3 \cdot \nabla_\parallel a_\theta}$; and a time-derivative correlation. Since $\overline{c_3 \gamma_\theta} = \mathcal{T}/(2\kappa) + \frac{1}{4}\mathcal{B}$ and $\overline{e_3 a_\theta} = -\mathcal{T}/(12\nu) + \frac{1}{12}\mathcal{B}$, the Laplacian correlation cancels identically in $\mathcal{C}_\perp$ and the time derivative is absent from $\mathcal{C}_\parallel$:

$$\mathcal{C}_\perp = \frac{1}{4}\mathcal{A} + \left(\frac{1}{2}\mathcal{P} - \frac{1}{4}\mathcal{A}_b\right)y - \frac{2\nu + \kappa}{4\nu\kappa} \mathcal{T} y^2 + O(y^3), \quad \mathcal{C}_\parallel = -\frac{1}{2}\mathcal{B} y^2 + O(y^3), \quad (61)$$

whose first two orders reproduce (41). The same coefficients have a two-point reading. With $\Phi_\alpha(\xi) \equiv \overline{b_\alpha(x) a_\theta(x + \xi)}$ the cross-correlation of the wall shear with the wall temperature, homogeneity gives for wall-parallel separation

$$Q_3(y; \xi_\parallel) = \frac{1}{2} y^2 \partial_{\xi_\alpha} \Phi_\alpha(\xi) + O(y^3) : \quad (62)$$

the near-wall correlation of wall-normal velocity and temperature is the divergence of the shear-temperature cross-correlation, its value at $\xi = 0$ being the flux $\frac{1}{2}\mathcal{A} y^2$ and its wall-parallel Laplacian at $\xi = 0$ the destruction $\mathcal{C}_\parallel = -\frac{1}{2}\mathcal{B} y^2$. The quotient $\mathcal{A}/\mathcal{B}$ is therefore a squared wall-parallel length of the wall-temperature field as seen through the shear field, a Taylor-type microscale of a_θ . The sign change of $\varepsilon_{3\theta,h}$ is the root of the truncated expansion, $\frac{1}{2}\mathcal{B}_e \delta_s^2 - \frac{1}{2}\mathcal{P}_e \delta_s - \frac{1}{4}\mathcal{A} = 0$, with $\mathcal{P}_e \equiv \mathcal{P} - \frac{1}{2}\mathcal{A}_b$ the linear coefficient of (61), equal to $\mathcal{P}$ at an imposed-flux wall, where $\mathcal{A}_b = 0$ exactly, and $\mathcal{B}_e \equiv \mathcal{B} + (2\nu + \kappa) \mathcal{T}/(2\nu\kappa)$ the Laplacian correlation augmented by the time-derivative term,

$$\delta_s = \frac{\mathcal{P}_e + (\mathcal{P}_e^2 + 2\mathcal{B}_e \mathcal{A})^{1/2}}{2\mathcal{B}_e}, \quad \delta_s \rightarrow \left(\frac{\mathcal{A}}{2\mathcal{B}}\right)^{1/2} \quad \text{for } \mathcal{A}_b = 0, \mathcal{P}, \mathcal{T} \rightarrow 0, \quad (63)$$

valid where δ_s lies within the range of the expansions. At an imposed-flux wall the source layer is thus as thick as the wall-parallel microscale of the wall-temperature field weighted by the shear. At $Pr = O(1)$ the wall temperature has structure on the viscous scales, $\mathcal{B}/\mathcal{A}$ is the inverse square of a viscous length and δ_s^+ is a few viscous units; at $Pr \ll 1$ the wall temperature is conductively smoothed over those scales, $\mathcal{B}$ is small and δ_s grows; at fixed low Pr the layer grows with the Reynolds number through the flux coefficient $\mathcal{A}$, which outgrows $\mathcal{B}_e$ as the thermal field scales with the outer variables (§4.3). The Prandtl-number dependence of the source layer is thereby traced to the wall-parallel smoothness of the wall temperature, the property that also sets the range of the streamwise asymptote (§4.3). For the Robin wall, $a_\theta = -Lb_\theta$ gives $\mathcal{A} = -L\mathcal{A}_b$ and $\mathcal{P} = -L\overline{c_\alpha \partial_\alpha b_\theta}$, and the linear truncation of (61) yields

$$\frac{1}{\delta_s} = \frac{2}{\ell_p} - \frac{1}{L}, \quad \ell_p \equiv -\frac{\overline{b_\alpha \partial_\alpha b_\theta}}{\overline{c_\alpha \partial_\alpha b_\theta}}, \quad (64)$$

with ℓ_p a length of the velocity field, the shear over the profile-curvature correlation with the wall heat-flux gradient: the crossover of the wall-normal ratio between its two anchors is fixed by the thermal length L and the mechanical length ℓ_p together, and tends to $\ell_p/2$ for $L \gg \ell_p$. The time-derivative correlation is expected to be small for streamwise-elongated wall structures, the frozen-pattern estimate $\mathcal{T} \simeq -U_c \overline{c_3 \partial_1 a_\theta}$ being a streamwise-gradient correlation; it is retained in (63), and $\frac{1}{2}\mathcal{B}_e$ as a whole is the $O(y^2)$ coefficient of $\varepsilon_{3\theta,h}/(\nu + \kappa)$, accessible from the profiles (§4.3).

3.6 Summary and remarks

Table 1 collects the wall limits. Three remarks. (i) *Evaluation*. All quantities are evaluable from

	Dirichlet ($a_\theta = 0$)		Fluctuating temperature ($a_\theta \neq 0$)	
	$i = 1$	$i = 3$	$i = 1$	$i = 3$
$\mathcal{R}_{i\theta,h}^u$	$\frac{2Pr}{1+Pr}$	$\frac{2Pr}{1+Pr}$	$\propto (y^+)^{-1}$	$-\frac{2Pr}{1+Pr}$
$\mathcal{R}_{i\theta,h}^\theta$	$\frac{2}{1+Pr}$	$\frac{2}{1+Pr}$	$\propto y^+ \rightarrow 0$	$\propto (y^+)^2 \rightarrow 0$
wall split of $\varepsilon_{i\theta}$	$\frac{1}{2} : \frac{1}{2}$ (value)	$\frac{1}{4} : \frac{3}{4}$ (slope)	finite parts, $\varepsilon_{1\theta} _w \neq 0$ ¹	finite parts cancel, $\varepsilon_{3\theta} _w = 0$
$\lambda_{i\theta}^2$ at the wall	$2C_{u\theta} y^2$	$2C_{u\theta} y^2$	$\propto y$ (either sign)	$-2C_{u\theta} y^2$

Table 1: Wall behavior of the homogeneous time-scale ratios (15) for the two flux components and the two classes of thermal wall condition. The fluctuating class comprises Neumann, Robin ($y^+ \ll L^+$) and conjugate walls; amplitudes refer to (56).

tabulated DNS statistics without recourse to the raw fields: $\varepsilon_{i\theta,ih}$ from the second derivative of the flux profile, and the cross-diffusion as the difference between the tabulated molecular-diffusion term and $\frac{1}{2}(\nu + \kappa)\Delta_x \overline{u_i \theta}$. In contrast to the variance case, the tabulated molecular diffusion cannot replace the profile curvature: the budget databases used in §4 store the exact form $\partial_j(\nu S_j + \kappa T_j)$ of (3), which differs from $\frac{1}{2}(\nu + \kappa)\Delta_x \overline{u_i \theta}$ by the cross-diffusion, and at $Pr = 0.01$ the $(\nu - \kappa) \approx -\kappa$ contribution is of leading order. The convention of each database is verified through the wall identities before use. (ii) *Signs*. The anchors are independent of the sign of the mean heat flux, since the wall correlations cancel; the sign structure of Table 1 is intrinsic. (iii) *Alternative grouping*. Moving the cross-diffusion into the inhomogeneous part, $\tilde{\varepsilon}_{i\theta,ih} \equiv \frac{1}{4}(\nu + \kappa)\Delta_x \overline{u_i \theta} + \frac{1}{2}(\nu - \kappa)\partial_j A_j$, absorbs the entire molecular remainder into $\frac{1}{4}(\nu + \kappa)\Delta_x \overline{u_i \theta}$ but renders the wall results Pr -dependent: the Dirichlet wall-normal fractions become $(3 - Pr)/[4(1 + Pr)]$ and $(5Pr + 1)/[4(1 + Pr)]$, and the fluctuating-wall wall-normal anchor $2Pr/(1 - 3Pr)$, with spurious sign changes at $Pr = 3$ and $Pr = \frac{1}{3}$. The natural split (11) is retained throughout: it is the exact two-point identity for $\varepsilon_{i\theta}$ itself, and its wall results are universal.

4 Assessment against DNS

4.1 Databases and conventions

Four DNS databases from independent codes are used. (i) The finite-difference channel-flow simulations of Abe, Kawamura, Antonia and Matsuo [25, 37, 38] at $Re_\tau = 180, 395$ and 640 with $Pr = 0.71$ and 0.025 , whose thermal forcing is the mixed-type condition of Kasagi et al. [5]: the condition seen by the fluctuating field is Dirichlet. The databases include the budget of $u_3 \theta$ at both Prandtl numbers and the budget of $\overline{u_1 \theta}$ at $Pr = 0.025$ only, so the streamwise results at $Pr = 0.71$ are drawn from the other three databases; the first grid point lies at $y^+ \approx 0.15$ – 0.2 , and $\overline{u_3 \theta}$ is stored on a staggered grid whose first point is at $y^+ \approx 0.3$ – 0.4 . (ii) The incompact3d channel-flow simulations of Flageul et al. [6] at $Re_\tau \simeq 150$, $Pr = 0.71$, with tabulated heat-flux budgets under Dirichlet, Neumann, Robin and three-dimensional conjugate conditions; the wall point is stored and the near-wall grid spacing is $\Delta y^+ \approx 0.5$. (iii) The pseudo-spectral liquid-sodium channel simulations of Tiselj and Cizelj [26] at $Pr = 0.01$, $Re_\tau = 180, 395$ and 590 , with non-fluctuating (Dirichlet for the fluctuation) and fluctuating (imposed-flux) conditions, full budgets and the wall point stored, the first grid point lying at $y^+ = 0.03$ – 0.05 . (iv) The spectral-element pipe-flow DNS of Straub et al. [27] at $Re_b = 5300$ ($Re_\tau \simeq 181$), $Pr = 0.71$ and 0.025 , under isothermal, mixed-type and isoflux conditions, with heat-flux budgets tabulated in the accompanying database [39], the wall point stored and the first grid point at $y^+ = 0.07$.

The isothermal and mixed-type conditions are both Dirichlet for the fluctuating field, so the pipe data provide the anchored class in two variants and the fluctuating class (isoflux) at both Prandtl numbers, in an independent geometry and code family. The well-resolved large-eddy simulations of that database at higher Reynolds numbers are not used: the dissipations are gradient statistics, and only fully resolved fields enter the assessment.

Conventions are fixed, as in Otić [24], by the exact wall identities rather than assumed. The mechanical and temperature-variance fields, k , ε_h , $\overline{\theta^2}$ and $\varepsilon_{\theta,h}$, are taken from the processing of Otić [24], extended to the pipe with the cylindrical Laplacian; the mechanical identity $\varepsilon_h|_w = \frac{1}{2}\varepsilon|_w$ holds to 0.2–5% in every case. The inhomogeneous part $\varepsilon_{i\theta,ih} = \frac{1}{4}(\nu + \kappa)\Delta_x \overline{u_i\theta}$ is evaluated from the curvature of the tabulated flux profiles by local polynomial differentiation on the native grids, with the wall values from asymptotic fits of the form (27) and (36) over $y^+ < 1.5$ ($Pr = 0.71$), $y^+ < 2.5$ ($Pr \leq 0.025$) or $y^+ < 3$ (incompact3d); the tabulated molecular-diffusion columns are verified to be the exact form $\partial_j(\nu S_j + \kappa T_j)$ through the wall balances $M_{1\theta}|_w = 0$ at Dirichlet walls and $M_{3\theta}|_w = \Delta_x \overline{u_3\theta}|_w$ at fluctuating walls (remark (i) of §3.6), and the sign convention of each dissipation column is fixed by budget closure. Budget-closure residuals, relative to the peak dissipation of the respective budget, are below 0.2% in the spectral channel data, below 2.5% in the streamwise budgets of the other databases, and reach 7–12% in the wall-normal budgets of the pipe and of the finite-difference channel at $Pr = 0.71$, whose dissipation is small against the pressure terms; the data are used as provided. The wall identities of §3, the equipartition (43), the 1:3 slope ratio (44) and the finite-part cancellation (42), serve as acceptance tests: they are satisfied to 0.1–1.5% by the two databases that resolve the wall, the spectral channel at $Pr = 0.01$ and the spectral-element pipe, to 1% for the streamwise and 1–2.5% for the wall-normal split by the finite-difference channel at $Pr = 0.025$, and to 1% for the streamwise split by the incompact3d data, whose near-wall spacing does not resolve the y^3 behavior of $\overline{u_3\theta}$; the wall-normal quantities of that database and of the finite-difference channel at $Pr = 0.71$ are therefore reported only where the profiles are resolved, $y^+ \gtrsim 1$. In the pipe the curvature terms of the vector Laplacian are $O(y^+/Re_\tau)$ at the wall and do not affect the anchors, so the pipe data test the geometry independence of Table 1.

4.2 Dirichlet-class walls

Figure 1 shows the decomposition for both components at the two ends of the Prandtl-number range, and Table 2 collects the wall values of the wall-resolved databases.

case	Pr	$\varepsilon_{1\theta,ih}/\varepsilon_{1\theta} _w$	$\varepsilon_{3\theta,ih} : \varepsilon_{3\theta}$ slope	$\mathcal{R}_{1\theta,h}^u _w$	$\mathcal{R}_{3\theta,h}^u _w$	$\mathcal{R}_{1\theta,h}^\theta _w$	$\mathcal{R}_{3\theta,h}^\theta _w$	$\mathcal{R}_{3\theta,h}^u _w$ (fluct.)
channel 180	0.01	0.499	0.752	0.0199	0.0201	1.97	2.00	−0.0199
channel 395	0.01	0.498	0.757	0.0200	0.0210	1.96	2.06	−0.0202
channel 590	0.01	0.498	0.746	0.0200	0.0197	1.97	1.94	−0.0200
analytical	0.01	0.5	0.75	0.0198	0.0198	1.980	1.980	−0.0198
pipe	0.025	0.499	0.750	0.0494	0.0497	1.94	1.95	−0.0497
analytical	0.025	0.5	0.75	0.0488	0.0488	1.951	1.951	−0.0488
pipe	0.71	0.500	0.761	0.833	0.886	1.166	1.24	−0.834
analytical	0.71	0.5	0.75	0.830	0.830	1.170	1.170	−0.830

Table 2: Wall values from the wall-resolved databases: the splits of (43) and (44), the anchors of (47) at Dirichlet-class walls, and the anchor (52) at walls with fluctuating temperature, with the analytical values. Channel: the pseudo-spectral data of Tiselj and Cizelj [26]; pipe: the spectral-element data of Straub et al. [27]; the mixed-type pipe cases coincide with the isothermal ones to within 1% and are omitted. Wall values are obtained from the asymptotic fits described in §4.1; the finite-difference channel at $Pr = 0.025$ gives $\mathcal{R}_{1\theta,h}^u|_w = 0.048$ and $\mathcal{R}_{3\theta,h}^u|_w = 0.051$ – 0.053 at $Re_\tau = 180$ – 640 , and the incompact3d channel at $Pr = 0.71$ gives $\mathcal{R}_{1\theta,h}^u|_w = 0.843$.

At the wall the two parts of the streamwise dissipation are equal in every case, the measured

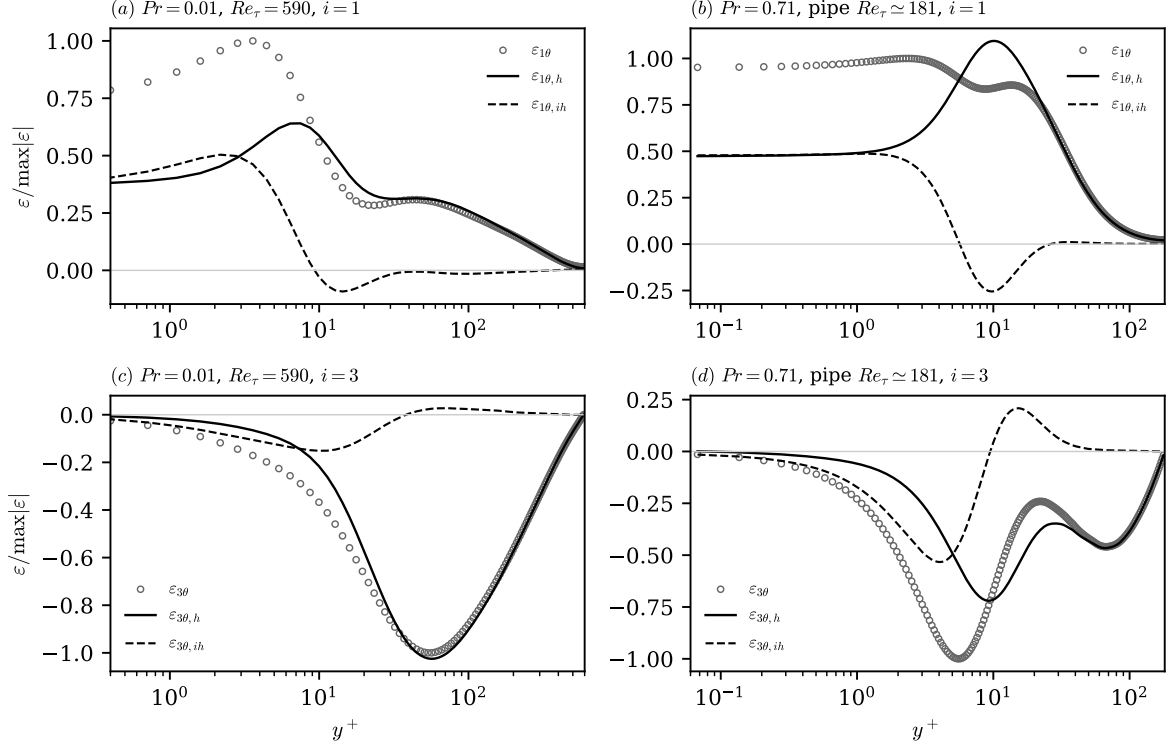


Figure 1: Decomposition (11) at Dirichlet-class walls, normalized by the maximum of the total: $\varepsilon_{i\theta}$ ($\circ$), $\varepsilon_{i\theta,h}$ (solid), $\varepsilon_{i\theta,ih}$ (dashed). (a,c) $Pr = 0.01$, $Re_\tau = 590$, non-fluctuating condition [26]; (b,d) $Pr = 0.71$, pipe, isothermal wall [27]. Top row: streamwise component, in which the two parts are equal at the wall; bottom row: wall-normal component, in which both parts vanish at the wall and their slopes are in the ratio 1:3.

inhomogeneous fractions being 0.496–0.500 across three codes and three Prandtl numbers, and the wall-normal slopes are in the ratio 1:3, the measured fractions being 0.746–0.761 in the two codes that resolve the wall. The two identities are independent of Pr , as derived, and they hold with the same accuracy for the finite-difference channel at $Pr = 0.025$ (0.497 and 0.759–0.769). The supplementary identity for the tabulated molecular diffusion of the wall-normal flux, whose wall slope must exceed that of the dissipation by the factor $(2\nu + \kappa)/(\nu + \kappa)$ because the exact form (3) weights the two gradient correlations differently, holds to 1% in the spectral channel data (1.013–1.022 against 1.010) and to 1–3% in the finite-difference channel (1.44–1.46 against 1.415 at $Pr = 0.71$, 1.04 against 1.024 at $Pr = 0.025$); in the pipe data the tabulated wall-normal molecular diffusion carries a small wall offset, 1.6% of its peak, and the slope ratio falls 7–9% below the analytical value. Away from the wall the inhomogeneous parts have the structure familiar from the variance analysis. The streamwise part remains equal to the homogeneous part through the conductive layer, changes sign at the inflection point of the $\overline{u_1\theta}$ profile, at $y^+ \approx 6$ for $Pr = 0.71$ and $y^+ \approx 8$ –9 for $Pr = 0.01$, and reaches a negative extremum in the buffer layer of 25% of the peak dissipation at $Pr = 0.71$ and 9–16% at $Pr = 0.01$, decreasing with the Reynolds number. The wall-normal part is three quarters of the total through the conductive layer and changes sign at $y^+ \approx 9$ at $Pr = 0.71$ (figure 1d) and at $y^+ \approx 38$ at $Pr = 0.01$, $Re_\tau = 590$ (figure 1c), beyond which it is small. At $Pr = 0.01$ the location of that sign change increases with the Reynolds number, from $y^+ \approx 26$ at $Re_\tau = 180$ to 38 at 590. The homogeneous part of the wall-normal dissipation does not change sign in any Dirichlet-class case and accounts for the entire $\varepsilon_{3\theta}$ beyond the buffer layer.

Figure 2 shows the two ratios. Both components are anchored at $2Pr/(1 + Pr)$: the wall-

resolved data give $\mathcal{R}_{1\theta,h}^u|_w$ within 0.4–1.5% and $\mathcal{R}_{3\theta,h}^u|_w$ within 0.5–7% of the analytical value at the three Prandtl numbers (Table 2), the larger uncertainty of the wall-normal anchor at $Pr = 0.71$ reflecting the thin layer over which the y^3 behavior of $\overline{u_3\theta}$ must be resolved. The thermal-side anchors follow to the same accuracy (1.94–1.97 and 1.166 against $2/(1 + Pr) = 1.980, 1.951$ and 1.170 for the streamwise component).

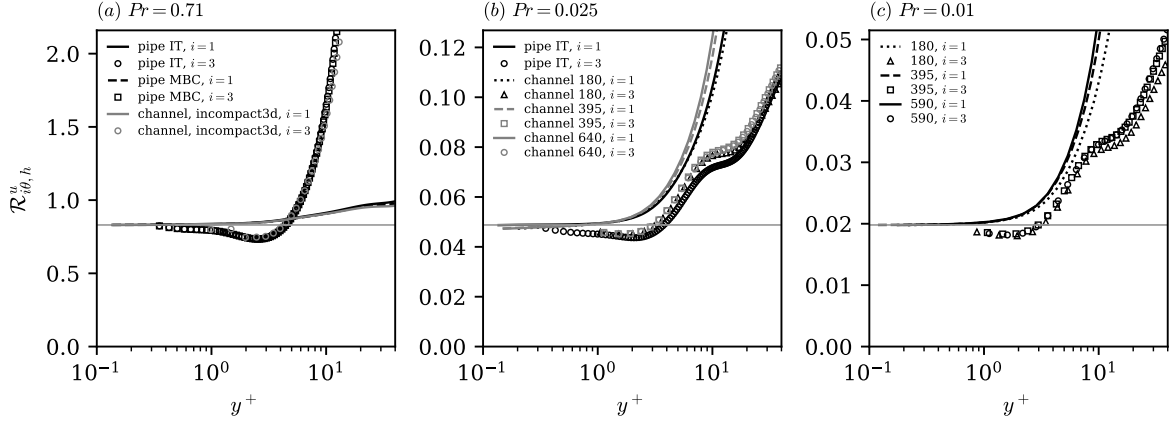


Figure 2: Homogeneous time-scale ratios $\mathcal{R}_{1\theta,h}^u$ (lines) and $\mathcal{R}_{3\theta,h}^u$ (symbols) at Dirichlet-class walls: (a) $Pr = 0.71$, pipe DNS [27] with isothermal and mixed-type conditions, and the incompact3d channel [6]; (b) $Pr = 0.025$, pipe DNS and the finite-difference channel [25] at $Re_\tau = 180, 395, 640$; (c) $Pr = 0.01$, channel [26] at $Re_\tau = 180, 395, 590$. Horizontal lines: the analytical anchor $2Pr/(1 + Pr)$. The wall-normal ratio is shown where the y^3 profile of $\overline{u_3\theta}$ is resolved.

The component independence of the anchors is thus confirmed, and with it the harmonic-mean composition (51), which is the content of the anchors. The approach to the anchors is different for the two components. The streamwise ratio departs from its anchor monotonically and slowly, by 5% at $y^+ = 5$ and 10–18% over the buffer layer at $Pr = 0.71$, and by 37–50% at $y^+ = 5$ at the low Prandtl numbers, where it then rises to several times the anchor because $\varepsilon_{1\theta,h}$ falls faster than the flux. The wall-normal ratio first falls below its anchor, to a minimum of 0.88–0.93 of the anchor at $y^+ \approx 1.5$ –2.5 in all cases and at all three Prandtl numbers, and then rises steeply beyond $y^+ \approx 3$. The location of the minimum scales in viscous units and is independent of Pr , as the crossing of R in Otić [24]; the pipe and the channels agree at matched y^+ to within a few percent for both components, the cross-code comparison at $Pr = 0.71$ between the pipe and the incompact3d channel showing no visible difference for $\mathcal{R}_{1\theta,h}^u$ over $y^+ < 40$. At the low Prandtl numbers the streamwise ratio in the buffer layer and beyond depends on the Reynolds number, rising at $y^+ = 30$ to 4.6, 6.5 and 7.8 times the anchor at $Re_\tau = 180, 395$ and 590, consistent with the Reynolds-number dependence of near-wall low- Pr temperature statistics noted by Tiselj and Cizelj [26].

4.3 Walls with fluctuating temperature

Figure 3(a,b) shows the finite-part cancellation (42) at the imposed-flux walls of the channel at $Pr = 0.01$ and of the pipe at $Pr = 0.71$. The total wall-normal dissipation vanishes at the wall, its wall value being below 0.2% of its maximum at the imposed-flux walls of the three codes and below 2% at the Robin and conjugate walls of the incompact3d data, while the two parts are finite and opposite: their wall values cancel to 0.03–0.6% of either part in the wall-resolved data. The finite value is $-\frac{1}{2}(\nu + \kappa)\overline{c_3a\theta}$, between 0.17 and 0.40 of the maximum of the total, and its molecular counterpart is confirmed independently by the total molecular term, which at

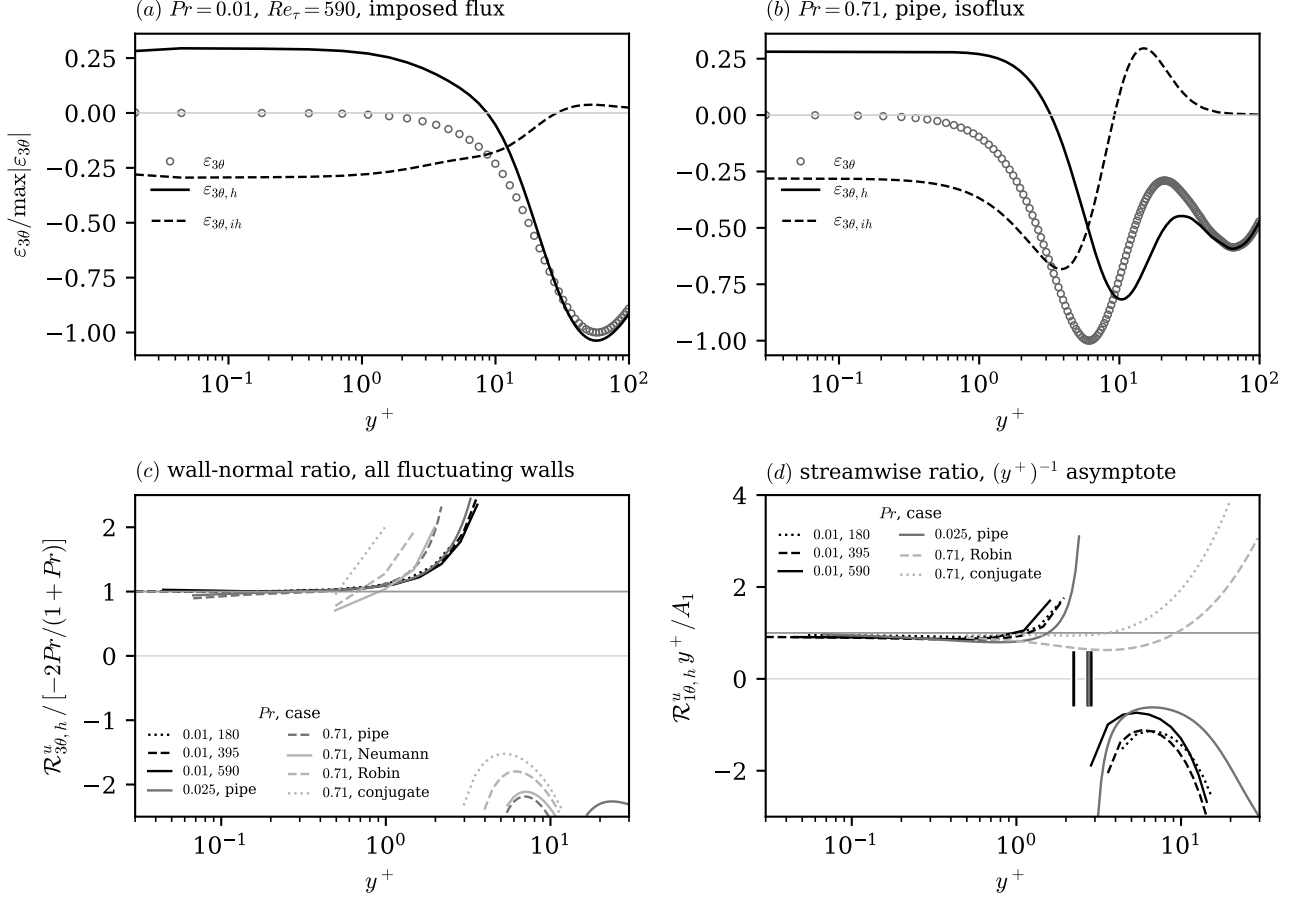


Figure 3: Walls with fluctuating temperature. (a,b) Finite-part cancellation (42): $\varepsilon_{3\theta}$ ($\circ$), $\varepsilon_{3\theta,h}$ (solid) and $\varepsilon_{3\theta,ih}$ (dashed), normalized by the maximum of the total, for (a) $Pr = 0.01$, $Re_\tau = 590$, imposed flux [26] and (b) $Pr = 0.71$, pipe, isoflux [27]. (c) Wall-normal ratio $\mathcal{R}_{3\theta,h}^u$ divided by the analytical anchor $-2Pr/(1+Pr)$ for all eight fluctuating-wall cases: imposed flux at $Pr = 0.01$ (three Reynolds numbers), isoflux pipe at $Pr = 0.025$ and 0.71 , and the Neumann, Robin and conjugate walls of Flageul et al. [6] at $Pr = 0.71$. (d) Streamwise ratio in the form $\mathcal{R}_{1\theta,h}^u y^+ / A_1$ with A_1 the parameter-free amplitude (56); the vertical bars on the zero line mark the sign change of $\varepsilon_{1\theta,h}$, the pole of the ratio.

the wall equals the curvature of the flux profile, $M_{3\theta}|_w = 2\nu \overline{a_\theta c_3}$, to within 1–7% in the channel and 5% in the pipe at $Pr = 0.71$; in the pipe at $Pr = 0.025$ the wall curvature of the flux is an order of magnitude smaller and the comparison is limited by noise to 40%. The correlation $\overline{a_\theta c_3}$ of the wall temperature with the wall-parallel divergence of the wall shear is therefore a robust, finite quantity at every wall with fluctuating temperature, and by (24) it measures the wall-parallel advection of the wall-temperature field by the fluctuating shear.

The negative anchor is shown in figure 3(c) for all eight fluctuating-wall cases, the ratio being divided by the analytical value so that the prediction is the common wall limit unity. The wall-resolved data give $\mathcal{R}_{3\theta,h}^u|_w = -0.0199$, -0.0202 and -0.0200 at $Pr = 0.01$, -0.0497 and -0.834 in the pipe at $Pr = 0.025$ and 0.71 , against -0.0198 , -0.0488 and -0.830 (Table 2); the incompact3d Neumann, Robin and conjugate walls, with coarser wall resolution, give -0.86 , -0.81 and -0.72 . The anchor is thus independent of the geometry, of the Reynolds number, of the Prandtl number beyond the explicit factor $2Pr/(1+Pr)$, and of the particular fluctuating condition, as (52) states. The approach is likewise universal in viscous units: the normalized ratio is 1.01–1.03 at $y^+ = 0.5$, 1.09–1.12 at $y^+ = 1$ for the imposed-flux walls at all three

Prandtl numbers, and 1.35–1.43 at $y^+ = 2$ at $Pr \leq 0.025$ against 1.9 at $Pr = 0.71$, where the source layer is thinner; it diverges where $\varepsilon_{3\theta,h}$ changes sign. That sign change bounds the layer in which the homogeneous dissipation acts as a source of the wall-normal flux. Its location, $y^+ = 7.2, 8.2$ and 8.7 at $Pr = 0.01$ for $Re_\tau = 180, 395$ and 590 , 6.3 at $Pr = 0.025$ and 3.2 at $Pr = 0.71$ for the imposed-flux walls, grows as the Prandtl number decreases, the ordering that (63) attributes to the wall-parallel smoothness of the wall temperature, and with the Reynolds number at fixed Pr , through the flux coefficient $\mathcal{A}$; it lies at 3.2, 2.5 and 1.8 for the Neumann, Robin and conjugate walls of Flageul et al. [6], ordered as the wall variance. For the Robin wall the length is $L^+ = (\overline{\theta^2}_w / \overline{b_\theta^2})^{1/2} = 10.6$ from the wall statistics, so the sign change occurs at one quarter of the Robin length, inside the Neumann-like region of the flux expansion (58): the crossover of the wall-normal ratio between its two anchors is not located by L^+ alone but by the two lengths of (64), which with $\delta_s^+ = 2.5$ and $L^+ = 10.6$ give $\ell_p^+ = 4.0$, a viscous-scale length of the near-wall velocity field, as §3.5 requires. Beyond the sign change the wall-normal ratio is positive and of the order of its Dirichlet-class value: the source layer is a near-wall feature, and the outer flux is destroyed, not amplified, by the homogeneous dissipation at every wall. The thickness formula (63) is tested with the same fits. The coefficients $\mathcal{A}$, $\mathcal{A}_b$ and $\mathcal{P}$ of (61) are one-point wall quantities obtained from the y^2 and y^3 coefficients of the flux and the wall slope of the dissipation, and $\mathcal{B}_e$ is twice the $O(y^2)$ coefficient of $\varepsilon_{3\theta,h}/(\nu + \kappa)$, fitted over $y^+ < 1.5$ with the zeroth- and first-order coefficients held at their exact values $-\frac{1}{4}\mathcal{A}$ and $\frac{1}{4}\mathcal{A}_b - \frac{1}{2}\mathcal{P}$ (over $y^+ < 2$ on the coarser grid of Flageul et al. [6]); the spread over the fit ranges $y^+ < 1$ – 2 is the quoted uncertainty, and a free quadratic fit recovers the imposed zeroth coefficient to a few percent where the grid resolves $y^+ < 1$, and the first-order one, an order of magnitude smaller, in sign and magnitude, an independent check of (61). At $Pr = 0.71$ the root of (63) gives $\delta_s^+ = 3.2, 2.7$ and 1.9 for the Neumann, Robin and conjugate walls against the observed 3.2, 2.5 and 1.8, and 3.2 for the isoflux pipe against 3.2; at $Pr \leq 0.025$ it gives 4.5–5.6 against 6.3–8.7, below the observed values, because there δ_s lies outside the range in which the curvature length is linear (§4.5) and the truncation of (61) underestimates the source. The coefficient $\mathcal{B}_e^+$ decreases by two orders of magnitude as the Prandtl number falls, from 2.3×10^{-4} at $Pr = 0.71$ to 4.6×10^{-6} at $Pr = 0.025$ and 1.0×10^{-6} at $Pr = 0.01$, $Re_\tau = 180$, as the conductive smoothing of the wall temperature requires; at fixed $Pr = 0.01$ it does not fall further with the Reynolds number but rises, 1.0 – 1.6×10^{-6} from $Re_\tau = 180$ to 590 , more slowly than the flux coefficient $\mathcal{A}^+$, which nearly doubles over the same range: the growth of the layer with the Reynolds number enters (63) through the numerator, the outer scaling of the low-Prandtl thermal field.

The streamwise component is shown in figure 3(d) in the form $\mathcal{R}_{1\theta,h}^u y^+ / A_1$, with A_1 the parameter-free amplitude (56) evaluated from the wall slope of the flux and the wall value of $\varepsilon_{1\theta,h}$. At the imposed-flux walls of the channel at $Pr = 0.01$ and of the pipe at $Pr = 0.025$ the total streamwise dissipation vanishes at the wall, to 0.005–0.2% of its maximum, as §3.5 requires, and the data follow the asymptote: the normalized ratio is 0.91–0.97 at the first grid point ($y^+ = 0.03$ – 0.07) and 0.82–0.93 at $y^+ = 0.2$ – 0.5 , for four cases and two codes, with amplitudes $A_1 = -0.59, -0.69, -1.02$ and -2.43 . The amplitude is negative: at these walls $\varepsilon_{1\theta,h}|_w = -\frac{1}{2}(\nu + \kappa) \overline{c_1 \overline{a_\theta}}$ has the sign opposite to the flux, so the homogeneous dissipation is also a source of the streamwise flux, over a layer bounded by the sign change of $\varepsilon_{1\theta,h}$ at $y^+ = 2.8, 2.7$ and 2.2 ($Pr = 0.01, Re_\tau = 180, 395, 590$) and 2.8 ($Pr = 0.025$, pipe), approximately one third of the wall-normal source layer. At the Robin and conjugate walls at $Pr = 0.71$ the amplitude is positive, $A_1 = 9.1$ and 4.9 , because the Dirichlet-class correlation $\overline{b_1 \overline{b_\theta}}$ dominates $\varepsilon_{1\theta,h}|_w$ in (35), and the asymptote is again followed on the coarser grid, to 5% at the conjugate wall and to 10–25% at the Robin wall over $y^+ = 0.5$ – 1.5 , in this case without a sign change: at these walls the streamwise flux is destroyed from the wall outward. At the isoflux and Neumann walls at $Pr = 0.71$, in the pipe and in the incompact3d channel, the $(y^+)^{-1}$ regime is not resolved. The wall value of $\varepsilon_{1\theta,h}$ is there two orders of magnitude below its buffer-layer values, the curvature of the nearly linear $\overline{u_1 \overline{\theta}}$ profile, which sets $\varepsilon_{1\theta,ih}|_w$, is not determined by the data to better than

a factor of two, and the sign change of $\varepsilon_{1\theta,h}$ lies below the first grid point, at $y^+ \approx 0.06$ in the pipe. The wall expansions identify the reason. At a Neumann wall

$$\varepsilon_{1\theta,h} = -\frac{1}{2}(\nu + \kappa) \overline{c_1 a_\theta} + (\nu + \kappa) \left[\frac{1}{2} \overline{b_1 \gamma_\theta} + \overline{\nabla_\parallel b_1 \cdot \nabla_\parallel a_\theta} - \frac{3}{2} \overline{d_1 a_\theta} \right] y + O(y^2), \quad (65)$$

from (31) and (28) with $b_\theta = 0$, and the fluctuating energy equation at the wall, $\partial_t a_\theta = \kappa(\nabla_\parallel^2 a_\theta + 2\gamma_\theta)$, converts the first bracketed term into $\frac{1}{4} \overline{b_1 \partial_t a_\theta} / \kappa + \frac{1}{4} \overline{\nabla_\parallel b_1 \cdot \nabla_\parallel a_\theta}$. The wall value is carried by the wall-pressure-gradient correlation $\overline{c_1 a_\theta}$ alone, while the slope contains, with the largest weight, the correlation of the wall-parallel gradients of the wall shear and of the wall temperature, together with the time-derivative correlation and $\overline{d_1 a_\theta}$. Where the gradient correlation dominates the slope, as its weight and the structure of the wall-temperature field suggest, the thickness of the streamwise source layer is the ratio of the two correlations, a wall-parallel length of the wall-temperature field relative to the shear field: at $Pr = O(1)$ the wall temperature has structure on the viscous scales and the layer is a small fraction of a viscous unit, at $Pr \ll 1$ the wall temperature is smooth over those scales and the layer is 2–3 viscous units thick. The $(y^+)^{-1}$ asymptote is therefore an exact limit whose range of validity is set by the wall-parallel smoothness of the wall temperature, and it is a low-Prandtl-number result in practice, in the same sense in which the $(y^+)^{-2}$ asymptote of $\mathcal{R}_h$ is attained at $Pr = 0.01$ and only approached at $Pr = 0.71$ [24].

4.4 Cross-diffusion

Figure 4 shows the cross-diffusion term of (13), obtained as the difference between the tabulated exact molecular diffusion and the symmetric diffusion $\frac{1}{2}(\nu + \kappa)\Delta_x \overline{u_i \theta}$.

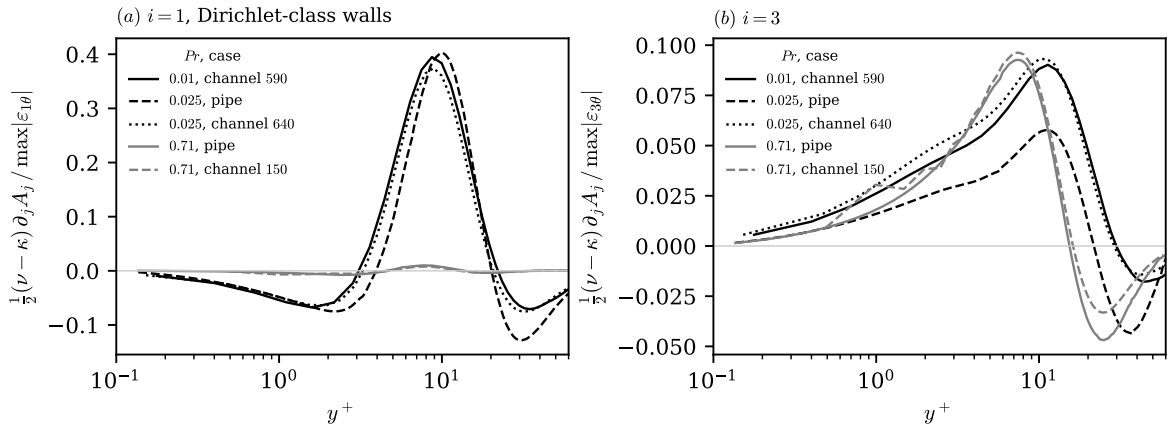


Figure 4: Cross-diffusion $\frac{1}{2}(\nu - \kappa) \partial_j A_j$ of (13), evaluated as the difference between the tabulated molecular diffusion and $\frac{1}{2}(\nu + \kappa)\Delta_x \overline{u_i \theta}$ and normalized by the maximum of the dissipation, at Dirichlet-class walls: (a) streamwise, (b) wall-normal component. Channel at $Pr = 0.01$ [26], pipe and channel at $Pr = 0.025$ [25, 27], pipe and channel at $Pr = 0.71$ [6, 27].

It vanishes at the wall for both components at Dirichlet walls, as (34) and (40) require, and it is confined to the conductive and buffer layers. Its magnitude depends on the Prandtl number as the prefactor $(\nu - \kappa)/(\nu + \kappa) = (Pr - 1)/(Pr + 1)$ suggests: for the streamwise component it is 1% of the peak dissipation at $Pr = 0.71$ and reaches 37–40% of it at $y^+ \approx 9$ –10 at $Pr = 0.025$ and 0.01, in three databases, the profiles at the two low Prandtl numbers collapsing in viscous units because the prefactor has saturated at -1 . At $y^+ \approx 5$ it amounts to 18–27% of the tabulated molecular diffusion at $Pr = 0.01$ and 11–15% at $Pr = 0.025$, so the frequently written symmetric form $\frac{1}{2}(\nu + \kappa)\Delta_x \overline{u_i \theta}$ misrepresents the molecular transport of the streamwise flux by that amount in the buffer layer of a liquid metal. For the wall-normal component the term is 6–10% of the

peak dissipation at all three Prandtl numbers, with its extremum at $y^+ \approx 7\text{--}11$, the smaller $(\nu - \kappa)$ prefactor at $Pr = 0.71$ being compensated by a larger antisymmetric correlation $A_3^{(3)}$. The cross-diffusion is thus the term through which the Prandtl number enters the molecular transport of the heat flux, and at liquid-metal Prandtl numbers it is of the same order as the symmetric diffusion in the layer where the flux is produced.

4.5 Composition of the heat-flux time scale

The composition of the heat-flux time scale from the mechanical and thermal ones is measured by the effective exponent $p_{\text{eff}}(y)$ of the power mean, $\tau_{1\theta,h} = M_{p_{\text{eff}}}(\tau_{u,h}, \tau_{\theta,h})$ with $M_p = [(\tau_{u,h}^p + \tau_{\theta,h}^p)/2]^{1/p}$, where $p = -1$, $p \rightarrow 0$ and $p = 1$ give the harmonic, geometric and arithmetic means, respectively. The harmonic-mean anchor (51) fixes $p_{\text{eff}} = -1$ on the wall. Its interior behavior, which the expansions do not determine, is shown in figure 5 for the streamwise component. The exponent is -1.0 to -1.1 across the conductive sublayer, $y^+ \leq 2$, at $Pr \leq 0.025$ in all codes, and -0.9 to -1.0 in the pipe at $Pr = 0.71$, where the two scales already differ by less than a factor of two. At $Pr = 0.71$ it is undetermined beyond $y^+ \approx 4$, because the two scales differ by only 30–40% there and all power means coincide at $\tau_{u,h} = \tau_{\theta,h}$: the blend is identifiable only at Prandtl numbers far from unity. At $Pr = 0.01$ and 0.025 , where $\tau_{u,h}$ exceeds $\tau_{\theta,h}$ by factors of 90 and 37 at $y^+ = 2$ and by 5–15 and 3–8 in the core, the exponent remains negative across the channel, between -0.7 and -2.5 for $5 \leq y^+ \leq 100$ in the channels and below -1.1 at $Pr = 0.01$, falling further towards the centerline and in the pipe: the heat-flux time scale remains at the harmonic mean or closer to the smaller of the two scales, well below the geometric mean, and the thermal control found at the wall persists.

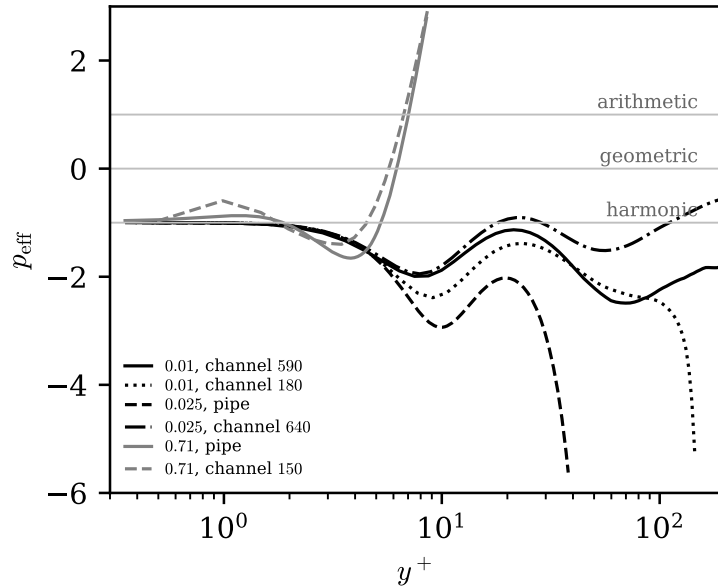


Figure 5: Effective power-mean exponent p_{eff} of the heat-flux time scale, $\tau_{1\theta,h} = M_p(\tau_{u,h}, \tau_{\theta,h})$, at Dirichlet-class walls; the harmonic ($p = -1$), geometric ($p \rightarrow 0$) and arithmetic ($p = 1$) values are marked, and the exponent is shown only where the two scales differ by more than 15%.

Figure 6 states this in the ratio $\mathcal{R}_{i\theta,h}^\theta = \tau_{i\theta,h}/\tau_{\theta,h}$ at $Pr = 0.01$. The streamwise ratio falls from its wall value $2/(1 + Pr) = 1.98$ to 1.5 at $y^+ = 5$ and remains between 1.3 and 1.8 over $10 \leq y^+ \leq 100$ at all three Reynolds numbers, while the mechanical-to-thermal ratio $\tau_{u,h}/\tau_{\theta,h} = 1/\mathcal{R}_h$ decreases from 90 to 7–14; the wall-normal ratio falls further, to 0.9–1.0

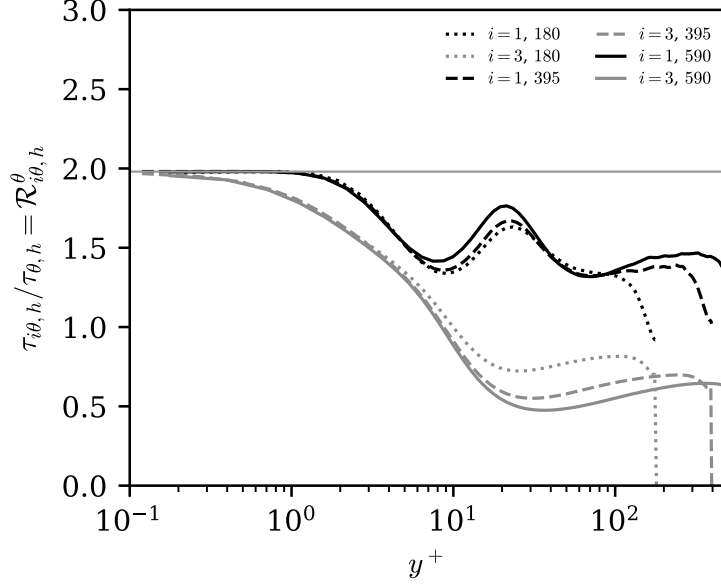


Figure 6: Ratio of the heat-flux to the thermal time scale, $\mathcal{R}_{\theta,h}^\theta$, at $Pr = 0.01$ for both components and three Reynolds numbers; horizontal line, the wall value $2/(1 + Pr)$.

at $y^+ = 10$ and 0.5–1.0 beyond, the time scale of the wall-normal flux being shorter than the thermal scale. The geometric mean $(\tau_{u,h}\tau_{\theta,h})^{1/2}$ overestimates the streamwise heat-flux time scale at the wall by the factor $(1 + Pr)/(2\sqrt{Pr})$, from (51) with $\tau_{\theta,h} \rightarrow Pr\tau_{u,h}$, which is $(2\sqrt{Pr})^{-1} = 5$ at $Pr = 0.01$, and by 3.6–4.2 at $y^+ = 10$ and 2–3 in the core, the wall-normal one by approximately twice these factors. At $Pr = 0.025$ the same structure is found, with $\mathcal{R}_{\theta,h}^\theta = 1.3$ –1.8 in the channel over $10 \leq y^+ \leq 100$ (rising to 2.0 near the centerline at $Re_\tau = 640$) and 0.4–1.6 in the pipe, the difference between the two geometries and between the two pipe conditions residing in $\mathcal{R}_h$. Figure 7 completes the assessment with the signed curvature length of §2.4, normalized so that its analytical wall limits are +1 at Dirichlet-class walls and –1 for the wall-normal component at fluctuating walls. Both limits are attained at $Pr = 0.01$ and 0.71; the normalized length remains within 10% of its wall value for $y^+ \lesssim 1$ and decays beyond, so the linear range of the curvature length is the conductive sublayer, as for the thermal microscale of Otić [24].

5 Constraints for heat-flux closures and data-driven representations

No closure is proposed here. The results are stated as constraints that any representation of $\varepsilon_{i\theta}$, a closure or a data-driven model, must satisfy, and as constraints on the data from which such representations are constructed. They follow from three exact elements: the transport equation (13), in which the inhomogeneous part is closed and the differential-diffusivity structure is isolated in one divergence term; the signed homogeneous part, a measure of small-scale anisotropy rather than a positive-definite destruction; and the wall identities of Table 1, free of wall statistics and explicit in the Prandtl number.

For closures the consequences are three. The inhomogeneous part, which near-wall closures represent through damping functions calibrated per thermal condition [14, 15], requires no model, and the weak-equilibrium truncation of algebraic heat-flux models [1, 40, 41], homogeneous by construction, is consistent with the homogeneous parts of the dissipations. The

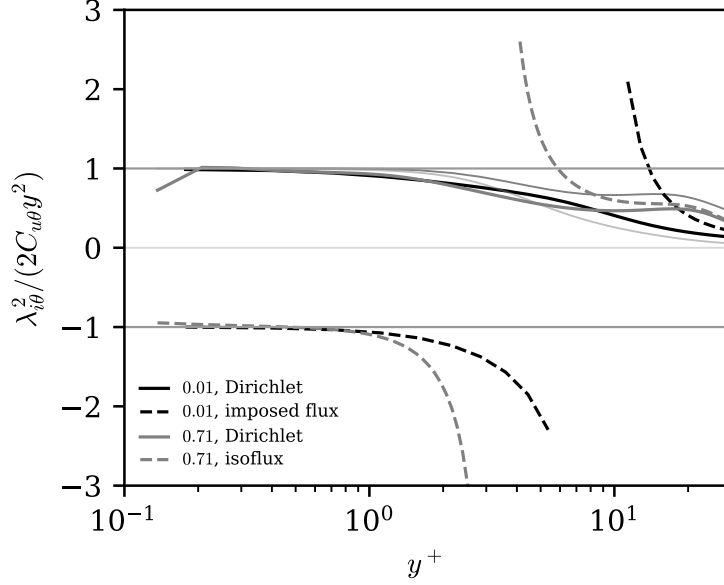


Figure 7: Signed curvature length $\lambda_{3\theta}^2/(2C_{u\theta}y^2)$ (thick lines) and $\lambda_{1\theta}^2/(2C_{u\theta}y^2)$ (thin lines, Dirichlet cases) at $Pr = 0.01$ (channel 590) and $Pr = 0.71$ (pipe), for the Dirichlet-class and the fluctuating wall; the analytical wall limits are $+1$ and, for $\lambda_{3\theta}$ at the fluctuating wall, -1 .

pressure scrambling $\Pi_{i\theta}$ remains a separate modeled term, so the constraints concern the dissipative constituent of any blended time scale. A destruction term $-C\overline{u_i\theta}/\tau$ with $C, \tau > 0$ fails identically at walls with fluctuating temperature, the configurations of liquid-metal systems [42], where (42) fixes the amplitude of the required sign change, and the cross-diffusion, absent from model forms calibrated on $Pr \approx 1$ fluids, is of leading order at $Pr \ll 1$ (§4.4). The composition of the time scales is derived rather than prescribed: at an isothermal wall the heat-flux time scale is the harmonic mean (51) of the thermal and mechanical scales, by the additive attenuation of one curvature length by both diffusivities (§2.4), whereas the geometric mean of Zeman and Lumley [7] is an interpolation without a mechanism. The harmonic mean is controlled by its smaller argument, so the exact limit $\tau_{i\theta,h}|_w \rightarrow 2\tau_{\theta,h}$ at $Pr \ll 1$ places on an analytical basis the a priori finding of Otić and Grötzbach [11] that the thermal time scale, not the geometric mean, is appropriate at low Prandtl number, and delimits it to that regime; the geometric mean overestimates the wall value by $(2\sqrt{Pr})^{-1}$, a factor of five at $Pr = 0.01$, and the thermal control persists across the channel (§4.5).

For data-driven representations of the scalar flux the same elements act at three levels, complementary to the invariance results of Pope [43] that constrain learned stress models [44, 45]: invariance constrains the admissible form, the present results constrain the values. (i) *The target*. The inhomogeneous part is closed in $\overline{u_i\theta}$ and the cross-diffusion is a divergence with known wall asymptotics (34) and (40), so the only unclosed dissipative quantity is $\varepsilon_{i\theta,h}$, equivalently the signed time scale (14); a representation trained on the total $\varepsilon_{i\theta}$ absorbs closed molecular transport, one half of the streamwise wall value, into a fitted term. The target changes sign at $y^+ \approx 2-9$ at every wall with fluctuating temperature (§4.3), so an architecture, output transformation or loss that enforces positivity is inconsistent with an exact feature of the field; the a priori test of figure 8, in which fits with identical inputs, training data and capacity differ only in the output constraint, shows the positivity-enforcing forms carrying their error inside the source layer while the signed fit reproduces the profile and the wall anchor. (ii) *Boundary values and asymptotes*. The anchors $2Pr/(1+Pr)$, $2/(1+Pr)$ and $-2Pr/(1+Pr)$, the equipartition and 1:3 splits, the linear curvature length (50) and the $(y^+)^{-1}$ asymptote (56) are exact, free of wall

statistics and explicit in Pr , and can be imposed as boundary conditions or penalty terms; they carry the most information where data are scarcest, at liquid-metal Prandtl numbers.

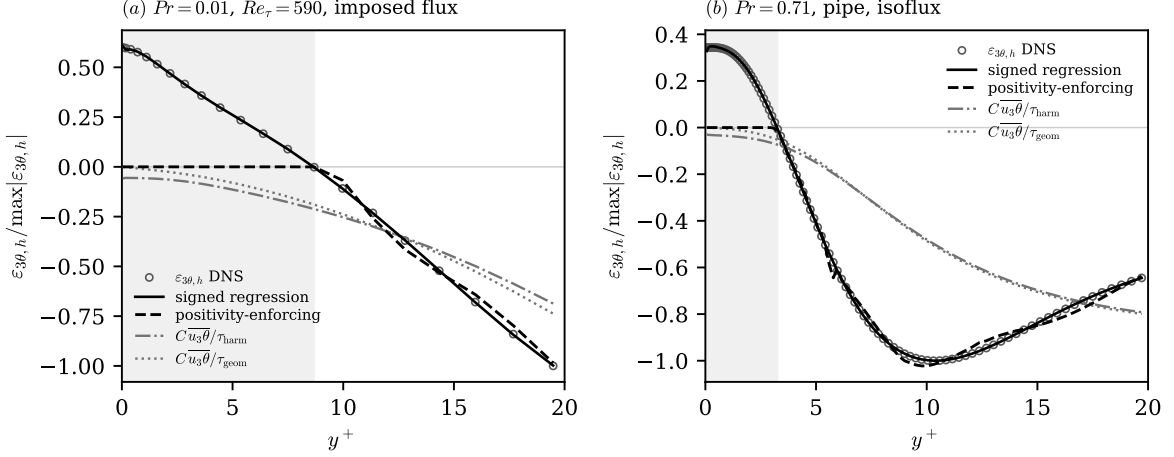


Figure 8: A priori test at walls with fluctuating temperature: $\varepsilon_{3\theta,h}$ from DNS (symbols) and fits with a positive-definite closure form $C \overline{u_3\theta}/\tau_m$, with τ_m the harmonic and the geometric mean of $\tau_{u,h}$ and $\tau_{\theta,h}$ and C by least squares over $y^+ < 50$, a positivity-enforcing regression and the same regression with signed output, identical inputs, training data and capacity. (a) $Pr = 0.01$, $Re_\tau = 590$, imposed flux; (b) $Pr = 0.71$, pipe, isoflux. Shaded: the source layer $0 < y^+ < \delta_s$. The positivity-enforcing fits carry their error inside the layer and vanish at the wall by construction; the signed fit reproduces the profile and the wall value, and with it the anchor $-2Pr/(1 + Pr)$, within a few percent.

The identity (16) removes one degree of freedom, representations of the heat-flux and variance time scales having to satisfy $\mathcal{R}_{i\theta,h}^u/\mathcal{R}_{i\theta,h}^\theta = \mathcal{R}_h$ pointwise, and the projection (17) provides the frame-invariant scalar for geometries without a distinguished wall frame. (iii) *The data.* The wall identities are acceptance tests for training data, in the role they have in §4.1: data that do not resolve the y^3 behavior of $u_3\theta$ fail the 1:3 split and the wall-normal anchor, filtered fields are excluded because the dissipations are gradient statistics, and the convention of the tabulated molecular diffusion must be verified through the wall balances, the difference between the exact and the symmetric form being of leading order at $Pr = 0.01$. The composition of the time scales is not identifiable from $Pr = O(1)$ data, where all power means coincide, and the streamwise ratio at $Pr \leq 0.025$ depends on the Reynolds number beyond the buffer layer (§4.2): a training set from which the composition is to be learned must span the Prandtl number, including $Pr \ll 1$, and the Reynolds number. Together with the exact equation and wall results for the temperature variance [24], these constraints form a unified set for the second moments of the scalar field; their assessment in buoyancy-driven configurations, for which Rayleigh–Bénard convection with its single flux component is the natural prototype, is left to future work.

6 Conclusion

The dissipation rate of the turbulent heat flux splits exactly, by the two-point correlation method, into a homogeneous part, the curvature of the velocity–temperature correlation at coincidence, and an inhomogeneous part proportional to the Laplacian of the flux, $\varepsilon_{i\theta,ih} = \frac{1}{4}(\nu + \kappa)\Delta_x u_i \theta$. The identity holds at any Prandtl number: the differential-diffusivity structure appears only in the molecular diffusion, which separates exactly into a symmetric part that absorbs the inhomogeneous dissipation and a cross-diffusion in divergence form, the mixed

centroid–separation derivative of the two-point correlation, vanishing at $Pr = 1$ and in homogeneous fields. The homogeneous part vanishes under local isotropy and measures the small-scale anisotropy of the velocity–scalar correlation; it is not sign-definite. The time scale formed from it defines two homogeneous ratios whose quotient equals the homogeneous thermal-to-mechanical ratio $\mathcal{R}_h$ identically. The transport equation formed with $\varepsilon_{i\theta,h}$ is exact, its inhomogeneous part being closed, and $\varepsilon_{i\theta}$ is thereby characterized consistently: a closed molecular-transport part, a signed measure of small-scale anisotropy and, at walls with fluctuating temperature, a source of the wall-normal flux.

The near-wall behavior follows analytically for the four thermal wall conditions. At an isothermal wall the two parts of the streamwise dissipation are equal, the slopes of the wall-normal parts are in the ratio 1:3, the signed curvature length is linear for both components, $\lambda_{i\theta} = \sqrt{2C_{u\theta}} y$, and both components share the anchors $\mathcal{R}_{i\theta,h}^u|_w = 2Pr/(1 + Pr)$ and $\mathcal{R}_{i\theta,h}^\theta|_w = 2/(1 + Pr)$: at the wall the heat-flux time scale is the harmonic mean of the thermal and mechanical time scales, reflecting the additive action of the two molecular diffusivities on a single curvature length, controlled at low Prandtl number by the thermal scale, $\tau_{i\theta,h}|_w \rightarrow 2\tau_{\theta,h}$. This is an exact statement about the hybrid time scale of heat-flux closures, immaterial at $Pr = O(1)$, where all power-mean blends coincide, and decisive at liquid-metal Prandtl numbers, where the geometric mean overestimates the heat-flux time scale by $(2\sqrt{Pr})^{-1}$: the composition of the time scales is thereby derived rather than prescribed. At walls with fluctuating temperature the wall-normal dissipation vanishes at the wall through the exact cancellation of finite homogeneous and inhomogeneous parts, the wall-normal ratio attains the universal anchor $-2Pr/(1 + Pr)$, the homogeneous dissipation acting as a source of the wall-normal flux and the curvature length reversing sign, $\lambda_{3\theta}^2 = -2C_{u\theta} y^2$, and the streamwise ratio diverges as $(y^+)^{-1}$ with a parameter-free amplitude. The source has a kinematic origin, the wall-normal velocity growing as y^2 against a wall temperature flat in y , and the thickness of the source layer is a wall-parallel microscale of the wall-temperature field weighted by the shear, which accounts for its growth as the Prandtl number decreases; the Robin wall connects the two regimes through a sign change of the homogeneous dissipation located by the Robin length and a mechanical length of the wall velocity field together.

The assessment against channel-flow and pipe-flow DNS from four independent codes, at $Pr = 0.71, 0.025$ and 0.01 and under Dirichlet, Neumann, Robin and conjugate conditions, confirms the analysis where the data resolve the wall and extends it where the expansions stop. The equipartition and the 1:3 slope ratio, the anchors of both components and both ratios, and the negative anchor are reproduced to 0.1–6% at $Pr \leq 0.025$ and to 0.4–7% at $Pr = 0.71$, in the channel and in the pipe, with the wall identities serving as acceptance tests for the data and fixing their conventions. The finite-part cancellation at walls with fluctuating temperature is exact to better than 1% of either part, and the correlation $\overline{a_\theta c_3}$ that carries it is a robust wall quantity confirmed independently by the total molecular term. The source layer of the wall-normal flux extends to $y^+ \approx 3$ at $Pr = 0.71$ and to $y^+ \approx 6$ –9 at liquid-metal Prandtl numbers; at imposed-flux walls the streamwise flux is amplified over approximately one third of that thickness, and the $(y^+)^{-1}$ asymptote with its parameter-free amplitude is followed to within 10% at the first grid points at $Pr \leq 0.025$, whereas at $Pr = 0.71$ the wall-parallel structure of the wall temperature confines it below the first grid point, a result the wall expansions explain. The cross-diffusion is negligible at $Pr = 0.71$ and 40% of the peak streamwise dissipation in the buffer layer at $Pr \leq 0.025$, the term through which the Prandtl number enters the molecular transport of the flux. The blend exponent is harmonic on the wall at every Prandtl number and remains negative across the channel at $Pr \leq 0.025$: the heat-flux time scale remains between 1.3 and 1.8 thermal time scales over the wall layer and the logarithmic region, and the wall-normal one at or below the thermal scale, while the mechanical scale is one to two orders of magnitude longer. The thermal control of the heat-flux destruction at low Prandtl number is thus a property of the whole layer, not of the wall alone, and the geometric-mean prescription

overestimates the destruction time scale by a factor of two to five at $Pr = 0.01$ throughout, and by two to three across the wall layer at $Pr = 0.025$.

For data-driven representations of the scalar flux the results fix the learning target, $\varepsilon_{i\theta,h}$ rather than $\varepsilon_{i\theta}$, together with its sign structure and its wall values, tie it to the variance time scale through the identity $\mathcal{R}_{i\theta,h}^u/\mathcal{R}_{i\theta,h}^\theta = \mathcal{R}_h$, and select the data: the wall identities are acceptance tests, and the composition of the time scale is identifiable only from data at Prandtl numbers far from unity. The assessment of these constraints in buoyancy-driven flows is left to future work.

Acknowledgments

This study was supported by the Scientific and Technological Research Council of Türkiye (TÜBİTAK) under project No. 123M535, and by the European Union co-funded partnership CONNECT-NM under Grant Agreement No. 101165375. The channel-flow data of Abe, Kawamura and co-workers were obtained from the JAXA DNS database; the data of Flageul and co-workers from the public repository of the Jožef Stefan Institute; the $Pr = 0.01$ database was kindly provided by I. Tiselj; and the pipe-flow database of Straub and co-workers was obtained from the KITopen/RADAR4KIT repository. The author reports no conflict of interest.

Data availability

The DNS databases analyzed here are available from the JAXA DNS database [25], from the repository <https://repo.ijs.si/CFLAG/incompact3d> [6], from the authors of Tiselj and Cizelj [26], and from the KITopen/RADAR4KIT repository, DOI 10.5445/IR/1000096346 [39]. The processing scripts and derived profiles are available from the author on reasonable request.

References

- [1] B. J. Daly and F. H. Harlow. Transport equations in turbulence. *Phys. Fluids*, 13: 2634–2649, 1970. doi: 10.1063/1.1692845.
- [2] J. O. Hinze. *Turbulence*. McGraw-Hill, 2 edition, 1975.
- [3] B. E. Launder. On the computation of convective heat transfer in complex turbulent flows. *Trans. ASME J. Heat Transfer*, 110:1112–1128, 1988. doi: 10.1115/1.3250614.
- [4] K. Hanjalić. One-point closure models for buoyancy-driven turbulent flows. *Annu. Rev. Fluid Mech.*, 34:321–347, 2002. doi: 10.1146/annurev.fluid.34.082801.161035.
- [5] N. Kasagi, Y. Tomita, and A. Kuroda. Direct numerical simulation of passive scalar field in a turbulent channel flow. *Trans. ASME J. Heat Transfer*, 114:598–606, 1992. doi: 10.1115/1.2911323.
- [6] C. Flageul, S. Benhamadouche, É. Lamballais, and D. Laurence. DNS of turbulent channel flow with conjugate heat transfer: effect of thermal boundary conditions on the second moments and budgets. *Int. J. Heat Fluid Flow*, 55:34–44, 2015. doi: 10.1016/j.ijheatfluidflow.2015.07.009.
- [7] O. Zeman and J. L. Lumley. Modeling buoyancy driven mixed layers. *J. Atmos. Sci.*, 33: 1974–1988, 1976. doi: 10.1175/1520-0469(1976)033<1974:MBDML>2.0.CO;2.
- [8] G. R. Newman, B. E. Launder, and J. L. Lumley. Modelling the behaviour of homogeneous scalar turbulence. *J. Fluid Mech.*, 111:217–232, 1981. doi: 10.1017/S002211208100236X.

- [9] Naoki Shikazono and Nobuhide Kasagi. Second-moment closure for turbulent scalar transport at various prandtl numbers. *International Journal of Heat and Mass Transfer*, 39(14): 2977–2987, 1996. ISSN 0017-9310. doi: 10.1016/0017-9310(95)00339-8.
- [10] Ronald MC So and Charles G Speziale. A review of turbulent heat transfer modeling. *Annual Review of Heat Transfer*, 10, 1999. doi: 10.1615/AnnualRevHeatTransfer.v10.70.
- [11] I. Otić and G. Grötzbach. Turbulent heat flux and temperature variance dissipation rate in natural convection in lead-bismuth. *Nuclear Science and Engineering*, 155(3):489–496, 2007. doi: 10.13182/NSE07-A2679.
- [12] M. Ziefuss, N. Karimi, and A. Mehdizadeh. Towards identification of a reliable framework to predict the thermal field in turbulent wall-bounded shear flows. *International Journal of Heat and Mass Transfer*, 180:121752, 2021. ISSN 0017-9310. doi: 10.1016/j.ijheatmasstransfer.2021.121752.
- [13] I. Otić, G. Grötzbach, and M. Wörner. Analysis and modelling of the temperature variance equation in turbulent natural convection for low-Prandtl-number fluids. *J. Fluid Mech.*, 525:237–261, 2005. doi: 10.1017/S0022112004002733.
- [14] Y. G. Lai and R. M. C. So. Near-wall modeling of turbulent heat fluxes. *Int. J. Heat Mass Transfer*, 33:1429–1440, 1990. doi: 10.1016/0017-9310(90)90040-2.
- [15] Y. Nagano and C. Kim. A two-equation model for heat transport in wall turbulent shear flows. *Trans. ASME J. Heat Transfer*, 110:583–589, 1988.
- [16] T. P. Sommer, R. M. C. So, and Y. G. Lai. A near-wall two-equation model for turbulent heat fluxes. *Int. J. Heat Mass Transfer*, 35:3375–3387, 1992. doi: 10.1016/0017-9310(92)90224-G.
- [17] F. Dehoux, S. Benhamadouche, and R. Manceau. An elliptic blending differential flux model for natural, mixed and forced convection. *Int. J. Heat Fluid Flow*, 63:190–204, 2017. doi: 10.1016/j.ijheatfluidflow.2016.09.003.
- [18] G. Mangeon, S. Benhamadouche, J.-F. Wald, and R. Manceau. Extension to various thermal boundary conditions of the elliptic blending model for the turbulent heat flux and the temperature variance. *J. Fluid Mech.*, 905:A1, 2020. doi: 10.1017/jfm.2020.683.
- [19] B. A. Kolovandin and I. A. Vatutin. Statistical transfer theory in non-homogeneous turbulence. *Int. J. Heat Mass Transfer*, 15:2371–2383, 1972. doi: 10.1016/0017-9310(72)90133-0.
- [20] J. Jovanović, Q.-Y. Ye, and F. Durst. Statistical interpretation of the turbulent dissipation rate in wall-bounded flows. *J. Fluid Mech.*, 293:321–347, 1995. doi: 10.1017/S002211209500173X.
- [21] P. Y. Chou. On velocity correlations and the solutions of the equations of turbulent fluctuation. *Quart. Appl. Math.*, 3:38–54, 1945. doi: 10.1090/qam/11999.
- [22] J. M. Burgers. Some considerations on turbulent flow with shear. In *Proc. K. Ned. Akad. Wet., Ser. B*, volume 6, pages 125–147, 1953.
- [23] S. Jakirlić and K. Hanjalić. A new approach to modelling near-wall turbulence energy and stress dissipation. *J. Fluid Mech.*, 459:139–166, 2002. doi: 10.1017/S0022112002007905.
- [24] I. Otić. A homogeneous thermal-to-mechanical time-scale ratio and its near-wall behavior. *KITopen preprint*, 2026. doi: 10.5445/IR/1000195654.

- [25] H. Abe, R. A. Antonia, and H. Kawamura. Correlation between small-scale velocity and scalar fluctuations in a turbulent channel flow. *J. Fluid Mech.*, 627:1–32, 2009. doi: 10.1017/S0022112008005569.
- [26] I. Tiselj and L. Cizelj. DNS of turbulent channel flow with conjugate heat transfer at Prandtl number 0.01. *Nucl. Eng. Des.*, 253:153–160, 2012. doi: 10.1016/j.nucengdes.2012.08.008.
- [27] Steffen Straub, Pourya Forooghi, Luca Marocco, Thomas Wetzel, Ricardo Vinuesa, Philipp Schlatter, and Bettina Frohnäpfel. The influence of thermal boundary conditions on turbulent forced convection pipe flow at two prandtl numbers. *International Journal of Heat and Mass Transfer*, 144:118601, 2019. ISSN 0017-9310. doi: 10.1016/j.ijheatmasstransfer.2019.118601.
- [28] A. S. Monin. On the symmetry of turbulence in the surface layer of air. *Izvestiya, Atmospheric and Oceanic Physics*, 1:45–54, 1965.
- [29] F. Harlow and C. Hirt. Generalized Transport Theory of Anisotropic Turbulence. Technical Report LA-4086, Los Alamos Scientific Laboratory, 1969.
- [30] M. Kholmyansky and A. Tsinober. Kolmogorov 4/5 law, nonlocality, and sweeping decorrelation hypothesis. *Phys. Fluids*, 20:041704, 2008. doi: 10.1063/1.2907201.
- [31] M. Kholmyansky, V. Sabelnikov, and A. Tsinober. Local versus nonlocal processes in turbulent flows, kinematic coupling and general stochastic processes. In M. Deville, T.-H. Lê, and P. Sagaut, editors, *Turbulence and Interactions*, pages 215–221. Springer, 2010. doi: 10.1007/978-3-642-14139-3_26.
- [32] G. I. Taylor. Statistical theory of turbulence. *Proc. R. Soc. Lond. A*, 151:421–444, 1935. doi: 10.1098/rspa.1935.0158.
- [33] H. Tennekes and J. L. Lumley. *A First Course in Turbulence*. MIT Press, 1972.
- [34] S. Jakirlić and J. Jovanović. On unified boundary conditions for improved predictions of near-wall turbulence. *J. Fluid Mech.*, 656:530–539, 2010. doi: 10.1017/S0022112010002442.
- [35] J. R. Ristorcelli. Passive scalar mixing: analytic study of time scale ratio, variance, and mix rate. *Phys. Fluids*, 18:075101, 2006. doi: 10.1063/1.2214704.
- [36] C. Flageul, S. Benhamadouche, É. Lamballais, and D. Laurence. On the discontinuity of the dissipation rate associated with the temperature variance at the fluid-solid interface for cases with conjugate heat transfer. *Int. J. Heat Mass Transfer*, 111:321–328, 2017. doi: 10.1016/j.ijheatmasstransfer.2017.04.005.
- [37] H. Kawamura, H. Abe, and Y. Matsuo. DNS of turbulent heat transfer in channel flow with respect to Reynolds and Prandtl number effects. *Int. J. Heat Fluid Flow*, 20:196–207, 1999. doi: 10.1016/S0142-727X(99)00014-4.
- [38] H. Abe, H. Kawamura, and Y. Matsuo. Surface heat-flux fluctuations in a turbulent channel flow up to $re_\tau = 1020$ with $pr = 0.025$ and 0.71 . *Int. J. Heat Fluid Flow*, 25:404–419, 2004. doi: 10.1016/j.ijheatfluidflow.2004.02.010.
- [39] S. Straub, P. Forooghi, L. Marocco, T. Wetzel, R. Vinuesa, P. Schlatter, and B. Frohnäpfel. Statistics on turbulent forced convection pipe flow with different thermal boundary conditions for two Prandtl numbers. KITopen/RADAR4KIT dataset, Karlsruhe Institute of Technology, 2019.

- [40] S. Kenjereš, S. B. Gunarjo, and K. Hanjalić. Contribution to elliptic relaxation modelling of turbulent natural and mixed convection. *Int. J. Heat Fluid Flow*, 26(4):569–586, 2005. doi: 10.1016/j.ijheatfluidflow.2005.03.007.
- [41] A. Shams, F. Roelofs, E. Baglietto, S. Lardeau, and S. Kenjereš. Assessment and calibration of an algebraic turbulent heat flux model for low-Prandtl fluids. *Int. J. Heat Mass Transfer*, 79:589–601, 2014. doi: 10.1016/j.ijheatmasstransfer.2014.08.018.
- [42] F. Roelofs, A. Shams, I. Otić, M. Böttcher, M. Duponcheel, Y. Bartosiewicz, D. Lakehal, E. Baglietto, S. Lardeau, and X. Cheng. Status and perspective of turbulence heat transfer modelling for the industrial application of liquid metal flows. *Nucl. Eng. Des.*, 290:99–106, 2015. doi: 10.1016/j.nucengdes.2014.11.006.
- [43] S. B. Pope. A more general effective-viscosity hypothesis. *J. Fluid Mech.*, 72:331–340, 1975. doi: 10.1017/S0022112075003382.
- [44] J. Ling, A. Kurzawski, and J. Templeton. Reynolds averaged turbulence modelling using deep neural networks with embedded invariance. *J. Fluid Mech.*, 807:155–166, 2016. doi: 10.1017/jfm.2016.615.
- [45] Deniz Günseren, Özgür Ertunç, Ismail Ari, Atakan Ataman Atik, Nitel Muhtaroglu, and Ivan Otić. Constructing a Reynolds stress model of decaying homogeneous isotropic turbulence using physics informed neural network. *J. Comput. Phys.*, 553:114725, 2026. doi: 10.1016/j.jcp.2026.114725.