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# On the integral simplicial volume of cyclic covers of mapping tori

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**Abstract**

In this paper, we investigate the asymptotic behavior of the integral simplicial volume of cyclic covers of manifolds that fiber over the circle with fiber given by an $n$-dimensional torus. By studying the integral filling volume—an invariant introduced by Frigerio and the first author—for the monodromy, we establish both lower and upper bounds for the limit of the integral simplicial volume of these covers, normalized by the degree of the covering. These bounds are expressed in terms of the action of the monodromy on the real homology of the fiber. As applications, we establish a close connection between the topological entropy and the integral filling volume of self-homeomorphisms of $n$-dimensional tori, we find new examples for which the $\Delta$-complexity and the integral simplicial volume are not equivalent, and we prove the nonvanishing of the filling volume for Anosov self-diffeomorphisms of infranilmanifolds.

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## 1 | INTRODUCTION

*Integral simplicial volume* is a homotopy invariant for oriented closed manifolds that measures their topological complexity in terms of integral singular chains. For an oriented closed $n$-dimensional manifold $M$, it is defined as the minimal weighted number of singular simplices appearing in an integral cycle representing the fundamental class for $M$:

$$\|M\|_{\mathbb{Z}} = \min\left\{ \sum_{i\in I} |a_i| \;\middle|\; \sum_{i\in I} a_i\sigma_i \text{ integral fundamental cycle for } M \right\}.$$

This invariant is the integral counterpart of the *real simplicial volume* (often simply called *simplicial volume*), which was introduced by Gromov in his influential paper [17]. The real simplicial volume enjoys many properties and can be studied via a dual theory known as *bounded cohomology*. In contrast, the integral simplicial volume satisfies fewer structural properties, and no analogous cohomological theory is currently available to assist in its study. This makes the integral simplicial volume a more challenging and elusive invariant: except for manifolds of dimension 1 and 2, where it is completely understood, its exact value is known only for a finite number of manifolds (up to homotopy equivalence) in each dimension [24]. Integral simplicial volume is often investigated via its stabilized version, the *stable integral simplicial volume*, defined as

$$\|M\|_{\mathbb{Z}}^{\infty} = \inf\left\{ \frac{\|N\|_{\mathbb{Z}}}{k} \;\middle|\; N \to M \text{ cover of degree } k \geqslant 1 \right\}.$$

On the contrary of integral simplicial volume, stable integral simplicial volume is stable under finite covers, meaning that if $N \to M$ is a cover of degree $k \geqslant 1$, then $\|N\|_{\mathbb{Z}}^{\infty} = k \cdot \|M\|_{\mathbb{Z}}^{\infty}$.

For any orientation-preserving self-homeomorphism $f : M \to M$ of an oriented closed manifold $M$, we consider the associated mapping torus

$$M_f := M \times [0,1]/\sim, \quad \text{where } (x,0) \sim (f(x),1).$$

In this paper, we investigate the asymptotic behavior of $\left\|M_{f^k}\right\|_{\mathbb{Z}}$ as $k \to \infty$, where $M_{f^k} \to M_f$ is a cyclic covering of degree $k$. With a slight abuse of terminology, we refer to these coverings as *the cyclic coverings* of $M_f$. Since only mapping tori are considered here, this terminology should not lead to any ambiguity.

Let us denote by $\mathbb{N}$ the set of positive integers, and for $n \in \mathbb{N}$ let $T^n = \mathbb{R}^n/\mathbb{Z}^n$ be the $n$-dimensional torus. The main result of this paper relates the growth of the integral simplicial volume of the cyclic covers of mapping tori associated to orientation-preserving self-homeomorphisms $f : T^n \to T^n$ to the action induced by $f$ on $H_1(T^n;\mathbb{R})$.

**Theorem 1.** *Fix $n \in \mathbb{N}$. There exists a constant $K_n > 0$ such that the following holds: if $f : T^n \to T^n$ is an orientation-preserving homeomorphism on the $n$-torus and $f_* : H_1(T^n;\mathbb{Z}) \to H_1(T^n;\mathbb{Z})$ denotes the induced map in homology, then*

$$\frac{1}{K_n}\log(\rho(f_*)) \leqslant \lim_{k\to+\infty} \frac{\left\|(T^n)_{f^k}\right\|_{\mathbb{Z}}}{k} \leqslant K_n \log(\rho(f_*)),$$

*where $\rho(f_*)$ denotes the spectral radius of the linear map $f_*$.*

See Section 2.4 for the definition of spectral radius.

To prove the upper bound of Theorem 1, we exploit an invariant of orientation-preserving self-homeomorphisms of oriented closed manifolds introduced by Frigerio and the first author [5]. This invariant, called *integral filling volume* and denoted by $\mathrm{FV}_{\mathbb{Z}}$, was originally defined by means of the action of $f : M \to M$ on fundamental cycles for $M$ (see Section 2.1) and only depends on the homotopy class of $f$. However, the two authors proved that

$$\mathrm{FV}_{\mathbb{Z}}(f) = \lim_{k\to\infty} \frac{\left\|M_{f^k}\right\|_{\mathbb{Z}}}{k}.$$

In particular, to prove the upper bound of Theorem 1 it is enough to furnish an upper bound on $\mathrm{FV}_{\mathbb{Z}}(f)$, where $f : T^n \to T^n$ is an orientation-preserving self-homeomorphism of the $n$-torus. This is done via an explicit construction in Section 4.

For the lower bound, we prove a more general result which can be also of interest for other manifolds (see, e.g., Corollary 7 below).

**Theorem 2.** *Let $n \in \mathbb{N}$. If $M$ is an oriented closed $n$-manifold and $f : M \to M$ is an orientation-preserving homeomorphism, then*

$$\mathrm{FV}_{\mathbb{Z}}(f) \geqslant \frac{2}{(n+1)(n+2)\log(n+2)} \sum_{|\lambda|>1} \log|\lambda|,$$

*where $\lambda$ varies among all the complex eigenvalues of the map $f_* : H_1(M;\mathbb{R}) \to H_1(M;\mathbb{R})$ induced by $f$ in homology.*

Theorem 1 completes the picture of what was already known for integral filling volume of 2-dimensional tori and extends it to all dimensions. Up to homotopy, orientation-preserving self-homeomorphisms of $T^2$ are either periodic, Dehn twists or Anosov maps, corresponding in homology to matrices in $\mathrm{SL}(2,\mathbb{Z})$, respectively, with trace less than 2, equal to 2 or bigger than 2. From the definition, it follows directly that periodic maps have vanishing integral filling volume. For Dehn twists, the integral filling volume was shown to vanish by Frigerio and the first author [6]. In the same paper, they showed that the integral filling volume for Anosov maps is positive and bounded from below by the logarithm of the spectral radius (up to a multiplicative constant). Thus, Theorem 1 extends this lower bound to all maps and dimensions, giving a complete description of $\mathrm{FV}_{\mathbb{Z}}$ on orientation-preserving self-homomorphisms of $T^n$ in terms of readily computable algebraic quantities.

One may ask whether it is possible to generalize Theorem 1 to other manifolds, but this is not the case in general: let us consider, for example, a hyperbolic surface $S_g$ with genus $g \geqslant 2$ and let $f : S_g \to S_g$ be a pseudo-Anosov homeomorphism. Then, the integral filling volume $\mathrm{FV}_{\mathbb{Z}}(f)$ is positive [6, Theorem 4], while there are many pseudo-Anosov homeomorphisms acting trivially in homology [13, Corollary 14.3], thus having $\log(\rho(f_*)) = 0$.

## Topological Entropy

Another invariant defined on self-homeomorphisms $f : M \to M$ of compact manifolds $M$ is the *topological entropy*, denoted by $h(f)$. This invariant was introduced by Adler, Konheim and

MacAndrew [1], and it measures the complexity of the actions of maps on coverings of manifolds. As the topological entropy of a self-homeomorphism $f : M \to M$ is not invariant under homotopy, it makes sense to consider the *minimal topological entropy*, defined on the homotopy class $[f]$ of the map $f$ by

$$h([f]) := \inf\{h(g) \mid [g] = [f]\}.$$

It is possible to state Theorem 1 in terms of minimal topological entropy of the monodromy.

**Corollary 3.** *Fix $n \in \mathbb{N}$. For any orientation-preserving self-homeomorphism $f : T^n \to T^n$ on the $n$-torus, it holds that*

$$\frac{1}{nK_n} h([f]) \leqslant \mathrm{FV}_{\mathbb{Z}}(f) \leqslant K_n h([f]),$$

*where $K_n$ is the constant of Theorem 1.*

*Proof.* It is a classical result that for a linear homeomorphism $f : T^n \to T^n$ one has

$$h(f) = \sum_{|\lambda|>1} \log|\lambda|,$$

where $\lambda$ ranges among all complex eigenvalues of the map $f_* : H_1(T^n;\mathbb{Z}) \to H_1(T^n;\mathbb{Z})$ induced in homology by $f$ (e.g., [8, Corollary 16]). The quantities $\sum_{|\lambda|>1} \log|\lambda|$ and $\rho(f_*)$ are related by the following basic inequalities:

$$\frac{1}{n} \sum_{|\lambda|>1} \log|\lambda| \leqslant \log(\rho(f_*)) \leqslant \sum_{|\lambda|>1} \log|\lambda|.$$

In particular, by Theorem 1 it holds that

$$\frac{1}{nK_n} h(f) \leqslant \mathrm{FV}_{\mathbb{Z}}(f) \leqslant K_n h(f).$$

Moreover, by a result of Manning, one has $h(g) \geqslant \log(\rho(g_*))$ for any homeomorphism $g : T^n \to T^n$ [27]. The statement then follows by combining these results and the fact that any orientation-preserving self-homeomorphism of the torus is homotopic to a linear map (see Section 2.4). □

To the best of the authors' knowledge, there are no results relating the topological entropy to the integral simplicial volume; in this sense, Corollary 3 provides the first result in this direction. Nevertheless, the simplicial volume is often studied in connection with the minimal volume entropy, an invariant of manifolds closely related to the topological entropy. In what follows, we aim to provide the reader with a brief overview of the mutual relationships among these invariants.

Let $(M, g)$ be a closed Riemannian manifold. One can consider the geodesic flow $\varphi_t$ defined on the unit sphere bundle $SM$ and define the *topological entropy* $h(M, g)$ of $(M, g)$ by taking the topological entropy of the flow for $t = 1$ [28, Section 3], that is

$$h(M, g) := h(\varphi_1).$$

To obtain an invariant of the manifold, one usually considers the *topological entropy of* $M$ [22, Section 2.4], denoted by $h(M)$ and defined by

$$h(M) := \inf\{h(M, g) \mid g \text{ Riemannian metric on } M \text{ with} \mathrm{Vol}(M, g) = 1\}.$$

Let $(\widetilde{M}, \widetilde{g})$ denote the (Riemannian) universal cover of a Riemannian manifold $(M, g)$. The *volume entropy* $\lambda(M, g)$ of $(M, g)$ is defined as the exponential growth rate of the volume of balls in $\widetilde{M}$ [22, Sections 2.3], that is,

$$\lambda(M, g) := \lim_{R\to+\infty} \frac{\log(\mathrm{Vol}_{\widetilde{g}}(B(p, R)))}{R},$$

where $B(p, R) \subseteq \widetilde{M}$ is the ball of radius $R$ and center $p \in \widetilde{M}$. This limit always exists and does not depend on $p$. As above, one defines the *minimal volume entropy* $\lambda(M)$ of $M$ by taking the infimum of $\lambda(M, g)$ normalized by volume, that is,

$$\lambda(M) := \inf\{\lambda(M, g) \mid g \text{ Riemannian metric on } M \text{ with} \mathrm{Vol}(M, g) = 1\}.$$

Manning proved that $\lambda(M, g) \leqslant h(M, g)$ for any closed oriented Riemannian $n$-manifold $(M, g)$ and thus $\lambda(M) \leqslant h(M)$ [28, Theorem 1], while Gromov showed that $C(n)\|M\| \leqslant \lambda(M)^n$, where $\|M\|$ is the *real simplicial volume* and $C(n)$ is a constant depending only on the dimension [17, Section 2.5]. Thus, the following inequalities hold:

$$C(n)\|M\| \leqslant \lambda(M)^n \leqslant h(M)^n.$$

Since then, the relationship between these invariants has attracted considerable interest, for example, [3, 9], also in the case of (4-dimensional) mapping tori [4]. Notice that for mapping tori with torus fiber and monodromy given by a self-homeomorphism of $T^n$ induced by a linear map their minimal topological entropy of the manifold vanishes, as these manifolds admit an $\mathcal{F}$-structure [16, Examples 19.8].

## Delta-complexity of mapping tori

Let $M$ be an oriented closed connected $n$-manifold. A *triangulation* of $M$ is a realization of $M$ as a $\Delta$-complex, that is, as a union of $n$-dimensional simplices with some facets identified in pairs along affine homeomorphisms. The $\Delta$-*complexity* $\Delta(M)$ is defined as the minimum number of $n$-simplices that appear in a triangulation of $M$.

The $\Delta$-complexity of mapping tori has been investigated as a stand-alone invariant by Lackenby and Purcell [25, 26] and by Frigerio and the first author in relation to integral simplicial volume [6, Theorem 1]. In particular, Frigerio and the first author showed that integral simplicial volume and $\Delta$-complexity are not bi-Lipschitz equivalent invariants on the set of cyclic covers of a mapping torus with a 2-torus as fiber and a Dehn twist as monodromy [6]. A consequence of Theorem 1 is that the same holds for mapping tori with an Anosov map as monodromy:

**Corollary 4.** *There exists a sequence $\{M_i\}$ of 3-dimensional mapping tori $M_i$ with a 2-torus as fiber and an Anosov map as monodromy such that*

$$\lim_{i\to\infty}\frac{\Delta(M_i)}{\|M_i\|_{\mathbb{Z}}}=+\infty.$$

*Proof.* For any $i\in\mathbb{N}$, we consider the matrix $A_i\in \mathrm{SL}(2,\mathbb{Z})$ given by

$$A_i=\begin{pmatrix} i+1 & i\\ 1 & 1\end{pmatrix},$$

and denote by $f_i : T^2\to T^2$ the map induced on the 2-dimensional torus $T^2=\mathbb{R}^2/\mathbb{Z}^2$ by $A_i$.

Recall that the group $\mathrm{PSL}(2,\mathbb{Z})=\mathrm{SL}(2,\mathbb{Z})/\{\pm\,\mathrm{Id}\}$ is isomorphic to the free product $\mathbb{Z}/2*\mathbb{Z}/3$, where the two factors are generated by $\overline{S}$ and $\overline{U}$ represented by the matrices [25, Theorem 1.4]

$$S=\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}\qquad U=\begin{pmatrix} 0 & -1\\ 1 & -1\end{pmatrix}.$$

We denote by $\overline{A}_i$ the element in $\mathrm{PSL}(2,\mathbb{Z})$ represented by the matrix $A_i$. As $\overline{A}_i=(\overline{U}^{-1}\overline{S})^i\overline{US}$ is a cyclically reduced alternating product in $\overline{S},\overline{U}$, it holds that $\ell(\overline{A}_i^j)=j(2i+2)$, where $\ell(\overline{A}_i)$ denotes the minimal length of a cyclically reduced word in the generators $\overline{S},\overline{U}$. Then, by the work of Lackenby and Purcell [25, Theorem 1.4], it holds that

$$\varkappa j(2i+2)\leqslant \Delta\left((T^2)_{f_i^j}\right)\leqslant j(i+1)+6,$$

where $\varkappa$ is a universal constant.

On the other hand, the spectral radius $\rho(A_i)$ of the matrix $A_i$ satisfies the inequalities

$$\log i<\rho(A_i)=\frac{1}{2}\left(i+2+\sqrt{i^2+4i}\right)<\log(i+2).$$

Therefore, by Theorem 1, for any $i$ there exists a $k_i\in\mathbb{N}$ such that

$$\frac{k_i}{K_2}\log(i)\leqslant\left\|(T^2)_{f_i^{k_i}}\right\|_{\mathbb{Z}}\leqslant k_iK_2\log(i+2).$$

The result then follows by setting $M_i:=(T^2)_{f_i^{k_i}}$. ☐

## Anosov diffeomorphisms on (infra)nilmanifolds

Let $M$ be a compact Riemannian $n$-manifold and $f : M\to M$ be a diffeomorphism. The tangent bundle $TM$ comes equipped with a Riemannian metric and the differential of $f$ induces a bundle map $Df : TM\to TM$.

**Definition 5** [30, Section 3]. We say that $f$ is *Anosov* if there exist $Df$-invariant continuous subbundles $E^s,E^u\subseteq TM$, constants $c>0$ and $\lambda\in(0,1)$ such that $TM=E^u\oplus E^s$ and for any $k\in\mathbb{N}$ it holds that

$$\left\|Df^k(v)\right\| \leqslant c\lambda^k \|v\|, \qquad \text{for any } v \in E^s,$$

$$\left\|Df^k(v)\right\| \geqslant c\lambda^{-k} \|v\|, \qquad \text{for any } v \in E^u.$$

The subbundles $E^s$ and $E^u$ are called *stable* and *unstable* bundles of $f$, respectively.

The property of being Anosov does not depend on the choice of the Riemannian metric on $M$.

An automorphism $f : T^n \to T^n$ induced by a linear map $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ with no eigenvalues on the unit circle $S^1 \subseteq \mathbb{C}$ is an Anosov diffeomorphism: indeed, the spaces $E^s, E^u$ are given by the generalized eigenspaces corresponding to eigenvalues $\lambda$ of $\widetilde{f}$ with $|\lambda| < 1$ and with $|\lambda| > 1$, respectively.

A more general class of manifolds admitting Anosov diffeomorphisms are *nilmanifolds*.

**Definition 6.** Let $G$ be a connected, simply connected Lie group. We say that $G$ is *nilpotent* if its Lie algebra $\mathfrak{g}$ is, that is if the lower central series

$$\mathfrak{g} \geqslant [\mathfrak{g}, \mathfrak{g}] \geqslant [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \geqslant [\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]] \geqslant \cdots$$

eventually gets to the zero Lie algebra. A *nilmanifold* is a manifold obtained as a quotient $G/\Gamma$ where $G$ is a connected, simply connected, nilpotent Lie group and $\Gamma \subseteq G$ is a discrete, finitely generated, cocompact lattice.

Connected, simply connected Lie groups admitting such a lattice are completely classified, and every finitely generated, nilpotent, torsion-free group is isomorphic to one of these lattices [30, Theorem 3.7]. On this class of manifolds, Anosov diffeomorphisms can be constructed algebraically, for example, [30, Section I.3].

An even more general class of manifolds is the one of *infranilmanifolds* [11]. It turns out that such manifolds are finitely covered by nilmanifolds [11, Section 2]. As in the case of nilmanifolds, it is possible to construct Anosov diffeomorphisms on infranilmanifolds algebraically [11, Sections 3, 5]. Let $M = G/\Gamma$ be a nilmanifold. Any simply connected, nilpotent Lie group being diffeomorphic to Euclidean space (via the exponential map), $M$ is aspherical and $\pi_1(M) \cong \Gamma$ is a finitely generated nilpotent group. If $M$ is an infranilmanifold it is also aspherical and $\pi_1(M)$ contains a nilpotent subgroup of finite index.

Let $f : M \to M$ be an orientation-preserving Anosov diffeomorphism on an infranilmanifold. By a result of Hirsch (which covers also the more general case of manifolds with virtually polycylic fundamental group) the map induced on homology $f_* : H_1(M;\mathbb{R}) \to H_1(M;\mathbb{R})$ does not admit eigenvalues that are roots of unity [20, Theorem 4]. This implies that there exists at least one eigenvalue $\lambda$ with $|\lambda| > 1$: indeed, if it were not the case, all eigenvalues would be on the unit circle $S^1 \subseteq \mathbb{C}$. For any eigenvalue $\lambda$, its Galois conjugates would still be eigenvalues, and thus be on the unit circle, and by Kronecker Theorem $\lambda$ would be a root of unity, for example, [19, Section 34, Lemma (a)]. It follows that $\rho(f_*) > 1$. The following statement is then an application of Theorem 2.

**Corollary 7.** *If $M$ is an infranilmanifold and $f : M \to M$ an orientation-preserving Anosov diffeomorphism, then* $\mathrm{FV}_{\mathbb{Z}}(f) > 0$.

Finitely generated nilpotent groups are residually finite and amenable. This means that fundamental groups of infranilmanifolds are amenable and residually finite. Thus, if $M$ is an infranilmanifold and $f : M \to M$ an Anosov diffeomorphism, the mapping torus $M_f$ is an aspherical manifold with $\pi_1(M_f) \cong \pi_1(M) \rtimes_{f_*} \mathbb{Z}$, which is also amenable and residually finite. By a result of Frigerio, Löh, Pagliantini and Sauer, the stable integral simplicial volume $\left\|M_f\right\|_{\mathbb{Z}}^{\infty}$ vanishes [12]. This gives a family of manifolds $M$ and maps $f : M \to M$ for which the stable integral simplicial volume $\left\|M_f\right\|_{\mathbb{Z}}^{\infty}$ vanishes while $\mathrm{FV}_{\mathbb{Z}}(f)$ does not, generalizing the examples already found by Frigerio and the first author [5, Theorem 1.12].

## Open problems and further questions

As previously mentioned, many maps induce the trivial map in homology while still exhibiting a positive integral filling volume. This suggests that the spectral radius considered in Theorem 1 may be insufficient to fully capture this phenomenon, and naturally leads to the question of whether there exists an invariant more closely aligned with the nonvanishing of the integral filling volume.

For 2-tori, the minimal topological entropy of a mapping class can serve as a meaningful invariant in this context: as shown in Corollary 3, for any self-homeomorphism $f : T^2 \to T^2$, the integral filling volume $\mathrm{FV}_{\mathbb{Z}}(f)$ can be bounded in terms of the minimal topological entropy of its homotopy class $h([f])$. In the case of a mapping class which admits an Anosov representative $f$, $h([f]) = \log(\rho(f_*)) = \log(\lambda([f]))$, where $\rho(f_*)$ denotes the spectral radius of the induced map $f_* : H_1(T^2;\mathbb{R}) \to H_1(T^2;\mathbb{R})$, and $\lambda(f)$ is the stretching factor of $f$ [13, Chapter 11].

For a mapping class represented by a pseudo-Anosov homeomorphisms $f : S_g \to S_g$ on surfaces of genus $g \geqslant 2$, the integral filling volume $\mathrm{FV}_{\mathbb{Z}}(f)$ is always positive, while $\log(\rho(f_*))$ may vanish. The minimal topological entropy, however, is always positive, and is given by $h([f]) = \log(\lambda(f))$, where $\lambda(f)$ again denotes the stretching factor [15, Section IV].

Moreover, in the proof of the upper bound in Theorem 1, we use the Gelfand formula for the spectral radius (see Proposition 2.9), which implies that for any matrix $A \in \mathrm{SL}(n,\mathbb{Z})$ and any vector $v \in \mathbb{R}^n$, the norm $\|A^k v\|$ grows like $\rho(A)^k$. A similar phenomenon occurs on hyperbolic surfaces: under iteration by a pseudo-Anosov homeomorphism, the lengths of simple closed curves grow like powers of the stretching factor [13, Theorem 14.23].

These analogies suggest that an upper bound for $\mathrm{FV}_{\mathbb{Z}}(f)$ in terms of a function of the stretching factor $\lambda(f)$ might also exist. Similarly to what Frigerio and the first author did for the *real* filling volume [5, Question 1.11], we ask the following question.

**Question 8.** Let $g > 1$ be a positive integer. Is there a constant $C_g$ such that for any closed oriented surface $S_g$ of genus $g$ and any pseudo-Anosov homeomorphism $f : S_g \to S_g$, it holds that

$$\frac{1}{C_g}\log(\lambda(f)) \leqslant \mathrm{FV}_{\mathbb{Z}}(f) \leqslant C_g \log(\lambda(f)),$$

where $\lambda(f)$ denotes the stretching factor of $f$?

At present, the only known lower bound for $\mathrm{FV}_{\mathbb{Z}}(f)$ comes from the hyperbolic volume of the associated mapping torus. However, no direct relation with $\log(\lambda(f))$ is known, since there exist sequences of pseudo-Anosov maps $f_n : S_g \to S_g$ such that $h(f_n) \to \infty$, while the volumes of the corresponding mapping tori remain bounded [23].

## Plan of the paper

In Section 2, we recall the main definitions and notational conventions used throughout the paper. The expert reader may safely skip this section, except for Sections 2.2 and 2.3, where some new notation is introduced. In Section 3, we establish the lower bound in Theorem 1, while the upper bound is proved in Section 4.

# 2 | PRELIMINARIES

## 2.1 | Integral filling volume and integral simplicial volume

Let $M$ be an oriented closed manifold. The $\ell^1$-norm on the space of singular chains $C_k(M;\mathbb{Z})$ is defined by

$$\|c\|_1 = \left\| \sum_{i\in I} a_i\sigma_i \right\|_1 = \sum_{i\in I} |a_i|,$$

where $c = \sum_{i\in I} a_i\sigma_i$ is an integral singular chain written as a reduced sum.

Via the $\ell^1$-norm, one can also define the *integral filling norm* on the space of singular boundaries $B_k(M;\mathbb{Z}) \subseteq C_k(M;\mathbb{Z})$ by setting, for $b \in B_k(M;\mathbb{Z})$,

$$\|b\|_{\mathrm{fill},\mathbb{Z}} = \min\{\|z\|_1 \mid z \in C_{k+1}(M;\mathbb{Z}),\ \partial z = b\}.$$

The $\ell^1$-norm being $\mathbb{Z}$-valued, for any $b \in B_k(M;\mathbb{Z})$ there exists a $z \in C_{k+1}(M;\mathbb{Z})$ such that $\partial z = b$ and $\|b\|_{\mathrm{fill},\mathbb{Z}} = \|z\|_1$.

Notation 2.1. In the following, whenever in a statement we write the integral filling norm $\|c\|_{\mathrm{fill},\mathbb{Z}}$ of a chain $c$, we are tacitly including the assertion that the chain $c$ is in fact a boundary.

If $n$ is the dimension of the manifold $M$, then the group $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$ is generated by the so-called *integral fundamental class*, denoted by $[M]_\mathbb{Z}$. A cycle $c \in Z_n(M;\mathbb{Z})$ representing the integral fundamental class is called an *integral fundamental cycle*.

Any continuous map $f : M \to N$ between manifolds induces a chain map $f_* : C_\bullet(M;\mathbb{Z}) \to C_\bullet(N;\mathbb{Z})$ on singular chains, which is norm nonincreasing for both the $\ell^1$-norm on chains and the filling norm on boundaries. If, moreover, $f : M \to M$ is an orientation-preserving homeomorphism and $z \in Z_n(M;\mathbb{Z})$ is an integral fundamental cycle, then for any $i \in \mathbb{N}$ the chain $f_*^i(z)$ is still an integral fundamental cycle, so that it makes sense to consider the filling norm of the boundary $z - f_*^i(z)$. We use this fact to define the integral filling volume of the map $f$.

**Definition 2.2** ([5, Definition 1.1]). Let $f : M \to M$ be an orientation-preserving homotopy equivalence and denote by $f_* : C_n(M;\mathbb{Z}) \to C_n(M;\mathbb{Z})$ the induced map on singular chains. Let $z \in$

$Z_n(M;\mathbb{Z})$ be an integral fundamental cycle. The *integral filling volume* of $f$ is defined as

$$\mathrm{FV}_{\mathbb{Z}}(f) := \lim_{i\to\infty} \frac{\|f_*^i(c) - c\|_{\mathrm{fill},\mathbb{Z}}}{i}.$$

The limit in this definition always exists, is independent of the choice of the integral fundamental cycle $z$, and depends only on the homotopy class of $f$ [6, Section 2].

The integral filling volume can be expressed in terms of integral simplicial volume, whose definition is as follows.

**Definition 2.3.** Let $M$ be an oriented closed $n$-manifold. The *integral simplicial volume* of $M$ is defined as

$$\|M\|_{\mathbb{Z}} := \min\{\|z\|_1 \mid [z] = [M]_{\mathbb{Z}} \in H_n(M;\mathbb{Z})\}.$$

Given an orientation-preserving homeomorphism $f : M \to M$, the *mapping torus* $M_f$ is the $(n+1)$-manifold given by the quotient

$$M_f = M \times [0,1]/\sim$$

with respect to the relation $(x,0) \sim (f(x),1)$ for any $x \in M$. In this case, the manifold $M$ is called *fiber* of the mapping torus and $f$ *monodromy*.

The role of the integral filling volume lies in its close relation to the integral simplicial volume of mapping tori.

**Theorem 2.4** ([6, Theorem 3]). *Let $M$ be an oriented closed manifold and $f : M \to M$ be an orientation-preserving homeomorphism, then*

$$\mathrm{FV}_{\mathbb{Z}}(f) = \lim_{k\to+\infty} \frac{\left\|M_{f^k}\right\|_{\mathbb{Z}}}{k}.$$

Notation 2.5. As in this work only integral coefficients appear, we will often omit the word “integral”; in particular, the terms “fundamental cycles” and “filling volume” will always refer to their integral version.

## 2.2 | Straight simplices on the torus

Fix $k \in \mathbb{N}$ and for $i = 0, \dots, k$ let $e_i \in \mathbb{R}^{k+1}$ be the $i$th vector in the canonical basis of $\mathbb{R}^{k+1}$. We denote by $\Delta^k \subset \mathbb{R}^{k+1}$ the convex hull (with respect to the Euclidean metric) of $e_0, \dots, e_k$, that is

$$\Delta^k = \left\{ \sum_{i=0}^k \lambda_i e_i \,\middle|\, 0 \leqslant \lambda_i \leqslant 1,\ \sum_{i=0}^k \lambda_i = 1 \right\} \subset \mathbb{R}^{k+1}.$$

Let us now consider another integer $n \in \mathbb{N}$ and let $p_0, \dots, p_k$ be points in $\mathbb{R}^n$. We define the *straight simplex with vertices* $p_0, \dots, p_k$ as the singular $k$-simplex in $\mathbb{R}^n$ given by the map

$$\begin{array}{rccc} \mathrm{str}_n(p_0, \dots, p_k) : & \Delta^k & \longrightarrow & \mathbb{R}^n \\ & \sum_{i=0}^k \lambda_i e_i & \longmapsto & \sum_{i=0}^k \lambda_i p_i, \end{array}$$

whose image is the (Euclidean) convex hull of the points $p_0, \dots, p_k$.

We denote by $T^n = \mathbb{R}^n/\mathbb{Z}^n$ the $n$-dimensional torus (or, simply, the $n$-torus) and by $\pi_n : \mathbb{R}^n \to T^n$ the quotient map. Any singular simplex in $\mathbb{R}^n$ projects to a simplex in $T^n$ via $\pi_n$. We are particularly interested in the projection of straight simplices, for which we introduce the following notation:

$$[p_0, \dots, p_k] := \pi_n(\mathrm{str}_n(p_0, \dots, p_k)) \in C_k(T^n; \mathbb{Z}).$$

*Remark* 2.6. As $T^n = \mathbb{R}^n/\mathbb{Z}^n$, we have that for every $v \in \mathbb{Z}^n$ the two symbols $[p_0, \dots, p_k]$ and $[p_0 + v, \dots, p_k + v]$ represent the same simplex in $C_k(T^n; \mathbb{Z})$.

## 2.3 | Prism operator on topological spaces

We recall the following construction, which allows us to see the product of a simplex and an interval as a singular chain. Let $I = [0, 1]$ and consider the space $\Delta^{k-1} \times I \subseteq \mathbb{R}^{k+1} = \mathbb{R}^k \times \mathbb{R}$. As before, denote by $e_0, \dots, e_k$ the canonical basis of $\mathbb{R}^{k+1}$. Then, the $k$-chain

$$p_k = \sum_{i=0}^{k-1} (-1)^i \mathrm{str}_{k+1}(e_0, \dots, e_{k-i-1}, e_{k-i-1} + e_k, \dots, e_{k-1} + e_k) \in C_k(\mathbb{R}^k \times \mathbb{R}; \mathbb{Z})$$

is supported exactly on the product $\Delta^{k-1} \times I$.

Let now $M, N$ be two manifolds and $f, g : M \to N$ be continuous maps that are homotopic via the homotopy $F : M \times [0, 1] \to N$ (i.e., $F(\cdot, 0) = f$ and $F(\cdot, 1) = g$). For any $k \in \mathbb{N}$, we define a *prism operator*

$$P_{k-1} : C_{k-1}(M; \mathbb{Z}) \to C_k(N; \mathbb{Z})$$

by sending any $(k-1)$-singular simplex $\sigma : \Delta^{k-1} \to M$ to the chain

$$P_{k-1}(\sigma) = (F \circ (\sigma \times \mathrm{Id}))_*(p_k),$$

and then extending the definition to the space $C_{k-1}(M; \mathbb{Z})$ by linearity.

These prism operators satisfy the following relation:

$$\partial_{k+1} \circ P_k - P_{k-1} \circ \partial_k = (-1)^k (g_* - f_*).$$

Moreover, $P_k$ has operator norm bounded from above by $k+1$, meaning that for any chain $c \in C_k(X; \mathbb{Z})$, it holds that $\|P_k(c)\|_1 \leqslant (k+1)\|c\|_1$.

In the following, if no confusion arises, we drop the index $k$ in the prism operator $P_k$.

*Remark* 2.7. The definition of prism operator that we gave slightly differs from what usually is found in the literature (e.g., [18]): our prism operator coincides with the classical one when $k$ is even, while it changes sign when $k$ is odd. In Section 4, we will see that this definition is more convenient for our purpose, as our operator $P_n$ applied to particular kinds of fundamental cycles of the $n$-torus (which will be called *parallelogram-like cycles*) yields fundamental cycles of the same kind of the $(n+1)$-torus.

## 2.4 | Matrices and self-homotopy equivalences of tori

Fix $n \in \mathbb{N}$. Any matrix $A \in \mathrm{SL}(n, \mathbb{Z})$ induces a linear map $\widetilde{f}_A : \mathbb{R}^n \to \mathbb{R}^n$, which descends to a homeomorphism $f_A : T^n \to T^n$ on the $n$-torus. In this context, we say that $f_A$ is the *map induced by $A$* and that $A$ is a *matrix associated to* the map $f_A$. Notice that if $(f_A)_* : H_1(T^n; \mathbb{Z}) \to H_1(T^n; \mathbb{Z})$ denotes the map induced by $f_A$ on the first homology group of $T^n$, then the matrix $A$ is the one representing $(f_A)_*$ with respect to the basis of $H_1(T^n; \mathbb{Z}) \cong \mathbb{Z}^n$ consisting of the homology classes represented by the 1-cycles $[\mathbf{0}, e_1], \dots, [\mathbf{0}, e_n]$ (where $\mathbf{0}$ denotes the zero vector of $\mathbb{R}^n$).

Since $T^n$ is a $K(\mathbb{Z}^n, 1)$-space, any automorphism in $\mathrm{Aut}(\mathbb{Z}^n) \cong \mathrm{GL}(n, \mathbb{Z})$ is induced by a self-homotopy equivalence of $T^n$, which is unique up to homotopy. In particular, the map $A \mapsto f_A$ is a noncanonical isomorphism between $\mathrm{SL}(n, \mathbb{Z})$ and the group of homotopy classes of orientation-preserving self-homeomorphism of the $n$-torus.

For any $n \times n$ matrix $A$, the *spectral radius* $\rho(A)$ is the maximum modulus of its eigenvalues:

$$\rho(A) = \max\{|\lambda| \mid \lambda \text{ is a complex eigenvalue of } A\}.$$

The spectral radius $\rho(f)$ of a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ is the spectral radius of any matrix associated to it. The eigenvalues of matrices being conjugacy invariant, $\rho(f)$ is well-defined and independent of the chosen basis.

*Remark* 2.8. For a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$, we have the following basic inequalities:

$$\log(\rho(f)) \leqslant \sum_{|\lambda|>1} \log|\lambda| \leqslant n \log(\rho(f)),$$

where $\lambda$ ranges among all the eigenvalues of $f$ with $|\lambda| > 1$.

We recall here the Gelfand formula, which is a classical result for the spectral radius that will become useful in the proof of the lower bound (Section 3).

**Proposition 2.9** (Gelfand formula). *Fix $n \in \mathbb{N}$. Let $A$ be an $n \times n$ matrix and $\|\cdot\|$ be any matrix norm. Then it holds that*

$$\rho(A) = \lim_{j \to \infty} \left\| A^j \right\|^{1/j}.$$

We denote by $\|\cdot\|_\infty$ the supremum norm on both the space of vectors $\mathbb{R}^n$ and the space of matrices $\mathbb{R}^{n \times n}$, defined for a vector $v \in \mathbb{R}^n$ or a matrix $M \in \mathbb{R}^{n \times n}$ as the maximum absolute value of its entries.

## 3 | LOWER BOUND

Let $f : T^2 \to T^2$ be an Anosov map on the 2-dimensional torus and $\lambda, \lambda^{-1}$ (with $|\lambda| > 1$) the eigenvalues of the induced map $f_* : H_1(T^2; \mathbb{Z}) \to H_1(T^2; \mathbb{Z})$. Then there exists a constant $C > 0$, not depending on the map $f$, for which $\mathrm{FV}_{\mathbb{Z}}(f) > C \log|\lambda|$ [5, Proposition 4.3]. In this section, we generalize this fact to all manifolds and all homeomorphisms.

**Theorem 2.** *Let $n \in \mathbb{N}$. If $M$ is an oriented closed $n$-manifold and $f : M \to M$ is an orientation-preserving homeomorphism, then*

$$\mathrm{FV}_{\mathbb{Z}}(f) \geqslant \frac{2}{(n+1)(n+2)\log(n+2)} \sum_{|\lambda|>1} \log|\lambda|,$$

*where $\lambda$ varies among all the complex eigenvalues of the map $f_* : H_1(M;\mathbb{R}) \to H_1(M;\mathbb{R})$ induced by $f$ in homology.*

The proof of Theorem 2 relies on the following three lemmas.

**Lemma 3.1.** *Let $n \in \mathbb{N}$. If $M$ is an oriented closed $n$-manifold and $f : M \to M$ is an orientation-preserving homeomorphism, then*

$$\mathrm{FV}_{\mathbb{Z}}(f) \geqslant \frac{2}{(n+1)(n+2)\log(n+2)} \cdot \left( \limsup_{k\to\infty} \frac{\log|\operatorname{tors} H_1(M_{f^k};\mathbb{Z})|}{k} \right),$$

*where $M_{f^k}$ is the mapping torus with fiber $M$ and monodromy $f^k$.*

Notice that this lemma is a generalization of a result by Frigerio and the first author for which the proof works analogously [5, Proposition 4.3].

*Proof.* For $k \in \mathbb{N}$, let $c_k$ be an integral fundamental cycle for $M_{f^k}$ realizing the integral simplicial volume, that is, $\|c_k\|_1 = \left\|M_{f^k}\right\|_{\mathbb{Z}}$. If $s_k$ denotes the number of singular simplices involved in the reduced expression of $c_k$, then a result of Sauer [29, Theorem 3.2] implies that

$$\log|\operatorname{tors} H_1(M_{f^k};\mathbb{Z})| \leqslant \frac{(n+1)(n+2)\log(n+2)}{2} \cdot s_k.$$

Thus, by Theorem 2.4 we get

$$\begin{aligned}\mathrm{FV}_{\mathbb{Z}}(f) &= \lim_{k\to\infty} \frac{\left\|M_{f^k}\right\|_{\mathbb{Z}}}{k} \\ &\geqslant \limsup_{k\to\infty} \frac{s_k}{k} \geqslant \limsup_{k\to\infty} \left( \frac{2}{(n+1)(n+2)\log(n+2)} \cdot \frac{\log|\operatorname{tors} H_1(M_{f^k};\mathbb{Z})|}{k} \right),\end{aligned}$$

as desired. □

Given a square matrix $M \in \mathrm{GL}(n,\mathbb{R})$ with (complex) eigenvalues $\lambda_1,\dots,\lambda_n$, we denote by $\widehat{\det}(M)$ the product of all the eigenvalues of $M$ outside the unit circle, that is

$$\widehat{\det} M = \prod_{|\lambda_i|>1} \lambda_i.$$

In view of the previous lemma, to give a lower bound for the filling volume of a map $f : M \to M$, one can study the growth of the torsion of the first homology groups of the mapping tori $M_{f^k}$. On the other hand, if $f_* : H_1(M;\mathbb{Z}) \to H_1(M;\mathbb{Z})$ is the map induced on the first homology group by $f$, then a Mayer–Vietoris argument yields the isomorphism $H_1(M_{f^k};\mathbb{Z}) = \mathrm{coker}(f_*^k - \mathrm{Id}_G) \oplus \mathbb{Z}$. This motivates the next lemma.

**Lemma 3.2.** *For any matrix $A \in \mathrm{SL}(n,\mathbb{Z})$, there exists a subsequence of natural numbers $\{k_h\}$ such that*

$$\lim_{h\to+\infty} \frac{\log|\,\mathrm{tors}(\mathrm{coker}(A^{k_h} - I))|}{k_h} = \log|\widehat{\det}(A)|.$$

*Proof.* For a fixed $k$, let $B_k = A^k - I$. We compute the torsion of the cokernel of $B_k$ seen as a morphism of $\mathbb{Z}^n$. Let $r$ be the rank of $B_k$. By the Smith normal form for integral matrices [2, Proposition 2.11], there exist matrices $P, Q \in \mathrm{GL}(n,\mathbb{Z})$ such that $PB_kQ$ is a diagonal $n \times n$ matrix with entries $d_1, \dots, d_n$, where $d_i > 0$ for any $1 \leqslant i \leqslant r$ and $d_{r+1} = \cdots = d_n = 0$. It follows that $\mathrm{Im}(PB_kQ) = d_1\mathbb{Z} \oplus \cdots \oplus d_r\mathbb{Z} \subseteq \mathbb{Z}^n$. As $P, Q$ are invertible, we have

$$\mathrm{coker}\,(B_k) \cong \mathrm{coker}(PB_kQ) \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_r\mathbb{Z} \oplus \mathbb{Z}^{n-r},$$

and so

$$|\,\mathrm{tors}(\mathrm{coker}(B_k))| = \prod_{i=1}^{r} d_i.$$

Assume at first that 1 is not an eigenvalue of $A$, so that there is a strictly increasing subsequence $(k_h)_{h\in\mathbb{N}}$ of positive integers for which $A^{k_h}$ does not admit 1 as eigenvalue. In particular, all eigenvalues of $B_{k_h}$ are nonzero, implying that $r = n$ and $|\det(PB_{k_h}Q)| = \prod_{i=1}^n d_i > 0$. In this case,

$$|\,\mathrm{tors}(\mathrm{coker}(B_{k_h}))| = \prod_{i=1}^{n} d_i = |\det(PB_{k_h}Q)| = |\det(B_{k_h})|,$$

so that [10, Lemma, Section 2.5]

$$\lim_{h\to\infty} \frac{\log|\,\mathrm{tors}(\mathrm{coker}(B_{k_h}))|}{k_h} = \lim_{h\to\infty} \frac{\log|\det(A^{k_h} - \mathrm{Id})|}{k_h} = |\widehat{\det} A|.$$

If 1 is an eigenvalue of $A$, then let us consider for any $k \in \mathbb{N}$ the operator $C_k : \mathrm{Im}\,(\mathrm{Id} - A) \to \mathrm{Im}\,(\mathrm{Id} - A)$ defined by

$$C_k(v - Av) := v - A^k v.$$

Then,

$$\det(C_k) = \prod_{\lambda\neq 1} \frac{|\lambda^k - 1|}{|\lambda - 1|},$$

where $\lambda$ ranges among the eigenvalues of $A$ [10, Section 2.6]. As before, we choose a strictly increasing sequence of positive integers $(k_h)_{h\in\mathbb{N}}$ such that $\det(C_{k_h}) \neq 0$, and by applying the Smith normal form we obtain

$$|\operatorname{tors}(\operatorname{coker}(C_{k_h}))| = |\det(C_{k_h})|.$$

Since $\operatorname{Im}(C_{k_h}) = \operatorname{Im}(\mathrm{Id} - A^{k_h})$, we have $\operatorname{coker}(C_{k_h}) \subseteq \operatorname{coker}(\mathrm{Id} - A^{k_h})$, so that

$$\prod_{\lambda\neq 1} \frac{|\lambda^{k_h} - 1|}{|\lambda - 1|} = |\operatorname{tors}(\operatorname{coker}(C_{k_h}))| \leqslant |\operatorname{tors}(\operatorname{coker}(\mathrm{Id} - A^{k_h}))|.$$

By considering the map induced by $\mathrm{Id} - A^{k_h}$ on $\mathbb{Z}^n / \ker(\mathrm{Id} - A)$, one gets an upper bound [10, Section 2.6]

$$|\operatorname{tors}(\operatorname{coker}(\mathrm{Id} - A^{k_h}))| \leqslant \prod_{\lambda\neq 1} |\lambda^{k_h} - 1|.$$

Thus, we have that

$$\sum_{\lambda\neq 1} \frac{\log|\lambda^{k_h} - 1| - \log|\lambda - 1|}{k_h} \leqslant \frac{\log|\operatorname{tors}(\operatorname{coker}(\mathrm{Id} - A^{k_h}))|}{k_h} \leqslant \sum_{\lambda\neq 1} \frac{\log|\lambda^{k_h} - 1|}{k_h}$$

from which we deduce [10, Lemma, Section 2.5] that

$$\lim_{h\to+\infty} \frac{\log|\operatorname{tors}(\operatorname{coker}(A^{k_h} - I))|}{k_h} = \lim_{h\to+\infty} \sum_{\lambda\neq 1} \frac{\log|\lambda^{k_h} - 1|}{k_h} = \log|\widehat{\det}(A)|. \qquad \square$$

If $H_1(M_{f^k}; \mathbb{Z})$ is torsion-free, then the last two lemmas are enough to prove Theorem 2. The following lemma takes care of the torsion case.

**Lemma 3.3.** *Let $G$ be an abelian group with torsion group $T = \operatorname{tors}(G)$. Let $h\colon G \to G$ be an endomorphism and $\overline{h}\colon G/T \to G/T$ the map induced by $h$ on $G/T$. Then, it holds that*

$$|\operatorname{tors}(\operatorname{coker}(\overline{h}))| \leqslant |\operatorname{tors}(\operatorname{coker}(h))|.$$

Notice that the map $\overline{h}$ is well-defined as any torsion element in a group is sent to a torsion element by any endomorphism.

*Proof.* Let us consider the quotient maps $\pi\colon G \to G/T$, $p\colon G \to \operatorname{coker} h$ and $\overline{p}\colon G/T \to \operatorname{coker}(\overline{h})$.

Since for any $g \in G$ we have that

$$\overline{p}(\pi(h(g))) = \overline{p}(\overline{h}(\pi(g))) = 0,$$

the inclusion $\operatorname{Im}(h) \subseteq \ker(\overline{p}\circ\pi)$ holds. In particular, there is a surjective homomorphism $\psi\colon \operatorname{coker} h \to \operatorname{coker} \overline{h}$ making the following diagram commute:

$$\begin{tikzcd} G \arrow[r, "h"] \arrow[d, two heads, "\pi"] & G \arrow[r, two heads, "p"] \arrow[d, two heads, "\pi"] & \operatorname{coker} h \arrow[d, two heads, "\psi"] \\ G/T \arrow[r, "\overline{h}"] & G/T \arrow[r, two heads, "\overline{p}"] & \operatorname{coker} \overline{h}. \end{tikzcd}$$

To prove the statement, it is sufficient to show that $\psi$ restricts to a surjective homomorphism between the torsion subgroups, which can be proved by some diagram chasing as follows.

Let $\alpha \in \operatorname{tors}(\operatorname{coker} \overline{h})$, so that $\alpha^n = 1_{\operatorname{coker} \overline{h}}$ for some $n \in \mathbb{N}$. Since $\overline{p} \circ \pi$ is surjective, there exists $a \in G$ satisfying $\overline{p}(\pi(a)) = \alpha$. As $\overline{p}(\pi(a^n)) = 1_{\operatorname{coker} \overline{h}}$, there exists $c \in G$ such that $\pi(a^n) = \overline{h}(\pi(c)) = \pi(h(c))$, meaning that $a^n = h(c)t$ for some $t \in T$. Let $m \in \mathbb{N}$ be such that $t^m = 1$. Then, $a^{nm} = h(c^m) \in \operatorname{Im} h$, implying that $p(a) \in \operatorname{tors}(\operatorname{coker} h)$. Moreover, by the commutativity of the above diagram, $\psi(p(a)) = \alpha$. This proves the surjectivity of $\psi$ on the torsion part and concludes the proof. $\square$

Now it is just a matter of putting all the ingredients together.

*Proof of Theorem* 2. Let $f : M \to M$ be as in the statement and $f_* : H_1(M; \mathbb{Z}) \to H_1(M; \mathbb{Z})$ the isomorphism induced by $f$ on the first homology group of $M$. If $M_{f^k}$ is the mapping torus with fiber $M$ and monodromy $f^k$, then a Mayer–Vietoris sequence yields an isomorphism $H_1(M_{f^k}; \mathbb{Z}) \cong \operatorname{coker}(f_*^k - \operatorname{Id}) \oplus \mathbb{Z}$, and thus

$$|\operatorname{tors}(H_1(M_{f^k}; \mathbb{Z}))| = |\operatorname{tors}(\operatorname{coker}(f_*^k - \operatorname{Id}_G))|.$$

Let us denote by $G = H_1(M; \mathbb{Z})$ the first homology group of $M$ and $T = \operatorname{tors} G$ the torsion subgroup of $G$. As in Lemma 3.3, the map $f_*^k$ descends to a map $\overline{f}_*^k : G/T \to G/T$. Since the map $(f_*^k - \operatorname{Id}_G)$ descends to the map $(\overline{f}_*^k - \operatorname{Id}_{G/T}) : G/T \to G/T$, by Lemma 3.3, it holds that

$$\left|\operatorname{tors}\left(\operatorname{coker}\left(\overline{f}_*^k - \operatorname{Id}_{G/T}\right)\right)\right| \leqslant \left|\operatorname{tors}\left(\operatorname{coker}\left(f_*^k - \operatorname{Id}_G\right)\right)\right|.$$

Since $G/T \cong \mathbb{Z}^d$ (for some $d \in \mathbb{N}$) is a finitely generated torsion-free abelian group, the map $\overline{f}_*$ can be represented by some matrix $A \in \operatorname{SL}(d, \mathbb{Z})$. By Lemmas 3.2 and 3.1, there exists a subsequence $\{k_h\}_h$ such that

$$\begin{aligned} \left(\frac{(n+1)(n+2)\log(n+2)}{2}\right) \cdot \operatorname{FV}_{\mathbb{Z}}(f) &\geqslant \limsup_{h\to\infty} \frac{\log|\operatorname{tors} H_1(M_{f^{k_h}}; \mathbb{Z})|}{k_h} \\ &= \limsup_{h\to\infty} \frac{\log|\operatorname{tors}(\operatorname{coker}(f_*^{k_h} - \operatorname{Id}_G))|}{k_h} \\ &\geqslant \lim_{h\to\infty} \frac{\log|\operatorname{tors}(\operatorname{coker}(\overline{f}_*^{k_h} - \operatorname{Id}_{G/T}))|}{k_k} \end{aligned}$$

$$= \lim_{h\to\infty} \frac{\log|\operatorname{tors}(\operatorname{coker}(A^{k_h} - \mathrm{Id}_{G/T}))|}{k_h}$$
$$= \log|\widehat{\det} A|.$$

The claim now follows by noticing that the matrix $A$ represents the map $H_1(M;\mathbb{R}) \to H_1(M;\mathbb{R})$ induced by $f$ on the first *real* homology group. Indeed,

$$H_1(M;\mathbb{R}) \cong G \otimes \mathbb{R} \cong (G/T \oplus T) \otimes \mathbb{R} \cong G/T \otimes \mathbb{R} \cong \mathbb{R}^d,$$

and under these isomorphisms the map $f_* \otimes \mathrm{Id}_{\mathbb{R}}$ (i.e., the map induced by $f$ on the first real homology group) corresponds to $\overline{f}_* \otimes \mathrm{Id}_{\mathbb{R}}$, which can still be represented by $A$. □

Thanks to Remark 2.8, this also concludes the proof of the lower bound of Theorem 1.

## 4 | UPPER BOUND

This section is devoted to proving the following theorem, which, combined with Theorem 2, completes the proof of Theorem 1.

**Theorem 4.1.** *Fix $n \in \mathbb{N}$. There exists a constant $K_n > 0$ such that the following holds: if $f : T^n \to T^n$ is an orientation-preserving homeomorphism on the $n$-torus and $f_* : H_1(T^n;\mathbb{Z}) \to H_1(T^n;\mathbb{Z})$ denotes the induced map in homology, then*

$$\mathrm{FV}_{\mathbb{Z}}(f) \leqslant K_n \log_2(\rho(f_*)).$$

To prove this theorem, we consider a fundamental cycle $c_n$ of $T^n$ and construct efficient fillings for $c_n - f_*^k(c_n)$ for any $k \in \mathbb{N}$. In order to do that, we deal with a particular kind of fundamental cycles in $T^n$, to which we refer as *parallelogram-like* cycles and whose lifts in $\mathbb{R}^n$ consist of chains with supports given by Euclidean parallelograms.

In the next subsection, we introduce these objects and present the key proposition (Proposition 4.7) to prove Theorem 4.1.

### 4.1 | Parallelogram and rectangle-like cycles

Let $n \in \mathbb{N}$. Given a vector $v \in \mathbb{Z}^n$, we consider the map $\widetilde{h}_v : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ defined by $\widetilde{h}_v(\mathbf{x}, t) = \mathbf{x} + tv$, which is a homotopy between the identity map of $\mathbb{R}^n$ and the translation by $v$. Since $\widetilde{h}_v(\cdot, t)$ is a translation for every $t \in [0,1]$, the map $\widetilde{h}_v$ descends to a homotopy $h_v : T^n \times [0,1] \to T^n$ on the $n$-torus with $h_v(\cdot, 0) = h_v(\cdot, 1) = \mathrm{Id}(\cdot)$.

Let

$$F^v_\bullet : C_\bullet(T^n;\mathbb{Z}) \to C_{\bullet+1}(T^n;\mathbb{Z})$$

be the prism operator associated to the homotopy $h_v$ (see Section 2.3). Since $h_v$ is a homotopy between $f = \mathrm{Id}$ and $g = \mathrm{Id}$, it holds that $\partial \circ F^v - F^v \circ \partial = 0$ and $\left\|F^v_j\right\| \leqslant j + 1$.

The following lemma ensures that the operator $F^v$ preserves the space of boundaries $B_\bullet(T^n;\mathbb{Z})$ in $T^n$.

**Lemma 4.2.** *Let $b \in B_j(T^n;\mathbb{Z})$ be a boundary in $T^n$ and $v \in \mathbb{Z}^n$. Then $F^v(b) \in B_{j+1}(T^n;\mathbb{Z})$ is a boundary and $\left\|F^v(b)\right\|_{fill,\mathbb{Z}} \leqslant (j+2)\|b\|_{fill,\mathbb{Z}}$.*

*Proof.* Let $d \in C_{j+1}(T^n;\mathbb{Z})$ with $\partial d = b$ and $\|d\| = \|b\|_{\mathrm{fill},\mathbb{Z}}$. Then

$$\partial(F^v(d)) = (F^v \circ \partial)(d) = F^v(b),$$

and $\left\|F^v(d)\right\| = (j+2)\|d\| = (j+2)\|b\|_{\mathrm{fill},\mathbb{Z}}$. □

We are now ready to introduce *parallelogram-like cycles*. They are defined inductively on the dimension, starting from dimension 1.

**Definition 4.3.** Let $\mathbf{0} \in \mathbb{Z}^n$ be the zero vector. The *parallelogram-like 1-cycle generated by the vector $v \in \mathbb{Z}^n$* in $T^n$ is the cycle given by

$$P_1^n(v) := [\mathbf{0}, v] \in Z_1(T^n;\mathbb{Z}).$$

For $k > 1$, having defined parallelogram-like cycles of dimension $k-1$, we define a *parallelogram-like $k$-cycle* $P_k^n(v_1, \dots, v_k)$ generated by the vectors $v_1, \dots, v_k \in \mathbb{Z}^n$ in $T^n$ as the chain

$$P_k^n(v_1, \dots, v_k) := F^{v_k}(P_{k-1}^n(v_1, \dots, v_{k-1})) \in Z_k(T^n;\mathbb{Z}).$$

In this case, we call $\left\|v_1\right\|_\infty, \dots, \left\|v_k\right\|_\infty$ the *sizes* of $P_k^n(v_1, \dots, v_k)$.

A *rectangle-like cycle with sizes* $|a_1|, \dots, |a_n|$, where $a_1, \dots, a_n \in \mathbb{Z}$, is a parallelogram-like cycle of the form

$$R_n(a_1, \dots, a_n) = P_n^n(a_1 e_1, \dots, a_n e_n) \in Z_n(T^n;\mathbb{Z}).$$

**Example 4.4.** By definition, parallelogram-like 1-cycles are just given by straight simplices. A parallelogram-like 1-cycle of $T^1 = S^1$ is, in particular, a rectangle-like cycle.

An explicit expression for a 2-dimensional parallelogram-like cycle in $T^n$ generated by $v, w \in \mathbb{Z}^n$ is given by

$$P_2^n(v, w) := [\mathbf{0}, v, v+w] - [\mathbf{0}, w, v+w] \in Z_2(T^n;\mathbb{Z})$$

(see Figure 1).

The following lemma ensures that parallelogram-like cycles are, indeed, cycles with bounded norm.

**Lemma 4.5.** *Every parallelogram-like cycle $P_k^n(v_1, \dots, v_k)$ is a cycle with $\left\|P_k^n(v_1, \dots, v_k)\right\|_1 \leqslant k!$.*

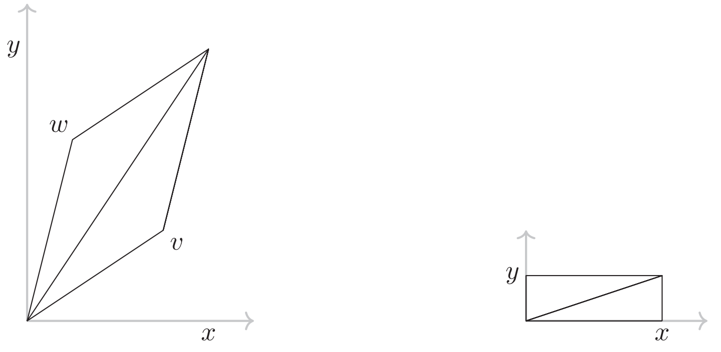


**FIGURE 1** Representations of the canonical lifts in $\mathbb{R}^2$ of the parallelogram-like cycle $P_2^2(v, w)$ in $T^2$ (on the left) and of a rectangle-like cycle (on the right).

*Proof.* The statement is clearly true for parallelogram-like cycles of dimension 1. If we assume by induction that the statement is true for $k-1$, with $k > 1$, then

$$\partial P_k^n(v_1, \dots, v_k) = \partial F^{v_k}(P_{k-1}^n(v_1, \dots, v_{k-1})) = F^{v_k}(\partial P_{k-1}^n(v_1, \dots, v_{k-1})) = 0,$$

and

$$\left\| P_k^n(v_1, \dots, v_k) \right\|_1 = \left\| F^{v_k}(P_{k-1}^n(v_1, \dots, v_{k-1})) \right\|_1 \leqslant k \cdot \left\| P_{k-1}^n(v_1, \dots, v_{k-1}) \right\|_1. \qquad \square$$

As $H_n(T^n; \mathbb{Z}) \cong \mathbb{Z}$, the previous lemma ensures that a parallelogram-like $n$-cycle in $T^n$ represents a multiple of the fundamental class in homology. The following lemma determines the multiple in terms of its defining vectors.

**Lemma 4.6.** *The rectangle-like cycle $R_n(1, \dots, 1)$ is a fundamental cycle for $T^n$. More generally, if $v_1, \dots, v_n \in \mathbb{Z}^n$, then the parallelogram-like cycle $P_n^n(v_1, \dots, v_n)$ represents the element*

$$\det A \cdot [T^n]_{\mathbb{Z}} \in H_n(T^n; \mathbb{Z}),$$

*where $A$ is the matrix whose columns are given by $v_1, \dots, v_n$.*

*Proof.* Being a cycle of degree 1 in every point of $T^n$, the parallelogram-like cycle $P_n^n(e_1, \dots, e_n)$ is a fundamental cycle for $T^n$.

If $A$ is the matrix whose columns are given by $v_1, \dots, v_n$, then the degree of the map $f : T^n \to T^n$ induced by the matrix $A$ is $\det(A)$. Thus,

$$\det A \cdot [T^n]_{\mathbb{Z}} = [f_*(P_n^n(e_1, \dots, e_n))] = [P_n^n(v_1, \dots, v_n)] \in H_n(T^n; \mathbb{Z}),$$

as desired. $\square$

A direct consequence of the previous lemma is the fact that for any $v_1, \dots, v_n \in \mathbb{Z}^n$, the parallelogram-like cycle $P_n^n(v_1, \dots, v_n)$ and the rectangle-like cycle $R_n(\det(A), 1, \dots, 1)$ are cobordant, where $A$ is the matrix with columns $v_1, \dots, v_n$. The key step for the proof of Theorem 4.1 is to bound the filling norm of their difference, as stated in the following proposition.

**Proposition 4.7.** *Let $n \in \mathbb{N}$. There exists a constant $K_n > 0$ such that for any $v_1, \dots, v_n \in \mathbb{Z}^n$ it holds that*

$$\|P_n^n(v_1, \dots, v_n) - R_n(\det(A), 1, \dots, 1)\|_{fill,\mathbb{Z}} \leqslant K_n \log_2(\|A\|_\infty) + K_n,$$

*where $A$ is the matrix whose columns are given by $v_1, \dots, v_n$.*

Recall that $c_n = R_n(1, \dots, 1)$ is a fundamental cycle for $T^n$ (Lemma 4.6). If $A \in \mathrm{SL}(n, \mathbb{Z})$ and $f : T^n \to T^n$ is the map induced by $A$, then $P_k = f_*^k(c_n)$ is a parallelogram-like fundamental cycle. The previous proposition furnishes an estimate for $\|P_k - R\|_{\mathrm{fill},\mathbb{Z}}$, where $R$ is the rectangle-like cycle given by $R = R_n(\det A, 1, \dots, 1) = c_n$. This estimate leads to a proof of Theorem 4.1.

*Proof of Theorem 4.1 assuming Proposition 4.7.* As $\mathrm{FV}_\mathbb{Z}$ is homotopy invariant, we can assume that $f : T^n \to T^n$ is a map induced by a matrix $A \in \mathrm{SL}(n, \mathbb{Z})$. Let us denote by $\rho(A)$ the spectral radius of $A$.

By the Gelfand formula for the spectral radius (Proposition 2.9), one has

$$\rho(A) = \lim_{j\to\infty} \left(\left\|A^j\right\|_\infty\right)^{1/j}.$$

For any $\varepsilon > 0$, there exists an $M > 0$ (depending on $\varepsilon$) such that for any $j > M$ it holds that

$$\left\|A^j\right\|_\infty \leqslant (1+\varepsilon)^j \rho(A)^j.$$

Fix $c_n = R_n(1, \dots, 1)$ a fundamental cycle for $T^n$ (Lemma 4.6). For $j \in \mathbb{N}$, we set $v_i^j = A^j e_i$, so that $f_*^j(c_n) = P_n^n\left(v_1^j, \dots, v_n^j\right)$. As $A \in \mathrm{SL}(n, \mathbb{Z})$ we have that

$$R_n\left(\det(A^j), 1, \dots, 1\right) = R_n(1, \dots, 1) = c_n.$$

By Proposition 4.7 and the definition of filling volume, one then obtains

$$\begin{aligned}
\mathrm{FV}_\mathbb{Z}(f) &= \lim_{j\to\infty} \frac{\left\|f_*^j(c_n) - c_n\right\|_{\mathrm{fill},\mathbb{Z}}}{j} \\
&= \lim_{j\to\infty} \frac{\left\|P_n^n\left(v_1^j, \dots, v_n^j\right) - R_n\left(\det(A^j), 1, \dots, 1\right)\right\|_{\mathrm{fill},\mathbb{Z}}}{j} \\
&\leqslant \lim_{j\to\infty} \frac{K_n \log_2(\|A^j\|_\infty) + K_n}{j} \\
&\leqslant \lim_{j\to\infty} \frac{K_n \log_2((1+\varepsilon)^j \rho(A)^j) + K_n}{j} \\
&= K_n \log_2((1+\varepsilon)\rho(A)).
\end{aligned}$$

The claim then follows from the arbitrariness of $\varepsilon$. $\square$

The rest of this section is devoted to the proof of Proposition 4.7. In order to understand the idea of the proof, we consider the graph whose vertices are given by integral cycles in $T^n$, and where two vertices are joined by an edge if and only if the corresponding cycles $a, b$ are cobordant and satisfy $\|a-b\|_{\text{fill},\mathbb{Z}} = 1$. In other words, this graph can be seen as a metric space whose distance function is induced by the filling norm. The final goal is to estimate the distance between $P^n_n(v_1, \dots, v_n)$ and $R_n(\det(A), 1, \dots, 1)$.

We proceed as follows: Section 4.2 consists in proving that we can split a parallelogram-like cycle into two cycles without going too far in our graph. To be more precise, we prove that for any $v_1', v_1'', v_2, \dots, v_k \in \mathbb{Z}^n$ the sum $P^n_k\left(v_1', v_2, \dots, v_k\right) + P^n_k\left(v_1'', v_2, \dots, v_k\right)$ is close to the parallelogram-like cycle $P^n_k\left(v_1' + v_1'', v_2, \dots, v_k\right)$ (Lemma 4.10).

In Section 4.3, we deal with “slim” parallelogram-like cycles, which are the parallelogram-like cycles whose generating vectors are linearly dependent. In particular, we control the distance between these cycles and the zero cycle in terms of their sizes (Lemma 4.14). Section 4.4 deals with rectangle-like cycles, proving that each of them is close to a rectangle-like cycle with all but one size equal to 1 (Lemma 4.16). Moreover, we show that any parallelogram-like cycle is close to a sum of rectangle-like cycles with a controlled number of summands (Lemma 4.15).

In Section 4.5, we put together all these results and get Proposition 4.7.

*Remark* 4.8. To simplify the notation, we denote all the constants appearing in the following lemmas by $C_n$, as they can be assumed to coincide up to taking the maximum. In the specific case in which we need to use the constant of a statement inside its proof (e.g., when the proof is done by induction), then we will refer to such a constant as $V_n$.

Here, the subscript $n$ in $C_n$ and $V_n$ denotes the dependence on the dimension.

We begin with the following lemma, which bounds the distance between two parallelogram-like cycles with the same generating vectors but arranged in different orders.

**Lemma 4.9.** *Let $k, n \in \mathbb{N}$ be two positive integers. There exists a constant $C_k$, depending only on $k$, such that for every $v_1, \dots, v_k \in \mathbb{Z}^n$ and every permutation $\theta$ on $\{1, \dots, k\}$ it holds that*

$$\left\| P^n_k(v_1, \dots, v_k) - \operatorname{sgn}\theta \cdot P^n_k\left(v_{\theta(1)}, \dots, v_{\theta(k)}\right) \right\|_{fill,\mathbb{Z}} \leqslant C_k$$

*and*

$$\left\| P^n_k(v_1, \dots, v_k) + P^n_k(-v_1, v_2, \dots, v_k) \right\|_{fill,\mathbb{Z}} \leqslant C_k.$$

Recall that the sign $\operatorname{sgn}\theta$ of a permutation $\theta : \{1, \dots, k\} \to \{1, \dots, k\}$ is defined to be $+1$ if the permutation is even and $-1$ if the permutation is odd.

*Proof.* Denote by $e_1, \dots, e_k$ the canonical basis for $\mathbb{R}^k$. Let us set $c = P^k_k(e_1, \dots, e_k)$, $c_- = P^k_k(-e_1, e_2, \dots, e_k)$, and $c_\sigma = P^k_k\left(e_{\sigma(1)}, \dots, e_{\sigma(k)}\right)$ for any permutation $\sigma : \{1, \dots, k\} \to \{1, \dots, k\}$. Let $A_\sigma$ (respectively, $A_-$) be the matrix whose columns are given by $e_{\sigma(1)}, \dots, e_{\sigma(k)}$ (respectively, $-e_1, e_2, \dots, e_k$). Clearly, $\det A_- = -1$. Moreover, $A_\sigma$ being the permutation matrix corresponding to $\sigma$, we have $\det A = \operatorname{sgn}\sigma$.

By Lemma 4.6, $c, -c_-$, and $\operatorname{sgn}\sigma \cdot c_\sigma$ represent the same class in homology. We set

$$C'_k = \max\left\{\|c - \operatorname{sgn}\sigma \cdot c_\sigma\|_{\mathrm{fill},\mathbb{Z}} \mid \sigma \in \mathrm{Sym}(\{1,\dots,k\})\right\},$$

and $C_k = \max\{C'_k, \|c + c_-\|_{\mathrm{fill},\mathbb{Z}}\}$.

Let $f : T^k \to T^n$ be the map induced by the linear map $\mathbb{R}^k \to \mathbb{R}^n$ sending $e_i$ to $v_i$ for any $i = 1,\dots,k$. Let us fix a permutation $\theta$ on $\{1,\dots,k\}$. As $P^n_k(v_1,\dots,v_k) = f_*(c)$, $P^n_k(-v_1, v_2,\dots,v_k) = f_*(c_-)$, and $P^n_k\left(v_{\theta(1)},\dots,v_{\theta(k)}\right) = f_*(c_\theta)$, it holds that

$$\left\|-P^n_k(v_1,\dots,v_k) - \operatorname{sgn}\theta \cdot P^n_k\left(v_{\theta(1)},\dots,v_{\theta(k)}\right)\right\|_{\mathrm{fill},\mathbb{Z}} = \|f_*(c - \operatorname{sgn}\theta\cdot c_\theta)\|_{\mathrm{fill},\mathbb{Z}} \leqslant C_k,$$

and

$$\left\|P^n_k(v_1,\dots,v_k) + P^n_k(-v_1, v_2,\dots,v_k)\right\|_{\mathrm{fill},\mathbb{Z}} = \|f_*(c + c_-)\|_{\mathrm{fill},\mathbb{Z}} \leqslant C_k. \qquad \square$$

## 4.2 | Splitting of parallelogram-like cycles

As mentioned above, a parallelogram-like cycle can be split into the sum of two parallelogram-like cycles as in the following lemma.

**Lemma 4.10.** *Let $k \in \mathbb{N}$. There exists a constant $C_k$ such that for any $v_1,\dots,v_k, v'_i, v''_i \in \mathbb{Z}^n$ with $v_i = v'_i + v''_i \in \mathbb{Z}^n$ it holds that*

$$\left\|P^n_k(v_1,\dots,v_k) - P^n_k\left(v_1,\dots,v'_i,\dots,v_k\right) - P^n_k\left(v_1,\dots,v''_i,\dots,v_k\right)\right\|_{fill,\mathbb{Z}} \leqslant C_k.$$

Before proving this statement, we need the following claim, which bounds the filling norm of a parallelogram-like cycle that has $\mathbf{0}$ among its generating vectors.

*Claim* 4.2.1. There exists a constant $C_k$, depending on $k \in \mathbb{N}$, such that for any $v_1,\dots,v_k \in \mathbb{Z}^n$ it holds that

$$\left\|P^n_{k+1}(v_1,\dots,v_k,\mathbf{0})\right\|_{\mathrm{fill},\mathbb{Z}} \leqslant C_k.$$

*Proof.* We denote by $\{e_1,\dots,e_k\}$ the canonical basis of $\mathbb{R}^k$. The parallelogram-like cycle $q_k = P^k_{k+1}(e_1,\dots,e_k,\mathbf{0})$ represents the trivial class in $H_{k+1}(T^k;\mathbb{Z}) = 0$. Set $C_k = \|q_k\|_{\mathrm{fill},\mathbb{Z}}$ and consider the map $\nu : T^k \to T^n$ induced by the linear map $\mathbb{R}^k \to \mathbb{R}^n$ sending $e_i$ to $v_i$ for every $i = 1,\dots,k$. Then, we obtain

$$\left\|P^n_{k+1}(v_1,\dots,v_k,\mathbf{0})\right\|_{\mathrm{fill},\mathbb{Z}} = \|\nu_*(q_k)\|_{\mathrm{fill},\mathbb{Z}} \leqslant C_k. \qquad \square$$

We also need to fix a canonical way to lift parallelogram-like cycles to $\mathbb{R}^n$; we do it as follows. If $c = \sum a_i\sigma_i \in C_k(\mathbb{R}^n;\mathbb{Z})$ is written in reduced form, then the support of $c$ is

$$\operatorname{supp} c = \bigcup_i \operatorname{Im}\sigma_i.$$

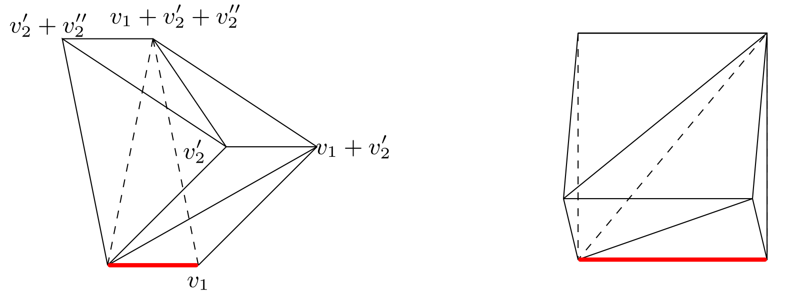


**FIGURE 2** On the left, a (3-dimensional) representation of a lift in $\mathbb{R}^3$ of the 3-chain $Q$ of Lemma 4.10. The boundary of this lift is made up of: the two forward parallelograms, representing lifts of $P_2^2(v_1, v_2')$ and $P_2^2(v_1, v_2'')$, the backward parallelogram, representing a lift of $P_2^2(v_1, v_2' + v_2'')$, and the two triangles on the left and on the right, given by $R(b_1^+)$ and $R(b_1^-)$, which simplify when projecting on $T^2$. The red segment is the support of $P_2^2(v_1, \mathbf{0})$. On the right, is represented the same picture in case of rectangle-like cycles (Lemma 4.12).

With this notation, a lift $P \in C_k(\mathbb{R}^n; \mathbb{Z})$ of $P_k^n(v_1, \dots, v_k)$ to $\mathbb{R}^n$ is *canonical* if

$$\operatorname{supp} P \subset \left\{ \sum_{i=1}^{k} \varepsilon_i v_i \mid \varepsilon_i \in [0,1] \right\},$$

that is, if the support of $P$ is contained in the Euclidean parallelogram with coordinate vectors $v_1, \dots, v_k$. Such a lift always exists and is unique.

*Remark* 4.11. The boundary of the canonical lift $P$ of a parallelogram-like cycle $P_k^n(v_1, \dots, v_k)$ is a sum

$$\partial P = \sum_{i=1}^{k} (-1)^i (b_i^+ - b_i^-),$$

where for every $i = 1, \dots, k$ one has that

- $b_i^+ \in C_{k-1}(\mathbb{R}^n; \mathbb{Z})$ is the canonical lift of the parallelogram-like cycle $P_k^n(v_1, \dots, \widehat{v_i}, \dots, v_k)$;
- $b_i^- = (\tau_{v_i})_*(b_i^+)$, where $\tau_{v_i} : \mathbb{R}^n \to \mathbb{R}^n$ is the translation by $v_i$ in $\mathbb{R}^n$.

In particular, $\pi_*(b_i^+ - b_i^-) = 0$, where $\pi_n : \mathbb{R}^n \to T^n$ denotes the quotient map.

*Proof of Lemma* 4.11. Let $\pi : \mathbb{R}^n \to T^n$ be the quotient map and denote by $\{e_1, \dots, e_{n+1}\}$ the canonical basis of $\mathbb{R}^{n+1}$. Up to applying Lemma 4.9, we can assume $i = k$. Let $v_1, \dots, v_k, v_k', v_k'' \in \mathbb{Z}^n$ be as in the statement.

We are going to define a chain $Q \in C_{k+1}(T^n; \mathbb{Z})$ (represented in Figure 2) with $\|Q\|_1 \leqslant (k+1)!$ and

$$\partial Q = (-1)^k \big( -P_k^n(v_1, \dots, v_{k-1}, v_k') - P_k^n(v_1, \dots, v_{k-1}, v_k'') + \\ + P_k^n(v_1, \dots, v_{k-1}, v_k' + v_k'') + P_k^n(v_1, \dots, v_{k-1}, \mathbf{0}) \big).$$

Let $\iota : \mathbb{R}^n \to \mathbb{R}^{n+1}$ be the inclusion into the first $n$ factors and $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection onto the first $n$ factors. Set $w'_k = \iota(v'_k) + e_{n+1}$.

We denote by $P$ the canonical lift of the parallelogram-like cycle $P^{n+1}_k\left(\iota(v_1), \dots, \iota(v_{k-1}), w'_k\right)$ and by $P'$ the one of $P^{n+1}_k\left(\iota(v_1), \dots, \iota(v_{k-1}), w'_k + \iota(v''_k)\right)$.

Consider the prism operator $R$ associated to the homotopy $f : \mathbb{R}^{n+1} \times [0,1] \to \mathbb{R}^{n+1}$ that at time $t$ is represented by the matrix

$$f_t = f(\cdot, t) = \begin{pmatrix} I_n & t \cdot v''_k \\ 0 & 1 \end{pmatrix}.$$

We set

$$Q = \pi_*(p_*(R(P))).$$

Since $f_0 = \mathrm{Id}$, $f_1(\iota(v_i)) = \iota(v_i)$ for $i = 1, \dots, k-1$ and $f_1(w'_k) = w'_k + \iota(v''_k)$ we have that $(f_0)_*(P) = P$ and $(f_1)_*(P) = P'$. By the properties of the prism operator (Section 2.3) it holds that

$$\begin{aligned}\partial Q &= \pi_*(p_*(\partial(R(P))))\\ &= \pi_*(p_*(R(\partial P))) + (-1)^k(\pi_*(p_*(P')) - \pi_*(p_*(P)))\\ &= \pi_*(p_*(R(\partial P))) + (-1)^k(P^n_k\left(v_1, \dots, v_{k-1}, v'_k + v''_k\right) - P^n_k\left(v_1, \dots, v_{k-1}, v'_k\right)).\end{aligned}$$

By Remark 4.11, we have that $\partial P = \sum_{i=1}^k (-1)^i (b_i^+ - b_i^-)$, where $b_k^- = (\tau_{w'_k})_*(b_k^+)$ and $b_i^- = (\tau_{\iota(v_i)})_*(b_i^+)$ for every $i = i, \dots, k-1$ (with $\tau_v : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ denoting the translation by $v$). Since for any $i = 1, \dots, k-1$ the translation $\tau_{\iota(v_i)}$ commutes with $f_t$, the induced map $(\tau_{\iota(v_i)})_*$ on the chain complex commutes with the operator $R$ and thus

$$\pi_*\left(p_*(R(b_i^-))\right) = \pi_*\left(p_*(R((\tau_{\iota(v_i)})_*(b_i^+)))\right) = \pi_*(p_*((\tau_{v_i})_*(R(b_i^+)))) = \pi_*(p_*(R(b_i^+))).$$

Thus, all terms in the sum $\pi_*(p_*(R(\partial P))) = \sum_{i=1}^k (-1)^i \pi_*(p_*(R(b_i^+ - b_i^-)))$ vanish except possibly the one corresponding to the index $i = k$. In this case, since $b_k^+$ is the canonical lift of $P^{n+1}_{k-1}(\iota(v_1), \dots, \iota(v_{k-1}))$, we have that $(f_t)_*(b_k^+) = b_k^+$. In particular, the homotopy $f$ acts on $b_k^+$ as $h_{\mathbf{0}}$ (defined in Section 4.1). Thus,

$$\pi_*(p_*(R(b_k^+))) = F^{\mathbf{0}}(\pi_*(p_*(b_k^+))) = P^n_k(v_1, \dots, v_{k-1}, \mathbf{0}).$$

Finally, since $b_k^- = (\tau_{w'_k})_*(b_k^+)$ is contained in the affine $(n-1)$-subspace $\mathcal{H} = \iota(\mathbb{R}^n) + w'_k$ and $f_t|_{\mathcal{H}} = \tau_{t\cdot\iota(v''_k)}|_{\mathcal{H}}$, we have that $(f_t)_*(b_k^-) = (\tau_{t\cdot\iota(v''_k)})_*(b_k^-) = (h_{t\cdot\iota(v''_k)})_*(b_k^-)$, getting

$$\pi_*(p_*(R(b_k^-))) = F^{v''_k}(P^n_{k-1}(v_1, \dots, v_{k-1})) = P^n_k\left(v_1, \dots, v_{k-1}, v''_k\right).$$

Putting everything together, we get

$$\pi_*(p_*(R(\partial P))) = (-1)^k(P^n_k(v_1, \dots, v_{k-1}, \mathbf{0}) - P^n_k\left(v_1, \dots, v_{k-1}, v''_k\right))$$

and, therefore,

$$\partial Q = (-1)^k\big(-P_k^n(v_1,\dots,v_{k-1},v_k') - P_k^n(v_1,\dots,v_{k-1},v_k'')+\\ +P_k^n(v_1,\dots,v_{k-1},v_k'+v_k'') + P_k^n(v_1,\dots,v_{k-1},\mathbf{0})\big).$$

As $v_k = v_k' + v_k''$, it follows that

$$\left\|P_k^n(v_1,\dots,v_{k-1},v_k) - P_k^n(v_1,\dots,v_{k-1},v_k'') - P_k^n(v_i,\dots,v_{k-1},v_k')\right\|_{\mathrm{fill},\mathbb{Z}}\\ \leqslant \|\partial Q\|_{\mathrm{fill},\mathbb{Z}} + \left\|P_k^n(v_1,\dots,v_{k-1},\mathbf{0})\right\|_{\mathrm{fill},\mathbb{Z}}.$$

The desired result then follows from the fact that $\left\|P_k^n(v_1,\dots,v_{k-1},\mathbf{0})\right\|_{\mathrm{fill},\mathbb{Z}} \leqslant C_{k-1}$, by Claim 4.2.1, and

$$\|\partial Q\|_{\mathrm{fill},\mathbb{Z}} \leqslant \|Q\|_1 = \|\pi_*(p_*(R(P)))\|_1 \leqslant (k+1)\left\|P_k^{n+1}(\iota(v_1),\dots,\iota(v_{k-1}),w_k')\right\|_1 \leqslant (k+1)!,$$

by Lemma 4.5. □

The following is a direct application of the previous lemma in the particular case in which parallelogram-like cycles are rectangle-like.

**Lemma 4.12.** *Let $n \in \mathbb{N}$. There exists a constant $C_n$ such that for any $a_1,\dots,a_n,a_i',a_i'' \in \mathbb{Z}$ with $a_i = a_i' + a_i''$ it holds that*

$$\left\|R_n(a_1,\dots,a_n) - R_n(a_1,\dots,a_i',\dots,a_n) - R_n(a_1,\dots,a_i'',\dots,a_n)\right\|_{fill,\mathbb{Z}} \leqslant C_n.$$

## 4.3 | Slim parallelogram-like cycles

In this subsection, we deal with parallelogram-like cycles $P_k^n(v_1,\dots,v_k)$ whose generating vectors $v_1,\dots,v_k \in \mathbb{Z}^n$ are linearly dependent. Another way to state this condition is by requiring that there exists a map $\iota\colon T^{k-1} \to T^n$ for which $P_k^n(v_1,\dots,v_k)$ is in the image of the map $\iota_*\colon Z_k(T^{k-1};\mathbb{Z}) \to Z_k(T^n;\mathbb{Z})$. As $H_k(T^{k-1};\mathbb{Z})$ is trivial, these kinds of cycles are boundaries. In this subsection, we control the filling norm of these parallelogram-like cycles in terms of their sizes.

We start with 2-dimensional cycles in $S^1$.

**Lemma 4.13.** *There exists a constant $C_2 > 0$ such that for any nonzero integers $a, l$ it holds that*

$$\left\|P_2^1(a,l)\right\|_{fill,\mathbb{Z}} \leqslant C_2\log_2(al) + C_2.$$

The following claims, together with Lemma 4.10 and Claim 4.2.1, provide paths with controlled length inside the graph of cycles from the parallelogram-like cycle $P_2^1(a,l)$ to the zero chain.

*Claim* 4.3.1. There exists a constant $C_2$ such that for every $x, y \in \mathbb{Z}$ and $k \in \{0,1,2,3\}$ it holds that $\left\|P_2^1(x,y) - P_2^1(x, y-kx)\right\|_{\mathrm{fill},\mathbb{Z}} \leqslant C_2$.

*Proof.* Set $c_2 = R_2(1,1)$, and consider the Dehn twist $\varphi : T^2 \to T^2$ on the torus $T^2$ along the curve $\gamma : S^1 \to T^2$ given by $\gamma(t) = (t, 0)$ (where we are using the identification $S^1 = \mathbb{R}/\mathbb{Z}$ and $T^2 = \mathbb{R}^2/\mathbb{Z}^2$). As $c_2$ and $(\varphi^{-k})_*(c_2)$ are fundamental cycles for $T^2$, they are cobordant. We set

$$C_2 = \max_{k=0,1,2,3}\left\{\left\|c_2 - (\varphi^{-k})_*(c_2)\right\|_{\mathrm{fill},\mathbb{Z}}\right\}.$$

If $p_{x,y} : T^2 \to S^1$ denotes the map defined by $p((s,t)) = xs + yt$, then

$$\left\|P_2^1(x,y) - P_2^1(x,y-kx)\right\|_{\mathrm{fill},\mathbb{Z}} = \left\|(p_{x,y})_*(c_2 - (\varphi^{-k})_*(c_2))\right\|_{\mathrm{fill},\mathbb{Z}} \leqslant C_2$$

holds for any $k = 0,1,2,3$. □

*Claim* 4.3.2. There exists a constant $C_2$ such that for every $x \in \mathbb{Z}$ and $y \in 2\mathbb{Z}$ it holds that $\left\|P_2^1(x,y) - P_2^1(2x, y/2)\right\|_{\mathrm{fill},\mathbb{Z}} \leqslant C_2$. Similarly, for every $x \in 2\mathbb{Z}$ and $y \in \mathbb{Z}$, the same also holds for the quantity $\left\|P_2^1(x,y) - P_2^1(x/2, 2y)\right\|_{\mathrm{fill},\mathbb{Z}}$.

*Proof of Claim* 4.3.2. This claim is essentially proved by Frigerio and the first author [6, proof of Step 2]. In the paper, the authors consider the difference $P_2^1\left(2^i, 2^{2k-i}\right) - P_2^1\left(2^{i+1}, 2^{2k-i-1}\right)$ for $k \in \mathbb{N}$ and for every $i = 0, \dots, 2k$. Notice that in the second cycle is obtained from the first by doubling one side and halving the other. They show that such a difference can be represented as a sum $(\varphi_k^i)_*(\partial\alpha) + (\varphi_k^{i+1})_*(\partial\beta)$ where $\varphi_k^i : T^2 \to S^1$ are continuous maps and $\alpha, \beta \in C_3(T^2;\mathbb{Z})$ do not depend on $k$ and $i$. Thus, one has that $\left\|P_2^1\left(2^i, 2^{2k-i}\right) - P_2^1\left(2^{i+1}, 2^{2k-i-1}\right)\right\|_{\mathrm{fill},\mathbb{Z}} \leqslant \|\alpha - \beta\|_1$, a constant that does not depend on $i$ or $k$. The claim above can be proved in the exact same way, considering the difference $P_2^1(x,y) - P_2^1(2x, y/2)$ where one starts with generic lengths $x \in \mathbb{Z}$ and $y \in 2\mathbb{Z}$ with a slight modification of the maps $\varphi_k^i$. □

Now we have all the ingredients to prove Lemma 4.13.

*Proof of Theorem* 4.13. Up to applying Lemma 4.9, we can assume that $a, l$ are two positive integers. We begin by showing that $\left\|P_2^1(1,l)\right\| \leqslant C_2 \log_2 l + C_1 + 2C_2$.

Set $x_0 = 1, y_0 = l$. For $i \geqslant 0$, if $y_i \geqslant x_i$ we set

$$(x_{i+1}, y_{i+1}) = \left(2x_i, \frac{y_i - k_i x_i}{2}\right),$$

where $k_i \in \{0,1,2,3\}$ and $k_i \equiv y_i/2^i \bmod 4$. We now explain that this is well-defined.

At each step $i$, we have $x_i = 2^i$ and $y_i \leqslant l/2^i$, so that there exists an integer $M \leqslant 1 + \frac{\log_2 l}{2}$ such that $y_M < x_M$.

Moreover, at the $i$th step of the iteration, $y_i$ is an integer divisible by $2^i$ (so that the sequence is well-defined): indeed, this is true for $i = 0$. Suppose by induction that for $1 \leqslant i \leqslant M$ the element $y_{i-1}$ is an integer divisible by $2^{i-1}$; by definition of $k_{i-1}$, there exists a nonnegative integer $h$ such that $y_{i-1} = (k_{i-1} + 4h)2^{i-1}$, so that

$$y_i = \frac{y_{i-1} - k_{i-1}x_{i-1}}{2} = 2^i h \geqslant 0.$$

In particular, as $0 \leqslant y_M < x_M = 2^M$ and $y_M$ is divisible by $2^M$, we have $y_M = 0$.

Now, by Claims 4.3.1 and 4.3.2 above we have that for every $i < M$ it holds that

$$\begin{aligned}&\left\|P_2^1(x_i, y_i) - P_2^1\left(x_{i+1}, y_{i+1}\right)\right\|_{\mathrm{fill},\mathbb{Z}} \leqslant \\ &\qquad \leqslant \left\|P_2^1(x_i, y_i) - P_2^1(x_i, y_i - k_i x_i)\right\|_{\mathrm{fill},\mathbb{Z}} + \left\|P_2^1(x_i, y_i - k_i x_i) - P_2^1\left(x_{i+1}, y_{i+1}\right)\right\| \leqslant 2C_2,\end{aligned}$$

and thus, by Claim 4.2.1, we have that

$$\begin{aligned}\left\|P_2^1(1,l)\right\|_{\mathrm{fill},\mathbb{Z}} &\leqslant \left(\sum_{i=0}^{M-1} \left\|P_2^1(x_i, y_i) - P_2^1\left(x_{i+1}, y_{i+1}\right)\right\|_{\mathrm{fill},\mathbb{Z}}\right) + \left\|P_2^1(x_M, 0)\right\|_{\mathrm{fill},\mathbb{Z}} \\ &\leqslant 2MC_2 + C_1 \leqslant C_2 \log_2 l + C_1 + 2C_2.\end{aligned}$$

Let us now consider the general case, with $a > 1$ and $l \geqslant 1$. We are going to prove that there is some positive integer $L \leqslant 2al$ for which $\left\|P_2^1(a,l) - P_2^1(1,L)\right\|_{\mathrm{fill},\mathbb{Z}}$ can be controlled (up to constants) by $\log_2(al)$. Then, since

$$\left\|P_2^1(a,l)\right\|_{\mathrm{fill},\mathbb{Z}} \leqslant \left\|P_2^1(a,l) - P_2^1(1,L)\right\|_{\mathrm{fill},\mathbb{Z}} + \left\|P_2^1(1,L)\right\|_{\mathrm{fill},\mathbb{Z}},$$

we can use the first part of the proof on $P_2^1(1,L)$ to conclude.

The idea is similar to the previous part of the proof: we consider a recursive sequence of parallelogram-like chains $s_i = P_2^1\left(a_i, 2^i l\right)$, where $a_i$ is the integral part $\lfloor a_{i-1}/2 \rfloor$ of $a_{i-1}/2$; if $a_{i-1}$ is odd we also get a remainder $r_i = P_2^1\left(1, 2^i l\right)$, intuitively obtained by cutting a "slim" parallelogram from $s_{i-1}$. At the end of this recursion, we get a chain obtained by the sum of the remainders $r_i$ with a chain $s_N$; all of them are parallelogram-like chains of type $P_2^1(1,x)$. Thanks to Lemma 4.10, the sum of these parallelogram-like cycles is close (with respect to the filling norm) to a single parallelogram-like cycle $P_2^1(1,L)$.

Consider the following sequence of numbers:

$$\begin{cases} a_0 = a, \\ a_i = \frac{a_{i-1}}{2} & \text{if } a_{i-1} \text{ is even,} \\ a_i = \frac{a_{i-1}-1}{2} & \text{if } a_{i-1} \text{ is odd.}\end{cases}$$

Let $N$ be the integer for which $a_N = 1$. Notice that $a_i \leqslant a/2^i$ for all $i$, and thus $N \leqslant \log_2(a)$. Let $I_e$ and $I_o$ be the sets of indices $i$ (with $0 \leqslant i < N$) for which $a_i$ is even or odd, respectively.

Let us set

$$s_i = P_2^1(a_i, 2^i l) \qquad \text{for } 0 \leqslant i \leqslant N,$$
$$r_i = P_2^1(1, 2^i l) \qquad \text{for } i \in I_o.$$

Note that by Claim 4.3.2 for $i \in I_e$ it holds that

$$\|s_i - s_{i+1}\|_{\text{fill},\mathbb{Z}} = \left\| P_2^1(a_i, 2^i l) - P_2^1\left(\frac{a_i}{2}, 2^{i+1} l\right) \right\|_{\text{fill},\mathbb{Z}} \leqslant C_2,$$

while for $i \in I_o$, by Lemma 4.10 and Claim 4.3.2, we have

$$\begin{aligned}\|s_i - r_i - s_{i+1}\|_{\text{fill},\mathbb{Z}} \leqslant & \left\| P_2^1(a_i, 2^i l) - P_2^1(1, 2^i l) - P_2^1(a_i - 1, 2^i l) \right\|_{\text{fill},\mathbb{Z}} \\ & + \left\| P_2^1(a_i - 1, 2^i l) - P_2^1\left(\frac{a_i - 1}{2}, 2^{i+1} l\right) \right\|_{\text{fill},\mathbb{Z}} \\ \leqslant & 2C_2.\end{aligned}$$

If we set

$$L := l\left(2^N + \sum_{i \in I_o} 2^i\right) < l \cdot 2^{N+1} \leqslant 2la,$$

then by inductively applying Lemma 4.10, one also gets

$$\left\| P_2^1(1, L) - s_N - \sum_{i \in I_o} r_i \right\|_{\text{fill},\mathbb{Z}} \leqslant C_2 \cdot |I_o|.$$

Thus, by noticing that

$$P_2^1(a, l) - P_2^1(1, L) = \sum_{i \in I_e} (s_i - s_{i+1}) + \sum_{i \in I_o} (s_i - s_{i+1} - r_i) - \left( P_2^1(1, L) - s_N - \sum_{i \in I_o} r_i \right),$$

we get

$$\begin{aligned}\left\| P_2^1(a, l) \right\|_{\text{fill},\mathbb{Z}} \leqslant & \left\| P_2^1(a, l) - P_2^1(1, L) \right\|_{\text{fill},\mathbb{Z}} + \left\| P_2^1(1, L) \right\|_{\text{fill},\mathbb{Z}} \\ \leqslant & \sum_{i \in I_e} \|s_i - s_{i+1}\|_{\text{fill},\mathbb{Z}} + \sum_{i \in I_o} \|s_i - s_{i+1} - r_i\|_{\text{fill},\mathbb{Z}} \\ & + \left\| P_2^1(1, L) - s_N - \sum_{i \in I_o} r_i \right\|_{\text{fill},\mathbb{Z}} + \left\| P_2^1(1, L) \right\|_{\text{fill},\mathbb{Z}} \\ \leqslant & C_2 \cdot |I_e| + 2C_2 \cdot |I_o| + C_2 \cdot |I_o| + C_2 \log_2 L + C_1 + 2C_2\end{aligned}$$

$$
\begin{aligned}
&\leqslant 4C_2N + C_2\log(2al) + C_1 + 2C_2\\
&\leqslant 4C_2\log_2 a + C_2\log(2al) + C_1 + 2C_2
\end{aligned}
$$

as desired. □

We now prove the same statement in any dimension.

**Lemma 4.14.** *There exists a constant $C_k > 0$ such that for any integers $k > n \geqslant 1$ and any $v_1, \dots, v_k \in \mathbb{Z}^n$ it holds that*

$$
\left\| P_k^n(v_1, \dots, v_k) \right\|_{fill,\mathbb{Z}} \leqslant C_k \cdot \log_2(\max\{\|v_1\|_\infty, \dots, \|v_k\|_\infty\}) + C_k.
$$

*Proof.* We first prove the statement when $k = n+1$ by induction on $n$: Lemma 4.13 already gives the statement on $T^1 = S^1$ since $\log_2(al) \leqslant 2\log_2(\max\{|a|, |l|\})$, so let us fix $n \geqslant 2$ and assume that there exists a constant $V_n$ as in the statement; the goal is to prove the result in dimension $n+1$.

*Claim* 4.3.3. If $v_1, \dots, \widehat{v_i}, \dots, v_{n+1}$ are linearly dependent for some $i = 1, \dots, n+1$, then

$$
\left\| P_{n+1}^n\left(v_1, \dots, v_{n+1}\right) \right\|_{\mathrm{fill},\mathbb{Z}} \leqslant (n+2)V_n \cdot \log_2(\max\{\|v_1\|_\infty, \dots, \|v_{n+1}\|_\infty\}) + (n+2)V_n + C_{n+1}.
$$

Indeed, in this case, the parallelogram-like cycle $P_n^n\left(v_1, \dots, \widehat{v_i}, \dots, v_{n+1}\right)$ is contained in some torus of dimension $n-1$ inside $T^n$. By inductive hypothesis,4.2 and 4.9, we have

$$
\begin{aligned}
\left\| P_{n+1}^n\left(v_1, \dots, v_{n+1}\right) \right\|_{\mathrm{fill},\mathbb{Z}} &\leqslant \left\| P_{n+1}^n\left(v_1, \dots, \widehat{v_i}, \dots, v_{n+1}, v_i\right) \right\|_{\mathrm{fill},\mathbb{Z}} + C_{n+1}\\
&= \left\| F^{v_i}(P_n^n\left(v_1, \dots, \widehat{v_i}, \dots, v_{n+1}\right)) \right\|_{\mathrm{fill},\mathbb{Z}} + C_{n+1}\\
&\leqslant (n+2)\left\| P_n^n\left(v_1, \dots, \widehat{v_i}, \dots, v_{n+1}\right) \right\|_{\mathrm{fill},\mathbb{Z}} + C_{n+1}\\
&\leqslant (n+2)V_n \cdot \log_2(\max\{\|v_1\|_\infty, \dots, \|v_{n+1}\|_\infty\}) + (n+2)V_n + C_{n+1},
\end{aligned}
$$

proving the claim.

Let us now consider $v_1, \dots, v_{n+1} \in \mathbb{Z}^n$, with $v_1, \dots, v_n$ linearly independent, and let $v_{n+1} = v_{n+1}^{/\!/} + v_{n+1}^{\perp}$, where $v_{n+1}^{/\!/}$ is parallel to $v_n$, while $v_{n+1}^{\perp}$ is contained in the subspace generated by $v_1, \dots, v_{n-1}$. Then, as the sets $\{v_1, \dots, v_{n-1}, v_{n+1}^{\perp}\}$ and $\{v_2, \dots, v_n, v_{n+1}^{/\!/}\}$ consist of linearly dependent vectors, one gets from Lemma 4.10, Claim 4.3.3, and the triangular inequality that

$$
\begin{aligned}
&\left\| P_{n+1}^n\left(v_1, \dots, v_{n+1}\right) \right\|_{\mathrm{fill},\mathbb{Z}} \leqslant\\
&\leqslant \left\| P_{n+1}^n\left(v_1, \dots, v_{n+1}\right) - P_{n+1}^n\left(v_1, \dots, v_n, v_{n+1}^{\perp}\right) - P_{n+1}^n\left(v_1, \dots, v_n, v_{n+1}^{/\!/}\right) \right\|_{\mathrm{fill},\mathbb{Z}}\\
&+ \left\| P_{n+1}^n\left(v_1, \dots, v_n, v_{n+1}^{\perp}\right) \right\|_{\mathrm{fill},\mathbb{Z}} + \left\| P_{n+1}^n\left(v_1, \dots, v_n, v_{n+1}^{/\!/}\right) \right\|_{\mathrm{fill},\mathbb{Z}}\\
&\leqslant C_{n+1} + 2(n+2)V_n \cdot \log_2(\max\{\|v_1\|_\infty, \dots, \|v_{n+1}\|_\infty\}) + 2(n+2)V_n + 2C_{n+1},
\end{aligned}
$$

as desired.

Consider now $k \geqslant n+1$, and by induction we suppose that there exists a constant $V_k$ as in the statement. Then, for any $v_1, \dots, v_{k+1} \in \mathbb{Z}^n$ one has

$$\begin{aligned}\left\|P^n_{k+1}(v_1, \dots, v_{k+1})\right\|_{\text{fill},\mathbb{Z}} &= \left\|F^{v_{k+1}}(P^n_k(v_1, \dots, v_k))\right\|_{\text{fill},\mathbb{Z}} \\ &\leqslant (k+2)\left\|P^n_k(v_1, \dots, v_k)\right\|_{\text{fill},\mathbb{Z}} \\ &\leqslant (k+2)V_k \log_2(\max\{\|v_1\|_\infty, \dots, \|v_k\|_\infty\}) + (k+2)V_k,\end{aligned}$$

as desired. □

## 4.4 | Rectangle-like cycles

In this subsection, we explore properties that hold specifically for rectangle-like cycles.

The next lemma shows that any $n$-dimensional parallelogram-like cycle in $T^n$ is close to a sum of rectangle-like cycles. Recall that the sizes of a rectangle-like cycle $R_n(a_1, \dots, a_n)$ are the integers $a_1, \dots, a_n \in \mathbb{Z}$.

**Lemma 4.15.** *For any positive integer $n \in \mathbb{N}$ there exists a constant $C_n$ such that the following holds: for any $v_1, \dots, v_n \in \mathbb{Z}^n$ there are $n!$ rectangle-like cycles $r_1, \dots, r_{n!}$ whose sizes are bounded from above by* $\max\{||v_1||_\infty, \dots, ||v_n||_\infty\}$ *and such that*

$$\left\|P^n_n(v_1, \dots, v_n) - \sum_{i=1}^{n!} \varepsilon_i r_i\right\|_{fill,\mathbb{Z}} \leqslant C_n \cdot \log_2(\max\{||v_1||_\infty, \dots, ||v_n||_\infty\}) + C_n,$$

*for some $\varepsilon_i \in \{\pm 1\}$.*

*Proof.* The argument is by induction on the dimension: the statement is clearly true in dimension 1. We fix $n > 1$ and suppose that there is a constant $V_m$ as in the statement in any dimension $1 \leqslant m < n$. Let $e_1, \dots, e_n$ be the canonical basis of $\mathbb{R}^n$, and let $v_1, \dots, v_n \in \mathbb{Z}^n$. For any $i = 1, \dots, n$, let $v_i^\perp$ be the projection of $v_i$ on the hyperplane orthogonal to $e_n$ and $v_i^{/\!/} = v_i - v_i^\perp$.

Let $\mathcal{P}_n$ be the power set of $\{1, \dots, n\}$ and let $\mathcal{U}_n \subset \mathcal{P}_n$ be the subset consisting only of one-element subsets. For any $S \in \mathcal{P}_n$, we denote by $\mathbf{v}^S$ the $n$-tuple $(v_1^{\theta_1}, \dots, v_n^{\theta_n})$, where $\theta_i = /\!/$ if $i \in S$, otherwise $\theta_i = \perp$.

For any $S_i := \{i\} \in \mathcal{U}_n$, the $n$-tuple $\mathbf{v}^{S_i}$ has $n-1$ vectors $v_1^\perp, \dots, v_{i-1}^\perp, v_{i+1}^\perp, \dots, v_n^\perp$ contained in the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ orthogonal to $e_n$. Thus, the parallelogram-like cycle $P^n_{n-1}\left(v_1^\perp, \dots, v_{i-1}^\perp, v_{i+1}^\perp, \dots, v_n^\perp\right)$ can be seen as a parallelogram-like cycle contained in the embedded torus $T^{n-1}$ inside $T^n$ corresponding to the subspace of $\mathbb{R}^n$ orthogonal to $e_n$. By applying the inductive hypothesis, we get a chain $c_i' \in Z_{n-1}(T^{n-1}; \mathbb{Z}) \subset Z_{n-1}(T^n; \mathbb{Z})$ that is a sum (or difference) of at most $(n-1)!$ rectangle-like cycles whose sizes are bounded from above by

$$\max\left\{||v_1^\perp||_\infty, \dots, ||v_{i-1}^\perp||_\infty, ||v_{i+1}^\perp||_\infty, \dots, ||v_n^\perp||_\infty\right\} \leqslant \max\{||v_1||_\infty, \dots, ||v_n||_\infty\}$$

and such that

$$\left\|P^n_{n-1}\left(v_1^\perp,\dots,v_{i-1}^\perp,v_{i+1}^\perp,\dots,v_n^\perp\right)-c_i'\right\|_{\mathrm{fill},\mathbb{Z}}\leqslant V_{n-1}\log_2(\max\{||v_1||_\infty,\dots,||v_n||_\infty\})+V_{n-1}.$$

Let us set $c_i:=-F^{v_i^{/\!/}}(c_i')$ for $i=1,\dots,n-1$ and $c_n:=F^{v_n^{/\!/}}(c_n')$. As $c_i'$ is a sum (or difference) of rectangle-like cycles of dimension $n-1$ whose canonical lifts are contained in the subspace orthogonal to $e_n$, the cycle $c_i$ is still a sum (or difference) of rectangle-like cycles for any $i=1,\dots,n$. By Lemmas 4.2 and 4.9, for $i\neq n$ we have that

$$\begin{aligned}\left\|P^n_n\left(\mathbf{v}^{S_i}\right)-c_i\right\|_{\mathrm{fill},\mathbb{Z}}\leqslant&+\left\|P^n_n\left(\mathbf{v}^{S_i}\right)+P^n_n\left(v_1^\perp,\dots,v_{i-1}^\perp,v_{i+1}^\perp,\dots,v_n^\perp,v_i^{/\!/}\right)\right\|_{\mathrm{fill},\mathbb{Z}}\\&+\left\|-P^n_n\left(v_1^\perp,\dots,v_{i-1}^\perp,v_{i+1}^\perp,\dots,v_n^\perp,v_i^{/\!/}\right)-c_i\right\|_{\mathrm{fill},\mathbb{Z}}\\\leqslant&C_n+\left\|-F^{v_i^{/\!/}}\left(P^n_{n-1}\left(v_1^\perp,\dots,v_{i-1}^\perp,v_{i+1}^\perp,\dots,v_n^\perp\right)-c_i'\right)\right\|_{\mathrm{fill},\mathbb{Z}}\\\leqslant&C_n+(n+1)\left\|P^n_{n-1}\left(v_1^\perp,\dots,v_{i-1}^\perp,v_{i+1}^\perp,\dots,v_n^\perp\right)-c_i'\right\|_{\mathrm{fill},\mathbb{Z}}\\\leqslant&C_n+(n+1)V_{n-1}\log_2(\max\{||v_1||_\infty,\dots,||v_n||_\infty\})+(n+1)V_{n-1},\end{aligned}$$

while for $i=n$ we have (analogously as before) that

$$\begin{aligned}\left\|P^n_n\left(\mathbf{v}^{S_n}\right)-c_n\right\|_{\mathrm{fill},\mathbb{Z}}\leqslant&\left\|P^n_n\left(v_1^\perp,\dots,v_{n-1}^\perp,v_n^{/\!/}\right)-c_n\right\|_{\mathrm{fill},\mathbb{Z}}\\\leqslant&(n+1)V_{n-1}\log_2(\max\{||v_1||_\infty,\dots,||mv_n||_\infty\})+(n+1)V_{n-1}.\end{aligned}$$

Thus, by setting $c:=\sum_{i=1}^n c_i$ and by applying iteratively the triangular inequality, one gets

$$\begin{aligned}&\left\|P^n_n(v_1,\dots,v_n)-c\right\|_{\mathrm{fill},\mathbb{Z}}=\\&\leqslant\left\|P^n_n(v_1,\dots,v_n)-\sum_{i=1}^n P^n_n\left(\mathbf{v}^{S_i}\right)\right\|_{\mathrm{fill},\mathbb{Z}}+\sum_{i=1}^n\left\|P^n_n\left(\mathbf{v}^{S_i}\right)-c_i\right\|_{\mathrm{fill},\mathbb{Z}}\\&\leqslant\left\|P^n_n(v_1,\dots,v_n)-\sum_{S\in\mathcal{P}_n}P^n_n\left(\mathbf{v}^S\right)\right\|_{\mathrm{fill},\mathbb{Z}}+\sum_{S\in\mathcal{P}_n\setminus\mathcal{U}_n}\left\|P^n_n\left(\mathbf{v}^S\right)\right\|_{\mathrm{fill},\mathbb{Z}}\\&\quad+nC_n+n(n+1)V_{n-1}\log_2(\max\{||v_1||_\infty,\dots,||v_n||_\infty\})+n(n+1)V_{n-1}.\end{aligned}$$

We deal with $\left\|P^n_n(v_1,\dots,v_n)-\sum_{S\in\mathcal{P}_n}P^n_n\left(\mathbf{v}^S\right)\right\|_{\mathrm{fill},\mathbb{Z}}$ by applying iteratively Lemma 4.10 to $P^n_n(v_1,\dots,v_n)=P^n_n\left(v_1^{/\!/}+v_1^\perp,\dots,v_n^{/\!/}+v_n^\perp\right)$ for each generating vector.

For any set $S\in\mathcal{P}_n\setminus\mathcal{U}_n$, the $n$-tuple $\mathbf{v}^S$ consists of linearly dependent vectors whose $\ell^\infty$-norm is bounded from above by $\max\{||v_1||_\infty,\dots,||v_n||_\infty\}$ and thus, by applying Lemma 4.14, we also get an upper bound for $\left\|P^n_n\left(\mathbf{v}^S\right)\right\|_{\mathrm{fill},\mathbb{Z}}$ for each $S\in P_n\setminus\mathcal{U}_n$.

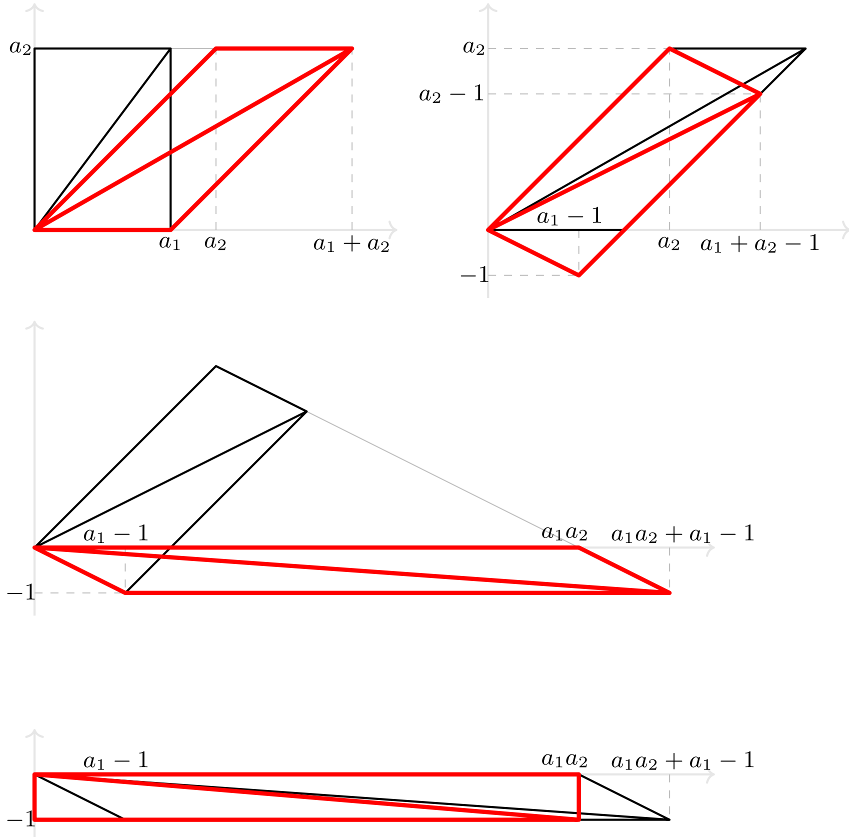


**FIGURE 3** To prove Lemma 4.16 for $n = 2$, we apply the slidings of Claim 4.4.1 four times as shown in the picture, where at each step the black parallelogram is the starting one, and the red parallelogram is the one we obtain after the application of Claim 4.4.1. Notice that in the case $a_1 = 1$, we already have a rectangle-like cycle at the third application of Claim 4.4.1.

Putting everything together we get the statement. □

Rectangle-like cycles are useful because they are close to rectangle-like cycles with all sizes but one equal to 1.

**Lemma 4.16.** *There exists a constant $C_n$ such that for any integers $a_1, \dots, a_n \in \mathbb{Z}$, not all zero, it holds that*

$$\left\| R_n(a_1, \dots, a_n) - R_n\left(\prod_{i=1}^{n} a_i, 1, \dots, 1\right) \right\|_{fill,\mathbb{Z}} \leqslant C_n \log_2(\max\{|a_1|, \dots, |a_n|\}) + C_n.$$

This lemma is proved via an inductive argument on the dimension $n$. In order to prove the base case $n = 2$ we need the following claim, which is a consequence of Lemma 4.10, and allows us to "slide" a parallelogram-like cycle along one of its sides (see Figure 3).

*Claim* 4.4.1. There exists a constant $C$ such that for any vectors $v, w \in \mathbb{Z}^2$ and any integers $n, d \in \mathbb{Z} \setminus \{0\}$ with $\frac{n}{d} v \in \mathbb{Z}^2$, it holds that

$$\left\| P_2^2\left(v, w + \frac{n}{d} v\right) - P_2^2(v, w) \right\|_{\mathrm{fill}, \mathbb{Z}} \leqslant C \log_2 |dn| + C$$

and

$$\left\| P_2^2(w, v) - P_2^2\left(w + \frac{n}{d} v, v\right) \right\|_{\mathrm{fill}, \mathbb{Z}} \leqslant C \log_2 |dn| + C.$$

*Proof.* Indeed, by the triangular inequality we have

$$\begin{aligned} &\left\| P_2^2\left(v, w + \frac{n}{d} v\right) - P_2^2(v, w) \right\|_{\mathrm{fill}, \mathbb{Z}} \leqslant \\ &\qquad \leqslant \left\| P_2^2\left(v, w + \frac{n}{d} v\right) - P_2^2(v, w) - P_2^2\left(v, \frac{n}{d} v\right) \right\|_{\mathrm{fill}, \mathbb{Z}} + \left\| P_2^2\left(v, \frac{n}{d} v\right) \right\|_{\mathrm{fill}, \mathbb{Z}}, \end{aligned}$$

where

$$\left\| P_2^2\left(v, w + \frac{n}{d} v\right) - P_2^2(v, w) - P_2^2\left(v, \frac{n}{d} v\right) \right\|_{\mathrm{fill}, \mathbb{Z}} \leqslant C_2$$

due to Lemma 4.10. Moreover, if we define $\iota_v : S^1 \to T^2$ by $\iota_v(t) = \frac{t}{d} v$ (where we are using the identifications $S^1 = \mathbb{R}/\mathbb{Z}$ and $T^2 = \mathbb{R}^2/\mathbb{Z}^2$), then $(\iota_v)_*(P_2^1(d, n)) = P_2^2\left(v, \frac{n}{d} v\right)$. By Lemma 4.13, we get

$$\left\| P_2^2\left(v, \frac{n}{d} v\right) \right\|_{\mathrm{fill}, \mathbb{Z}} \leqslant \left\| P_2^1(d, n) \right\|_{\mathrm{fill}, \mathbb{Z}} \leqslant C_2 \log_2 |dn| + C_2.$$ □

We can now proceed with the proof of Lemma 4.16.

*Proof of Lemma* 4.16. The statement is trivial for $n = 1$. Moreover, notice that if $a_i = 0$ for some $i$, then the statement is a consequence of Lemma 4.9 and Claim 4.2.1. Let us then suppose $n \geqslant 2$ and $a_1, \ldots, a_n \in \mathbb{Z} \setminus \{0\}$.

As said before, the proof is an induction on $n$ with base case $n = 2$. Let us start by proving the result for $n = 2$ in the case $a_1 \neq 1$. We apply Claim 4.4.1 four times as follows (see Figure 3):

(1) with the notation of Claim 4.4.1, consider $v = a_1 e_1$, $w = a_2 e_2$, $n = a_2$, and $d = a_1$; then, recalling that $R_2(a_1, a_2) = P_2^2(a_1 e_1, a_2 e_2)$, we have that

$$\left\| R_2(a_1, a_2) - P_2^2(a_1 e_1, a_2 e_1 + a_2 e_2) \right\|_{\mathrm{fill}, \mathbb{Z}} \leqslant C \log_2 |a_1 a_2| + C;$$

(2) consider $v = a_2 e_1 + a_2 e_2$, $w = a_1 e_1$, $n = -1$, and $d = a_2$; then

$$\left\| P_2^2(a_1 e_1, a_2 e_1 + a_2 e_2) - P_2^2((a_1 - 1) e_1 - e_2, a_2 e_1 + a_2 e_2) \right\|_{\mathrm{fill}, \mathbb{Z}} \leqslant C \log_2 |a_2| + C;$$

(3) consider $v = (a_1 - 1)e_1 - e_2$, $w = a_2e_1 + a_2e_2$, $n = a_2$ and $d = 1$; then

$$\left\| P_2^2((a_1-1)e_1 - e_2, a_2e_1 + a_2e_2) - P_2^2((a_1-1)e_1 - e_2, a_1a_2e_1) \right\|_{\mathrm{fill},\mathbb{Z}} \leqslant C\log_2|a_2| + C;$$

(4) consider $v = a_1a_2e_1$, $n = -a_1 + 1$, $w = (a_1 - 1)e_1 - e_2$ and $d = a_1a_2$; then

$$\left\| P_2^2((a_1-1)e_1 - e_2, a_1a_2e_1) - P_2^2(-e_2, a_1a_2e_1) \right\|_{\mathrm{fill},\mathbb{Z}} \leqslant C\log_2|(1-a_1)a_1a_2| + C.$$

Moreover, by applying twice Lemma 4.9 we have that

$$\left\| P_2^2(-e_2, a_1a_2e_1) - P_2^2(a_1a_2e_1, e_2) \right\|_{\mathrm{fill},\mathbb{Z}} \leqslant 2C_2.$$

Putting everything together, by the triangle inequality we get

$$\begin{aligned}
&\|R_2(a_1,a_2) - R_2(a_1a_2,1)\|_{\mathrm{fill},\mathbb{Z}} \leqslant \\
&\qquad \leqslant + \left\| R_2(a_1,a_2) - P_2^2(a_1e_1, a_2e_1 + a_2e_2) \right\|_{\mathrm{fill},\mathbb{Z}} \\
&\qquad\quad + \left\| P_2^2(a_1e_1, a_2e_1 + a_2e_2) - P_2^2((a_1-1)e_1 - e_2, a_2e_1 + a_2e_2) \right\|_{\mathrm{fill},\mathbb{Z}} \\
&\qquad\quad + \left\| P_2^2((a_1-1)e_1 - e_2, a_2(e_1 + e_2)) - P_2^2((a_1-1)e_1 - e_2, a_1a_2e_1) \right\|_{\mathrm{fill},\mathbb{Z}} \\
&\qquad\quad + \left\| P_2^2((a_1-1)e_1 - e_2, a_1a_2e_1) - P_2^2(-e_2, a_1a_2e_1) \right\|_{\mathrm{fill},\mathbb{Z}} \\
&\qquad\quad + \left\| P_2^2(-e_2, a_1a_2e_1) - R_2(a_1a_2, 1) \right\|_{\mathrm{fill},\mathbb{Z}} \\
&\qquad \leqslant 4C\log_2|(1-a_1)a_1a_2| + 4C + 2C_2 \\
&\qquad \leqslant 4C\log_2(2|a_1a_2|^2) + 4C + 2C_2,
\end{aligned}$$

proving the statement for $a_1 \neq 1$ and $n = 2$.

On the other hand, if $a_1 = 1$ and $n = 2$, then we can apply the same proof without the fourth application of Claim 4.4.1 (that is Item 4).

Let us now consider $n \geqslant 3$, suppose there is a constant $V_m$ in any dimension $m < n$ as in the statement, and consider $a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}$. Then, it holds that

$$\begin{aligned}
&\left\| R_n(a_1,\dots,a_n) - R_n\left(\prod_{i=1}^n a_i, 1, \dots, 1\right) \right\|_{\mathrm{fill},\mathbb{Z}} \leqslant \\
&\qquad \leqslant + \|R_n(a_1,\dots,a_n) - R_n(a_1a_n, a_2, \dots, a_{n-1}, 1)\|_{\mathrm{fill},\mathbb{Z}} \\
&\qquad\quad + \left\| R_n(a_1a_n, a_2, \dots, a_{n-1}, 1) - R_n\left(\prod_{i=1}^n a_i, 1, \dots, 1\right) \right\|_{\mathrm{fill},\mathbb{Z}}.
\end{aligned}$$

By the 2-dimensional case (where we regard $P_2^n(a_1e_1, a_ne_n)$ and $P_2^n(a_1a_ne_1, e_n)$ as rectangle-like cycles contained in a 2-dimensional torus embedded in $T^n$), Lemma 4.9 and an iterative application of the prism operator (Lemma 4.5) we have that

$$\begin{aligned}
&\|R_n(a_1,\dots,a_n) - R_n(a_1a_n, a_2,\dots,a_{n-1},1)\|_{\mathrm{fill},\mathbb{Z}} \leqslant \\
&\qquad \leqslant + \|R_n(a_1,\dots,a_n) + P_n^n(a_1e_1, a_ne_n, a_2e_2,\dots,a_{n-1}e_{n-1})\|_{\mathrm{fill},\mathbb{Z}} \\
&\qquad\quad + \|P_n^n(a_1a_ne_1, e_n, a_2e_2,\dots,a_{n-1}e_{n-1}) - P_n^n(a_1e_1, a_ne_n, a_2e_2,\dots,a_{n-1}e_{n-1})\|_{\mathrm{fill},\mathbb{Z}} \\
&\qquad\quad + \|-P_n^n(a_1a_ne_1, e_n, a_2e_2,\dots,a_{n-1}e_{n-1}) - R_n(a_1a_n, a_2,\dots,a_{n-1},1)\|_{\mathrm{fill},\mathbb{Z}} \\
&\qquad \leqslant 2C_n + \left\|F^{a_{n-1}e_{n-1}}\left(\cdots\left(F^{a_2e_2}(P_2^n(a_1e_1, a_ne_n) - P_2^n(a_1a_ne_1, e_n))\right)\cdots\right)\right\|_{\mathrm{fill},\mathbb{Z}} \\
&\qquad \leqslant 2C_n + (n+1)!\cdot V_2\log_2(\max\{|a_1|,|a_n|\}) + (n+1)!\cdot V_2.
\end{aligned}$$

On the other hand, as above, the parallelogram-like cycles $P_{n-1}^n(a_1a_ne_1, a_2e_2,\dots,a_{n-1}e_{n-1})$ and $P_{n-1}^n(a_1\cdots a_ne_1, e_2,\dots,e_{n-1})$ can be regarded as rectangle-like cycles contained in a $(n-1)$-dimensional torus embedded in $T^n$. By applying the inductive hypothesis one gets

$$\begin{aligned}
&\|R_n(a_1a_n, a_2,\dots,a_{n-1},1) - R_n(a_1\cdots a_n, 1,\dots,1)\|_{\mathrm{fill},\mathbb{Z}} = \\
&\qquad = \left\|F^{e_n}\left(P_{n-1}^n(a_1a_ne_1, a_2e_2,\dots,a_{n-1}e_{n-1}) - P_{n-1}^n(a_1\cdots a_ne_1, e_2,\dots,e_{n-1})\right)\right\|_{\mathrm{fill},\mathbb{Z}} \\
&\qquad \leqslant (n+1)V_{n-1}\log_2(\max\{|a_1a_n|,|a_2|,\dots,|a_n|\}) + (n+1)V_{n-1}.
\end{aligned}$$

Putting the three inequalities above together, we get the desired bound. □

## 4.5 | Proof of Proposition 4.7

Now that we have all the ingredients, we are ready to draw the path in the graph of cycles that goes from the parallelogram-like cycle $P_n^n(v_1,\dots,v_n)$ to the rectangle-like cycle $R_n(\det(A), 1,\dots,1)$, where $A$ is the matrix whose columns are given by $v_1,\dots,v_n$. This will prove Proposition 4.7, which we recall in the following for the convenience of the reader.

**Proposition 4.7.** *Let $n \in \mathbb{N}$. There exists a constant $K_n > 0$ such that for any $v_1,\dots,v_n \in \mathbb{Z}^n$ it holds that*

$$\|P_n^n(v_1,\dots,v_n) - R_n(\det(A),1,\dots,1)\|_{\mathit{fill},\mathbb{Z}} \leqslant K_n\log_2(\|A\|_\infty) + K_n,$$

*where $A$ is the matrix whose columns are given by $v_1,\dots,v_n$.*

The steps of this journey go as follows: we start from $P_n^n(v_1,\dots,v_n)$ and go to a sum of rectangle-like cycles (Lemma 4.15); then, from each of these rectangle-like cycles we go to rectangle-like cycles of unit-height (Lemma 4.16), that we can sum to get a single rectangle-like cycle of unit-height (Lemma 4.12). The following proof makes the journey just described more precise.

*Proof.* By Lemma 4.15, there exists a sum $\sum_{i=1}^{n!} \varepsilon_i r_i'$, where $\varepsilon_i \in \{\pm 1\}$ and $r_i'$ is a rectangle-like cycle with sizes bounded from above by $\max\{||v_1||_\infty, \dots, ||v_n||_\infty\}$ for any $i = 1, \dots, n!$, such that

$$\left\| P_n^n(v_1, \dots, v_n) - \sum_{i=1}^{n!} \varepsilon_i r_i' \right\|_{\mathrm{fill},\mathbb{Z}} \leqslant C_n \cdot \log_2(\max\{||v_1||_\infty, \dots, ||v_n||\_\infty\}) + C_n.$$

By Lemma 4.16, for any $r_i'$ there exists a rectangle-like cycle

$$r_i := R_n(\ell_i, 1, \dots, 1),$$

for which

$$\left\| r_i' - r_i \right\|_{\mathrm{fill},\mathbb{Z}} \leqslant C_n \log_2(\max\{||v_1||_\infty, \dots, ||v_n||\_\infty\}) + C_n.$$

Moreover, by applying Lemma 4.12 iteratively to the sum $\sum_{i=1}^{n!} \varepsilon_i r_i$, we obtain a rectangle-like cycle

$$R := R_n(L, 1, \dots, 1),$$

with $L = \sum_{i=1}^{n!} \varepsilon_i \ell_i$, for which

$$\left\| R - \sum_{i=1}^{n!} \varepsilon_i r_i \right\|_{\mathrm{fill},\mathbb{Z}} \leqslant n! C_n.$$

Thus, by the triangular inequality, we have

$$\begin{aligned}
\left\| P_n^n(v_1, \dots, v_n) - R \right\|_{\mathrm{fill},\mathbb{Z}} \leqslant & + \left\| P_n^n(v_1, \dots, v_n) - \sum_{i=1}^{n!} \varepsilon_i r_i' \right\|_{\mathrm{fill},\mathbb{Z}} \\
& + \sum_{i=1}^{n!} \left\| \varepsilon_i (r_i' - r_i) \right\|_{\mathrm{fill},\mathbb{Z}} + \left\| \sum_{i=1}^{n!} \varepsilon_i r_i - R \right\|_{\mathrm{fill},\mathbb{Z}} \\
& \leqslant (1 + n!) C_n \log_2(\max\{||v_1||_\infty, \dots, ||v_n||_\infty\}) + (1 + 2n!) C_n.
\end{aligned}$$

In particular, $P_n^n(v_1, \dots, v_n)$ and $R$ are cobordant, so that if $A$ (respectively, $B$) denotes the matrix with columns $v_1, \dots, v_n$ (respectively, $Le_1, e_2 \dots, e_n$), then by Lemma 4.6 we get $\det(A) = \det(B) = L$. This concludes the proof. □

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