

Article

Exact Schur-Stability Regions and Computational Controller Selection for Delayed Scalar Discrete-Time Systems

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Abstract

Delayed feedback converts local stabilization of a scalar map with delay τ into a degree- $(\tau + 1)$ Schur-stability problem. Using the classical Kuruklis criterion as the analytical starting point, we derive exact Schur-stability regions in the multiplier-gain plane for fixed-point feedback control (FPFC), time-delayed feedback control (TDFC), and nonlinear predictive feedback control (NFC) as pullbacks of a universal Schur region. A direct unit-circle parametrization yields the curved boundary and supports exact results on same-parity nesting, large-delay intersections, stabilizable-multiplier projections, delay budgets, and necessary-and-sufficient common-gain conditions under interval uncertainty. These results lead to reproducible algorithms for certified pointwise classification, $O(N)$ boundary generation, endpoint-based robust-gain testing, and gain-constrained controller selection from complete feasible-gain intervals. Non-explicit calculations reduce to monotone scalar bisection, avoiding repeated characteristic-root calculations in routine region construction. Independent root computations at 114,000 validation points, including delays up to $\tau = 100$, show no classification mismatch. Logistic-map examples and a non-polynomial tumour-growth map illustrate the local stabilization and controller-selection implications. The resulting framework provides an exact analytical and algorithmic treatment of arbitrary-delay local stabilization in scalar discrete-time systems.

Keywords: Schur stability; delayed feedback; controller selection; stability-region algorithm; discrete-time systems; difference equations; robust gain

MSC: 39A30; 39A60; 93D15; 37C75



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1. Introduction

Delayed feedback provides a simple mechanism for changing the local stability of invariant states in discrete dynamical systems. The classical Pyragas principle introduced delayed self-feedback for non-invasive stabilization [1], and subsequent work developed delayed and multiple-delay schemes for unstable fixed points and periodic orbits of discrete maps [2,3]. Predictive and adaptive variants provide complementary ways of modifying the local multiplier while retaining the uncontrolled target as a fixed point [4,5]. Recent discrete-time applications continue to use delayed feedback to recover local stability in chaotic systems; for example, delayed feedback has recently been applied to a nonlinear discrete Cournot model to restore local asymptotic stability beyond the uncontrolled stability range [6].

For a scalar map, the uncontrolled fixed point is characterized locally by one real multiplier. Introducing a delay τ enriches the controlled problem: the augmented state has dimension $\tau + 1$, and the associated characteristic polynomial grows with the delay. This makes a compact analytical description particularly valuable for systematic exploration of the multiplier–gain plane, where repeated generic Jury or Schur–Cohn calculations become less convenient. For the sparse three-term polynomial arising here, Kuruklis obtained an exact necessary-and-sufficient Schur criterion [7]; related necessary-and-sufficient stability questions for higher-order difference equations have continued to be studied in the difference-equation literature [8]. Schur-based delayed-feedback analysis also has a direct control-theoretic precedent. In particular, Morgül [9] associated delayed feedback for one-dimensional discrete-time systems with a characteristic polynomial and used the Schur–Cohn criterion to obtain stabilizing gain bounds. Together, these results provide a rigorous Schur-theoretic foundation for the present analysis.

Complete stability geometry broadens the design question from testing one prescribed gain at one prescribed multiplier to characterizing the full set of admissible gains, its variation with delay, the unstable multipliers that remain reachable, and the existence of a single gain for a family of nearby multipliers. Related delay-parity and region-contraction phenomena have appeared in other settings, including diffusively coupled map networks [10]. Geometric stability-region methods have also been developed for periodic linear difference equations with delayed feedback [11]. Robustness has likewise been studied for delayed-feedback mechanisms under different uncertainty descriptions and controller structures. Piccirillo et al. [12] considered probabilistic robustness of time-delayed and extended time-delayed feedback under parametric and stochastic uncertainty, while Dmitrishin et al. [13] characterized robustly stabilizable sets of Jacobian eigenvalues for a multi-tap linear delayed-feedback law.

The common-gain problem studied here focuses on a complementary deterministic setting: the uncertainty is a real interval of scalar fixed-point multipliers, one constant gain is required to work throughout that interval, and exact necessary-and-sufficient conditions are obtained separately for fixed-point feedback control (FPFC), time-delayed feedback control (TDFC), and nonlinear predictive feedback control (NFC). This positioning places the present common-gain formulation alongside robustness to additive noise considered in multiple-delay control studies such as [2].

A related feasibility-first viewpoint appears in recent algebraic-control work on stable inversion, where necessary-and-sufficient solvability conditions are established before approximation or performance-oriented design is introduced [14]. Although the underlying systems and objectives differ, both frameworks separate exact feasibility from subsequent design criteria.

Earlier controller-specific stability-region calculations and numerical verification for FPFC, TDFC, and NFC were presented in [15]. The present work develops this foundation into a unified Schur-region framework and extends it through the structural delay results, exact common-gain theory, certified computational procedures, and gain-constrained controller-selection formulation developed below.

The development therefore proceeds through three levels: the universal Schur geometry of the sparse delayed polynomial; its controller-specific pullbacks; and the resulting structural, robustness, and computational consequences. With this analytical structure in place, the main contributions of the paper are as follows:

- (i) A direct unit-circle parametrization and monotone scalar representation of the curved Kuruklis boundary are derived, providing an explicit computational realization of the universal region tailored to controller calculations.

- (ii) The exact FPFC, TDFC, and NFC multiplier–gain domains are written as controller-specific pullbacks of the universal region for every positive integer delay, providing one common parameter-plane description for the three feedback laws.
- (iii) Building on these domains, same-parity nesting is proved and the intersection over unbounded odd or even delays is characterized exactly, including the boundary segments that remain stable for every finite delay.
- (iv) Exact projections onto the uncontrolled multiplier axis give stabilizability intervals, delay-budget formulas, and a parity-shift identity relating odd-delay TDFC and the following even-delay NFC.
- (v) The pointwise domains are extended to deterministic interval uncertainty, yielding necessary-and-sufficient conditions for a single gain to stabilize every multiplier in a prescribed interval.
- (vi) The formulas are converted into reproducible computational procedures for certified pointwise stability classification, direct boundary generation, robust common-gain testing, and gain-constrained controller selection from complete feasible-gain intervals, thereby avoiding repeated computation of all roots of a degree- $(\tau + 1)$ polynomial.

The analytical results are also implemented as lightweight algorithms based on monotone scalar bisection, direct parametric evaluation, and explicit endpoint calculations, offering an efficient alternative to repeated high-order-root calculations. Direct characteristic-root calculations serve as an independent verification. The logistic map illustrates stabilization in chaotic and periodic regimes, while a normalized discrete tumour-growth map demonstrates transfer of the local multiplier-based theory to a non-polynomial system. These examples link the exact stability geometry to representative nonlinear dynamics within the local fixed-point framework.

2. Problem Formulation and Controller Embeddings

Let

$$x_{k+1} = f(x_k), \quad k \in \mathbb{N}_0, \quad (1)$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable in a neighbourhood of a fixed point x^* satisfying $f(x^*) = x^*$. Denote its local multiplier (equivalently, the period-one Floquet multiplier) by

$$a = f'(x^*). \quad (2)$$

The uncontrolled fixed point is locally asymptotically stable when $|a| < 1$ and unstable when $|a| > 1$.

Let $b \in \mathbb{R}$ be the feedback strength and let $\tau \in \mathbb{N}_0$ be an integer delay. We consider the following controllers.

Fixed-point feedback control (FPFC)

$$x_{k+1} = f(x_k) - b(x_{k-\tau} - x^*). \quad (3)$$

The target value appears explicitly. The sign convention is chosen so that the linearized delayed coefficient is $-b$.

Time-delayed feedback control (TDFC)

$$x_{k+1} = f(x_k) + b(x_{k-\tau} - x_k). \quad (4)$$

For period-one stabilization, $\tau = 1$ is the classical Pyragas choice. We retain arbitrary positive delays to determine their parity-dependent effect.

Nonlinear predictive feedback control (NFC)

$$x_{k+1} = (1 - b)f(x_k) + bx_{k-\tau} = f(x_k) + b(x_{k-\tau} - f(x_k)). \quad (5)$$

The controller compares the delayed state with the one-step uncontrolled prediction $f(x_k)$.

All three feedback terms vanish at $x_k = x^*$. Introduce the error $e_k = x_k - x^*$. Linearization gives

$$e_{k+1} = \alpha e_k - \beta e_{k-\tau}, \quad (6)$$

with controller-dependent coefficients

controller	$\alpha(a, b)$	$\beta(a, b)$
FPFC	a	b
TDFC	$a - b$	$-b$
NFC	$a(1 - b)$	$-b$

(7)

For $\tau \geq 1$, define the augmented state

$$E_k = (e_k, e_{k-1}, \dots, e_{k-\tau})^T \in \mathbb{R}^{\tau+1}.$$

Although the nonlinear state variable is scalar, the delayed local problem is therefore a $(\tau + 1)$ -dimensional autonomous system. Increasing the delay raises the degree of the characteristic polynomial and the dimension of the local state simultaneously. Then,

$$E_{k+1} = A_\tau(\alpha, \beta)E_k, \quad (8)$$

where

$$A_\tau(\alpha, \beta) = \begin{pmatrix} \alpha & 0 & \cdots & 0 & -\beta \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \quad (9)$$

Its characteristic polynomial is

$$p_\tau(s; \alpha, \beta) = s^{\tau+1} - \alpha s^\tau + \beta. \quad (10)$$

Proposition 1 (Linearized stability). *Assume that the controlled augmented map is continuously differentiable near $(x^*, \dots, x^*)$. Then,*

1. *If all roots of (10) lie in $\mathbb{D} = \{s \in \mathbb{C} : |s| < 1\}$, the fixed point is locally exponentially stable;*
2. *If at least one root has modulus greater than one, the fixed point is unstable;*
3. *If at least one root lies on the unit circle and none lies outside, the linearization is inconclusive for asymptotic stability.*

Proof. The derivative of the augmented controlled map at the fixed point is $A_\tau(\alpha, \beta)$. The result follows from the standard discrete-time linearization principle [16]. $\square$

Remark 1 (Scope of the stability statements). *Throughout the paper, the terms stability, stabilization, stabilizing gain, and stabilizable multiplier refer to local fixed-point stability in the sense of Proposition 1, unless explicitly stated otherwise. Thus, if (a, b) belongs to one of the derived Schur domains, the controlled fixed point is locally exponentially stable and trajectories starting from a sufficiently small neighbourhood of the augmented fixed point $(x^*, \dots, x^*)$ converge to it. The analysis is therefore centred on the local stability neighbourhood. Basin geometry, attraction from arbitrary initial histories, and nonlinear transients away from the fixed point belong to the global nonlinear dynamics and provide natural directions for further study. The simulations below complement the local theory by illustrating convergence from selected initial histories.*

3. Universal Schur Region and Boundary Geometry

Definition 1 (Universal Schur region). For $\tau \geq 1$, define

$$\mathcal{R}_\tau = \{(\alpha, \beta) \in \mathbb{R}^2 : p_\tau(\cdot; \alpha, \beta) \text{ is Schur stable}\}. \quad (11)$$

For convenience write

$$c_\tau = \frac{\tau + 1}{\tau}. \quad (12)$$

The next theorem states the Kuruklis polynomial criterion in the notation of (10) and serves as the universal analytical foundation used throughout. The subsequent controller domains are obtained by pulling this region back through the controller maps in Section 2, providing the basis for the structural and computational results developed thereafter.

Theorem 1 (Kuruklis Schur criterion [7]). Let $\tau \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}$. All roots of (10) lie in $\mathbb{D}$ if and only if

$$|\alpha| < c_\tau \quad (13)$$

and the corresponding parity-dependent conditions hold. If τ is odd,

$$|\alpha| - 1 < \beta < R_\tau(\alpha), \quad (14)$$

whereas if τ is even,

$$|\beta - \alpha| < 1, \quad |\beta| < R_\tau(\alpha). \quad (15)$$

For $\alpha = 0$, set $R_\tau(0) = 1$. For $0 < |\alpha| < c_\tau$,

$$R_\tau(\alpha) = \left(\alpha^2 + 1 - 2|\alpha| \cos \varphi_\tau(\alpha)\right)^{1/2}, \quad (16)$$

where $\varphi_\tau(\alpha)$ is the unique solution of

$$\frac{\sin((\tau + 1)\varphi)}{\sin(\tau\varphi)} = |\alpha|, \quad 0 < \varphi < \frac{\pi}{\tau + 1}. \quad (17)$$

3.1. A Direct Parametrization of the Curved Boundary

Proposition 2 (Unit-circle parametrization). Let $\tau \in \mathbb{N}$ and let $s = e^{i\theta}$ with $0 < \theta < \pi/(\tau + 1)$. The positive curved branch of the Schur boundary is parameterized by

$$\alpha_\tau(\theta) = \frac{\sin((\tau + 1)\theta)}{\sin(\tau\theta)}, \quad \beta_\tau(\theta) = \frac{\sin \theta}{\sin(\tau\theta)}. \quad (18)$$

Consequently, if $u = |\alpha|$ and $\theta = \varphi_\tau(\alpha)$, then

$$R_\tau(\alpha) = \frac{\sin \theta}{\sin(\tau\theta)}. \quad (19)$$

The remaining curved branches follow from the polynomial symmetry $(\alpha, \beta) \mapsto (-\alpha, (-1)^{\tau+1}\beta)$.

Proof. On the stability boundary, a characteristic root lies on the unit circle. Substituting $s = e^{i\theta}$ into $p_\tau(s) = s^{\tau+1} - \alpha s^\tau + \beta = 0$ and dividing by $e^{i\tau\theta}$ gives

$$e^{i\theta} - \alpha + \beta e^{-i\tau\theta} = 0.$$

Separating real and imaginary parts yields $\beta \sin(\tau\theta) = \sin \theta$ and $\alpha = \cos \theta + \beta \cos(\tau\theta)$, from which (18) follows. The first component is precisely (17); substitution gives (19). Finally, $(-1)^{\tau+1}p_\tau(-s; \alpha, \beta) = p_\tau(s; -\alpha, (-1)^{\tau+1}\beta)$. $\square$

Remark 2 (Choice of the unit-circle parameter interval). *The interval*

$$0 < \theta < \frac{\pi}{\tau + 1}$$

is the first upper-half-plane unit-circle arc that generates the positive curved Schur boundary without duplication. Since the polynomial has real coefficients, the conjugate roots $e^{i\theta}$ and $e^{-i\theta}$ give the same real boundary point, so it is sufficient to take $\theta > 0$. On the interval above,

$$0 < \tau\theta < \pi, \quad 0 < (\tau + 1)\theta < \pi,$$

and therefore both sine factors in (18) are positive. Moreover,

$$\lim_{\theta \downarrow 0} (\alpha_\tau(\theta), \beta_\tau(\theta)) = \left(\frac{\tau + 1}{\tau}, \frac{1}{\tau} \right), \quad \lim_{\theta \uparrow \pi/(\tau+1)} (\alpha_\tau(\theta), \beta_\tau(\theta)) = (0, 1). \quad (20)$$

The function $\alpha_\tau(\theta)$ is strictly decreasing on this interval (the same $x \cot x$ monotonicity argument used in Lemma 1 gives this directly). Hence the interval maps one-to-one onto $0 < \alpha < c_\tau$ and supplies exactly the branch used to define $\varphi_\tau(\alpha)$ and $R_\tau(\alpha)$. Extending θ beyond $\pi/(\tau + 1)$ changes the sign of $\sin((\tau + 1)\theta)$ and no longer continues this positive branch; the complementary curved boundary required for $\partial\mathcal{R}_\tau$ is obtained from the symmetry below.

Corollary 1 (Parity-dependent symmetry of the universal Schur region). *Define*

$$S_\tau(\alpha, \beta) = (-\alpha, (-1)^{\tau+1}\beta). \quad (21)$$

Then

$$(\alpha, \beta) \in \mathcal{R}_\tau \iff S_\tau(\alpha, \beta) \in \mathcal{R}_\tau, \quad (22)$$

and the same transformation leaves the boundary $\partial\mathcal{R}_\tau$ invariant. Consequently,

$$S_\tau(\alpha, \beta) = \begin{cases} (-\alpha, \beta), & \tau \text{ odd}, \\ (-\alpha, -\beta), & \tau \text{ even}. \end{cases}$$

Thus the universal stability region is mirror-symmetric about the β -axis for odd delay and centrally symmetric about the origin for even delay.

Proof. The polynomial identity

$$(-1)^{\tau+1}p_\tau(-s; \alpha, \beta) = p_\tau(s; S_\tau(\alpha, \beta))$$

shows that the roots for the transformed coefficients are the negatives of the roots for (α, β) . The map $s \mapsto -s$ preserves modulus and therefore preserves the open unit disk. Schur stability, and hence its boundary by continuity, is consequently invariant under S_τ . $\square$

This parity-dependent symmetry explains why a single curved branch in Proposition 2 suffices for boundary construction: the second branch is generated by (21). The same symmetry is visible in the universal-region plots and is used explicitly in Algorithm 2.

3.2. Boundary Identities and Delay Dependence

Lemma 1 (Boundary identities). *Let $u = |\alpha| \in (0, c_\tau)$ and set $\theta = \varphi_\tau(\alpha)$. Then*

$$u + R_\tau(\alpha) < 1 + \frac{2}{\tau}. \quad (23)$$

For every $u > 0$,

$$R_\tau(u) > |1 - u|. \quad (24)$$

Moreover, $R_1 \equiv 1$, while for $\tau \geq 2$ the function $u \mapsto R_\tau(u)$ is strictly decreasing on $[0, c_\tau)$, $R_\tau(u) < 1$, and

$$\frac{u}{1 - R_\tau(u)} < \frac{\tau + 1}{\tau - 1}. \quad (25)$$

Proof. Using Proposition 2,

$$u + R_\tau = \frac{\sin((\tau + 1)\theta) + \sin \theta}{\sin(\tau\theta)} = \frac{\sin((\tau + 2)\theta/2)}{\sin(\tau\theta/2)}.$$

Since $\sin x/x$ is strictly decreasing on $(0, \pi)$, (23) follows. Furthermore,

$$R_\tau^2 - (1 - u)^2 = 2u(1 - \cos \theta) > 0,$$

which proves (24).

For monotonicity put $g(x) = x \cot x$. On $(0, \pi)$, $g'(x) = (\sin x \cos x - x)/\sin^2 x < 0$. Along the parametrization,

$$\frac{d}{d\theta} \log u = \frac{g((\tau + 1)\theta) - g(\tau\theta)}{\theta} < 0,$$

whereas, for $\tau \geq 2$,

$$\frac{d}{d\theta} \log R_\tau = \frac{g(\theta) - g(\tau\theta)}{\theta} > 0.$$

Hence R_τ decreases with u . For $\tau = 1$, (19) gives $R_1 \equiv 1$. If $\tau \geq 2$, then $\sin(\tau\theta) > \sin \theta$ on the relevant interval, so $R_\tau < 1$. Finally,

$$\frac{u}{1 - R_\tau} = \frac{\sin((\tau + 1)\theta/2)}{\sin((\tau - 1)\theta/2)},$$

and another use of the monotonicity of $\sin x/x$ gives (25). $\square$

Lemma 2 (Monotonicity with respect to delay). Fix $u \geq 0$. Whenever the solution of (17) exists for two positive delays $\tau_1 < \tau_2$, one has

$$\varphi_{\tau_2}(u) < \varphi_{\tau_1}(u), \quad R_{\tau_2}(u) \leq R_{\tau_1}(u), \quad (26)$$

with strict radius inequality for $u > 0$.

Proof. Treat the delay parameter temporarily as a positive real variable t and let $\phi(t)$ solve $\sin((t + 1)\phi)/\sin(t\phi) = u$. Implicit differentiation gives

$$\phi'(t) = \frac{\phi \sin \phi}{-t \sin \phi + \cos((t + 1)\phi) \sin(t\phi)}. \quad (27)$$

The denominator equals $\frac{1}{2}[\sin((2t + 1)\phi) - (2t + 1)\sin \phi]$ and is negative on $0 < \phi < \pi/(t + 1)$. Indeed, for $q > 1$ and $0 < x \leq 2\pi/(q + 1)$, if $h(x) = q \sin x - \sin(qx)$, then

$$h'(x) = 2q \sin\left(\frac{q + 1}{2}x\right) \sin\left(\frac{q - 1}{2}x\right) > 0,$$

with $h(0) = 0$. Hence the denominator in (27) is negative and $\phi'(t) < 0$. The radius formula (16) then shows that $R_t(u)$ decreases with t for $u > 0$; at $u = 0$ it equals one for all delays. $\square$

Figure 1 visualizes the two monotonicity properties proved above. For each fixed delay, $\varphi_\tau(u)$ decreases from $\pi/(\tau + 1)$ at $u = 0$ to 0 as $u \uparrow c_\tau$; for a fixed admissible u , increasing the delay shifts the angle downward, in agreement with Lemma 2. Through (18), the same curves therefore describe how the curved Schur boundary contracts as the delay grows.

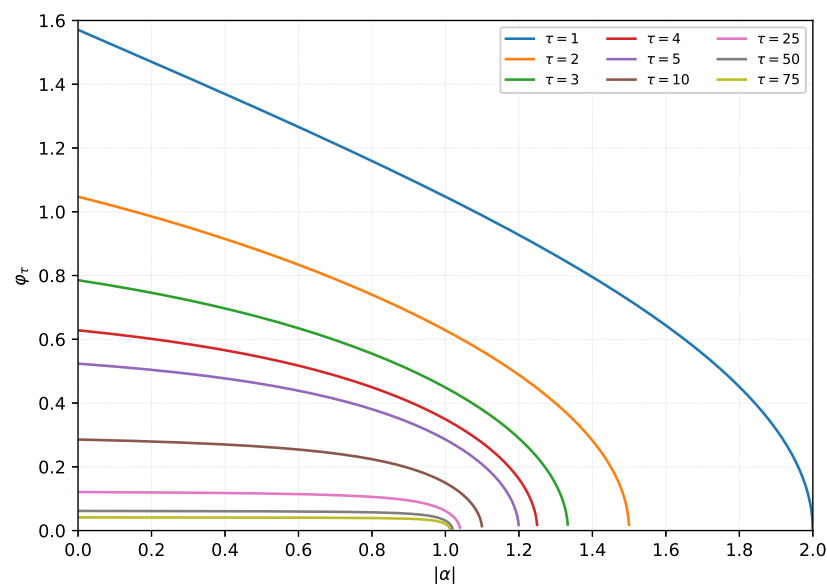


Figure 1. The boundary angle φ_τ for selected delays. The curves also parameterize the curved Schur boundary through (18).

Remark 3 (Zero delay). For $\tau = 0$, Equation (6) becomes $e_{k+1} = (\alpha - \beta)e_k$. Hence

$$\mathcal{R}_0 = \{(\alpha, \beta) : |\alpha - \beta| < 1\}. \quad (28)$$

Thus $\tau = 0$ is naturally treated as a first-order scalar stability problem, separately from the positive-delay Schur problem. The corresponding controller-level multiplier projections are therefore stated directly in Proposition 3, while Theorem 1 applies to positive delays.

The delay-independent sufficient condition

$$|\alpha| + |\beta| < 1 \quad (29)$$

defines the universal minimal region

$$\mathcal{R}_{\min} = \{(\alpha, \beta) \in \mathbb{R}^2 : |\alpha| + |\beta| < 1\}. \quad (30)$$

It is contained in $\mathcal{R}_\tau$ for every positive integer delay [17].

4. Controller-Specific Stability Domains

Define the controller maps

$$\begin{aligned}\Psi_F(a, b) &= (a, b), \\ \Psi_T(a, b) &= (a - b, -b), \\ \Psi_N(a, b) &= (a(1 - b), -b).\end{aligned}\tag{31}$$

For $j \in \{F, T, N\}$, the corresponding stability domain is

$$\mathcal{D}_\tau^j = \Psi_j^{-1}(\mathcal{R}_\tau).\tag{32}$$

4.1. FPFC

Corollary 2 (Exact FPFC stability domain). *Let $\tau \in \mathbb{N}$. The fixed point of (3) is locally exponentially stable if and only if*

$$|a| < \frac{\tau + 1}{\tau}\tag{33}$$

and, for odd τ ,

$$|a| - 1 < b < R_\tau(a),\tag{34}$$

while for even τ ,

$$|b - a| < 1, \quad |b| < R_\tau(a).\tag{35}$$

For $\tau = 0$, the exact condition is $|a - b| < 1$.

Proof. Use $(\alpha, \beta) = \Psi_F(a, b) = (a, b)$ in Theorem 1 and Remark 3. $\square$

The minimal FPFC domain is

$$\mathcal{D}_{\min}^F = \{(a, b) : |a| + |b| < 1\}.\tag{36}$$

For positive delay, condition (33) shows that an unstable multiplier can be stabilized only if

$$1 < |a| < 1 + \frac{1}{\tau}.\tag{37}$$

Both signs occur in the FPFC domain. Thus, FPFC can act after a flip bifurcation ($a < -1$) or after a loss of stability through $a > 1$, although the available multiplier interval contracts as τ increases.

For $\tau = 1$, the identity $2 \cos \varphi = |a|$ reduces (34) to

$$|a| - 1 < b < 1, \quad |a| < 2.\tag{38}$$

The zero-delay domain is the unbounded strip $|a - b| < 1$, while positive even delays add the curved restriction in (35).

4.2. TDFC

Corollary 3 (Exact TDFC stability domain). *Let $\tau \in \mathbb{N}$. The fixed point of (4) is locally exponentially stable if and only if*

$$|a - b| < \frac{\tau + 1}{\tau}\tag{39}$$

and, for odd τ ,

$$|a - b| - 1 < -b < R_\tau(a - b),\tag{40}$$

while for even τ ,

$$|a| < 1, \quad |b| < R_\tau(a - b). \quad (41)$$

For $\tau = 0$, the exact condition is $|a| < 1$.

Proof. Substitute $(\alpha, \beta) = \Psi_T(a, b) = (a - b, -b)$ into Theorem 1. In the even case, $|\beta - \alpha| = |-b - (a - b)| = |a|$. $\square$

The minimal TDFC domain is

$$\mathcal{D}_{\min}^T = \{(a, b) : |a - b| + |b| < 1\}. \quad (42)$$

Condition $|a| < 1$ in (41) proves that a positive even delay cannot stabilize an unstable fixed point. For odd delays, the lower bound in (40) excludes $a > 1$, while a negative unstable interval remains available. For $\tau = 1$, the projection onto the multiplier axis is

$$-3 < a < -1, \quad -1 < b < \frac{a+1}{2}. \quad (43)$$

Thus, TDFC is particularly suited to fixed points destabilized through a flip bifurcation.

4.3. NFC

Corollary 4 (Exact NFC stability domain). *Let $\tau \in \mathbb{N}$. The fixed point of (5) is locally exponentially stable if and only if*

$$|a(1 - b)| < \frac{\tau + 1}{\tau} \quad (44)$$

and, for odd τ ,

$$|a(1 - b)| - 1 < -b < R_\tau(a(1 - b)), \quad (45)$$

while for even τ ,

$$|b + a(1 - b)| < 1, \quad |b| < R_\tau(a(1 - b)). \quad (46)$$

For $\tau = 0$, the exact condition is $|a(1 - b) + b| < 1$.

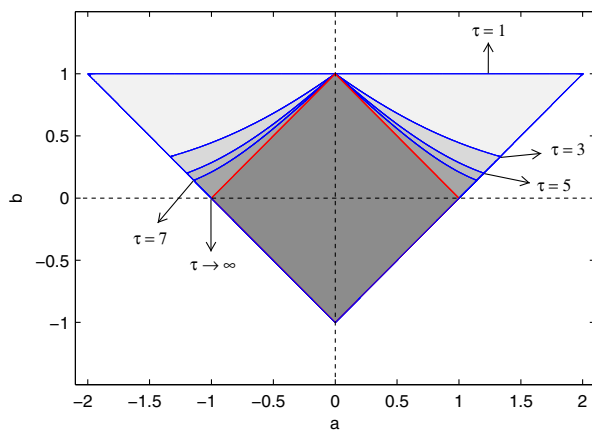
Proof. Substitute $(\alpha, \beta) = \Psi_N(a, b) = (a(1 - b), -b)$ into Theorem 1. For even delay, $|\beta - \alpha| = |-b - a(1 - b)| = |b + a(1 - b)|$. $\square$

The minimal NFC domain is

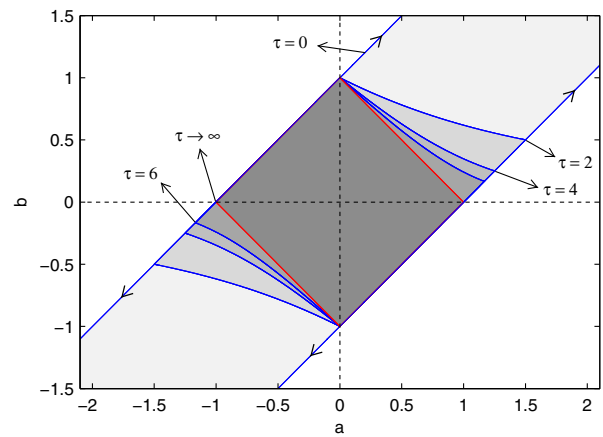
$$\mathcal{D}_{\min}^N = \{(a, b) : |a(1 - b)| + |b| < 1\}. \quad (47)$$

For positive odd delay, (45) implies $|a| < 1$, so an unstable fixed point cannot be stabilized. Positive even delays provide a complementary region for negative unstable multipliers. The zero-delay case is qualitatively different and may also contain positive unstable multipliers; it is governed by the scalar condition $|a(1 - b) + b| < 1$ rather than by Theorem 1.

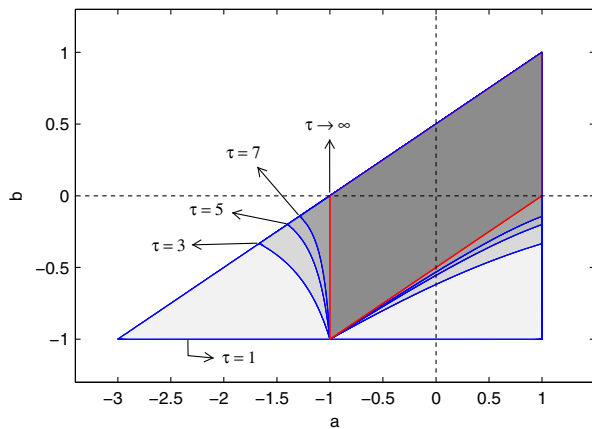
Figure 2 places the three controller pullbacks side by side so that the parity effect can be read directly from the (a, b) geometry. FPFC retains stabilizing regions for either positive delay parity, whereas the unstable negative-multiplier extension appears for TDFC only at odd delays and for NFC only at even delays. Within each fixed parity, the visible contraction of the positive-delay regions is the controller-level image of the same-parity nesting theorem.



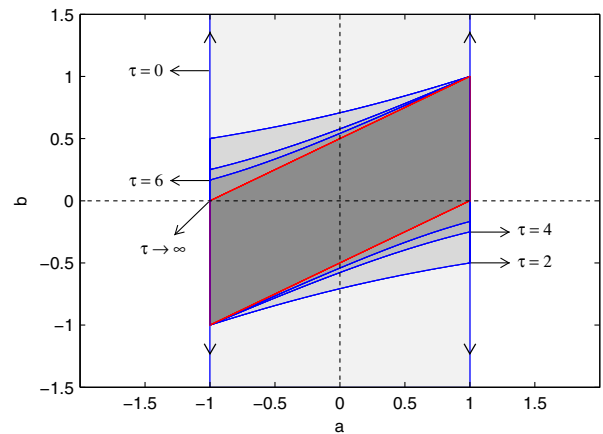
(a) FPFC, selected positive odd delays.



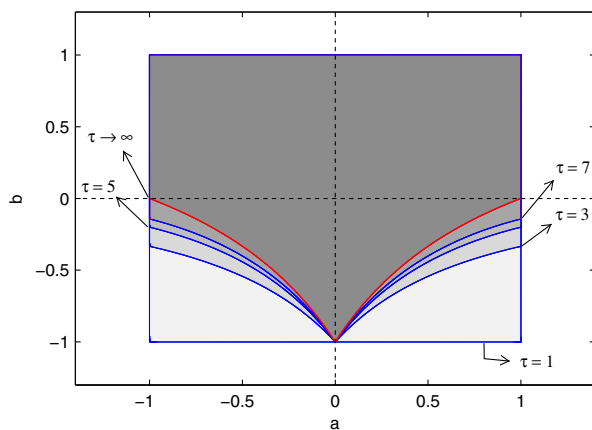
(b) FPFC, zero and selected positive even delays.



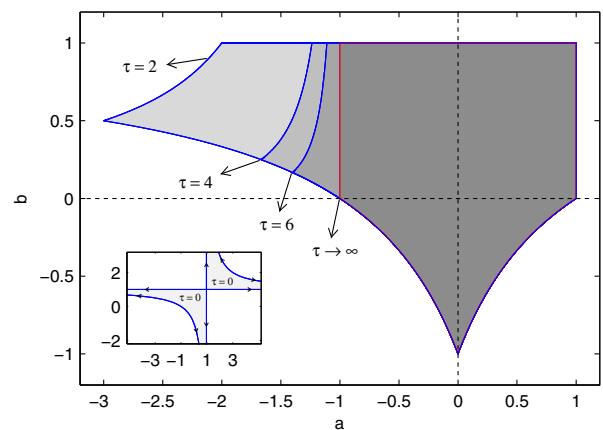
(c) TDFC, selected positive odd delays.



(d) TDFC, zero and selected positive even delays.



(e) NFC, selected positive odd delays.



(f) NFC, zero and selected positive even delays.

Figure 2. Controller-specific multiplier-gain stability domains. The shaded regions denote Schur-stable parameter pairs, with the different shades corresponding to the delays indicated in each panel. The paired odd/even panels display the delay-parity dependence for FPFC, TDFC, and NFC; for each parity, the positive-delay domains contract as the delay increases.

5. Delay Parity and Consequences of the Exact Regions

5.1. Same-Parity Nesting

Theorem 2 (Nesting for delays of the same parity). *Let $\tau_1, \tau_2 \in \mathbb{N}$ satisfy $\tau_1 < \tau_2$ and $\tau_1 \equiv \tau_2 \pmod{2}$. Then,*

$$\mathcal{R}_{\tau_2} \subset \mathcal{R}_{\tau_1}. \quad (48)$$

Proof. The coefficient bound $c_\tau = 1 + 1/\tau$ decreases with the delay, and Lemma 2 gives $R_{\tau_2}(\alpha) \leq R_{\tau_1}(\alpha)$ wherever both are defined. For delays of one parity, the remaining linear lower bound in the odd case and the strip condition in the even case are independent of τ . Thus, every point satisfying the conditions for τ_2 satisfies those for τ_1 . $\square$

Corollary 5 (Controller-domain nesting). *For $j \in \{F, T, N\}$ and positive delays of one parity,*

$$\mathcal{D}_{\tau_2}^j \subset \mathcal{D}_{\tau_1}^j \quad (\tau_1 < \tau_2). \quad (49)$$

Proof. Take inverse images of (48) under Ψ_j . $\square$

5.2. Exact Large-Delay Intersections

Define

$$\Gamma_o = \{(\alpha, \beta) : 0 < |\alpha| < 1, \beta = 1 - |\alpha|\}, \quad (50)$$

$$\Gamma_e = \{(\alpha, \beta) : 0 < |\alpha| < 1, \beta = \operatorname{sgn}(\alpha)(1 - |\alpha|)\}. \quad (51)$$

Theorem 3 (Exact intersection over unbounded delays). *Let $\mathbb{N}_o = \{1, 3, 5, \dots\}$ and $\mathbb{N}_e = \{2, 4, 6, \dots\}$. Then,*

$$\bigcap_{\tau \in \mathbb{N}_o} \mathcal{R}_\tau = \mathcal{R}_{\min} \cup \Gamma_o, \quad (52)$$

$$\bigcap_{\tau \in \mathbb{N}_e} \mathcal{R}_\tau = \mathcal{R}_{\min} \cup \Gamma_e. \quad (53)$$

The same intersections are obtained from any unbounded increasing sequence of positive delays of the corresponding parity.

Proof. Every point of $\mathcal{R}_{\min}$ is stable for every delay. Let $u = |\alpha| \in (0, 1)$. Lemma 1 gives $R_\tau(\alpha) > 1 - u$ for every finite delay. Hence, for odd τ , $\beta = 1 - u$ satisfies $u - 1 < \beta < R_\tau(\alpha)$, and Γ_o belongs to every odd-delay region.

Conversely, if a point lies in every odd-delay region, then $|\alpha| \leq 1$ because $c_\tau \downarrow 1$. Since $\varphi_\tau(\alpha) \rightarrow 0$, one has $R_\tau(\alpha) \rightarrow 1 - |\alpha|$. The odd inequalities therefore imply $|\alpha| - 1 < \beta \leq 1 - |\alpha|$. Interior points lie in $\mathcal{R}_{\min}$; equality at the upper bound produces precisely Γ_o . The endpoint cases $|\alpha| = 1$ and $\alpha = 0$ do not satisfy the required strict inequalities on the boundary.

For even delay, the same limiting argument gives $|\beta| \leq 1 - |\alpha|$, while $|\beta - \alpha| < 1$ remains strict. On the boundary $|\beta| = 1 - |\alpha|$, this strip condition holds exactly when α and β have the same nonzero sign, yielding Γ_e . Finally, same-parity nesting implies that an unbounded subsequence has the same intersection as the whole parity class. $\square$

Corollary 6 (Controller-specific large-delay intersections). For $j \in \{F, T, N\}$,

$$\bigcap_{\tau \in \mathbb{N}_o} \mathcal{D}_\tau^j = \Psi_j^{-1}(\mathcal{R}_{\min} \cup \Gamma_o), \quad (54)$$

$$\bigcap_{\tau \in \mathbb{N}_e} \mathcal{D}_\tau^j = \Psi_j^{-1}(\mathcal{R}_{\min} \cup \Gamma_e). \quad (55)$$

Proof. Inverse images commute with arbitrary intersections. $\square$

5.3. Exact Multiplier Projections for Local Stabilization

In view of Remark 1, these projections describe the multipliers for which some gain yields *local* fixed-point stabilization. Define the multiplier projection of controller j by

$$\mathcal{A}_\tau^j = \{a \in \mathbb{R} : \exists b \in \mathbb{R} \text{ with } (a, b) \in \mathcal{D}_\tau^j\}. \quad (56)$$

Theorem 4 (Exact multiplier projections). For every positive integer delay,

$$\mathcal{A}_\tau^F = (-c_\tau, c_\tau). \quad (57)$$

For TDFC,

$$\mathcal{A}_\tau^T = \begin{cases} \left(-1 - \frac{2}{\tau}, 1\right), & \tau \text{ odd}, \\ (-1, 1), & \tau \text{ even}, \end{cases} \quad (58)$$

and for NFC,

$$\mathcal{A}_\tau^N = \begin{cases} (-1, 1), & \tau \text{ odd}, \\ \left(-\frac{\tau+1}{\tau-1}, 1\right), & \tau \text{ even}. \end{cases} \quad (59)$$

Proof. For FPFC, $a = \alpha$, so (57) is exactly the coefficient bound in Theorem 1.

For TDFC, $a = \alpha - \beta$. If τ is even, the strip condition gives $|a| < 1$, and conversely $b = 0$ stabilizes every $|a| < 1$. If τ is odd, $\beta > |\alpha| - 1$ implies $a < 1 + \alpha - |\alpha| \leq 1$. On the negative side, write $u = -\alpha > 0$. The upper boundary gives $a > -u - R_\tau(u) > -1 - 2/\tau$ by (23). The universal odd region is connected and its image under the continuous map $(\alpha, \beta) \mapsto \alpha - \beta$ is therefore an interval. The upper endpoint 1 is approached at $\alpha = 0$, $\beta \downarrow -1$, and the lower endpoint is approached on the negative curved boundary as $u \uparrow c_\tau$. Equation (23) gives the stated limiting value.

For NFC, $a = \alpha/(1 + \beta)$ wherever the controller is stable. For odd delay, $\beta > |\alpha| - 1$ gives $1 + \beta > |\alpha|$, hence $|a| < 1$; again $b = 0$ gives the converse. For even delay, Lemma 1 gives $|\beta| < R_\tau(\alpha) < 1$, so $1 + \beta > 0$. The strip condition implies $a < 1$. If $\alpha = -u < 0$, then $\beta > -R_\tau(u)$ yields

$$a = -\frac{u}{1 + \beta} > -\frac{u}{1 - R_\tau(u)} > -\frac{\tau + 1}{\tau - 1}$$

by (25). Connectedness and the boundary limits give all intermediate values and the stated lower endpoint. $\square$

Proposition 3 (Zero-delay multiplier projections). For $\tau = 0$, the three controller projections are

$$\mathcal{A}_0^F = \mathbb{R}, \quad \mathcal{A}_0^T = (-1, 1), \quad \mathcal{A}_0^N = \mathbb{R} \setminus \{1\}. \quad (60)$$

Proof. For FPFC, zero delay gives $|a - b| < 1$, and choosing $b = a$ works for every $a \in \mathbb{R}$. For TDFC, the zero-delay feedback cancels identically in the linearization and the condition is $|a| < 1$. For NFC the condition is

$$|a(1 - b) + b| = |a + b(1 - a)| < 1.$$

If $a \neq 1$, the choice $b = -a/(1 - a)$ makes the expression zero. If $a = 1$, the expression is identically equal to one for every gain, so stabilization is impossible. $\square$

Corollary 7 (Delay budgets and parity shift). *Let $|a| > 1$.*

- (a) *FPFC can locally stabilize the fixed point with positive delay if and only if $1 < |a| < 2$, and the admissible delays satisfy*

$$\tau < \frac{1}{|a| - 1}. \quad (61)$$

- (b) *TDFC can locally stabilize an unstable fixed point if and only if $-3 < a < -1$; the admissible delays are the odd positive integers satisfying*

$$\tau < \frac{2}{-a - 1}. \quad (62)$$

- (c) *NFC can locally stabilize an unstable fixed point if and only if $-3 < a < -1$; the admissible delays are the even positive integers satisfying*

$$\tau < 1 + \frac{2}{-a - 1}. \quad (63)$$

Moreover, for every odd τ ,

$$\mathcal{A}_\tau^T = \mathcal{A}_{\tau+1}^N = \left(-1 - \frac{2}{\tau}, 1\right). \quad (64)$$

Proof. The delay inequalities are rearrangements of Theorem 4. Substitution of $\tau + 1$ into the even-NFC lower endpoint gives $-(\tau + 2)/\tau = -1 - 2/\tau$. $\square$

Corollary 8 (Bifurcation-selective local stabilizability). *For positive delay, a loss of fixed-point stability through the real multiplier $a = -1$ may be reversed by FPFC and, according to delay parity, by TDFC or NFC within the intervals of Theorem 4. After a crossing through $a = +1$, positive-delay TDFC and NFC cannot recover local stability, whereas FPFC can recover local stability for $1 < a < 1 + 1/\tau$.*

Proof. This is the unstable part of the three projection sets in Theorem 4 on the two sides of the unit interval. $\square$

Figure 3 translates Theorem 4 into delay-dependent multiplier bounds. The FPFC interval narrows toward $(-1, 1)$ as τ increases, while the TDFC and NFC lower bounds extend below -1 only on their effective odd/even delay subsequences and approach -1 with increasing delay. The shared upper bound $a_{\max} = 1$ for TDFC and NFC makes the parity-shift relation visually explicit. Table 1 condenses the unstable portions of these projections into a practical screening rule and adds the information required by each controller.

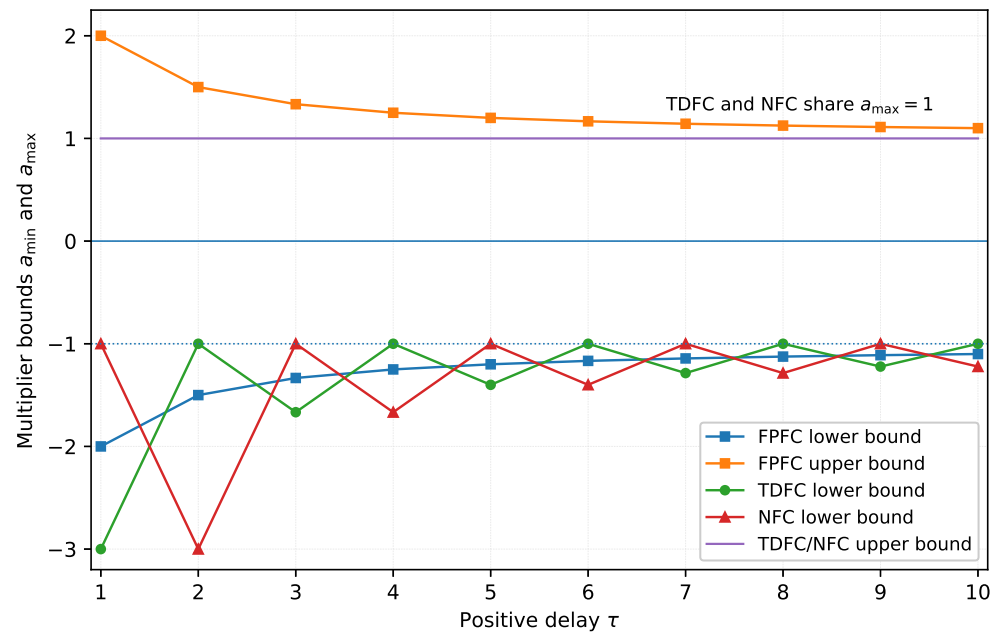


Figure 3. Exact multiplier bounds for local stabilization with positive delays $\tau = 1, \dots, 10$. The curves are the projections in Theorem 4; the zero-delay projections are qualitatively different and are stated separately in Proposition 3.

Table 1. Exact positive-delay stabilizability of unstable fixed points.

Controller	Effective Delays	Stabilizable Unstable a	Information Required
FPFC	Any parity	$1 < a < 1 + 1/\tau$	Target fixed point x^*
TDFC	Odd	$-1 - 2/\tau < a < -1$	Current and delayed states
NFC	Even	$-(\tau + 1)/(\tau - 1) < a < -1$	Delayed state and $f(x_k)$

Admissible gain constraints

A practical gain restriction integrates naturally with the Schur analysis. Let

$$G = [b_{\min}, b_{\max}] \subset \mathbb{R}$$

be the admissible feedback-gain interval. For controller $j \in \{F, T, N\}$ define the gain-constrained stability domain

$$\mathcal{D}_{\tau, G}^j = \mathcal{D}_{\tau}^j \cap (\mathbb{R} \times G), \quad (65)$$

the corresponding pointwise feasible set

$$\mathcal{G}_{\tau, G}^j(a) = \{b \in G : (a, b) \in \mathcal{D}_{\tau}^j\}, \quad (66)$$

and the constrained multiplier projection

$$\mathcal{A}_{\tau, G}^j = \{a \in \mathbb{R} : \mathcal{G}_{\tau, G}^j(a) \neq \emptyset\}. \quad (67)$$

Thus, a hard gain bound clips the unconstrained (a, b) stability domain by the horizontal strip $b_{\min} \leq b \leq b_{\max}$, while the Schur boundary remains the one inherited from the underlying controller domain. In particular,

$$\mathcal{A}_{\tau, G}^j \subseteq \mathcal{A}_{\tau}^j,$$

and if $G_1 \subseteq G_2$, then

$$\mathcal{D}_{\tau, G_1}^j \subseteq \mathcal{D}_{\tau, G_2}^j, \quad \mathcal{A}_{\tau, G_1}^j \subseteq \mathcal{A}_{\tau, G_2}^j.$$

Consequently, the exact projection formulas in Theorem 4 remain a fast unconstrained feasibility screen, while the final practical decision is the nonemptiness test $\mathcal{G}_{\tau, G}^j(a) \neq \emptyset$.

A bound on the *control signal* is conceptually different from a bound on the scalar gain. Writing the additive feedback corrections in (3)–(5) as

$$\begin{aligned} u_k^F &= -b(x_{k-\tau} - x^*), \\ u_k^T &= b(x_{k-\tau} - x_k), \\ u_k^N &= b(x_{k-\tau} - f(x_k)), \end{aligned}$$

all three satisfy $u_k^j = 0$ at the target fixed point. If a hard saturation $\text{sat}_U(u)$ is inserted with $U > 0$ and is equal to the identity in a neighbourhood of $u = 0$, then the local linearization at x^* is unchanged and the Schur domains derived here remain the local stability conditions. Saturation may also influence the nonlinear basin of attraction and transient behaviour outside that unsaturated neighbourhood; these global effects form a natural extension beyond the present local theory. If the actuator law is already active or has a nonunit local slope at the target, the controller must instead be re-linearized before applying the universal Schur criterion.

Figure 4 compares the three controllers on common axes rather than one controller at a time. Panels (a) and (b) show that the controller domains overlap but have different extensions for representative odd and even delays, while panel (c) isolates the delay-independent minimal cores. This comparison explains why controller selection cannot be reduced to a universal ranking: feasibility depends jointly on the multiplier, delay parity, and available gain range.

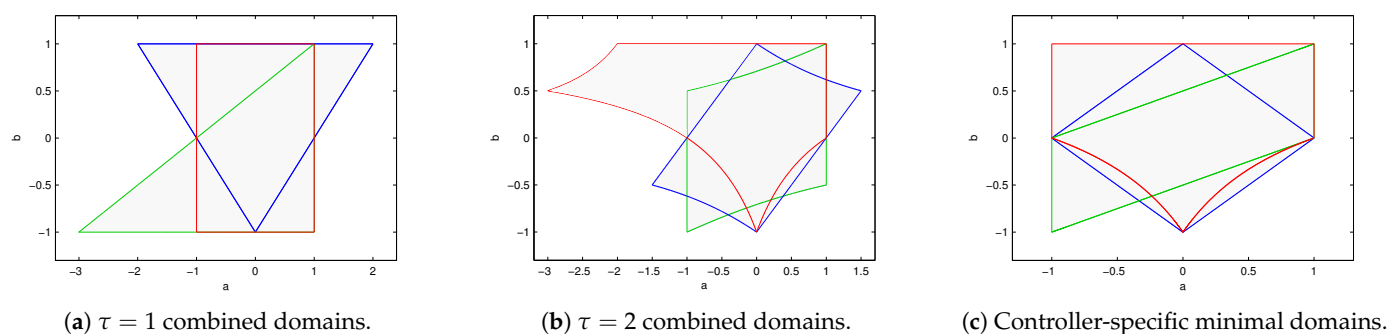


Figure 4. Embedded controller-specific stability regions. Blue, green, and red denote the FPFC, TDFC, and NFC stability regions, respectively. Panels (a,b) compare the three controller pullbacks for representative odd and even positive delays; panel (c) shows the corresponding pullbacks of $\mathcal{R}_{\min}$.

5.4. Common-Gain Stabilization Under Multiplier Uncertainty

Let the uncontrolled local multiplier be known only to lie in

$$I = [a_-, a_+], \quad a_- \leq a_+. \quad (68)$$

Define

$$\mathcal{B}_{\tau}^j(I) = \{b \in \mathbb{R} : (a, b) \in \mathcal{D}_{\tau}^j \text{ for every } a \in I\}. \quad (69)$$

Under the practical gain restriction $b \in G$, the robust feasible-gain set is simply

$$\mathcal{B}_{\tau, G}^j(I) = \mathcal{B}_{\tau}^j(I) \cap G. \quad (70)$$

Hence, a constrained robust design exists if and only if the exact common-gain set from Theorem 5 has nonempty intersection with G . The present analysis focuses on deterministic common-gain robustness and thereby complements broader delayed-feedback robustness formulations. Related results, such as the multi-tap eigenvalue-set problem in [13], consider different controller structures and uncertainty sets. Here, the uncertainty is one real scalar multiplier ranging over an interval, and the objective is an exact necessary-and-sufficient test for one constant gain that works throughout that interval for each of FPFC, TDFC, and NFC. Set

$$A_I = \max\{|a_-|, |a_+|\}, \quad (71)$$

$$M_T(b) = \max\{|a_- - b|, |a_+ - b|\}, \quad (72)$$

$$M_N(b) = |1 - b|A_I, \quad (73)$$

$$S_N(b) = \max\{|b + a_-(1 - b)|, |b + a_+(1 - b)|\}. \quad (74)$$

Theorem 5 (Exact common-gain test). *Let $I = [a_-, a_+]$ and $\tau \in \mathbb{N}$.*

(a) *For FPFC with odd delay,*

$$b \in \mathcal{B}_\tau^F(I) \iff A_I < c_\tau, \quad A_I - 1 < b < R_\tau(A_I). \quad (75)$$

For even delay,

$$b \in \mathcal{B}_\tau^F(I) \iff \begin{cases} A_I < c_\tau, \\ a_+ - 1 < b < a_- + 1, \\ |b| < R_\tau(A_I). \end{cases} \quad (76)$$

(b) *For TDFC with odd delay,*

$$b \in \mathcal{B}_\tau^T(I) \iff \begin{cases} M_T(b) < c_\tau, \\ M_T(b) + b < 1, \\ -b < R_\tau(M_T(b)). \end{cases} \quad (77)$$

For even delay,

$$b \in \mathcal{B}_\tau^T(I) \iff \begin{cases} A_I < 1, \\ M_T(b) < c_\tau, \\ |b| < R_\tau(M_T(b)). \end{cases} \quad (78)$$

(c) *For NFC with odd delay,*

$$b \in \mathcal{B}_\tau^N(I) \iff \begin{cases} M_N(b) < c_\tau, \\ M_N(b) + b < 1, \\ -b < R_\tau(M_N(b)). \end{cases} \quad (79)$$

For even delay,

$$b \in \mathcal{B}_\tau^N(I) \iff \begin{cases} M_N(b) < c_\tau, \\ S_N(b) < 1, \\ |b| < R_\tau(M_N(b)). \end{cases} \quad (80)$$

Proof. Intersect the pointwise controller inequalities over $a \in I$. The maximum absolute value of an affine function on a compact interval occurs at an endpoint, giving the quantities in (74). By Lemma 1, R_τ is non-increasing in $|\alpha|$ (constant only for $\tau = 1$), so the smallest

admissible radius over each transformed interval occurs at the endpoint with largest absolute transformed multiplier. Substitution gives (75)–(80); no relaxation is used, so the conditions are necessary and sufficient. $\square$

Remark 4. Theorem 5 distinguishes pointwise stabilizability from common-gain stabilizability. Even when every $a \in I$ belongs to the projection $\mathcal{A}_\tau^I$, the pointwise gain intervals need not possess a nonempty common intersection.

6. Algorithmic Construction and Controller Selection

The exact inequalities are useful not only as analytical descriptions but also as computational procedures. A direct numerical approach would construct p_τ for every candidate point and compute all $\tau + 1$ characteristic roots. The sparse structure of the present problem allows this repeated polynomial-root step to be avoided. This section formulates four procedures that implement the proved results directly, including an explicit controller-feasibility and gain-selection procedure.

6.1. Pointwise Stability Classification

For $0 < |\alpha| < c_\tau$, define

$$F_{\tau,\alpha}(\phi) = \frac{\sin((\tau+1)\phi)}{\sin(\tau\phi)} - |\alpha|.$$

The proof of Lemma 2 shows that this function is strictly decreasing on $(0, \pi/(\tau+1))$. Hence, the angle $\varphi_\tau(\alpha)$ can be enclosed by a guaranteed scalar bisection. For a finite-precision implementation, the strict curved-boundary inequality must not be evaluated using a single approximate value of R_τ . Define

$$r_\tau(\phi) = \frac{\sin \phi}{\sin(\tau\phi)}, \quad r_\tau(0) = \frac{1}{\tau}, \quad r_\tau\left(\frac{\pi}{\tau+1}\right) = 1. \quad (81)$$

For $\tau \geq 2$, r_τ is strictly increasing on the relevant angle interval. Therefore, if a bisection bracket $[\phi_-, \phi_+]$ contains $\varphi_\tau(\alpha)$, then

$$R^- := r_\tau(\phi_-) \leq R_\tau(\alpha) \leq r_\tau(\phi_+) =: R^+. \quad (82)$$

A strict comparison $q < R_\tau(\alpha)$ is thus certified true when $q < R^-$ and certified false when $q \geq R^+$. If neither conclusion follows, the bracket is refined. At a prescribed finite tolerance, an unresolved comparison is returned as *indeterminate*; this includes exact curved-boundary points and prevents an approximate radius from being used in a strict inequality.

If K_ϵ denotes the maximum number of bisection refinements before an unresolved comparison is declared indeterminate, then

$$K_\epsilon \leq \left\lceil \log_2 \left(\frac{\pi}{(\tau+1)\epsilon} \right) \right\rceil. \quad (83)$$

Thus, Algorithm 1 uses $O(\log(1/\epsilon))$ scalar trigonometric evaluations and $O(1)$ memory. Every *Schur-stable* or *non-Schur-stable* return is certified by the exact Schur inequalities; the third outcome records only that the chosen finite angular tolerance is insufficient to separate the point from the curved boundary. For every point strictly off that boundary, continued refinement eventually yields a determinate decision. The degree $\tau + 1$ of the characteristic polynomial does not create an all-roots computation in the classifier.

Algorithm 1 Certified criterion-based pointwise stability classifier.**Input:** controller $j \in \{F, T, N\}$, multiplier a , gain b , delay $\tau \geq 0$, angular tolerance $\varepsilon > 0$.**Output:** *Schur stable, not Schur stable, or indeterminate.*

1. Compute $(\alpha, \beta) = \Psi_j(a, b)$ from (31).
2. If $\tau = 0$, return *Schur stable* when $|\alpha - \beta| < 1$ and *not Schur stable* otherwise.
3. Set $u = |\alpha|$ and $c_\tau = (\tau + 1)/\tau$. If $u \geq c_\tau$, return *not Schur stable*.
4. Apply the exact non-curved inequalities first:
 - (a) if τ is odd and $\beta \leq u - 1$, return *not Schur stable*; otherwise set $q = \beta$;
 - (b) if τ is even and $|\beta - \alpha| \geq 1$, return *not Schur stable*; otherwise set $q = |\beta|$.
5. If $u = 0$ or $\tau = 1$, use $R_\tau(u) = 1$ exactly and return *Schur stable* when $q < 1$, otherwise *not Schur stable*.
6. Set $\phi_- = 0$ and $\phi_+ = \pi/(\tau + 1)$. Repeat:
 - (a) set $R^- = r_\tau(\phi_-)$ and $R^+ = r_\tau(\phi_+)$;
 - (b) if $q < R^-$, return *Schur stable*;
 - (c) if $q \geq R^+$, return *not Schur stable*;
 - (d) if $\phi_+ - \phi_- \leq \varepsilon$, return *indeterminate*;
 - (e) set $\phi_m = (\phi_- + \phi_+)/2$. If $F_{\tau, \alpha}(\phi_m) > 0$, set $\phi_- = \phi_m$; if $F_{\tau, \alpha}(\phi_m) < 0$, set $\phi_+ = \phi_m$; if equality holds, evaluate $R = r_\tau(\phi_m)$ and return *Schur stable* when $q < R$, otherwise *not Schur stable*.

6.2. Direct Generation of the Stability Boundary

Proposition 2 removes even the scalar bisection when a stability boundary is to be drawn. Sampling the parameter θ directly gives exact points on the curved boundary. The sampling resolution affects only the plotted polyline, not the mathematical boundary formula. The straight components are appended analytically from Theorem 1.

For FPFC and TDFC, the inverse controller maps are globally defined. For NFC, however,

$$\Psi_N^{-1}(\alpha, \beta) = \left(\frac{\alpha}{1 + \beta}, -\beta \right) \quad (84)$$

is valid only for $\beta \neq -1$. The exceptional universal-boundary point $(\alpha, \beta) = (0, -1)$ has the nonunique fibre

$$\{(a, b) : \Psi_N(a, b) = (0, -1)\} = \{(a, 1) : a \in \mathbb{R}\},$$

because $\Psi_N(a, 1) = (0, -1)$ for every a . Only a finite portion of this fibre belongs to the boundary of the NFC stability domain.

Lemma 3 (Singular NFC boundary fibre). *For positive delay,*

$$\partial \mathcal{D}_\tau^N \cap \{b = 1\} = \begin{cases} \{(a, 1) : -1 \leq a \leq 1\}, & \tau \text{ odd}, \\ \left\{ (a, 1) : -\sec\left(\frac{\pi}{\tau + 1}\right) \leq a \leq 1 \right\}, & \tau \text{ even}. \end{cases} \quad (85)$$

Proof. At $b = 1$, the NFC map collapses every a to the universal coefficient point $(0, -1)$, which lies on the universal Schur boundary. The boundary fibre in the controller plane is obtained by limits of nonsingular NFC boundary points as $\beta \rightarrow -1$.

For odd τ , the straight universal boundary $\beta = |\alpha| - 1$ approaches $(0, -1)$. Its two branches give the limiting values $a = \alpha/(1 + \beta) = \pm 1$, and the intervening part of the fibre gives $-1 \leq a \leq 1$.

For even τ , one endpoint again comes from the strip boundary and gives $a = 1$. The other comes from the curved branch. Writing $u = -\alpha > 0$ and letting the curved boundary approach $(0, -1)$,

$$a = -\frac{u}{1 - R_\tau(u)}.$$

The unit-circle parametrization gives

$$\lim_{u \downarrow 0} \frac{u}{1 - R_\tau(u)} = \sec\left(\frac{\pi}{\tau + 1}\right),$$

which yields the stated lower endpoint. The intervening fibre is continuous. $\square$

Algorithm 2 requires $O(N)$ scalar evaluations; the singular NFC segment is an $O(1)$ addition. The construction therefore remains one-dimensional while generating the complete controller boundary without an undefined division by $1 + \beta$.

Algorithm 2 Direct stability-boundary generator.

Input: controller $j \in \{F, T, N\}$, delay $\tau \geq 1$, number of curved-branch samples $N \in \mathbb{N}$.

1. Choose angles θ_i , $i = 1, \dots, N$, satisfying $0 < \theta_i < \pi/(\tau + 1)$; for very large τ , use adaptive refinement near $\theta = \pi/(\tau + 1)$ if approximately uniform resolution in α is required, as discussed below.
2. For each θ_i , evaluate

$$\alpha_i = \frac{\sin((\tau + 1)\theta_i)}{\sin(\tau\theta_i)}, \quad \beta_i = \frac{\sin\theta_i}{\sin(\tau\theta_i)}.$$

3. Generate the complementary curved branch with the parity-dependent symmetry S_τ in (21), and append the straight universal-boundary pieces prescribed by Theorem 1.
4. Map universal boundary points to the controller plane using

$$\Psi_F^{-1}(\alpha, \beta) = (\alpha, \beta), \quad \Psi_T^{-1}(\alpha, \beta) = (\alpha - \beta, -\beta).$$

For NFC, map every nonsingular point $\beta \neq -1$ by (84); do not apply this formula at $(\alpha, \beta) = (0, -1)$.

5. If $j = N$, append the singular $b = 1$ segment in (85) explicitly.
 6. Clip the straight and curved components at their analytical intersections.
-

Large-delay evaluation and conditioning

For very large τ , the formulas in (18) remain analytically regular on the open parameter interval, but their direct floating-point evaluation deserves care near the two endpoints because both numerator and denominator may be small. Let

$$\theta_0 = \frac{\pi}{\tau + 1}, \quad \text{sinc}(z) = \begin{cases} \sin z / z, & z \neq 0, \\ 1, & z = 0. \end{cases}$$

Near $\theta = 0$, the same boundary coefficients can be evaluated as

$$\alpha_\tau(\theta) = \frac{\tau + 1}{\tau} \frac{\text{sinc}((\tau + 1)\theta)}{\text{sinc}(\tau\theta)}, \quad \beta_\tau(\theta) = \frac{1}{\tau} \frac{\text{sinc}(\theta)}{\text{sinc}(\tau\theta)}. \quad (86)$$

Near the opposite endpoint, write $\delta = \theta_0 - \theta$. Then,

$$\alpha_\tau(\theta) = \frac{\sin((\tau + 1)\delta)}{\sin(\theta_0 + \tau\delta)}, \quad \beta_\tau(\theta) = \frac{\sin(\theta_0 - \delta)}{\sin(\theta_0 + \tau\delta)}. \quad (87)$$

Equations (86) and (87), together with the exact endpoint limits in (20), avoid evaluating a numerically unnecessary 0/0 ratio or a small sine through an argument close to π .

A separate issue is the geometric conditioning of the parametrization. With the normalized variable

$$x = \frac{\theta}{\theta_0} = \frac{(\tau+1)\theta}{\pi} \in (0,1),$$

the upper-endpoint expansion is

$$\alpha_\tau(x\theta_0) = \frac{\pi(1-x)}{\sin(\pi/(\tau+1))} + O((1-x)^2), \quad x \uparrow 1. \quad (88)$$

Hence,

$$\left| \frac{d\alpha_\tau}{dx} \right|_{x \uparrow 1} = \frac{\pi}{\sin(\pi/(\tau+1))} \sim \tau + 1.$$

Thus, the coefficient evaluation itself can be stabilized, but a uniform θ - or x -grid becomes geometrically nonuniform in α as the delay grows. For large-delay plotting, Algorithm 2 should therefore use adaptive angle refinement near $x = 1$ when uniform visual resolution is desired. Alternatively, one may prescribe an α -grid and recover the corresponding angle by the certified scalar bisection of Algorithm 1; this changes the boundary-generation cost to $O(N \log(1/\varepsilon))$ but gives direct control of multiplier-space resolution.

The endpoint-stable formulas were checked against an 80-digit high-precision evaluation for $\tau = 10^2, 10^4$, and 10^6 , using normalized parameters extending to 10^{-12} from both endpoints. The maximum absolute discrepancy in either boundary coefficient was 2.22×10^{-16} in double precision. This check separates floating-point evaluation error from the intrinsic large- τ geometric sensitivity quantified by (88).

6.3. Robust Common-Gain Classification

Theorem 5 also has a direct computational consequence. A direct sampling-based uncertainty calculation would sample many multipliers $a \in [a_-, a_+]$ for every candidate gain. Because the transformed coefficients are affine in a and R_τ decreases with the magnitude of its argument, the continuous multiplier interval reduces exactly to endpoint maxima. For each controller/parity case, write the robust criterion as a collection P of exact non-curved inequalities together with at most one strict curved comparison

$$q < R_\tau(u),$$

where u is the relevant endpoint maximum. Specifically,

case	u	q
FPFC, odd	A_I	b
FPFC, even	A_I	$ b $
TDFC, odd	$M_T(b)$	$-b$
TDFC, even	$M_T(b)$	$ b $
NFC, odd	$M_N(b)$	$-b$
NFC, even	$M_N(b)$	$ b $.

(89)

The exact inequalities in P are checked first. If they hold, the remaining curved comparison is certified by the same radius enclosure (82) used in Algorithm 1.

For a fixed candidate gain, Algorithm 3 has $O(\log(1/\varepsilon))$ scalar cost and no dependence on the number of uncertainty samples because no discretization of I is required. As in Algorithm 1, every determinate return is certified by the exact strict inequalities. If the

complete robust gain set is desired numerically, Theorem 5 reduces the problem to a one-dimensional search in b .

Algorithm 3 Certified common-gain classifier over a multiplier interval.

Input: controller $j \in \{F, T, N\}$, interval $I = [a_-, a_+]$, candidate gain b , delay $\tau \geq 1$, and angular tolerance $\varepsilon > 0$.

Output: *robustly stable*, *not robustly stable*, or *indeterminate*.

1. Compute A_I , $M_T(b)$, $M_N(b)$, and $S_N(b)$ from (74) as required.
2. From the applicable condition (75)–(80), identify the collection P of exact non-curved inequalities and the pair (u, q) from (89), so that the remaining curved condition is $q < R_\tau(u)$.
3. If any inequality in P fails, return *not robustly stable*.
4. If $u = 0$ or $\tau = 1$, use $R_\tau(u) = 1$ exactly and return

$$\begin{cases} \text{robustly stable,} & q < 1, \\ \text{not robustly stable,} & q \geq 1. \end{cases}$$

5. Otherwise, bisect the defining angle for $R_\tau(u)$ and propagate the angle bracket to $R^- \leq R_\tau(u) \leq R^+$ using (82). Return

$$\begin{cases} \text{robustly stable,} & q < R^-, \\ \text{not robustly stable,} & q \geq R^+. \end{cases}$$

6. If neither inequality holds, continue refining.
 6. If the current angular bracket $[\phi_-, \phi_+]$ satisfies $\phi_+ - \phi_- \leq \varepsilon$ before either decision in Step 5 is certified, return *indeterminate*.
-

6.4. Gain-Constrained Controller Selection

The multiplier projections identify which feedback structures can possibly stabilize a prescribed multiplier, but they do not by themselves choose a controller. Such a choice also depends on implementation information and on admissible gains. Let $\mathcal{J}_{\text{avail}} \subseteq \{F, T, N\}$ denote the controller structures available in the application, according to the information requirements summarized in Table 1, and let $G = [b_{\min}, b_{\max}]$ be the admissible gain interval. The exact feasible controller set is

$$\mathcal{J}_{\text{feas}}(a, \tau; G) = \left\{ j \in \mathcal{J}_{\text{avail}} : \mathcal{G}_{\tau, G}^j(a) \neq \emptyset \right\}. \quad (90)$$

For positive delay and an unstable multiplier, Theorem 4 gives the following exact unconstrained prescreen:

controller	necessary and sufficient unconstrained condition
F	$1 < a < 1 + 1/\tau$,
T	τ odd and $-1 - 2/\tau < a < -1$,
N	τ even and $-(\tau + 1)/(\tau - 1) < a < -1$.

(91)

For fixed $|a| > 1$, define the unconstrained gain fibre

$$\mathcal{H}_\tau^j(a) = \{b \in \mathbb{R} : (a, b) \in \mathcal{D}_\tau^j\}, \quad \mathcal{G}_{\tau, G}^j(a) = \mathcal{H}_\tau^j(a) \cap G. \quad (92)$$

After the prescreen (91), every nonempty $\mathcal{H}_\tau^j(a)$ is a single open interval. For FPFC,

$$\mathcal{H}_\tau^F(a) = \begin{cases} (|a| - 1, R_\tau(a)), & \tau \text{ odd,} \\ (\max\{a - 1, -R_\tau(a)\}, \min\{a + 1, R_\tau(a)\}), & \tau \text{ even,} \end{cases} \quad (93)$$

for $1 < |a| < 1 + 1/\tau$.

For the surviving TDFC and NFC cases, introduce

$$\begin{aligned} h_{\tau}^T(\theta) &= \frac{\sin((\tau+2)\theta/2)}{\sin(\tau\theta/2)}, & h_{\tau}^T(\theta_T(a)) &= -a, \\ h_{\tau}^N(\theta) &= \frac{\sin((\tau+1)\theta/2)}{\sin((\tau-1)\theta/2)}, & h_{\tau}^N(\theta_N(a)) &= -a. \end{aligned} \quad (94)$$

On $0 < \theta < \pi/(\tau+1)$ these functions are strictly decreasing; this follows from the monotonicity of $x \cot x$ used in Lemma 1. Hence, the relevant roots in (94) are unique and may be bracketed by bisection. The corresponding gain intervals are

$$\mathcal{H}_{\tau}^T(a) = \left(-\beta_{\tau}(\theta_T(a)), \frac{a+1}{2} \right), \quad -1 - \frac{2}{\tau} < a < -1, \quad (95)$$

for odd τ , and, for even τ ,

$$\mathcal{H}_{\tau}^N(a) = \left(\frac{a+1}{a-1}, U_N(a) \right), \quad U_N(a) = \begin{cases} \beta_{\tau}(\theta_N(a)), & a < -\sec\left(\frac{\pi}{\tau+1}\right), \\ 1, & -\sec\left(\frac{\pi}{\tau+1}\right) \leq a < -1, \end{cases} \quad (96)$$

with $-(\tau+1)/(\tau-1) < a < -1$. The value $U_N = 1$ is the singular boundary of Lemma 3.

Writing $\mathcal{H}_{\tau}^j(a) = (L_j, U_j)$, the gain constraint gives

$$\mathcal{G}_{\tau,G}^j(a) = [b_{\min}, b_{\max}] \cap (L_j, U_j), \quad \ell_j = \max\{b_{\min}, L_j\}, \quad u_j = \min\{b_{\max}, U_j\}. \quad (97)$$

Thus, the feasible set is empty if $\ell_j \geq u_j$ and otherwise has one connected component with relative closure $[\ell_j, u_j]$. Curved endpoints are retained as certified enclosures; if these do not certify feasibility or a later margin comparison at the requested tolerance, the corresponding case remains *indeterminate* rather than being assigned to either side of the stability boundary.

Because controller preference is application-dependent when several schemes are feasible, the algorithm below separates *feasibility* from *preference*. If the application supplies a controller cost or priority C_j (for example, availability of x^* , delayed measurements, or a model evaluation $f(x_k)$), that priority is applied first. If no such preference is supplied, a neutral numerical tie-breaker is to select the controller with the largest admissible gain-perturbation margin. For a resolved feasible controller, let $[\ell_j, u_j]$ be the relative closure in (97) and define

$$b_j^* = \frac{\ell_j + u_j}{2}, \quad m_j = \frac{u_j - \ell_j}{2}. \quad (98)$$

The quantity m_j provides a neutral gain-robustness margin. When the nonlinear convergence rate, control effort, or other application-specific performance objectives are available, they can be incorporated through the controller priority/cost rule C_j .

Algorithm 4 therefore reduces controller selection to explicit endpoint evaluations and at most one monotone scalar solve for each surviving controller; no gain scan or characteristic-root calculation is required. The interval-uncertainty extension follows by replacing $\mathcal{G}_{\tau,G}^j(a)$ with $\mathcal{B}_{\tau,G}^j(I)$ from (70).

Table 2 summarizes the computational role of the analytical results across the four tasks developed above. Pointwise classification reduces to a certified scalar solve, boundary tracing to a one-parameter construction, interval robustness to endpoint tests, and controller selection to projection and explicit gain-endpoint construction. The direct alternative in every row requires repeated characteristic-root calculations or dense parameter sampling.

Algorithm 4 Gain-constrained controller-feasibility and selection procedure.

Input: multiplier a with $|a| > 1$, delay $\tau \geq 1$, available-controller set $\mathcal{J}_{\text{avail}} \subseteq \{\text{F}, \text{T}, \text{N}\}$, admissible gain interval $G = [b_{\min}, b_{\max}]$, endpoint tolerance $\eta > 0$, and optionally a controller priority/cost rule.

Output: feasible controllers and their admissible gain sets; if all required decisions are resolved and at least one controller is feasible, a selected controller–gain pair (j^*, b^*) .

1. Apply the exact projection pre-screen (91) and define

$$\mathcal{J}_{\text{cand}} = \{j \in \mathcal{J}_{\text{avail}} : j \text{ satisfies (91)}\}.$$

2. For each $j \in \mathcal{J}_{\text{cand}}$, construct

$$\mathcal{H}_{\tau}^j(a) = (L_j, U_j)$$

from (93)–(96). Evaluate straight endpoints explicitly and bracket curved endpoints by bisection of (17) or (94) to tolerance η .

3. For each $j \in \mathcal{J}_{\text{cand}}$, compute $\ell_j = \max\{b_{\min}, L_j\}$ and $u_j = \min\{b_{\max}, U_j\}$ as in (97), and classify

$$j \in \begin{cases} \mathcal{J}_{\text{feas}}, & \ell_j < u_j \text{ is certified,} \\ \mathcal{J}_{\text{rej}}, & \ell_j \geq u_j \text{ is certified,} \\ \mathcal{J}_{\text{unres}}, & \text{otherwise.} \end{cases}$$

4. If $\mathcal{J}_{\text{feas}} = \emptyset$ and $\mathcal{J}_{\text{unres}} = \emptyset$, return *no feasible controller under the stated gain and information constraints*; if $\mathcal{J}_{\text{unres}} \neq \emptyset$, return the certified gain information and report the feasibility decision as *unresolved*.
5. For each $j \in \mathcal{J}_{\text{feas}}$, compute (b_j^*, m_j) from (98). If an application-specific priority/cost rule is supplied, use it to choose $j^* \in \mathcal{J}_{\text{feas}}$; otherwise choose

$$j^* \in \arg \max_{j \in \mathcal{J}_{\text{feas}}} m_j.$$

In either case, set $b^* = b_{j^*}^*$.

6. If the required controller ranking cannot be certified at tolerance η , return the certified feasible gain sets and report the selection as *unresolved*. Otherwise return

$$\mathcal{J}_{\text{feas}}, \quad \{\mathcal{G}_{\tau, G}^j(a) : j \in \mathcal{J}_{\text{feas}}\}, \quad (j^*, b^*).$$

Table 2. Computational consequences of the exact stability formulas.

Task	Analytical Procedure Used Here	Direct Numerical Alternative
Classify one (a, b, τ) point	Certified scalar bisection bounds plus explicit inequalities; $O(\log(1/\varepsilon))$ work	Construct a degree- $(\tau + 1)$ polynomial and compute all characteristic roots
Trace a stability boundary	N evaluations of the unit-circle parametrization plus analytical straight/singular components; $O(N)$	Dense two-parameter scan with a root calculation at every grid point
Check one gain for $a \in [a_-, a_+]$	Endpoint maxima plus the same certified radius comparison; no multiplier discretization	Sample the multiplier interval and repeat the pointwise stability calculation
Select a controller under gain bounds	Projection prescreen, explicit feasible-gain endpoints and clipping by G , then application-specific ranking or gain-margin tie-breaker	Scan controllers and gains and repeat characteristic-root calculations

A reference Python implementation of Algorithms 1–4 is supplied as Supplementary File S1 (algorithms_stability.py), with its machine-readable main-validation output as Supplementary File S2 (algorithms_stability_results.json). The implementation

includes the certified pointwise and common-gain classifiers, direct boundary generation, and the endpoint-based gain-constrained controller-selection procedure of Algorithm 4. The same implementation contains the independent polynomial-root comparator used in Section 7.

Supporting large-delay verification and computational checks are supplied as Supplementary File S3 (`algorithms_large_delay_benchmark.py`) and Supplementary File S4 (`algorithms_large_delay_results.json`). The nonlinear chaotic/periodic regime diagnostic and its machine-readable output comprise Supplementary File S5 (`nonlinear_regime_diagnostics.py` and `nonlinear_regime_diagnostics.json`). Files S1–S5 are packaged together in a single supplementary archive.

7. Computational Verification and Logistic-Map Applications

The analytical results determine complete local stability domains and their boundaries in the (a, b) plane. Numerical calculations are used here for two purposes: to check the implementation of the exact inequalities against direct characteristic roots and to illustrate convergence from selected nonlinear initial histories for parameter points inside the local domains. The nonlinear applications in this and the following section use the logistic map and a normalized tumor-growth map, respectively. Both are scalar systems with rich nonlinear dynamics, including periodic and chaotic regimes. The trajectory plots therefore provide representative nonlinear illustrations of the corresponding local convergence.

7.1. Reproducible Characteristic-Root Verification

The certified pointwise classifier of Algorithm 1 is independently compared with all roots of $p_\tau(s; \alpha, \beta)$. The boundary angle is computed by monotone bisection with angular tolerance 10^{-13} , while the characteristic roots are obtained from the polynomial coefficients using `numpy.roots`. The random generator uses the fixed seed 20260719.

For each controller and each delay $\tau = 1, \dots, 6$, 5000 points are sampled uniformly from

$$a \in [-4, 4], \quad b \in [-2, 2],$$

giving 90,000 domain-wide tests. Boundary sensitivity is tested separately in the universal (α, β) plane using 250 admissible values of α per controller and delay, with points displaced by 10^{-4} on both sides of the lower and upper active β -boundaries. Mapping these points through the controller transformations gives an additional 18,000 tests and avoids testing floating-point equality directly on the unit circle.

For all 108,000 test points, the chosen tolerance produced determinate analytical decisions, and these agreed with the direct root condition $\rho_{\text{sp}}(A_\tau) < 1$, with no mismatches. This validates the pointwise classification away from the unresolved tolerance band; certification for arbitrary inputs instead follows from the radius enclosure (82) and the explicit indeterminate outcome of Algorithm 1.

Large-delay verification and timing comparison

The verification was extended to $\tau = 10, 20, 50$, and 100. For each delay and each of FPFC, TDFC, and NFC, 500 independent points were drawn uniformly from $a \in [-4, 4]$ and $b \in [-2, 2]$, giving 1500 controller-domain tests per delay and 6000 additional tests overall. With random seed 20260908 and angular tolerance 10^{-13} , every analytical decision was determinate and agreed with the direct characteristic-root calculation. Thus the complete validation comprises 114,000 points with no mismatches.

Figure 5 provides a visual counterpart to the pointwise validation for one representative delay of each parity. The independently computed spectral-radius boundary $\rho_{\text{sp}}(A_\tau) = 1$ coincides with the analytical boundary from Theorem 1 at the plotting resolu-

tion, while the boundary-sensitive test points fall on the predicted stable and unstable sides. This agreement provides a visual confirmation of the exact analytical stability regions and is consistent with the complete 114,000-point validation set, for which no mismatch was found. The large-delay benchmark and nonlinear-regime diagnostic are supplied as S3–S5 as specified above.

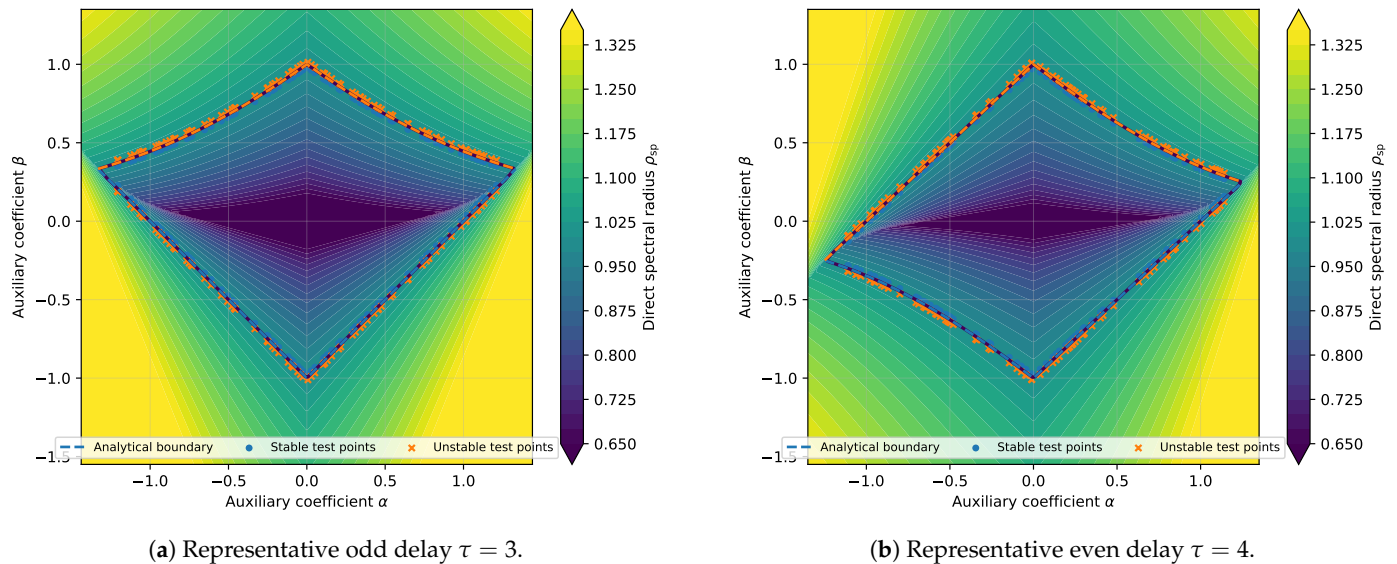


Figure 5. Representative numerical verification of the exact universal Schur boundaries. The background field is the spectral radius $\rho_{sp}(A_\tau)$ obtained from direct computation of all characteristic roots. The solid contour denotes $\rho_{sp}(A_\tau) = 1$, whereas the dashed curves are the analytical boundaries from Theorem 1. Circles denote boundary-sensitive test points classified as stable and crosses denote points classified as unstable.

A separate timing benchmark, implemented in Supplementary File S3, compared Algorithm 1 with `numpy.roots` on the same 1000 universal coefficient pairs at each delay, sampled uniformly from $\alpha, \beta \in [-1.2, 1.2]$. Reported times are medians of three repetitions using one CPU thread on an AMD EPYC 9V74 processor with Python 3.13.5 and NumPy 2.3.5; the corresponding machine-readable results are provided in Supplementary File S4. Table 3 reports the total time for classifying all 1000 points. The criterion-based cost remains nearly constant over these delays, whereas the all-roots calculation grows rapidly with the polynomial degree.

Table 3. Large-delay validation and pointwise timing comparison. The validation column refers to 500 random (a, b) points for each of the three controllers at each delay. Timing columns refer to a separate set of 1000 universal (α, β) points and give the median of three single-threaded repetitions.

τ	Validation Tests	Mismatches	Criterion [ms]	Roots [ms]	Speedup
10	1500	0	1.721	40.900	23.8×
20	1500	0	1.966	84.674	43.1×
50	1500	0	2.225	391.488	175.9×
100	1500	0	2.433	5516.442	2267.8×

The trend in Table 3 is pronounced: the criterion time stays in the millisecond range for 1000 points, whereas the direct all-roots time rises from 40.900 ms at $\tau = 10$ to 5516.442 ms at $\tau = 100$. The measured speedup therefore increases from 23.8× to 2267.8×, with zero validation mismatches at every reported delay.

7.2. Logistic-Map Setup and Trajectory Protocol

Consider

$$f_{\mu}(x) = \mu x(1 - x), \quad 0 \leq x \leq 1, \quad 0 < \mu \leq 4. \quad (99)$$

Its positive fixed point and multiplier are

$$x^* = \frac{\mu - 1}{\mu}, \quad a = f'_{\mu}(x^*) = 2 - \mu. \quad (100)$$

Thus x^* loses stability at $\mu = 3$ and has $-2 \leq a < -1$ for $3 < \mu \leq 4$.

Theorem 4 converts these multiplier projections directly into model-parameter ranges. For the unstable positive fixed point and positive delay,

$$3 < \mu < 3 + \frac{1}{\tau} \quad \text{for FPFC}, \quad (101)$$

$$3 < \mu < 3 + \frac{2}{\tau} \quad \text{for TDFC with odd } \tau, \quad (102)$$

$$3 < \mu < 3 + \frac{2}{\tau - 1} \quad \text{for NFC with even } \tau. \quad (103)$$

Within the logistic-map range $\mu \leq 4$, these intervals are understood after intersection with that model bound. Thus the projection theorem supplies an exact model-level interpretation of the stability-region geometry before any trajectory simulation is performed.

Logistic trajectory protocol

The logistic FPFC examples use $x_0 = 10^{-2}$, while the TDFC and NFC examples use $x_0 = 10^{-1}$. In every displayed logistic trajectory, the first 80 plotted values belong to the uncontrolled orbit and feedback begins at the $k = 80$ threshold. Since every displayed delay satisfies $\tau \leq 6 < 80$, all required delayed samples are already available from the preceding uncontrolled orbit. No transient iterations are discarded from the displayed trajectories. For the logistic-map examples, the reported chaotic and periodic regimes are verified directly from the corresponding *uncontrolled* orbits. For reproducibility of this diagnostic, we use the fixed reference initial condition $x_0 = 0.1$, discard

$$N_{\text{tr}} = 5000$$

iterations, and analyze the next

$$N_{\text{an}} = 5000.$$

For a one-parameter map $x_{k+1} = f_{\eta}(x_k)$, the finite-time largest Lyapunov exponent is computed as

$$\lambda_{N_{\text{an}}} = \frac{1}{N_{\text{an}}} \sum_{k=N_{\text{tr}}}^{N_{\text{tr}}+N_{\text{an}}-1} \log |f'_{\eta}(x_k)|. \quad (104)$$

A positive value is classified as chaotic. For negative values, the least period $p \leq 16$ is identified by the recurrence test

$$\max_k |x_{k+p} - x_k| < 10^{-10} \quad (105)$$

over the final 1000 analyzed iterates. The resulting regime classifications, together with the controller parameters and independently computed spectral radii for the selected trajectory examples, are summarized in Table 4.

Table 4. Parameters and direct spectral-radius verification for the logistic-map examples.

Controller	μ	a	τ	b	$\rho_{sp}(A_\tau)$	Regime
FPFC	3.70	−1.70	1	0.800	0.8944	chaotic
FPFC	3.45	−1.45	2	−0.452	0.9669	period four
TDFC	3.70	−1.70	1	−0.400	0.8000	chaotic
TDFC	3.59	−1.59	3	−0.300	0.9726	chaotic
TDFC	3.15	−1.15	5	−0.150	0.9212	period two
NFC	4.00	−2.00	2	0.500	0.9406	chaotic
NFC	3.61	−1.61	4	0.234	0.9868	chaotic
NFC	3.25	−1.25	6	0.125	0.9536	period two

All selected controlled cases satisfy $\rho_{sp}(A_\tau) < 1$, including examples close to the stability boundary such as TDFC at $\tau = 3$ and NFC at $\tau = 4$. The set deliberately spans both chaotic and periodic uncontrolled regimes and the parity-specific delay sequences used for TDFC and NFC.

7.3. FPFC Examples

The two FPFC cases in Table 4 converge to the positive fixed points $x^* = 0.7297$ for $\mu = 3.70$ and $x^* = 0.7101$ for $\mu = 3.45$. Figure 6a,b show suppression of a chaotic and a period-four trajectory, respectively, and illustrate FPFC with both delay parities.

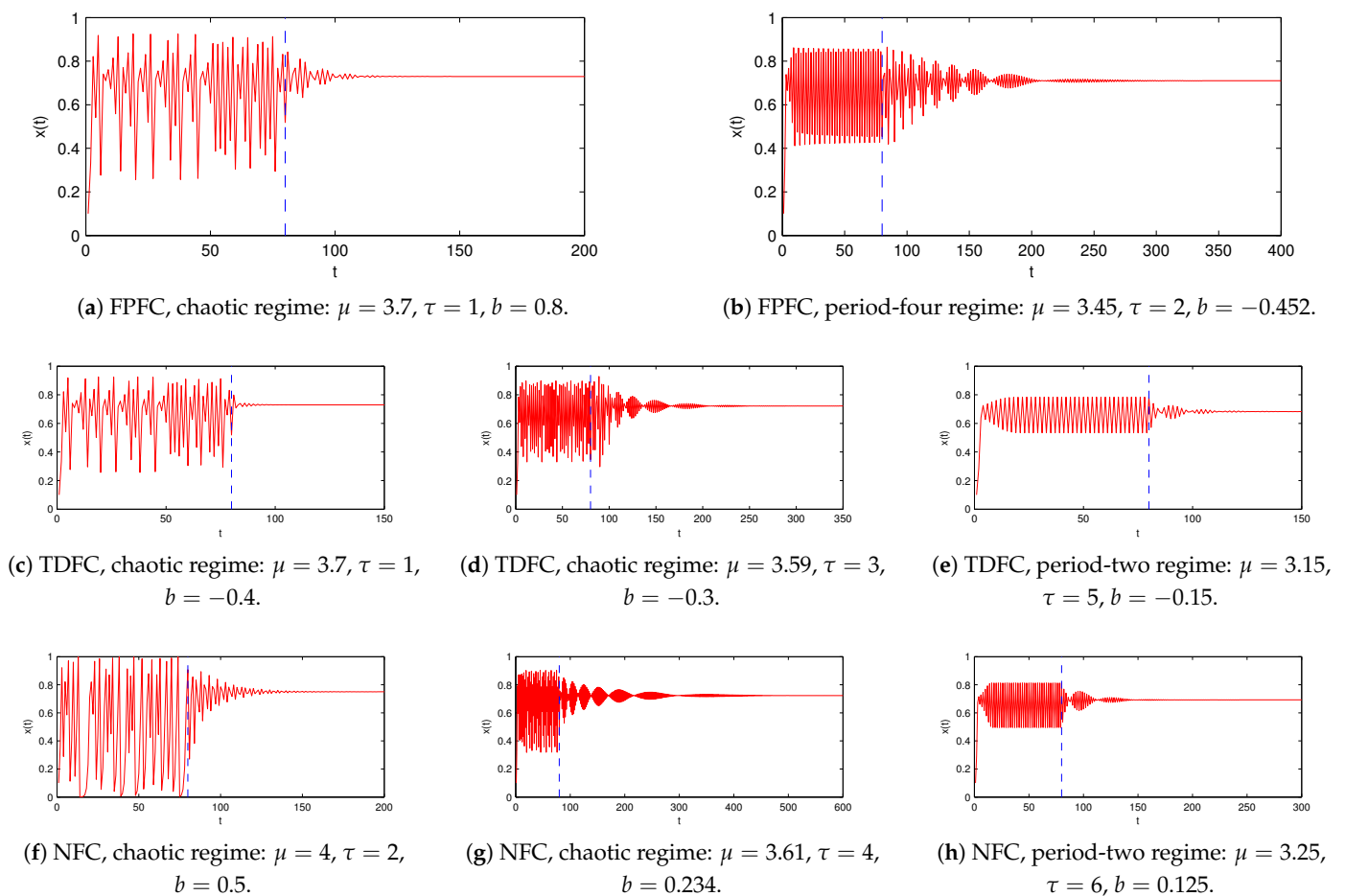


Figure 6. Logistic-map local stabilization examples: FPFC in panels (a,b), TDFC in panels (c–e), and NFC in panels (f–h). The first 80 plotted values are uncontrolled and feedback begins at $k = 80$; no displayed transient is discarded.

7.4. TDFC Examples

For $\mu = 3.70$ and $\tau = 1$, the characteristic roots are -0.8 and -0.5 . The three TDFC cases converge to $x^* = 0.7297, 0.7214$, and 0.6825 for $\mu = 3.70, 3.59$, and 3.15 , respectively. Figure 6c–e cover the admissible odd delays 1, 3, and 5, including chaotic and period-two uncontrolled regimes; the corresponding spectral radii are listed in Table 4.

7.5. NFC Examples

The NFC cases converge to $x^* = 0.7500, 0.7230$, and 0.6923 for $\mu = 4.00, 3.61$, and 3.25 , respectively. Figure 6f–h cover the complementary even delays 2, 4, and 6. The $\tau = 4$ case is close to the stability boundary, with $\rho_{\text{sp}}(A_4) = 0.9868$, while the other spectral radii and parameters are given in Table 4. Figure 6 collects the eight examples in a single controller-organized comparison. The post-activation convergence in each panel is consistent with the spectral-radius values in Table 4 and with the parity-specific projection results.

8. Application to a Discrete Tumour-Growth Map

8.1. Model Origin and Local Dynamics

A second application is provided by a discrete malignant tumour-growth map. The model is related to the von Bertalanffy growth law:

$$\frac{dN}{dt} = \hat{\gamma}N^{2/3} - \hat{\delta}N, \quad (106)$$

where the $N^{2/3}$ term models growth limited by nutrient or oxygen supply through the tumour surface and the linear term represents loss proportional to tumour volume [18,19]. A forward discrete approximation gives

$$N_{k+1} = \gamma N_k^{2/3} + (1 - \delta)N_k. \quad (107)$$

After the rescaling,

$$x_k = \left(\frac{\delta - 1}{\gamma}\right)^3 N_k, \quad \rho = \frac{\delta - 1}{6.75},$$

one obtains the normalized scalar map

$$x_{k+1} = f_\rho(x_k) = 6.75\rho(x_k^{2/3} - x_k). \quad (108)$$

Here, $0 \leq x_k \leq 1$ and $0 \leq \rho \leq 1$. This one-dimensional formulation and its chaotic regimes were studied by Codreanu and Danca [20]. Here, it is used only as a mathematical growth model: the feedback term is not identified with a specific clinical intervention.

In addition to $x = 0$, the map has the positive fixed point

$$x_\rho^* = \left(\frac{6.75\rho}{1 + 6.75\rho}\right)^3. \quad (109)$$

For $x > 0$,

$$f'_\rho(x) = 4.5\rho x^{-1/3} - 6.75\rho, \quad (110)$$

and evaluation at (109) gives

$$a = f'_\rho(x_\rho^*) = \frac{2}{3} - \frac{9}{4}\rho. \quad (111)$$

Consequently, the positive fixed point is stable for $-1 < a < 1$, whereas

$$a < -1 \iff \rho > \frac{20}{27} \approx 0.7407. \quad (112)$$

The instability is therefore of the negative-multiplier type throughout the relevant post-bifurcation range. The zero fixed point is not used as a target in the present linearization because $x^{2/3}$ is not differentiable at $x = 0$.

The corresponding exact positive-delay model-parameter windows are

$$\frac{20}{27} < \rho < \frac{20}{27} + \frac{4}{9\tau} \quad \text{for FPFC,} \quad (113)$$

$$\frac{20}{27} < \rho < \frac{20}{27} + \frac{8}{9\tau} \quad \text{for TDFC with odd } \tau, \quad (114)$$

$$\frac{20}{27} < \rho < \frac{20}{27} + \frac{8}{9(\tau-1)} \quad \text{for NFC with even } \tau, \quad (115)$$

subject in each case to $\rho \leq 1$. These formulas follow by substituting the closed-form multiplier into Theorem 4. The simulations below use selected points from these domains. Figure 7 shows the bifurcation diagram and Lyapunov exponent of (108), providing the parameter-space context for the periodic and chaotic regimes considered below.

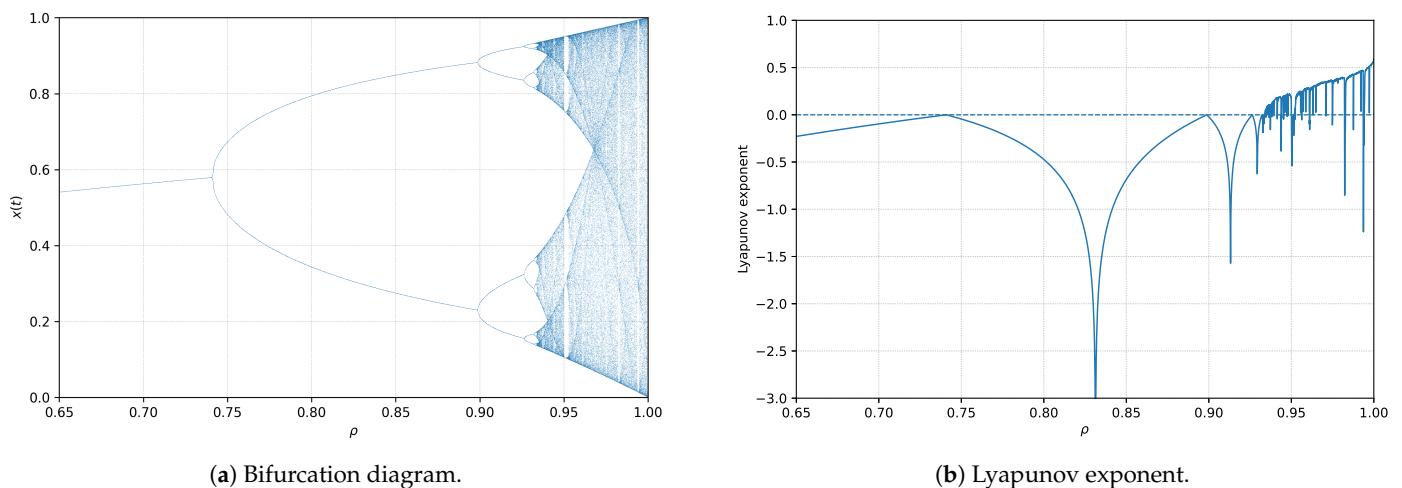


Figure 7. Bifurcation diagram and Lyapunov exponent of the normalized tumor-growth map (108). Panel (a) shows the changes in asymptotic orbit structure as ρ varies, with period-doubling beginning near the exact fixed-point threshold (112), while panel (b) provides the corresponding sign-based chaos diagnostic. The regime labels of the selected cases are verified independently using the uncontrolled-orbit diagnostic (104)–(105).

Tumour-map trajectory protocol

The FPFC and TDFC tumour examples use $x_0 = 10^{-8}$, while the NFC examples use $x_0 = 10^{-1}$. As in the logistic simulations, the first 80 plotted values are uncontrolled, feedback begins at $k = 80$, and no displayed transient is discarded.

8.2. FPFC Verification

For the tumour map, FPFC takes the form

$$x_{k+1} = 6.75\rho \left(x_k^{2/3} - x_k \right) - b \left(x_{k-\tau} - x_\rho^* \right). \quad (116)$$

The four FPFC cases lie inside Corollary 2 and span both delay parities: two chaotic cases, one period-four case, and one period-two case. All converge locally to the positive fixed point listed in Table 5 after feedback activation. The table also gives the independently computed spectral radii; all are below one, with the $\tau = 2$ chaotic case closest to the boundary at 0.9932.

Table 5. FPFC tumour-map simulations taken from the application study.

Regime	ρ	x_ρ^*	a	τ	b	$\rho_{sp}(A_\tau)$
Chaotic	0.976	0.6545	−1.5293	1	0.600	0.7746
Chaotic	0.958	0.6496	−1.4888	2	−0.490	0.9932
Period four	0.926	0.6407	−1.4168	2	−0.425	0.9468
Period two	0.830	0.6110	−1.2008	3	0.240	0.9166

Figure 8 provides the trajectory counterpart of Table 5, showing convergence for chaotic and periodic cases with both delay parities.

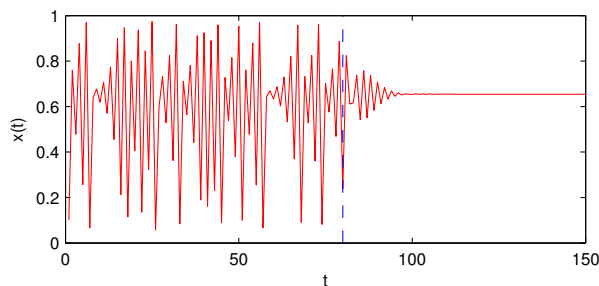
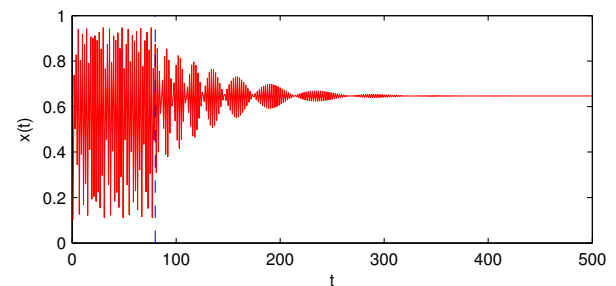
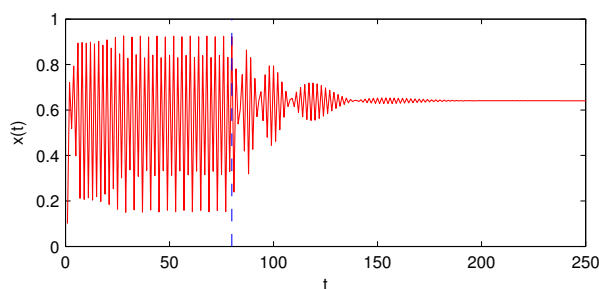
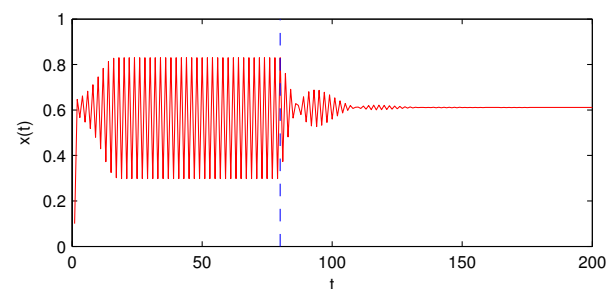
(a) FPFC, chaotic regime: $\rho = 0.976$, $\tau = 1$, $b = 0.6$.(b) FPFC, chaotic regime: $\rho = 0.958$, $\tau = 2$, $b = -0.49$.(c) FPFC, period-four regime: $\rho = 0.926$, $\tau = 2$, $b = -0.425$.(d) FPFC, period-two regime: $\rho = 0.83$, $\tau = 3$, $b = 0.24$.

Figure 8. FPFC local stabilization of the positive tumour-map fixed point for two chaotic, one period-four, and one period-two case. Feedback begins at $k = 80$ (dashed line); no displayed transient is discarded.

8.3. TDFC Verification

For TDFC, the controlled tumour map is

$$x_{k+1} = 6.75\rho\left(x_k^{2/3} - x_k\right) + b(x_{k-\tau} - x_k). \quad (117)$$

The TDFC examples use the admissible odd delays 1, 3, and 5, in agreement with Corollaries 3 and 7; two are chaotic and one is period-four. Table 6 confirms $\rho_{sp}(A_\tau) < 1$ for all three cases. The radius increases from 0.8944 to 0.9833 as the selected delay grows, consistent with the slower convergence and same-parity contraction.

Table 6. TDFC tumour-map simulations taken from the application study.

Regime	ρ	x_ρ^*	a	τ	b	$\rho_{sp}(A_\tau)$
Chaotic	0.989	0.6579	−1.5586	1	−0.800	0.8944
Chaotic	0.995	0.6594	−1.5721	3	−0.289	0.9644
Period four	0.900	0.6331	−1.3583	5	−0.182	0.9833

8.4. NFC Verification

For NFC, the controlled tumour map is

$$x_{k+1} = (1 - b) 6.75\rho \left(x_k^{2/3} - x_k \right) + b x_{k-\tau}. \quad (118)$$

The exact window (115) restricts local NFC stabilization of negative unstable multipliers to positive even delays. Figure 9d–f uses $\rho = 0.9$ with $(\tau, b) = (2, 0.6)$, $(4, 0.215)$, and $(6, 0.1535)$; for the common multiplier $a = -1.3583$, the spectral radii are 0.9286, 0.9407, and 0.9876.

Figure 9 places the odd-delay TDFC and even-delay NFC sequences on common axes, directly illustrating the complementary parity restrictions.

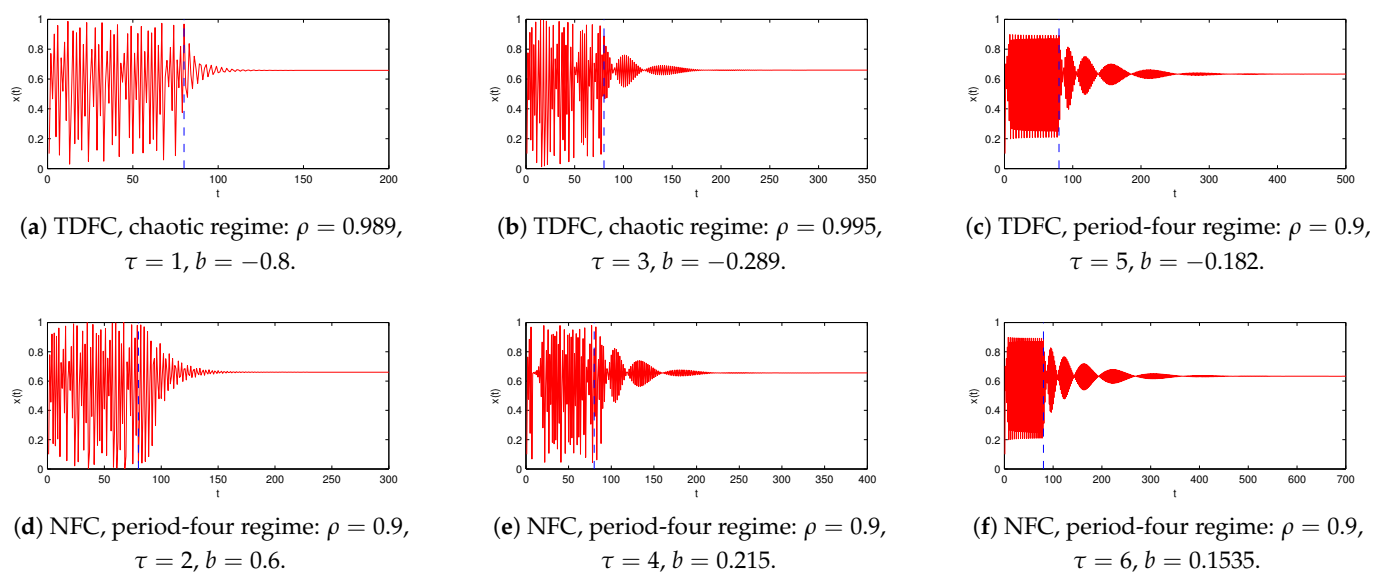


Figure 9. Tumour-map local stabilization: TDFC at the admissible odd delays 1, 3, 5 in panels (a–c), and NFC at the admissible even delays 2, 4, 6 in panels (d–f). Feedback begins at $k = 80$ (dashed line); no displayed transient is discarded.

8.5. Interpretation and Scope

The tumor application confirms that controller feasibility depends on the local multiplier rather than on the explicit map formula: substituting $a(\rho) = 2/3 - 9\rho/4$ into the exact domains selects the admissible gains directly. The simulations also display the predicted parity pattern—FPFC with either parity, TDFC with odd delays, and NFC with even delays. These trajectories illustrate local fixed-point stabilization of the mathematical model; global basin geometry and clinical interpretation remain outside the present local analysis.

9. Discussion

The analysis is organized around the distinction between the universal delayed polynomial and the three controller parametrizations. This makes the scalar setting analytically powerful: arbitrary delay raises the local dimension and polynomial degree, while the sparse structure still permits a complete exact description. Proposition 2 shows directly how the curved Schur boundary is generated by a unit-circle root, and the controller pullbacks transform this universal geometry into the physical multiplier–gain plane.

The pointwise stability domains form the first design layer. The set $\mathcal{A}_\tau^j$ determines whether any stabilizing gain exists for a given multiplier, whereas $\mathcal{D}_\tau^j$ retains the complete admissible-gain geometry. The delay-budget formulas invert the same information and quantify the contraction of stabilizability with increasing delay. Same-parity nesting and

the exact large-delay intersections sharpen this picture: specific parity-dependent boundary segments remain stable for every finite delay. The parity-shift identity further explains why odd-delay TDFC and the following even-delay NFC have the same multiplier reach despite their different gain regions and information requirements. Related parity and contraction effects have been reported in other delayed-map settings, including coupled networks [10], whereas here they arise from the universal scalar Schur region and are inherited by three distinct controller pullbacks.

The common-gain theorem adds a second design layer. Pointwise stability asks whether a gain can be chosen once the multiplier is known exactly; interval stability asks whether one gain works uniformly for every multiplier in a prescribed range. Because the transformed coefficients depend affinely on the multiplier and R_τ is monotone in its magnitude, this deterministic robustness problem reduces exactly to endpoint quantities rather than to a dense parameter scan. The resulting conditions are therefore necessary-and-sufficient common-gain feasibility tests, rather than sampled approximations of the uncertain multiplier interval.

Controller selection adds a third layer by separating feasibility from preference. Projection membership first identifies which controller structures can stabilize the prescribed unstable multiplier; gain restrictions then intersect the exact feasible-gain intervals with the admissible set G , while implementation requirements determine which remaining structures are available. For each surviving controller, the complete feasible-gain interval is obtained from explicit boundary endpoints or, where necessary, from a certified monotone scalar solve, so the selection procedure does not rely on gain sampling. Because FPFC, TDFC, and NFC use different information, performance ranking remains application-dependent; the midpoint gain-margin rule (98) provides only a neutral tie-breaker when no such preference is supplied. This feasibility-first organization is also consistent with the broader control-theoretic viewpoint in [14], where necessary-and-sufficient solvability conditions for stable inversion are established before subsequent performance-oriented design, although the underlying control problems are technically different.

The analytical reductions are implemented by four complementary computational procedures. Algorithm 1 performs certified pointwise classification using a monotone scalar solve, Algorithm 2 generates the stability boundary directly from its unit-circle parametrization, Algorithm 3 reduces common-gain testing over a continuum of multipliers to endpoint quantities, and Algorithm 4 constructs the feasible-gain sets from explicit or certified boundary endpoints. Finite bisection uncertainty is retained explicitly: when a strict curved-boundary comparison cannot be certified at the prescribed tolerance, the result remains indeterminate rather than being forced into a classification.

Direct polynomial roots therefore serve primarily as an independent validation method. Across the complete 114,000-point validation set no mismatch was observed. The 6000 large-delay tests at $\tau = 10, 20, 50$, and 100 likewise produced no mismatch, while the separate timing benchmark showed that the computational advantage over repeated all-roots calculations grows strongly with delay. At large delay the boundary parametrization also becomes geometrically steep near $(0, 1)$; the endpoint-stable forms (86) and (87) and adaptive sampling address the associated floating-point and geometric-resolution issues.

The logistic and tumour-growth maps serve complementary illustrative roles. The former displays convergence to the locally stabilized fixed point from selected chaotic and periodic initial histories for all three controllers, whereas the latter shows that the same local multiplier-based classification transfers to a non-polynomial map. These examples retain the local scope of the theorem: once $a = f'(x^*)$ is known, the exact Schur domain determines local fixed-point stability as a function of a , b , and τ . Global basin geometry,

convergence from arbitrary initial histories, nonlinear transient performance away from the fixed point, and noise sensitivity form natural nonlinear extensions of this local analysis.

The framework focuses on a real scalar multiplier, one constant gain, and one delayed sample. This focused setting makes exact arbitrary-delay geometry possible. Natural extensions include optimization of the dominant-root modulus within the proved gain sets, quantitative basin estimates, and feedback laws containing several independently weighted delayed samples.

10. Conclusions

Exact multiplier–gain stability domains have been derived for FPFC, TDFC, and NFC with arbitrary integer delay. The three local problems reduce to the same sparse delayed characteristic polynomial, so a single Schur criterion generates the controller domains through explicit pullbacks. The unit-circle parametrization provides a direct representation of the curved boundary and the identities used throughout the structural analysis.

The resulting domains reveal a systematic delay-parity structure. They obey same-parity nesting, have exact parity-dependent large-delay intersections, and yield closed-form stabilizable-multiplier projections and delay budgets. In particular, positive-delay FPFC can stabilize unstable real multipliers on either side of the unit interval within its delay-dependent reach, whereas TDFC and NFC stabilize negative unstable multipliers only for odd and even delays, respectively. Odd-delay TDFC and the following even-delay NFC have the same multiplier projection despite their different gain regions and information requirements.

For interval uncertainty, the theory gives necessary-and-sufficient common-gain conditions and leads directly to certified procedures for pointwise classification, boundary generation, robust gain testing, and gain-constrained controller selection. The controller-selection procedure constructs the complete feasible-gain interval from explicit or certified boundary endpoints and preserves an indeterminate outcome whenever finite precision does not permit a strict decision. Across 114,000 characteristic-root validation points, including tests at delays up to $\tau = 100$, no classification mismatch was observed; the large-delay benchmarks further demonstrate the computational advantage over repeated all-roots calculations.

The logistic and tumour-growth maps, both exhibiting periodic and chaotic regimes over their parameter ranges, demonstrate convergence to the locally stabilized fixed point from selected initial histories and illustrate how the same multiplier-based theory applies to both polynomial and non-polynomial discrete-time maps. Together with the exact stability geometry and certified computational procedures, these examples complete a unified analytical and algorithmic framework for arbitrary-delay local stabilization of scalar discrete-time systems.

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Abbreviations

FPFC	Fixed-point feedback control
TDFC	Time-delayed feedback control
NFC	Nonlinear predictive feedback control

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