

On the steady and breathing propagation of 2D flame-balls

Jorge Yanez  ; Mike Kuznetsov 



Chaos 36, 093149 (2026)

<https://doi.org/10.1063/5.0340083>



Articles You May Be Interested In

Numerical modeling and suppression of combustion instabilities in a partially premixed combustor

Physics of Fluids (September 2024)

Large eddy simulation of vortex structure evolution, flow, and heat transfer characteristics in hydrocarbon fuel boundary layer combustion

Physics of Fluids (June 2026)

Birds breathe at an aerodynamic resonance

Chaos (December 2021)



Chaos

**Special Topics Open
for Submissions**

[Learn More](#)

On the steady and breathing propagation of 2D flame-balls

Cite as: Chaos 36, 093149 (2026); doi: 10.1063/5.0340083

Submitted: 22 April 2026 · Accepted: 26 August 2026 ·

Published Online: 28 September 2026



View Online



Export Citation



CrossMark

Jorge Yanez^{a)} and Mike Kuznetsov

AFFILIATIONS

Institute for Thermal Energy Technology and Safety, Karlsruhe Institute of Technology, Hermann-von-Helmholtz-Platz 1, 76344 Eggenstein-Leopoldshafen, Germany

^{a)}Author to whom correspondence should be addressed: jorge.yanez@kit.edu

ABSTRACT

In this article, we examine the combustion of 2D flame-balls inside of the Hele-Shaw cell. Experiments have shown that the propagation of such structures occurs in a plethora of different regimes. Most of them remain unexplained. Here, we investigate the steady and cyclic dynamics numerically. First, we use a parametric study to numerically derive the stable manifold in the plane defined by the imposed inflow velocity and the heat-loss intensity. The conditions of observability are identified, involving a minimal velocity of propagation. Based on these results, the topology of the space of phases around these conditions is explored utilizing an eigenvalue analysis. It was concluded that cyclic solutions are a transient regime. In addition, its appearance results from a subcritical Hopf bifurcation. These conclusions are also sustained through qualitative analytic discussions.

© 2026 Author(s). All article content, except where otherwise noted, is licensed under a Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>). <https://doi.org/10.1063/5.0340083>

The Hele-Shaw cell is a cornerstone of nonlinear science, renowned for revealing complex pattern formation in fluid dynamics. When applied to gaseous premixed combustion, this device produces intricate flame structures that emerge as isolated reactive fronts as the system nears the quenching limit. These flamelets represent two-dimensional analogs to three-dimensional flame-balls—a fundamental structure in combustion—yet their underlying physics remains poorly understood. This work constitutes an attempt to bridge the gap between synergetics and combustion, applying methods of the former to flame dynamics and linking combustion phenomena to the broader and more universal behavior of dissipative nonlinear systems. We explore the physical conditions, resulting in the steady propagation of the flamelets and examine the surrounding phase space to identify the onset of drift bifurcations. Furthermore, we unveil the conditions under which Hopf bifurcations appear, to understand the terms in which the cyclic solutions observed in experiments appear. Resolving the latter constitutes a necessary first step toward addressing more complex patterns visualized in the experiments.

1. INTRODUCTION

Flame-balls were unprecedentedly observed by Ronney in 1990,^{1,2} but their existence was postulated much earlier by Zeldovich.³ After its discovery, the regime was subsequently analyzed by Buckmaster and his co-authors in a set of classic papers.^{4–9} In this work, we focus on a similar phenomenon in which isolated flamelets are propagated in a narrow gap between two parallel plates. Such 2D analogs to flame-balls were discovered by Veiga *et al.*¹⁰ in 2020 (see Fig. 1).

Combustion in enclosures has been analyzed among others by Kurdyumov and Matalon.¹² These authors utilized an approach based on asymptotic analysis. In successive steps, their model was refined^{13,14} to be applicable for two-dimensional problems¹⁵ with heat losses.¹⁶ Utilizing this derivation, Martinez-Ruiz and his co-authors¹⁷ reproduced the isolated spots. Notably, they replicated the mono- and bi-cellular cell patterns in a 2D configuration, analyzing their shapes and structures and even including the effect of gravity.¹⁸

Let us come back to the taxonomy of the experiments of Ref. 10, where a multiplicity of patterns were detected.¹⁹ Among the regimes reported in Refs. 11 and 19, in this paper, we are interested in the

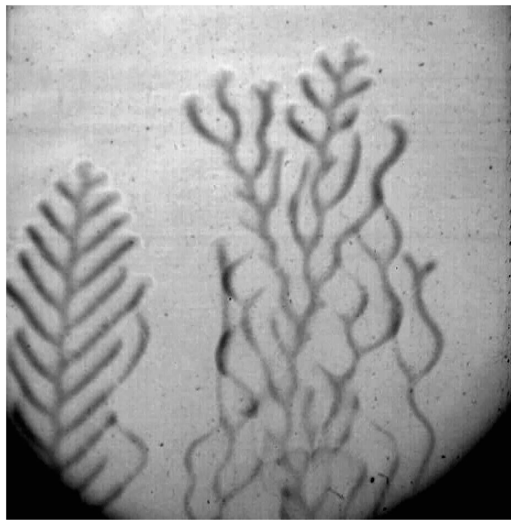


FIG. 1. Unconventional propagation photographed in a Hele-Shaw cell. Experimental configuration and observations are described in Refs. 10 and 11. Figure reproduced from Yanez *et al.*, *Fire* 9(7), 288 (2026). Copyright 2026 Author(s), licensed under the Creative Commons Attribution (CC BY) license.

types shown in Fig. 2. There, the flamelets propagate in a cyclic or quasi-steady regime.

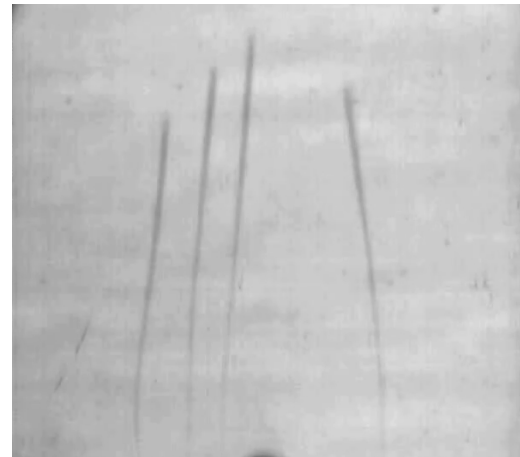
Steady propagation was analyzed in Refs. 20–22. In Ref. 20, the authors were able to determine theoretically, utilizing asymptotic analysis, the existence and limits of self-propagating 2D flame-balls.

The statistical analysis of the experiments undertaken in Ref. 11 showed the crucial role of heat losses for unconventional propagation. Experimental evidence indicates that steady regimes may appear as a limit for very intense heat loss. It also suggests that cyclic ones appear when heat losses approach, but are slightly lower, to the conditions of steady propagation.

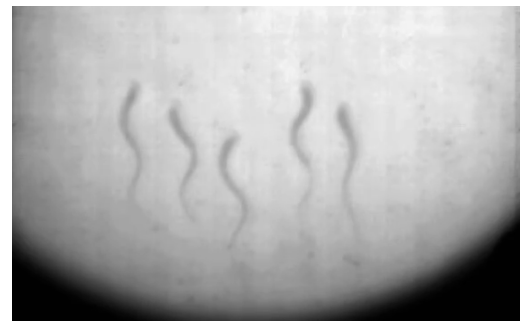
We endeavor our investigation utilizing techniques that are common in the investigation of *solitons*. These²³ are isolated waves appearing in conservative systems with dispersion.²⁴ The spreading of such structures, to form a continuous front, is prevented by the physical properties of the matter and the geometrical characteristics of the facility. That is, by an equilibrium between dissipation due to heat and viscous losses on one hand and energy provided by the chemical reaction on the other.

In the specialized literature,²⁵ cyclic regimes in which structures change their shape periodically are known as *breathing* solitons. Those have been detected, among others, in (a) laser systems,²⁶ (b) current filaments,²⁷ (c) chemical reactors,²⁸ (d) neural media,²⁹ (e) photonic crystals,³⁰ and (f) Bose–Einstein condensates.³¹

Such regimes have been analyzed using different approaches: an algebraic procedure was followed by Koga and Kuramoto,³² who utilized a linear null-cline and a set of simplified equations to study the conditions for the existence of Hopf bifurcation. Building on this work, Nishiura and Mimura³³ significantly enhanced the depth of the analysis. Utilizing asymptotic expansions, these authors derived an expression for the critical onset of the breathing behavior. They



(a) Quasi-steady



(b) Cyclic

FIG. 2. Flame-balls propagating in a Hele-Shaw cell. Quasi-steady and cyclic regimes are described in Refs. 10, 11, and 19. Figure reproduced from Yanez *et al.*, *Fire* 9(7), 288 (2026). Copyright 2026 Author(s), licensed under the Creative Commons Attribution (CC BY) license. (a) Quasi-steady and (b) cyclic.

also analyzed the comportment of the eigenvectors of the system of equations. Finally, Kerner and Osipov³⁴ also describe and analyze breathing extensively in 1D.

Breathing was also studied numerically. Hagberg utilized a FitzHung–Nagumo model.³⁵ Suzuki *et al.*³⁶ followed a analytic-numerical path to derive a map of the conditions for breathing and wiggling.

Finally, breathing was also investigated by Gurevich in Ref. 37, an approach that has provided the inspiration for this work. There, the structure undergoes a bifurcation from a stationary to a cyclic state, by the effect of an unstable breathing mode. Notably, an equation for the amplitude of the oscillations was derived coinciding with the normal form of a Hopf bifurcation. Therefore, and without further delay, let us start our investigation based on the results of Ref. 37.

II. MODEL

A. General

We consider the problem of an isolated flamelet propagating in a reduced domain, whose size is of the order of several hundred flame thicknesses. In this reduced area of a Hele-Shaw cell, the structure propagates at a speed U through a large amount of fresh gases injected at a speed V , Fig. 3(a).

In our analysis, we utilize the thermo-diffusive model⁹ similar to the one adopted by Brailovsky and Sivashinsky³⁸ and in Refs. 21 and 20. The system of equations reads

$$T_t + V \cdot \nabla T = \nabla^2 T + (1 - \sigma)W - Q, \quad (1)$$

$$C_t + V \cdot \nabla C = Le^{-1} \nabla^2 C - W, \quad (2)$$

$$W = (1 - \sigma)^2 Le^{-1} N^2 C \exp(N(1 - T^{-1}))/2, \quad (3)$$

where T is the dimensionless temperature; C is the concentration of fuel of the scarcest species; Q is the dimensionless volumetric heat loss; $Le = \chi/D$ is the Lewis number, with χ being the thermal diffusivity of the mixture and D being the molecular diffusivity of the deficient fuel in the mixture; σ is the ratio of the temperature of the fresh mixture to the adiabatic temperature of the burned products; N is the activation energy; and W is the species consumption. The changes

$$T = \sigma + (1 - \sigma)\theta, \quad (4)$$

$$Q = (1 - \sigma)q\theta, \quad (5)$$

$$\beta = (1 - \sigma)N \quad (6)$$

allow us to rewrite the equations as

$$\theta_t + V \cdot \nabla \theta = \nabla^2 \theta + \Omega - q\theta, \quad (7)$$

$$C_t + V \cdot \nabla C = Le^{-1} \nabla^2 C - \Omega, \quad (8)$$

$$\Omega = Le^{-1} \beta^2 C \exp[\beta(\theta - 1)/(\sigma + (1 - \sigma)\theta)]/2, \quad (9)$$

in which q is the scaled heat-loss intensity.

At the lower boundary, a purely fresh mixture $\theta = 0$, $C = 1$ is injected at a constant velocity V perpendicular to the boundary. The top border assumes that the gradients of all quantities are zero, $\nabla \theta = \nabla C = 0$. In the lateral boundaries, Dirichlet boundary conditions corresponding to the fresh mixture are set $\theta = 0$, $C = 1$. The latter is a conventional choice, and similarly, zero gradient or periodic boundary conditions may have been selected for our purposes.

B. Drift instability

By *drift instability*, we understand the conditions at which the steady propagation state [Fig. 3(a)] will be destabilized by noise^{39,40} [Fig. 3(b)]. Note that this definition *assumes* the existence of a stationary or steady state. In our analysis, we will investigate the behavior of the solution near the locus of the *drift bifurcation*. To do this, we will determine the values of the parameters (q , V) where U

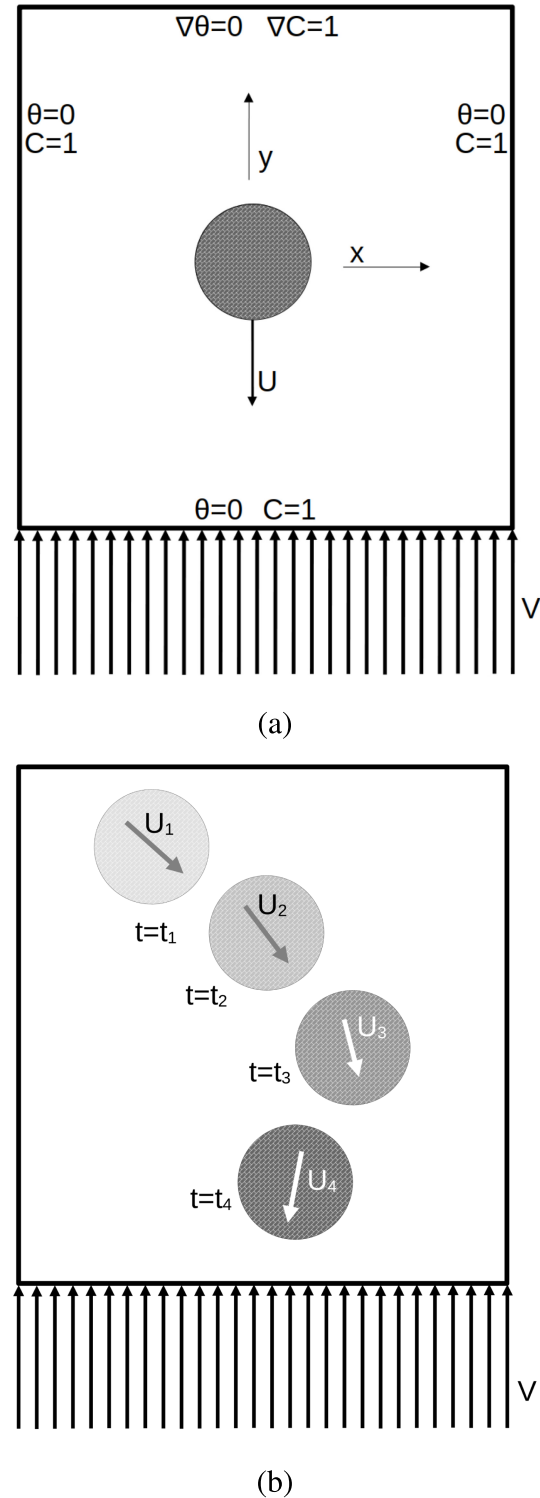


FIG. 3. Propagation regimes. (a) Steady—stable and (b) non-steady—unstable.

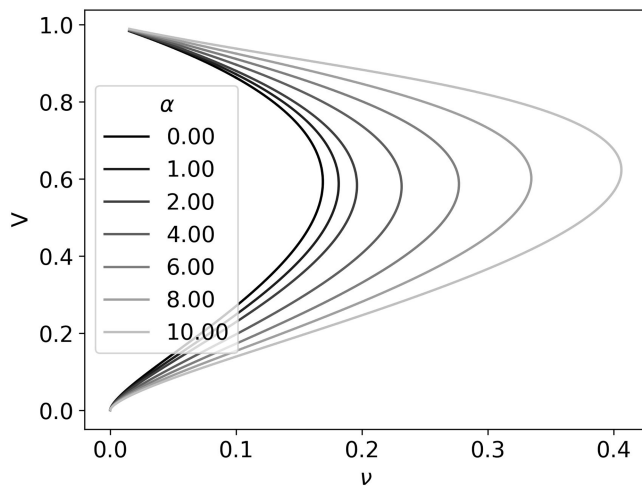


FIG. 4. $V - \nu$ locus for steady propagation. Taken with permission from Yanez *et al.*, *Combust. Flame* **258**, 113059 (2023). Copyright 2023 Elsevier.²⁰

starts to be nonzero. This methodology was also utilized previously in Refs. 39, 41, and 42.

These conditions are sketched in Fig. 3. There, we see two velocities, V and U , meaning of which will be clarified now.

By V , we understand the velocity of the gases injected in the lower boundary. In the thermo-diffusive approach, the velocity is decoupled from temperature and concentration. Thus, combustion does not induce changes in momentum. This constitutes a very significant simplification, inherent to the model and widely utilized in the past.^{9,38,43} Therefore, we can regard V as constant and vertical in the whole domain.

Although velocity is constant, temperature and concentration fields are not; see, e.g., $\theta = \theta(x, y)$ dependence in Eqs. (7) and (8). Advancing in the narrative, see the results of Fig. 5. In the conditions of investigation, hotspots will be formed. The position and size of them change in time. We calculate the center of a selected hotspot and track its temporal evolution; this post-processing procedure defines U , as detailed in Sec. III C.

C. Conditions of the drift instability

Considering the results available in the literature, we may examine the expected conditions of the *drift instability* by recalling the locus of the steady propagation derived in Ref. 20, reproduced in Fig. 4.

We obtained this picture in terms of dimensionless units. The dimensions are expressed in terms of ℓ , the thermal width of the flame, where $\ell = \chi/U_b$ with U_b being the speed of a planar adiabatic flame. The time t was scaled in units of χ/U_b^2 . The heat-loss intensity, q , was defined as $q = (\pi \ell/d)^2$, a magnitude governed by the width of the Hele-Shaw gap, d , and by the mixture composition, % H_2 , which, in turn, determines the thermal flame width ℓ , as discussed in Refs. 10, 44, and 45. To compare with theoretical findings,²⁰ we re-scaled the latter, defining $\nu = q\beta$, in terms

of the Zeldovich number β . Finally, we define the Lewis number parameter, $\alpha = (Le^{-1} - 1)\beta$.

Let us interpret Fig. 4 with regard to drift bifurcation. The choice of a gaseous mixture defines a Lewis number. In turn, this fixes α determining a single curve in the plot. The conditions of the curve apply to steady single flamelets. Thus, to the right of the curve, the flame should be extinguished due to too high heat loss. *Drift instability* occurs as a limit state near the extinction boundary. It appears due to a small reduction in heat loss; see Sec. V D. Those regimes correspond to conditions immediately to the left of the curve.

In Ref. 20, we restricted our analysis to simplified steady conditions. Our objective is to investigate the vicinity of this region, showing steady, quasi-steady, or cyclic propagation (ranges of ν for a certain V).

Due to the conditions of the theoretical analysis of Ref. 20, in Fig. 4, the steady propagation curve appears as a bi-valued function. This defines two branches: an upper one and a lower one. The observability of each of the branches—that is, if they are physical solutions—is still under discussion.

III. ANALYSIS

A. Conditions of the analysis

The conditions and shape of flamelet solution in the critical conditions of destabilization are not known. Approaches for an estimate of the size, velocity, and shape of the hotspot are the theoretical analysis of Refs. 20 and 46 (Fig. 4). Pitifully, the conditions of asymptotic derivation are too coarse to be utilized for numerical modeling.

The conditions of drift instability appear as a perturbation of the steady locus. This is characterized by the tuple $(\theta(x, y), C(x, y), q, V)$. Its determination is, thus, a boundary value problem. Nevertheless, the attempts to find a solution of the boundary problem numerically failed. Instead of this, we find such conditions though an initial value problem, integrating for sufficiently long time.

Thus, we consider an initial value problem of the system described by Eqs. (7)–(9), with the boundary conditions defined in Sec. II A. We regard significant conditions for sets of pairs (V, q) and assume the hotspots remain sufficiently far away of the borders of our simulation so that the boundary conditions remain valid.

We may consider a set of values for $(V_1, V_2, \dots, V_n)$ and trace the $|U| - q$ dependence for each V_i , a method similar to those used in Refs. 39 and 41. Let us fix V_i and consider a value for q, q_j . The flamelet may, or not, reach a steady solution after a transient. If it does, we have found a certain solution $(\theta_i(x, y, t = \infty), C_i(x, y, t = \infty), V_i, q_j)$ in which $U = 0$. If no steady state is reached, the flamelet is either blown downstream, indicating that V_i is too high for the selected q_j , or it approaches the inflow boundary, indicating that V_i is too low. Running the necessary amount of simulations, we may find a q_j matching the V_i , thus finding the conditions of the locus of the *drift bifurcation*, although the steady state may or may not be observable.

After a preliminary parameter sweep following a “trial and error” strategy with local refinement around the transition regions, we selected the pairs $V - q$ listed in Table I. We regard this trial-and-error strategy as consistent with the exploratory character of

TABLE I. Pairs (V, q) utilized in the numerical experiments.

	Heat loss, q (–)
$V = 0.1$	0.150, 0.160, 0.170, 0.180, 0.182, 0.184, 0.186, 0.188, 0.190, 0.200
$V = 0.3$	0.150, 0.160, 0.170, 0.180, 0.190, 0.192, 0.194, 0.196, 0.198, 0.200
$V = 0.4$	0.198, 0.200, 0.202, 0.204, 0.208, 0.212, 0.216, 0.220
$V = 0.5$	0.150, 0.160, 0.170, 0.180, 0.190, 0.200, 0.210, 0.220, 0.222, 0.224, 0.226, 0.228, 0.230, 0.232, 0.234, 0.236, 0.238, 0.240, 0.250
$V = 0.6$	0.200, 0.220, 0.224, 0.228, 0.232, 0.236, 0.240, 0.260
$V = 0.7$	0.230, 0.236, 0.238, 0.240, 0.242, 0.244, 0.250
$V = 0.8$	0.230, 0.236, 0.238, 0.240, 0.242, 0.244, 0.250
$V = 0.9$	0.210, 0.212, 0.214, 0.218, 0.220, 0.224, 0.226, 0.228, 0.230, 0.232, 0.234, 0.236, 0.238, 0.240, 0.242, 0.244, 0.246, 0.248, 0.250, 0.252, 0.254, 0.256, 0.258, 0.260, 0.262, 0.264, 0.266, 0.268, 0.270

this stage of the investigation. Nevertheless, we are aware that more refined follow-up analyses should rely on more systematic strategies, such as Bayesian optimization, greedy adaptive refinement, bracketed searches in q for fixed V , active-learning approaches, or continuation methods.^{47–49} The purpose of such strategies would be to automate the search for transition points in the (V, q) plane and reduce the number of required time integrations.

The entries in Table I represent the set of numerical experiments performed. For each imposed value of V , several heat-loss intensities q were considered in order to bracket and refine the transition between drifting, downstream-advection, and nearly steady regimes. This will become clearer through the discussion in Sec. III D and the analysis of Fig. 9.

B. Numerical results

For the calculations, we consider a squared domain with resolution 960×960 with the same conditions and boundary conditions shown in Sec. II A. We utilize a second-order center finite differences for the spacial derivatives. Time integration is carried out utilizing an explicit Dormand–Prince Runge–Kutta method of order 5.⁵⁰ We will locate an ignition point near the center of the domain, at a position (x_0, y_0) very slightly eccentric to avoid imposing symmetry to the solution.⁵¹ We adopt an initial hotspot of shape, $\theta(t=0) = 1 - C(t=0) = \exp[-((x-x_0)^2 + (y-y_0)^2)/\ell^2]$. To limit the amount of parameters in this study and for the rest of the paper, we will consider $N=10$ and $Le=0.3$.

For $V=0.9$ and $q=0.252$, the results of integration can be seen in Fig. 5. We see here a short transient diagrams [Figs. 5(a)–5(d)], followed by a quasi-steady state plots [Figs. 5(e) and 5(f)].

After ignition, the flame forms an initial pattern taking the shape of flower with four petals, Fig. 5(b). The amount of leafs and their shape were discussed in Dominguez-Gonzalez *et al.*⁵² Those authors suggested that the amount of petals was a function of the ignition energy.

Note that the shape, amount of flamelets, degree of boundedness among them, existence of symmetry, etc., of the results depend on (q, V) . It also showed a large degree of variability (Fig. 6). In addition, in many cases, a steady state was not achieved.

C. Basic post-processing

We post-process the results⁵³ of the parametric study to derive U for each (q, V) . We track the position of one of the flamelets advancing contra-current, calculating the *center of gravity* of the field θ . From its evolution, we may derive the velocity $U(t)$. Finally, to check that, we have reached a defined propagation regime and integrated the total field $\int \theta d\Omega$. The last constitute the so-called *mass* of the structure.

For illustration purposes, we display this for a single velocity $V=0.5$, although we have carried out this procedure for all conditions in our database. In Fig. 7, we show the mass time evolution for different heat losses. We observe that for most of the calculations, a regime is found after 125 dimensional time units. This lasts until the finalization of the simulations. After a relatively short transient, the flamelet achieves a constant size and temperature.

We turn now our attention to the velocity U , plotted in Fig. 8. We observe that during the first 50–75 time units, the transient associated with the ignition produces a velocity peak. Nevertheless, we disregard this initial part. We are interested in the propagation in the limit of long times.

After 75 time units, a *cruise* velocity is reached. We may divide the results in three groups.

1. Low heat losses, $q \leq 0.2$. U reaches a constant value quite fast. The structure propagates indefinitely in the direction opposite to the injected stream, eventually approaching the boundaries of the domain of simulation, a moment in which our simulation is terminated.
2. Intermediate heat losses, $0.2 < q \leq 0.226$. U reaches a sensible lower value. It also shows significant oscillations. The flame-ball does propagate, but in a lateral direction, perpendicular to the injected current. The hotspot remains steady in the direction of the injected current.
3. Large heat losses, $q > 0.226$. U approaches zero after a long transient, but the hotspot still does not quench, since in Fig. 7, a constant nonzero value is displayed. These are steady or quasi-steady conditions.

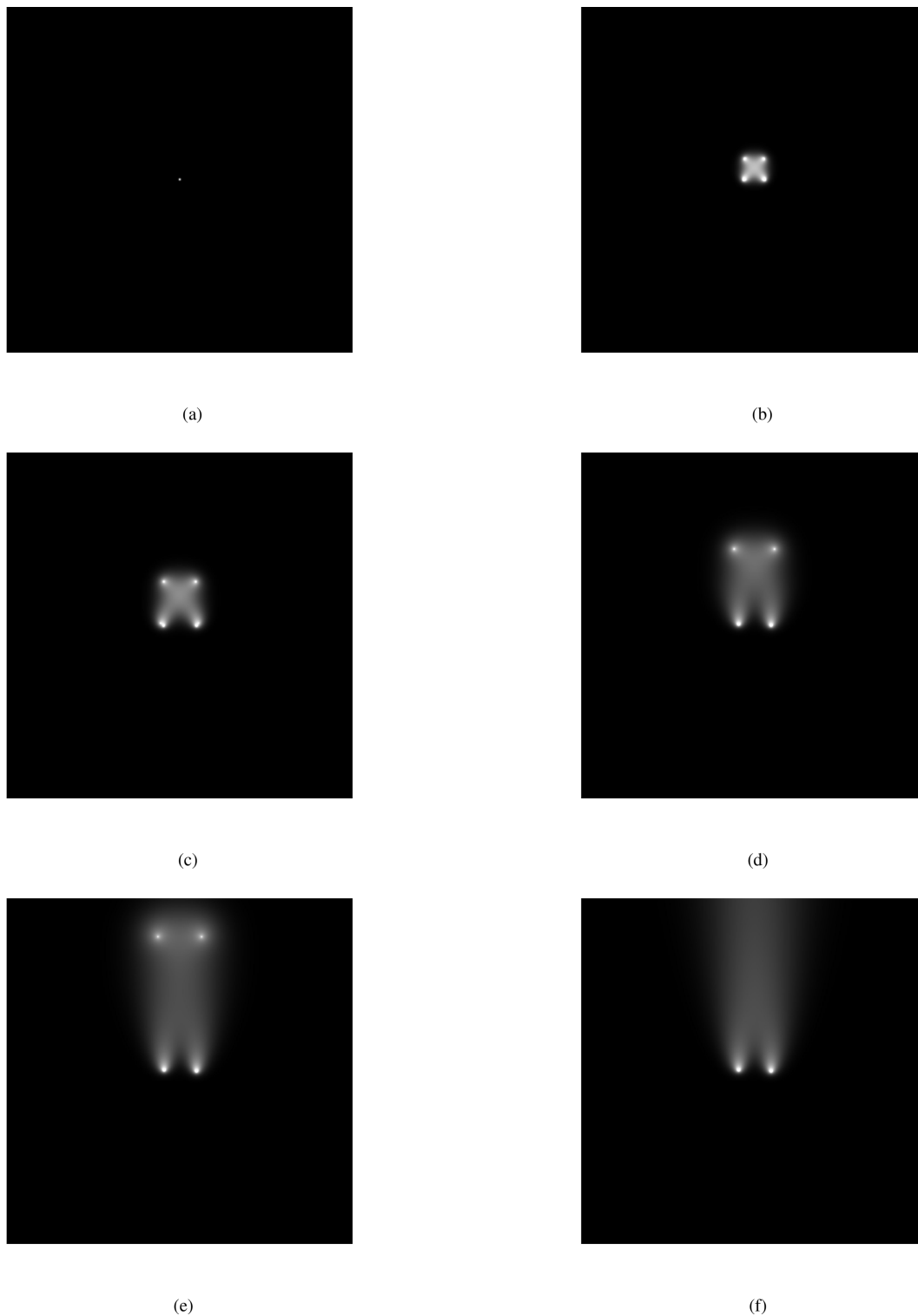


FIG. 5. Integration of the problem, dimensionless normalized temperature θ , with $N = 10$, $Le = 0.3$, $V = 0.9$, and $q = 0.252$. (a) $t = 0$, (b) $t = 20$, (c) $t = 50$, (d) $t = 100$, (e) $t = 200$, (f) $t = 400$.

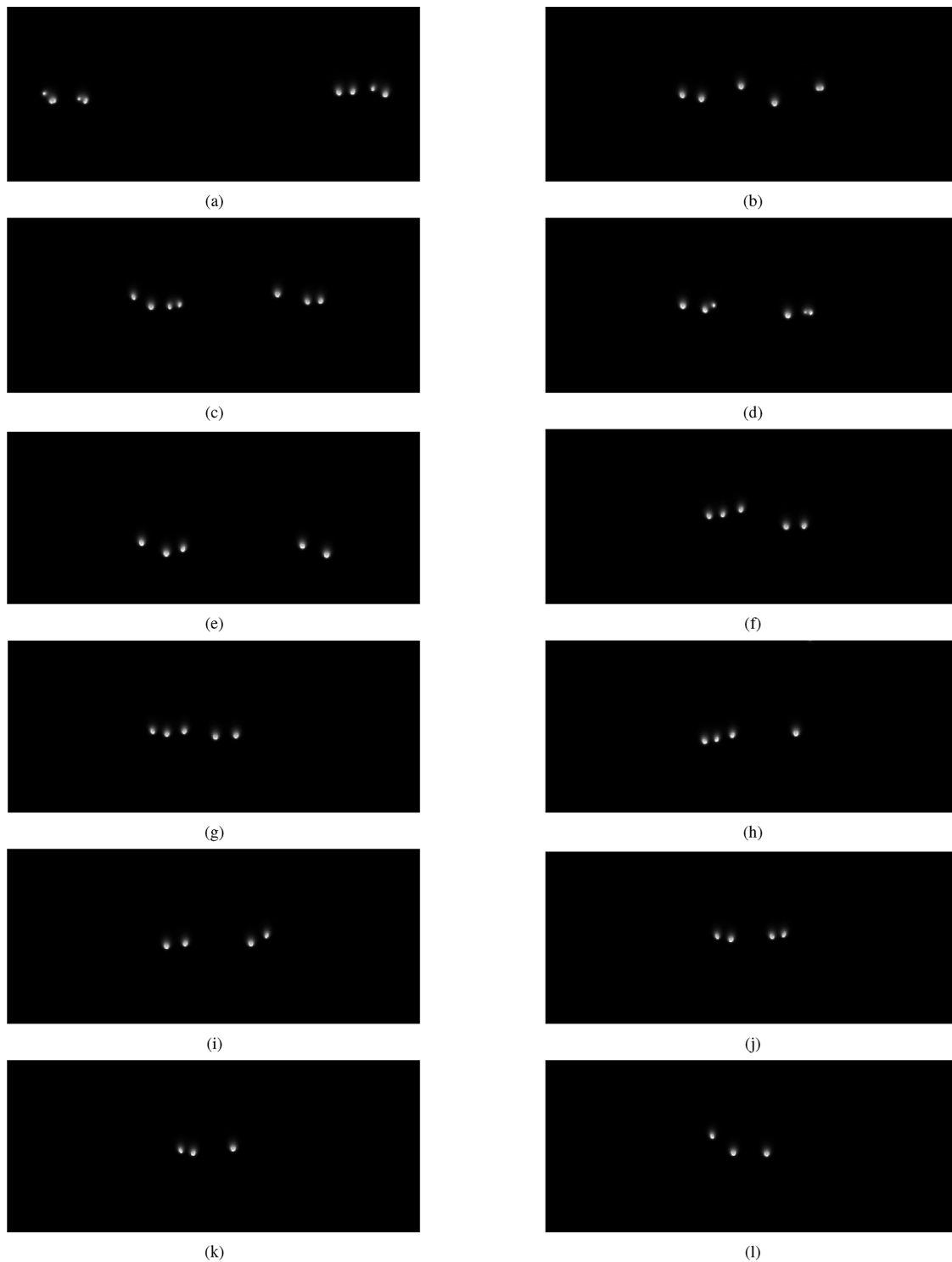


FIG. 6. Variations in the topology of the flamelets, dimensionless normalized temperature θ , for $N = 10$, $Le = 0.3$, $V = 0.9$ and different heat losses. Reduced domain. (a) $q = 0.218$, (b) $q = 0.224$, (c) $q = 0.226$, (d) $q = 0.228$, (e) $q = 0.230$, (f) $q = 0.232$, (g) $q = 0.234$, (h) $q = 0.236$, (i) $q = 0.238$, (j) $q = 0.240$, (k) $q = 0.246$, (l) $q = 0.250$.

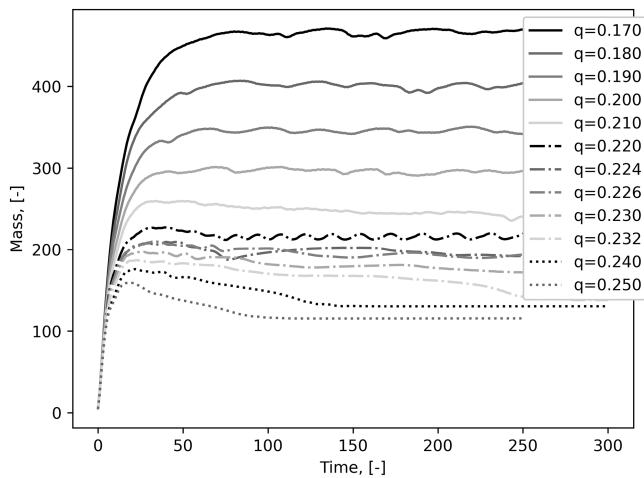


FIG. 7. Mass of the θ field, for $V = 0.5$ for different heat losses.

D. $U - q - V$ dependence

We may provide a synthesis of the results of Fig. 8, including at the same time for all V . We are not interested in the initial transient. Thus, we collect the average value of the velocity after a quasi-steady regime has been reached. Note that the instant at which that is reached changes for each $V - q$ condition.

In Fig. 9(a), we have represented the dependency of $|U|$ on heat loss for different values of V . In Fig. 9(b), we do the same for $|U|^2$. We have indicated the values of V in different shades as described in the legend. For the simplicity of diagrams, we have assigned velocity zero to those calculations in which the propagation occurs downstream. That is, when the flamelet is blown away, we are no longer interested in its velocity, and thus, we set it to zero. This

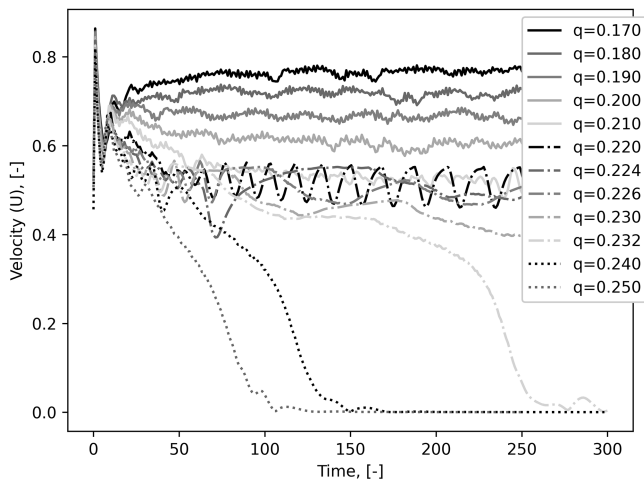
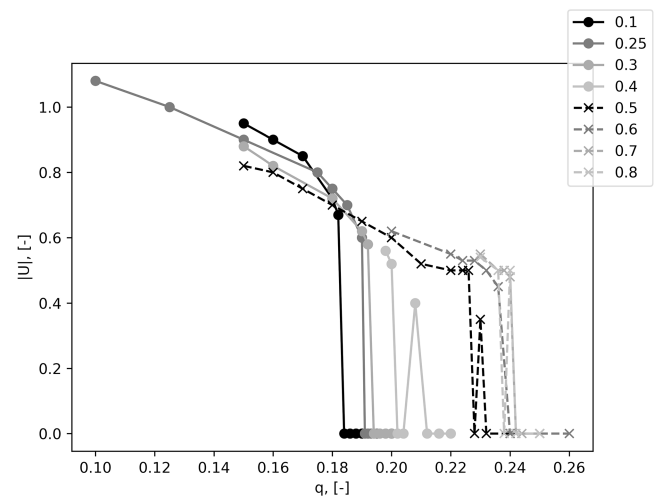
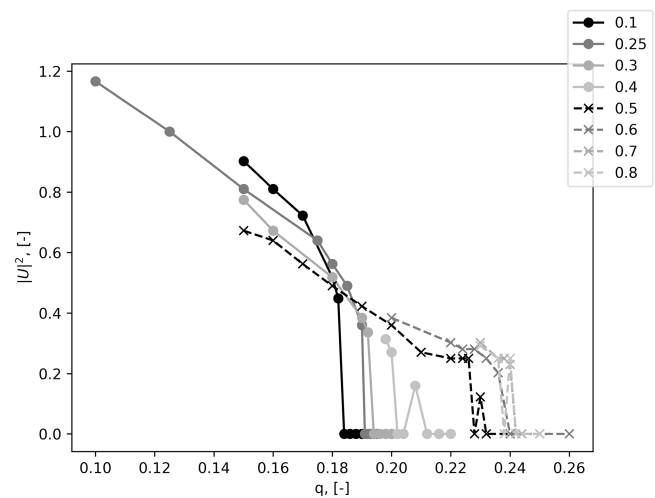


FIG. 8. Velocity of the center of gravity of the lower left flamelet obtained from the θ field, for $V = 0.5$.



(a) $|U|$



(b) $|U|^2$

FIG. 9. $|U| - q$ dependencies for the different values of V appearing in the legend. For downstream-advection cases, the plotted value $U = 0$ is a graphical convention used to mark simulations in which the flamelet is blown downstream; it should not be interpreted as a measured stationary flamelet. (a) $|U|$ and (b) $|U|^2$.

assigned zero value should, therefore, be understood as a marker used in the search for the transition location, not necessarily as a physically stationary flamelet. At this stage of the analysis, our objective is to identify where the relevant branch either reaches $U = 0$ or changes sign; for this purpose, both situations mark the vicinity of the transition.

In Fig. 9, we observe the expected decrease of $|U|$ with increased heat losses. Thus, heat loss tends to stabilize the structures, as we predicted by observing the results of Fig. 4 in Sec. II C. Note that each curve of Fig. 9 corresponds to a horizontal line in Fig. 4.

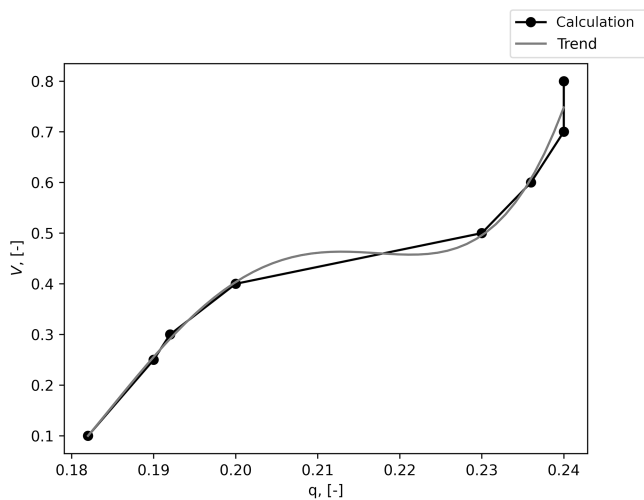


FIG. 10. $V - q$ diagram. Steady propagation.

Figure 9 includes the results represented for both $|U|$ and $|U|^2$. This allows us to explore the nature of the drift bifurcation. Saddle-node and Hopf bifurcation show a $q^{1/2}$ proportionality. Transcritical and Pitchfork bifurcation show instead a q dependence (linear) near the bifurcation point, Ref. 54, p. 16.

Close to the bifurcation position, we do not see a linear dependence neither of $|U|$ nor of $|U|^2$ on q . Near that point where $|U|$ starts to be nonzero, we do see an abrupt change for both $|U|$ and $|U|^2$. This can be attributed to a subcritical Hopf bifurcation; see the Appendix.

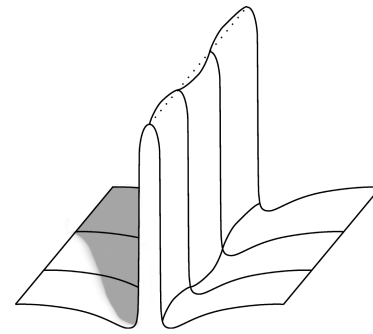
In Fig. 9, we also see that some transitions are not clean, e.g., for $V = 0.4$ and $V = 0.5$, zero velocity is reached, but then, for a higher value of q , U is nonzero again. We attribute this to a broad range of heat losses and geometrical configurations (size, number of flamelets, and disposition, Fig. 6) for which the stable manifold can be reached. Similar patterns can be seen in Fig. 14.

E. Steady $q - V$ dependence

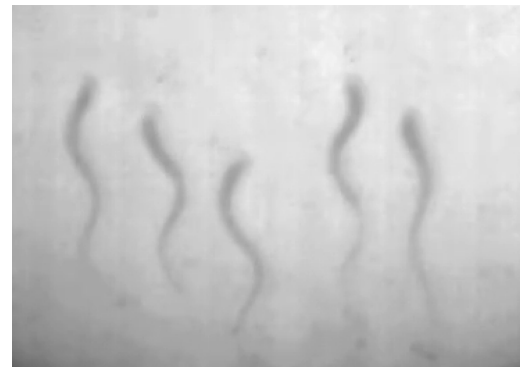
We may collect the minimum values of q for which $|U| = 0$ for the whole set of V . We have depicted this in Fig. 10. This constitutes the $q - V$ locus of steady propagation, a numerical analog to Fig. 4.

We may draw some conclusions from the observation of this picture.

1. First, q has a surprisingly small domain of variation, especially when compared to Fig. 4.
2. Note that in mathematical terms, the curve of Fig. 10 represents our numerical equivalent to the center manifold.
3. For imposed velocities smaller than $V < 0.4$, the steady state was not observable. Flame-balls either continue to grow or are blown away by the current.
4. The steady regime appears in $V \geq 0.4$.
5. This constitutes an indication that there may be a bifurcation inside the center manifold. This is a non-hyperbolic stationary point in the vicinity of $V \geq 0.4$.



(a) Description.



(b) Experimental photograph

FIG. 11. Winding solitons. Figures adapted from Yanez *et al.*, Fire 9(7), 288 (2026). Copyright 2026 Author(s), licensed under the Creative Commons Attribution (CC BY) license. (a) Description. (b) Experimental photograph.

6. Regarding the described behavior, the bifurcation point may be a sub-critical Pitchfork. This will appear as more probable when observing the diagram of Fig. 14.

IV. EIGENVALUE ANALYSIS

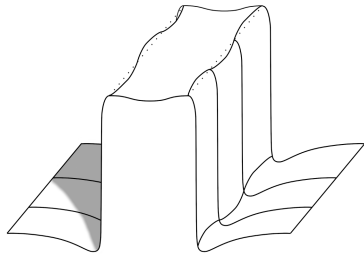
A. Discussion

We now turn our attention to a second topic, intimately linked with the derivation of Fig. 10, that of breathing solitons shown in Figs. 11 and 12.

Considering the behavior described in Kuramoto⁵⁴ for other physical systems, our hypothesis is that these regimes appear in the immediate vicinity of the curve in Fig. 10, to the left and due to a small reduction in the heat losses. Such conjecture can also be sustained by the results of Fig. 9. This constitutes a similar bifurcation pattern as we had investigated above, just this time we have a steady-breathing bifurcation.

B. Mathematical procedure

Inspired by the ideas of Kuramoto, we start our analysis by linearizing the system (7)–(9) near a known state, indicated with the sub-index 0. Like in Ref. 55, we do not require θ_0 or C_0 to be



(a) Description.



(b) Experimental photograph

FIG. 12. Rocking solitons. Figure reproduced from Yanez *et al.*, *Fire* **9**(7), 288 (2026). Copyright 2006 Author(s), licensed under the Creative Commons Attribution (CC BY) license. (a) Description. (b) Experimental photograph.

constant. Denoting the small perturbation with tildes, our variables become

$$\theta = \theta_0 + \tilde{\theta}, \quad (10)$$

$$C = C_0 + \tilde{C} \quad (11)$$

to form the linear system,

$$\begin{bmatrix} \tilde{\theta}_t \\ \tilde{C}_t \end{bmatrix} + L \begin{bmatrix} \tilde{\theta} \\ \tilde{C} \end{bmatrix} = 0. \quad (12)$$

This is now a well known procedure (Refs. 37 and 55, Sec. 1.14). Here, we have

$$L = [L_1, L_2], \quad (13)$$

with

$$L_1 = \begin{bmatrix} V \cdot \nabla - \nabla^2 + q - \partial_\theta \Omega|_{\theta_0, C_0} \\ \partial_\theta \Omega|_{\theta_0, C_0} \end{bmatrix} \quad (14)$$

and

$$L_2 = \begin{bmatrix} -\partial_C \Omega|_{\theta_0, C_0} \\ V \cdot \nabla - Le^{-1} \nabla^2 + \partial_C \Omega|_{\theta_0, C_0} \end{bmatrix}. \quad (15)$$

There, also

$$\partial_\theta \Omega = AC(\sigma + (1 - \sigma)\theta)^{-2} \exp[\beta(\theta - 1)/(\sigma + (1 - \sigma)\theta)], \quad (16)$$

$$\partial_C \Omega = A \exp[\beta(\theta - 1)/(\sigma + (1 - \sigma)\theta)]/\beta, \quad (17)$$

$$A = Le^{-1} \beta^3/2. \quad (18)$$

We solve the eigenvalue problem,

$$-Lf = \lambda f, \quad (19)$$

where f is the eigenvector and λ is the eigenvalue.

C. Numerical treatment

For the eigenvalue problem, we consider the same frames as in the calculations of Sec. III B and boundary conditions of Sec. II A, that is, the 960×960 numerical domain represented in Fig. 3 with a fresh mixture in the laterals $\theta = 0$, $C = 1$, injection of a fresh mixture $\theta = 0$, $C = 1$ in the bottom and zero gradients on top, $\nabla \theta = \nabla C = 0$. For the perturbations, this translates into $\tilde{\theta} = \tilde{C} = 0$ in the laterals and the bottom and $\nabla \tilde{\theta} = \nabla \tilde{C} = 0$ at the top.

For the base state θ_0 , C_0 , we utilize the final conditions of the calculations described in Sec. III B. For example, for $V = 0.9$, $q = 0.252$, we will use the state of Fig. 5(f) for θ_0 , or in the $V - q$ conditions of Fig. 6, the complete domain partially represented in the later figure for θ_0 . We calculate the eigen-problem with the pertinent θ_0 , C_0 fields for each of the conditions of Table I.

We address the solution of Eq. (19) numerically. Since the operators ∇ and ∇^2 are two-dimensional, it is advantageous to solve such a problem utilizing a matrix-less formulation. The *Implicitly Restarted Arnoldi Method*⁵⁶ handily available in the *Sparse* library of *Scipy*⁵⁷ was utilized to provide the results.

In the conditions mentioned in the previous paragraph, $V = 0.9$ and $q = 0.252$ [case of Fig. 5(f)], the details of the shape of the eigenvectors can be visualized in Fig. 13.

D. Analysis

The qualitative theory of multi-dimensional dynamics systems⁵⁸ provides the frames for the study of steady and periodic movements, bifurcations, and the respective regions of attraction. Nevertheless, the investigation of the behavior of the system in the global space of the parameters is an overwhelming task. Therefore, we focus on the local exploration of the space of phases, and this in the vicinity of the curve of Fig. 10.

In this local approach, we analyze the behavior of the system utilizing the so-called *Slaving Principle*,^{55,59,60} a concept popularized by H. Haken in the 70s. Under this approach, we divide the eigenvalues into two groups. The first one is constituted by the eigenvalues with a positive real part. Near the conditions of stability, the gray



FIG. 13. First three eigenvectors. Only a detailed reduced subdomain is shown. Eigenvalues obtained in the conditions of Fig. 5(f). (a) First eigenvector, real part. (b) First eigenvector, imaginary part. (c) Second eigenvector, real part. (d) Second eigenvector, imaginary part. (e) Third eigenvector, real part. (f) Third eigenvector, imaginary part.

curve in Fig. 14, those are few and are treated as the independent variables of the problem. The rest, having negative real parts, converge fast to a certain state. The latter, in turn, can be made dependent on the first group only. That is, the second category can be excluded from the analysis. This constitutes a tremendous simplification, since according to this principle, the behavior of the system can be understood by studying the—few—eigenvalues with a positive real part.

Thus, we have represented the three eigenvalues with the largest real parts (circle, star, and open triangle markers) in Fig. 14. There, we plot them in the coordinates $V - q$ in which the 2D calculations were carried out. Note that stars and open triangles are obtained for the same (q, V) coordinates as the circles—correctly located—and are shifted vertically to allow for a clean representation. The color code and the marker definitions used in Fig. 14 are specified in the figure caption.

For each velocity V (horizontal line), it is easy to identify the qualitative effect of q on the behavior of the dynamical system. It is sufficient to look for the change in the color of the eigenvalues represented. For example, for $V = 0.1$, we observe that for small heat losses, we have all major eigenvalues with $Re > 0$ and zero imaginary part (red markers). By increasing the losses, the second and third eigenvalues have a nonzero imaginary part (change from red to blue). For higher heat-loss values, we recover all red markers, or again secondary symbols change to magenta, $Re < 0$.

E. Lyapunov exponents

Let us now separate our points into two categories $Re < 0$ and $Re > 0$ to study the exponents of Lyapunov.⁶¹ Thus, we first assimilate the red and blue points. Second, we collapse green and magenta. We utilize the discussion of Haken⁵⁵ and that of Vallejo *et al.*⁶² Combinations of blue and red symbols denote an unstable focus or node. Symmetrically, green and magenta symbols produce a stable pattern.

We see that our results contain two separate regions separated by $V = 0.45$. Below it, we find red symbols—specifically for the main eigenvector—to the right and left of the gray curve. Negative Lyapunov exponents—magenta points—exist only scarcely and in secondary eigenvalues.

Above $V = 0.45$, we observe that points with positive Lyapunov exponents are mostly located to the left of the gray curve. Similarly, points with negative Lyapunov exponents are located to the right. The border between these two categories is not sharp. Between both, there are some points with mixed positive–negative exponents.

From this, we conclude that steady states are observable for $V > 0.45$.

F. Cyclic regimes

Stable cyclic regimes are characterized by at least a zero Lyapunov exponent.⁵⁵ Nevertheless, we see in Fig. 14 that no zero eigenvalue is present. Therefore, henceforth, we discuss the alternative ways in which such patterns can be achieved.

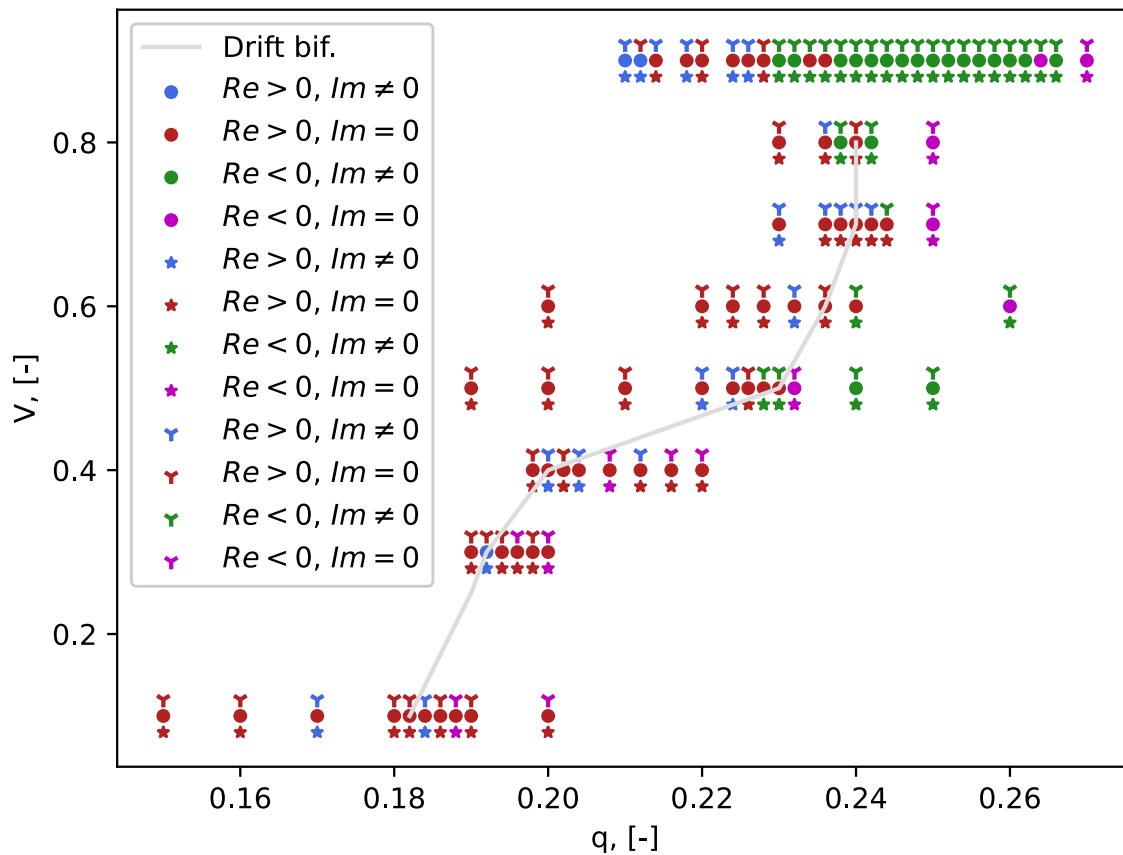


FIG. 14. Eigenvalues in the coordinates $V - q$. Point: largest real part eigenvalue. Star: second largest real part eigenvalue. Open triangle: third largest real part eigenvalue. These three eigenvalues with the largest real part are divided into four categories. Color code: blue $Re > 0, Im \neq 0$; red $Re > 0, Im = 0$; green $Re < 0, Im \neq 0$; magenta $Re < 0, Im = 0$. For the simplicity of comparison and completeness, the curve of Fig. 10, describing the drift bifurcation, is shown in gray.

Naïvely, one may believe that the system with breathing appears for $Re(\lambda) = 0, Im(\lambda) \neq 0$ exclusively. Nevertheless, the results of non-linear analysis—inspired by the *Slaving principle* of Haken—show that breathing may appear associated with $Re(\lambda) > 0, Re(\lambda) \approx 0, Im(\lambda) \neq 0$ eigenvalues;³⁷ see the Appendix. This mechanism implies that after a transient, the non-linearity is able to equilibrate the action of a positive but relatively small eigenvalue. This mechanism is analyzed in detail in, e.g., Ref. 63.

Previous behavior is associated with one—or few—eigenvalues that become positive and complex due to the change in the heat loss. In Fig. 14, these conditions are represented by a transition from a red, green, and magenta to a blue dot (main eigenvalue). Nevertheless, there are very scarce states in which $Re > 0, Im \neq 0$. Blue dots appear only for $(V, q) = (0.4, 0.19)$ and $(V, q) = (0.9, 0.21)$ exclusively. The latter cases typically represent a stable cyclic regime such as the one in Fig. 11(b).

We observe that complex eigenvalues appear very often superimposed on a purely real larger one. Blue stars or open triangles appear mostly superimposed on red dots virtually for the whole range of velocities V from 0.1 to 0.9. Such conditions are located

in the vicinity and to the left of the gray curve. This confirms the hypothesis that breathing appears for a reduced heat loss in the vicinity of steady regimes.

Since non-zero complex eigenvalues appear in secondary modes, we may be dealing with *secondary bifurcation*. That is, conditions of change from red, green, or magenta to blue in subaltern eigenvalues may indicate a secondary Hopf bifurcation. Since these blue symbols appear only in subordinate modes (second or third), breathing is mostly a transient regime.

Note that the regime $Re < 0, Im \neq 0$ (green markers) appears only for flamelets propagating with $V > 0.4$ (see the discussion of Sec. III E). We may interpret the emergence of this regime⁶⁴ as connected with the possible existence of a singular non-hyperbolic stationary point in the vicinity of coordinates $(V = 0.4, q = 0.22)$.

Finally, red-to-green or magenta-to-green changes also appear in the upper range of velocities $V \in [0.8, 0.9]$. In this range, we also encounter most of the main eigenvalues—green dots—for which $Im \neq 0$. We believe these are associated with the complex periodic interactions among the larger number of hotspots associated with several of the topologies of Fig. 6.

V. ALGEBRAIC INTERPRETATION

A. Ansatz

We define $\Psi = (\tilde{\theta}, \tilde{C})^T$. Let us substitute the plane wave perturbation ansatz,

$$\Psi = \hat{\Psi} \exp(\omega t + i\mathbf{k} \cdot \mathbf{x}), \quad (20)$$

into the linearized system. The wavevector is $\mathbf{k} = (k_x, k_y)$ and its module $k^2 = k_x^2 + k_y^2$. Introducing the ansatz into the linear differential operators, the linear system becomes

$$\begin{bmatrix} \omega + ik_x V + k^2 + q & 0 \\ 0 & \omega + ik_x V + Le^{-1}k^2 \end{bmatrix} \begin{pmatrix} \hat{\theta} \\ \hat{C} \end{pmatrix} - \begin{pmatrix} \Omega_\theta & \Omega_C \\ -\Omega_\theta & -\Omega_C \end{pmatrix} \begin{pmatrix} \hat{\theta} \\ \hat{C} \end{pmatrix} = 0. \quad (21)$$

Let $\lambda = \omega + i k_x V$ be the shifted eigenvalue incorporating the advection. The corresponding characteristic polynomial is

$$\lambda^2 + \text{Tr}\lambda + \det = 0, \quad (22)$$

where

$$\text{Tr} = (1 + Le^{-1})k^2 + q - \Omega_\theta + \Omega_C, \quad (23)$$

$$\det = (k^2 + q)Le^{-1}k^2 + \Omega_C(k^2 + q) - \Omega_\theta Le^{-1}k^2. \quad (24)$$

Thus,

$$\lambda_{\pm} = \left(-\text{Tr} \pm \sqrt{\text{Tr}^2 - 4\det} \right) / 2, \quad (25)$$

and, finally,

$$\omega_{\pm} = -ik_x V + (-\text{Tr} \pm \sqrt{\text{Tr}^2 - 4\det}) / 2. \quad (26)$$

B. Ideology of the assumption

Investigating the Hopf bifurcation, we focus on the conditions in which the trace of the Jacobian is zero. This suggests the assumption, $q = \Omega_\theta - \Omega_C$. By this hypothesis, we set the linear growth rate to zero. This is a critical state where (a) $q > \Omega_\theta - \Omega_C$, the system is mathematically heavily damped (flame quenches); (b) $q < \Omega_\theta - \Omega_C$, the system has a positive growth rate. Apart from its mathematical advantages, this hypothesis has some degree of incongruence. q is a constant, while $\Omega_\theta = \Omega_\theta(x, y)$, $\Omega_C = \Omega_C(x, y)$. Let us discuss this when $\beta \rightarrow \infty$. In this limit, the reaction rate Ω is negligible everywhere except in an infinitesimally thin layer. Inside the reaction zone, θ and C are effectively pinned to specific values. Because the chemistry is restricted to this thin sheet, Ω_θ and Ω_C are evaluated only at these fixed values, making their difference a constant, dependent of the fuel–air mixture rather than a variable of the flow field. Thus, we conclude the assumption has also physical sense.

C. Neutral limit assumption

1. Description

Substituting $q = \Omega_\theta - \Omega_C$, the trace simplifies to

$$\text{Tr} = (1 + Le^{-1})k^2. \quad (27)$$

The determinant is

$$\det = Le^{-1}k^4 + \Omega_C(1 - Le^{-1})k^2 + \Omega_C(\Omega_\theta - \Omega_C), \quad (28)$$

and the growth rate is

$$\begin{aligned} \omega_{\pm} = & -ik_x U - \frac{(1 + Le^{-1})k^2}{2} \\ & \pm \frac{1}{2} \sqrt{(1 - Le^{-1})^2 k^4 - 4\Omega_C(1 - Le^{-1})k^2 - 4\Omega_C(\Omega_\theta - \Omega_C)}. \end{aligned} \quad (29)$$

2. Small wavenumber

To approximate $\omega_{\pm}$ for small wavenumbers, we perform a Taylor expansion on the square root and ignore $O(k^4)$ terms. The discriminant, Δ , simplifies to

$$\Delta \approx -4\Omega_C(\Omega_\theta - \Omega_C) - 4\Omega_C(1 - Le^{-1})k^2. \quad (30)$$

Using the identity $\sqrt{a + bx} \approx \sqrt{a} + bx/(2\sqrt{a})$ with $a = -4\Omega_C(\Omega_\theta - \Omega_C)$, we find that

$$\sqrt{\Delta} \approx 2\sqrt{-\Omega_C(\Omega_\theta - \Omega_C)} \left(1 + \frac{(1 - Le^{-1})k^2}{2(\Omega_\theta - \Omega_C)} \right). \quad (31)$$

Substituting this back into the expression for $\omega_{\pm}$, it yields

$$\begin{aligned} \omega_{\pm} \approx & -ik_x U - \frac{(1 + Le^{-1})k^2}{2} \\ & \pm \left(\sqrt{\Omega_C(\Omega_C - \Omega_\theta)} + i \frac{(1 - Le^{-1})\sqrt{\Omega_C}k^2}{2\sqrt{\Omega_C - \Omega_\theta}} \right). \end{aligned} \quad (32)$$

If $\Omega_\theta > \Omega_C$, the term under the square root is imaginary. This results in an oscillatory decay or growth dominated by the k^2 diffusion terms. If $\Omega_\theta < \Omega_C$, the square root is real. The positive branch of ω will have fast growth even as $k \rightarrow 0$. From the derivatives of Ω , we obtain

$$\Omega_C - \Omega_\theta = \frac{\beta^2 f(\theta_0)}{2Le} \left(1 - \frac{\beta C_0}{[\sigma + (1 - \sigma)\theta_0]^2} \right), \quad (33)$$

with $f(\theta) = \exp[\beta(\theta - 1)/(\sigma + (1 - \sigma)\theta)]$. So, $\beta C_0/[\sigma + (1 - \sigma)\theta_0]^2 \gtrless 1$ determines the behavior of the system.

D. Near-limit assumption

1. Description

Let us suppose that we approach the neutral limit without reaching it. That is, $q = \Omega_\theta - \Omega_C - \varepsilon\mu$, with ε being a small parameter and μ being a constant of order one. The trace and determinant

become

$$\text{Tr} = (1 + Le^{-1})k^2 - \varepsilon\mu, \quad (34)$$

$$\det = Le^{-1}k^4 + k^2 [\Omega_C - Le^{-1}(\Omega_C + \varepsilon\mu)] - \Omega_C\varepsilon\mu + \Omega_\theta\Omega_C. \quad (35)$$

The growth rate ω is

$$\omega_{\pm} = -ik_x U + \frac{-(1 + Le^{-1})k^2 - \varepsilon\mu \pm \sqrt{\Delta}}{2}, \quad (36)$$

and with $\Delta = \text{Tr}^2 - 4\det$, we get

$$\Delta = (1 - Le^{-1})^2 k^4 - 2\varepsilon\mu(1 + Le^{-1})k^2 + \varepsilon^2\mu^2 - 4k^2 [\Omega_C - Le^{-1}(\Omega_C + \varepsilon\mu)] - 4\Omega_C(\Omega_\theta - \Omega_C - \varepsilon\mu). \quad (37)$$

2. Small wavenumber

We perform a Taylor expansion on the discriminant, ignoring terms of $O(k^4)$. We get,

$$\Delta \approx -4\Omega_C(\Omega_\theta - \Omega_C - \varepsilon\mu) + \varepsilon^2\mu^2 - 4k^2 \left[\Omega_C - Le^{-1}(\Omega_C + \varepsilon\mu) + \frac{\varepsilon\mu}{2}(1 + Le^{-1}) \right]. \quad (38)$$

We utilize the shorthand $a = \varepsilon^2\mu^2 - 4\Omega_C(\Omega_\theta - \Omega_C - \varepsilon\mu)$. By grouping all terms that are linear with respect to k^2 , we define b as

$$b = -2\varepsilon\mu(1 + Le^{-1}) - 4[\Omega_C - Le^{-1}(\Omega_C + \varepsilon\mu)]. \quad (39)$$

We manipulate b . Combining $\varepsilon\mu Le^{-1}$ and factoring out the shared $(Le^{-1} - 1)$ terms, it yields

$$b = (4\Omega_C + 2\varepsilon\mu)(Le^{-1} - 1). \quad (40)$$

Using the binomial identity, we get $\sqrt{a + bk^2} \approx \sqrt{a} + bk^2/(2\sqrt{a})$. Thus, we have $\sqrt{\Delta} \approx \sqrt{a}(1 + bk^2/(2a))$. Substituting $\text{Tr} = (1 + Le^{-1})k^2 - \varepsilon\mu$ and the approximated $\sqrt{\Delta}$ into the eigenvalue solution, we obtain

$$\omega_{\pm} \approx -ik_x U - \frac{(1 + Le^{-1})k^2 - \varepsilon\mu}{2} \pm \frac{1}{2} \left(\sqrt{a} + \frac{bk^2}{2\sqrt{a}} \right). \quad (41)$$

We now expand a , $\sqrt{a}$, and b in powers of ε , keeping terms up to $O(\varepsilon)$,

$$a \approx -4\Omega_C(\Omega_\theta - \Omega_C) + 4\Omega_C\varepsilon\mu + O(\varepsilon^2). \quad (42)$$

Let us define $a_0 = -4\Omega_C(\Omega_\theta - \Omega_C)$. Then,

$$\sqrt{a} \approx \sqrt{a_0} + \varepsilon\mu\Omega_C/\sqrt{a_0}. \quad (43)$$

Expanding b ,

$$b \approx 4\Omega_C(Le^{-1} - 1) + 2\varepsilon\mu(Le^{-1} - 1). \quad (44)$$

Finally, substituting into ω , and taking common factor,

$$\omega_{\pm} \approx \frac{\varepsilon\mu}{2} - \frac{(1 + Le^{-1})k^2}{2} - ik_x U \pm \frac{\sqrt{a_0}}{2} \left[1 + \frac{\varepsilon\mu\Omega_C}{a_0} + \frac{2\Omega_C(Le^{-1} - 1)k^2}{a_0} \right]. \quad (45)$$

E. Case study

Let us study the growth rate $\omega_{\pm}$, based on the properties of the discriminant $a_0 = 4\Omega_C(\Omega_C - \Omega_\theta)$, since the rest of the terms are always real:

1. $a_0 > 0$, the system exhibits pure exponential growth or decay without oscillation. The positive branch shows rapid growth because the term $(\varepsilon\mu + \sqrt{a_0})/2$ is strictly positive. This exists even at $k=0$, indicating that the entire flamelet reacts similarly. The k^2 term (effective diffusivity) determines for $Le < 1$ that higher-frequency wrinkles are enhanced.
2. $a_0 < 0$ results in an oscillatory behavior. Perturbations manifest as traveling waves due to the combination of the advection term $-ik_x V$ and the imaginary part of the growth rate.

The real part of the growth rate is dominated by $\varepsilon\mu/2 - k^2(1 + Le^{-1})/2$. If μ is small, the k^2 diffusion terms lead to oscillatory decay. If μ is sufficiently large, it can overcome diffusive damping, leading to oscillatory growth (breathing).

VI. CONCLUSIONS

In our analysis, we derived the conditions in which the flameballs propagate steadily. This has been achieved numerically, determining the limit case in which the structures are able to overcome heat losses without quenching. If this state is observable, the limit conditions at which the flamelet overcomes heat losses also correspond to those in which it does not have any motion. Those are the results of Fig. 9.

Movement reappears only for reduced heat losses. In the limit case of a small reduction, it forms cyclic solutions. This results from the eigenvalue analysis of Fig. 14.

The propagation diagram applied for steady solutions shows a surprisingly low range of heat-loss variations. Its shape corresponds to $dV/dq > 0$, lower branch of Fig. 4, that is, a regime in which more reactants are consumed due to enhanced velocity to counteract increased heat losses. This is a controversial result that seems contra-intuitive. Certainly, a negative dependence of speed and heat losses $dV/dq < 0$ was expected.

We want to indicate that the locus $V - q$ does not represent a flame that increases its velocity with heat losses. It is a limit locus, showing the conditions of steady propagation. In the terminology of this article $dV/dq|_{U=0} > 0$. A generic velocity-heat-loss dependence will include regimes that are not steady, a $d|V + U|/dq$, and thus exceed the representation of the $V - q$ curve, of Figs. 4, 10, and 14. Note that for the cases represented in Fig. 9 for which $V \ll U$, we recover the intuitive $d|V + U|/dq < 0$ dependence.

We also need to signal that an eventual comparison of Figs. 10 and 4 should be carried out with caution. The latter was obtained for a single spot. The flame topologies shown in Fig. 6 and of the rest of the database do not include that configuration. Only periodic solutions have appeared in our numerical analysis, notably including the *two heads finger* appearing in experiments of Ref. 10 shown in Fig. 2, right.

Thus, we underline the necessity of further confirmation and explanation with regard to this $dV/dq|_{U=0} > 0$ result. The locus of velocities obtained in Ref. 46, recently known to the author, confirms this necessity.

Utilizing the procedure shown above, we obtained an approximation of the center manifold. However, this appears to be not observable for lower speeds. The analysis of the eigenvalues of Sec. IV has allowed us to infer that the steady state is not observable for $V < 0.4$. For such, flamelets either grow indefinitely or are blown away, never reaching a steady state. The latter appears for a minimum speed of $V \gtrsim 0.45$.

Observing the eigenvalues obtained in our parametric space, we conclude that breathing is mostly a transient regime. It runs for a while, but it is dominated by a main non-steady motion. Note that typically, steady breathing appears for $\text{Re}(\lambda) > 0$, $\text{Im}(\lambda) \neq 0$ eigenvalues, where the non-linearity is able to equilibrate positive but small eigenvalues. This regime almost never occurs. This is also sustained by our brief analytical exploration.

In this paper, we have opted for a numerical approach. In spite of the power of clarification of the algebraic formulation, its concrete application is limited for this problem. First, the formulation is dependent on the basis states. As we see in Fig. 6, these can be topologically complex. Second, the assumption regarding heat loss is disputable. Therefore, we have opted to utilize analytics as a clarification of the numerical observations. In conceptual terms, we also want to underline that we believe it is of the highest importance to interpret mathematically 2D flame-balls as solitons. Through this, one establishes a connection between combustion and synergetic science, a topic brilliantly pioneered by Haken.^{55,60} This link may enable to bring the results of the latter discipline into combustion, potentially allowing to interpret some combustion phenomena under a new perspective. In addition, it will bring into the field related topics to synergetics, like catastrophe theory and the well known results on reaction–diffusion systems.⁶⁵

ACKNOWLEDGMENTS

The authors thank Professor Svetlana Gurevich of the Institute for Theoretical Physics of the University of Münster for the discussions and her inspiration and support in the realization of this article.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Jorge Yanez: Investigation (equal). **Mike Kuznetsov:** Project administration (lead); Visualization (equal).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

APPENDIX: AMPLITUDE EQUATION

To derive the amplitude equation near the Hopf bifurcation is simple.^{25,37,66,67} We employ a multiple-scale expansion⁶⁸ by introducing a small parameter ε . We scale as follows: (a) solution:

$\Psi = \varepsilon^{1/2}\Psi_1 + \varepsilon\Psi_2 + \varepsilon^{3/2}\Psi_3 + \dots$; (b) time: slow time scale $T = \varepsilon t$; (c) heat loss: $q = q_c - \varepsilon\mu$, where $q_c = \Omega_\theta - \Omega_C$.

1. Order $\mathcal{O}(\varepsilon^{1/2})$

At first order, we recover the linearized system at the critical point,

$$\mathcal{L}_c\Psi_1 = 0. \quad (\text{A1})$$

We combine the spatial transport terms and the time derivative into the shifted eigenvalue $\lambda = \omega + ik_x U$. The matrix becomes

$$\mathcal{L}_c(\lambda) = \begin{pmatrix} \lambda + k^2 - \Omega_C & -\Omega_C \\ \Omega_\theta & \lambda + Le^{-1}k^2 + \Omega_C \end{pmatrix}. \quad (\text{A2})$$

The solution of (A1) is

$$\Psi_1 = A(T)e^{i\omega_c t + i\mathbf{k} \cdot \mathbf{x}} \mathbf{v} + \bar{A}(T)e^{-(i\omega_c t + i\mathbf{k} \cdot \mathbf{x})} \bar{\mathbf{v}}. \quad (\text{A3})$$

Here, $A(T)$ is the complex amplitude, and $\mathbf{v}(x, y)$ is the spatial eigenmode $\mathbf{v} = (\Omega_C, i\omega_c + k^2 - \Omega_C)^T$ corresponding to the critical frequency ω_c . At this stage, $A(T)$ is yet unknown.

2. Order $\mathcal{O}(\varepsilon)$

The non-linear terms from Taylor expansion of Ω appear at this order. The total equation is

$$\mathcal{L}_c\Psi_2 = \mathbf{R} \left(\frac{1}{2}\Omega_{jk}\Psi_{1j}\Psi_{1k} \right), \quad (\text{A4})$$

with $\mathbf{R} = (1, -1)^T$. Substituting Ψ_1 , the RHS is

$$\frac{1}{2}\Omega_{jk} \left(A^2 e^{2i\omega_c t} v_j v_k + \bar{A}^2 e^{-2i\omega_c t} \bar{v}_j \bar{v}_k + 2A\bar{A} v_j \bar{v}_k \right). \quad (\text{A5})$$

This forcing creates two distinct temporal components that Ψ_2 must satisfy. They will be treated separately.

3. Mean flow

The mean flow represents a stationary distortion of the base state. The term $2A\bar{A}v_j\bar{v}_k$ is proportional to $|A|^2$. In addition, it has no exponential time dependence, $\lambda = 0$. Thus,

$$\mathcal{L}_c(0) = \begin{pmatrix} k^2 - \Omega_C & -\Omega_C \\ \Omega_\theta & Le^{-1}k^2 + \Omega_C \end{pmatrix}. \quad (\text{A6})$$

Let us define the solution $\Psi_{2,\text{mean}} = |A|^2 \mathbf{w}$ and introduce it into Eq. (A4); we get

$$\mathcal{L}_c(0)\mathbf{w} = \mathbf{R}(\Omega_{\theta\theta}v_\theta\bar{v}_\theta + \Omega_{\theta C}(v_\theta\bar{v}_C + \bar{v}_\theta v_C)). \quad (\text{A7})$$

$\mathcal{L}_c(0)$ is non-singular. Thus, solution can be achieved, e.g., by direct inversion.

4. Second harmonic

For the oscillations at twice the critical frequency,

$$\Psi_{2,\text{harmonic}} = A^2(T)e^{2i\omega_c t}\mathbf{z} + \bar{A}^2(T)e^{-2i\omega_c t}\bar{\mathbf{z}}. \quad (\text{A8})$$

The vector $\mathbf{z}$ is found by solving

$$\mathcal{L}_c(2i\omega_c)\mathbf{z} = \mathbf{R}\left(\frac{1}{2}\Omega_{\theta\theta}v_\theta^2 + \Omega_{\theta C}v_\theta v_C\right). \quad (\text{A9})$$

$\mathcal{L}_c(2i\omega_c)$ is again non-singular. The total solution at this order is the sum of these two effects,

$$\Psi_2 = |A|^2\mathbf{w} + (A^2e^{2i\omega_c t}\mathbf{z} + \bar{A}^2e^{-2i\omega_c t}\bar{\mathbf{z}}). \quad (\text{A10})$$

5. Order $\mathcal{O}(\epsilon^{3/2})$

We gather terms that oscillate at $e^{i\omega_c t}$. The resulting equation is

$$\begin{aligned} \mathcal{L}_c\Psi_3 = & -\partial_T\Psi_1 + \mu\mathcal{Q}\Psi_1 \\ & + \mathbf{R}\left(\Omega_{jk}\Psi_{1j}\Psi_{2k} + \frac{1}{6}\Omega_{jkl}\Psi_{1j}\Psi_{1k}\Psi_{1l}\right), \end{aligned} \quad (\text{A11})$$

with $\mathcal{Q} = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$. Let us treat the terms separately as follows:

1. Slow time derivative: the solution $\Psi_1 = A(T)e^{i\omega_c t}\mathbf{v} + \text{c.c.}$. Thus, the derivative with respect to the slow time T acts only on the amplitude A ,

$$\frac{\partial\Psi_1}{\partial T} = \frac{dA}{dT}e^{i\omega_c t}\mathbf{v} + \text{c.c.} \quad (\text{A12})$$

2. Shift parameter: acts only on the temperature component $\tilde{\theta}_1$. Thus,

$$\mu\mathcal{Q}\Psi_1 = \mu A e^{i\omega_c t} \begin{pmatrix} v_\theta \\ 0 \end{pmatrix} + \text{c.c.} \quad (\text{A13})$$

3. Non-linearities: the first-order mode interacts with the mean flow shift $\mathbf{w}$ and the second harmonic $\mathbf{z}$. Thus,

$$\begin{aligned} \Omega_{jk}\Psi_{1j}\Psi_{2k} \rightarrow & A|A|^2e^{i\omega_c t}[\Omega_{\theta\theta}(v_\theta w_\theta + \bar{v}_\theta z_\theta) \\ & + \Omega_{\theta C}(v_\theta w_C + \bar{v}_\theta z_C + v_C w_\theta + \bar{v}_C z_\theta)]. \end{aligned} \quad (\text{A14})$$

The third-order derivative of the reaction rate results in

$$\begin{aligned} \frac{1}{6}\Omega_{jkl}\Psi_{1j}\Psi_{1k}\Psi_{1l} \rightarrow & A|A|^2e^{i\omega_c t}\left[\frac{1}{2}\Omega_{\theta\theta\theta}|v_\theta|^2v_\theta \right. \\ & \left. + \frac{1}{2}\Omega_{\theta\theta C}(v_\theta^2\bar{v}_C + 2|v_\theta|^2v_C)\right]. \end{aligned} \quad (\text{A15})$$

At $i\omega_c$, the operator $\mathcal{L}_c$ is singular. The RHS must be orthogonal to the adjoint null-vector $\mathbf{v}^\dagger$ (Fredholm alternative⁶⁹) if Ψ_3 exists. This

yields the Landau equation,⁷⁰

$$\langle\mathbf{v}^\dagger, \mathbf{v}\rangle \frac{dA}{dT} = \mu\langle\mathbf{v}^\dagger, \mathcal{Q}\mathbf{v}\rangle A + \langle\mathbf{v}^\dagger, \mathbf{R}[\mathcal{N}_2 + \mathcal{N}_3]\rangle A|A|^2, \quad (\text{A16})$$

with

$$\begin{aligned} \mathcal{N}_2 = & \Omega_{\theta\theta}(v_\theta w_\theta + \bar{v}_\theta z_\theta) \\ & + \Omega_{\theta C}(v_\theta w_C + \bar{v}_\theta z_C + v_C w_\theta + \bar{v}_C z_\theta), \end{aligned} \quad (\text{A17})$$

$$\mathcal{N}_3 = \frac{1}{2}\Omega_{\theta\theta\theta}|v_\theta|^2v_\theta + \frac{1}{2}\Omega_{\theta\theta C}(v_\theta^2\bar{v}_C + 2|v_\theta|^2v_C). \quad (\text{A18})$$

We call $\sigma = \langle\mathbf{v}^\dagger, \mathcal{Q}\mathbf{v}\rangle/\langle\mathbf{v}^\dagger, \mathbf{v}\rangle$ and $\gamma = -\langle\mathbf{v}^\dagger, \mathbf{R}[\mathcal{N}_2 + \mathcal{N}_3]\rangle/\langle\mathbf{v}^\dagger, \mathbf{v}\rangle$ and write the Landau equation in conventional terms,

$$\frac{dA}{dT} = \sigma\mu A - \gamma|A|^2A. \quad (\text{A19})$$

6. Case study

Depending on the values of σ and γ , the system has different behavior.

1. Supercritical Hopf bifurcation occurs when $\text{Re}(\sigma) > 0$ and $\text{Re}(\gamma) > 0$. As the parameter q traverses a threshold, the steady regime becomes unstable. However, the nonlinearities (captured by γ) are restorative. The amplitude A grows initially but eventually saturates at a stable limit cycle with radius $|A_{\text{sat}}| = \sqrt{\frac{\text{Re}(\sigma)\mu}{\text{Re}(\gamma)}}$. The flame-ball exhibits steady, periodic pulsations.
2. Subcritical Hopf bifurcation happens when $[\text{Re}(\sigma) > 0$ and $\text{Re}(\gamma) < 0]$. In this scenario, the non-linearities are reinforcing rather than restorative. Once q crosses the threshold, the term $\text{Re}(\gamma)|A|^2$ becomes negative, meaning that the further the flame oscillates, the faster the oscillations grow. There is no local saturation. The amplitude grows exponentially. Full description of the phenomena requires considering asymptotic expansions up to quintic order (a task too extensive for this small annex).
3. Linear stability happens when $\text{Re}(\sigma) < 0$, regardless of the value of γ . Any perturbation to the flame is damped out over time. The flame remains in its base state.

REFERENCES

- ¹P. D. Ronney, "Near-limit flame structures at low Lewis number," *Combust. Flame* **82**, 1–14 (1990).
- ²P. D. Ronney, M.-S. Wu, H. G. Pearlman, and K. J. Weiland, "Experimental study of flame balls in space: Preliminary results from STS-83," *AIAA J.* **36**, 1361–1368 (1998).
- ³Y. B. Zeldovich, *Selected Works of Yakov Borisovich Zeldovich, Volume I: Chemical Physics and Hydrodynamics* (Princeton University Press, 1992).
- ⁴J. Buckmaster, G. Joulin, and P. Ronney, "The structure and stability of nonadiabatic flame balls," *Combust. Flame* **79**, 381–392 (1990).
- ⁵J. Buckmaster, G. Joulin, and P. Ronney, "The structure and stability of nonadiabatic flame balls: II. Effects of far-field losses," *Combust. Flame* **84**, 411–422 (1991).
- ⁶J. Buckmaster, M. Smooke, and V. Giovangigli, "Analytical and numerical modeling of flame-balls in hydrogen-air mixtures," *Combust. Flame* **94**, 113–124 (1993).
- ⁷J. Buckmaster and G. Joulin, "Flame balls stabilized by suspension in fluid with a steady linear ambient velocity distribution," *J. Fluid Mech.* **227**, 407–427 (1991).

- ⁸J. Buckmaster and G. Joulin, "Influence of boundary-induced losses on the structure and dynamics of flame-balls," *Combust. Sci. Technol.* **89**(1–4), 57–69 (1993).
- ⁹B. Matkowsky and G. Sivashinsky, "An asymptotic derivation of two models in flame theory associated with the constant density approximation," *SIAM J. Appl. Math.* **37**, 686–699 (1979).
- ¹⁰F. Veiga-López, M. Kuznetsov, D. Martínez-Ruiz, E. Fernández-Tarrazo, J. Grune, and M. Sánchez-Sanz, "Unexpected propagation of ultra-lean hydrogen flames in narrow gaps," *Phys. Rev. Lett.* **124**, 174501 (2020).
- ¹¹J. Y. Escanciano, M. Kuznetsov, and F. Veiga-López, "Characterization of unconventional hydrogen flame propagation in narrow gaps," *Phys. Rev. E* **103**, 033101 (2021).
- ¹²V. N. Kurdyumov and M. Matalon, "Flame acceleration in long narrow open channels," *Proc. Combust. Inst.* **34**, 865–872 (2013).
- ¹³V. N. Kurdyumov and M. Matalon, "Self-accelerating flames in long narrow open channels," *Proc. Combust. Inst.* **35**, 921–928 (2015).
- ¹⁴V. N. Kurdyumov and M. Matalon, "Critical conditions for flame acceleration in long adiabatic channels closed at their ignition end," *Proc. Combust. Inst.* **36**, 1549–1557 (2017).
- ¹⁵D. Fernández-Galisteo, V. N. Kurdyumov, and P. D. Ronney, "Analysis of premixed flame propagation between two closely-spaced parallel plates," *Combust. Flame* **190**, 133–145 (2018).
- ¹⁶V. N. Kurdyumov, "Propagation of premixed isobaric flames in narrow channels with heat-losses: The asymptotic analysis revised and reliance on the flame-sheet model," *Combust. Flame* **206**, 138–149 (2019).
- ¹⁷D. Martínez-Ruiz, F. Veiga-López, D. Fernández-Galisteo, V. N. Kurdyumov, and M. Sánchez-Sanz, "The role of conductive heat losses on the formation of isolated flame cells in Hele-Shaw chambers," *Combust. Flame* **209**, 187–199 (2019).
- ¹⁸A. Domínguez-González, D. Martínez-Ruiz, and M. Sánchez-Sanz, "Stable circular and double-cell lean hydrogen-air premixed flames in quasi two-dimensional channels," *Proc. Combust. Inst.* **39**, 1731–1741 (2023).
- ¹⁹J. Yanez, M. Kuznetsov, L. Kagan, and G. Sivashinsky, "2D flameballs: An enhanced classification based on soliton theory," *Fire* **9**(7), 288 (2026).
- ²⁰J. Yanez, L. Kagan, G. Sivashinsky, and M. Kuznetsov, "Modeling of 2D self-drifting flame-balls in Hele-Shaw cells," *Combust. Flame* **258**, 113059 (2023).
- ²¹J. Yanez, M. Kuznetsov, and F. Veiga-López, "On the velocity, size, and temperature of gaseous dendritic flames," *Phys. Fluids* **34**, 113601 (2022).
- ²²J. Yanez, L. Kagan, M. Kuznetsov, and G. Sivashinsky, "On disintegration of lean hydrogen flames in narrow gaps," [arXiv:2403.07734](https://arxiv.org/abs/2403.07734) [physics.flu-dyn] (2024).
- ²³P. G. Drazin and R. S. Johnson, *Solitons: An Introduction* (Cambridge University Press, 1989), Vol. 2.
- ²⁴N. Akhmediev and A. Ankiewicz, "Three sources and three component parts of the concept of dissipative solitons," in *Dissipative Solitons: From Optics to Biology and Medicine* (Springer, 2008), pp. 1–28.
- ²⁵H.-G. Purwins, H. Bödeker, and S. Amiranashvili, "Dissipative solitons," *Adv. Phys.* **59**, 485–701 (2010).
- ²⁶J. M. Soto-Crespo, M. Grapinet, P. Grelu, and N. Akhmediev, "Bifurcations and multiple-period soliton pulsations in a passively mode-locked fiber laser," *Phys. Rev. E* **70**, 066612 (2004).
- ²⁷F.-J. Niedernostheide, B. Kerner, and H.-G. Purwins, "Spontaneous appearance of rocking localized current filaments in a nonequilibrium distributive system," *Phys. Rev. B* **46**, 7559 (1992).
- ²⁸G. Li, Q. Ouyang, and H. L. Swinney, "Transitions in two-dimensional patterns in a ferrocyanide-iodate-sulfite reaction," *J. Chem. Phys.* **105**, 10830–10837 (1996).
- ²⁹P. C. Bressloff, S. E. Folias, A. Prat, and Y.-X. Li, "Oscillatory waves in inhomogeneous neural media," *Phys. Rev. Lett.* **91**, 178101 (2003).
- ³⁰S. F. Mingaleev and Y. S. Kivshar, "Self-trapping and stable localized modes in nonlinear photonic crystals," *Phys. Rev. Lett.* **86**, 5474 (2001).
- ³¹A. Trombettoni and A. Smerzi, "Discrete solitons and breathers with dilute Bose-Einstein condensates," *Phys. Rev. Lett.* **86**, 2353 (2001).
- ³²S. Koga and Y. Kuramoto, "Localized patterns in reaction-diffusion systems," *Prog. Theor. Phys.* **63**, 106–121 (1980).
- ³³Y. Nishiura and M. Mimura, "Layer oscillations in reaction-diffusion systems," *SIAM J. Appl. Math.* **49**, 481–514 (1989).
- ³⁴B. S. Kerner and V. V. Osipov, *Autosolitons: A New Approach to Problems of Self-organization and Turbulence* (Springer Science and Business Media, 1994), Vol. 61.
- ³⁵A. Hagberg and E. Meron, "Pattern formation in non-gradient reaction-diffusion systems: The effects of front bifurcations," *Nonlinearity* **7**, 805 (1994).
- ³⁶M. Suzuki, T. Ohta, M. Mimura, and H. Sakaguchi, "Breathing and wiggling motions in three-species laterally inhibitory systems," *Phys. Rev. E* **52**, 3645 (1995).
- ³⁷S. Gurevich, S. Amiranashvili, and H.-G. Purwins, "Breathing dissipative solitons in three-component reaction-diffusion system," *Phys. Rev. E* **74**, 066201 (2006).
- ³⁸I. Brailovsky and G. Sivashinsky, "On stationary and travelling flame balls," *Combust. Flame* **110**, 524–529 (1997).
- ³⁹A. Liehr, *Dissipative Solitons in Reaction Diffusion Systems* (Springer, 2013), Vol. 70.
- ⁴⁰H.-G. Purwins, H. Bödeker, and A. Liehr, "Dissipative solitons in reaction-diffusion systems," in *Dissipative Solitons* (Springer, 2005), pp. 267–308.
- ⁴¹S. Gurevich, H. Bödeker, A. Moskalenko, A. Liehr, and H.-G. Purwins, "Drift bifurcation of dissipative solitons due to a change of shape: Experiment and theory," *Physica D* **199**, 115–128 (2004).
- ⁴²M. Or-Guil, M. Bode, C. Schenk, and H.-G. Purwins, "Spot bifurcations in three-component reaction-diffusion systems: The onset of propagation," *Phys. Rev. E* **57**, 6432 (1998).
- ⁴³L. Kagan and G. Sivashinsky, "Self-fragmentation of nonadiabatic cellular flames," *Combust. Flame* **108**, 220–226 (1997).
- ⁴⁴M. Kuznetsov, J. Yanez, and F. Veiga-López, "Near limit flame propagation in a thin layer geometry at low Peclet numbers," in *Proceedings on 18th International Conference on Flow Dynamics (ICFD 2021)*, OS18-8 (ICFD, 2022), pp. 688–692.
- ⁴⁵P. V. Moskalev, V. P. Denisenko, and I. A. Kirillov, "Classification and dynamics of ultralean hydrogen-air flames in horizontal cylindrical Hele-Shaw cells," *J. Exp. Theor. Phys.* **137**, 104–113 (2023).
- ⁴⁶A. Domínguez-González, V. N. Kurdyumov, A. L. Sánchez, and D. Martínez-Ruiz, "On the existence of propagating circular flames in narrow channels," *Combust. Flame* **281**, 114415 (2025).
- ⁴⁷R. Garnett, *Bayesian Optimization* (Cambridge University Press, 2023).
- ⁴⁸A. I. J. Forrester, A. Sobester, and A. J. Keane, *Engineering Design via Surrogate Modelling: A Practical Guide* (John Wiley and Sons, Chichester, 2008).
- ⁴⁹R. U. Seydel, *Practical Bifurcation and Stability Analysis* (Springer Science and Business Media, 2009), Vol. 5.
- ⁵⁰J. R. Dormand and P. J. Prince, "A family of embedded Runge-Kutta formulae," *J. Comput. Appl. Math.* **6**, 19–26 (1980).
- ⁵¹D. Fernández-Galisteo, C. Jiménez, M. Sánchez-Sanz, and V. N. Kurdyumov, "Effects of stoichiometry on premixed flames propagating in narrow channels: Symmetry-breaking bifurcations," *Combust. Theory Modell.* **21**, 1050–1065 (2017).
- ⁵²A. Dominguez-Gonzalez, M. Encinar, and D. Martinez-Ruiz, "Numerical study on the causality of solution multiplicity in ultra-lean H₂-air premixed flames," in *76th Annual Meeting of the APS Division of Fluid Dynamics (APS, 2023)*.
- ⁵³A. Liehr, H. Bödeker, M. Röttger, T. Frank, R. Friedrich, and H. Purwins, "Drift bifurcation detection for dissipative solitons," *New J. Phys.* **5**, 89 (2003).
- ⁵⁴Y. Kuramoto, *Chemical Turbulence* (Springer, 1984).
- ⁵⁵H. Haken, *Advanced Synergetics: Instability Hierarchies of Self-organizing Systems and Devices* (Springer Science and Business Media, 2012), Vol. 20.
- ⁵⁶R. B. Lehoucq and D. C. Sorensen, "Deflation techniques for an implicitly restarted Arnoldi iteration," *SIAM J. Matrix Anal. Appl.* **17**, 789–821 (1996).
- ⁵⁷P. Virtanen, R. Gommers, T. E. Oliphant, M. Haberland, T. Reddy, D. Cournapeau, E. Burovski, P. Peterson, W. Weckesser, J. Bright, S. J. Van der Walt, M. Brett, J. Wilson, K. J. Millman, N. Mayorov, A. R. J. Nelson, E. Jones, R. Kern, E. Larson, C. J. Carey, Í. Polat, Y. Feng, E. W. Moore, J. VanderPlas, D. Laxalde, J. Perktold, R. Cimrman, I. Henriksen, E. A. Quintero, C. R. Harris, A. M. Archibald, A. H. Ribeiro, F. Pedregosa, and P. van Mulbregt, and SciPy 1.0 Contributors, "SciPy 1.0: Fundamental algorithms for scientific computing in Python," *Nat. Methods* **17**, 261–272 (2020).
- ⁵⁸N. V. Butenin, Yu. I. Neimark, and N. A. Fufaev, *Introduction to the Theory of Nonlinear Oscillations* (Nauka, Moscow, 1987), p. 384.
- ⁵⁹H. Haken, "Slaving principle revisited," *Physica D* **97**, 95–103 (1996).

⁶⁰H. Haken, "Synergetics. An introduction: Nonequilibrium phase transitions and self-organization in physics, chemistry and biology," in *Synergetics: Introduction and Advanced Topics* (Springer, 2004).

⁶¹H. Haken and A. Wunderlin, *Die Selbststrukturierung der Materie: Synergetik in der Unbelebten Welt* (Springer, 1991).

⁶²J. C. Vallejo and M. A. Sanjuan, *Predictability of Chaotic Dynamics* (Springer, 2017).

⁶³R. Friedrich, "Group theoretic methods in the theory of pattern formation," in *Collective Dynamics of Nonlinear and Disordered Systems* (Springer, 2005), pp. 61–84.

⁶⁴P. Glendinning, *Stability, Instability and Chaos: An Introduction to the Theory of Nonlinear Differential Equations* (Cambridge University Press, 1994).

⁶⁵A. M. Turing, "The chemical basis of morphogenesis," *Bull. Math. Biol.* **52**, 153–197 (1990).

⁶⁶M. Cross and H. Greenside, *Pattern Formation and Dynamics in Nonequilibrium Systems* (Cambridge University Press, 2009).

⁶⁷L. M. Pismen, *Patterns and Interfaces in Dissipative Dynamics* (Springer, 2006), Vol. 706.

⁶⁸M. H. Holmes, *Introduction to Perturbation Methods* (Springer Science and Business Media, 2012).

⁶⁹M. L. Krasnov, A. Kiselev, G. I. Makarenko, and G. Yankovsky, *Problems and Exercises in Integral Equations* (Mir publishers, Moscow, 1971).

⁷⁰J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer Science and Business Media, 2013).