

Next-to-next-to-leading QCD corrections to the $B^+-B_d^0$, D^+-D^0 , and $D_s^+-D^0$ lifetime ratios

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ABSTRACT: The total decay widths of heavy mesons can be systematically calculated in terms of an expansion in the two parameters $1/m_Q$ and $\alpha_s(m_Q)$, where $Q = c, b$ denotes the heavy quark. The dominant contributions to meson lifetime splittings stem from terms which are suppressed by $1/m_Q^3$ with respect to the leading universal contribution to the total decay width. We calculate three-loop contributions of order α_s^2/m_q^3 to the lifetime ratios $\tau(B^+)/\tau(B_d^0)$, $\tau(D^+)/\tau(D^0)$, and $\tau(D_s^+)/\tau(D^0)$ in the limit of exact isospin and V-spin symmetry, respectively. Furthermore, we present new α_s/m_q^3 corrections to the Cabibbo-suppressed terms in $\tau(B^+)/\tau(B_d^0)$. Combining our perturbative coefficients with hadronic matrix elements calculated from Heavy Quark Effective Theory sum rules, we find $\tau(B^+)/\tau(B_d^0) = 1.072 \pm 0.024$. Using hadronic matrix elements from a recent lattice QCD calculation we find $\tau(D^+)/\tau(D^0) = 2.344 \pm 0.170$ and $\tau(D_s^+)/\tau(D^0) = 1.289 \pm 0.042$. We find good agreement of our predictions with experimental data, which constitutes a successful probe of the calculations of hadronic matrix elements and permits estimates of the unknown $1/m_Q^4$ contributions as well as the V-spin breaking terms in $\tau(D_s^+)/\tau(D^0)$.

KEYWORDS: Bottom Quarks, Higher-Order Perturbative Calculations

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1 Introduction

Lifetimes of heavy hadrons H_Q can be calculated with the *Heavy Quark Expansion (HQE)*, which is an operator product expansion with the heavy quark mass m_Q as the hard scale [1–5]. The HQE results in expressions of the schematic form

$$\begin{aligned} \Gamma(H_Q) = & \Gamma_3 \langle \mathcal{O}_3 \rangle + \Gamma_5 \frac{\langle \mathcal{O}_5 \rangle}{m_Q^2} + \Gamma_6 \frac{\langle \mathcal{O}_6 \rangle}{m_Q^3} + \Gamma_7 \frac{\langle \mathcal{O}_7 \rangle}{m_Q^4} + \dots \\ & + 16\pi^2 \left[\tilde{\Gamma}_6 \frac{\langle \tilde{\mathcal{O}}_6 \rangle}{m_Q^3} + \tilde{\Gamma}_7 \frac{\langle \tilde{\mathcal{O}}_7 \rangle}{m_Q^4} + \dots \right] \end{aligned} \quad (1.1)$$

for the total width $\Gamma(H_Q)$. Here $\langle \mathcal{O}_n \rangle \equiv \langle H_Q | \mathcal{O}_n | H_Q \rangle$ denotes a hadronic matrix element of a local operator of dimension n and $\Gamma_n, \tilde{\Gamma}_n$ represent Wilson coefficients which are calculable in perturbation theory to the desired order in the coupling constant α_s of Quantum Chromodynamics (QCD). The first term $\Gamma_3 \langle \mathcal{O}_3 \rangle$ equals the QCD-corrected quark decay rate and is thus a universal contribution, entailing that to first approximation all b -flavoured hadrons have the same lifetime. Higher dimensions n corresponds to higher powers of the QCD scale parameter Λ_{QCD} , so that the HQE expansion parameters are $\Lambda_{\text{QCD}}/m_b \sim 0.1$ and $\Lambda_{\text{QCD}}/m_c \sim 0.3$ for the two heavy-quark systems of interest. The Wilson coefficients involve $\alpha_s(m_b) \sim 0.2$ and $\alpha_s(m_c) \sim 0.3$, respectively. Whether the expansions in these two parameters actually converge to a satisfactory level can only be found out if $\Gamma(H_Q)$ is calculated to sufficiently high orders in these parameters.

An important feature of eq. (1.1) is the appearance of the numerical enhancement factor $16\pi^2$ in the second line of eq. (1.1), emerging from contributions in which a valence quark in H_Q participates in the weak decay amplitude. While Γ_3 stems from three-body decays like

$b \rightarrow c\bar{u}d$, $\tilde{\Gamma}_6(B_d^0)$ receives contributions from e.g. $\bar{b}d \rightarrow c\bar{u}$ and the two-particle phase space comes with one power of $1/(16\pi^2)$ less than the terms in the first line of eq. (1.1). There are no terms with enhancement factors of higher powers of $16\pi^2$, so that the terms in the second line of eq. (1.1) do not spoil the convergence of the HQE. However, in the case of charmed hadrons one has $16\pi^2\Lambda_{\text{QCD}}^3/m_c^3 \sim 1$, so that lifetime differences of order 1 are allowed.

Lifetime differences emerge from the feature that the matrix elements of operators with $n \geq 5$ are different for different hadrons H_q . The lifetime splittings in the isospin doublets (B^+, B_d^0) and (Ξ_b^0, Ξ_b^-) and their counterparts in the charm system are especially interesting, because isospin symmetry simplifies the calculation in two ways:

- (i) The hadronic matrix elements of isospin partners are related to each other in a trivial way. In particular, the terms in the first line of eq. (1.1) drop out from the lifetime splittings in doublets in the symmetry limit. Since strong isospin is an excellent symmetry of QCD, holding with an accuracy of 2%, isospin-breaking effects in the hadronic matrix elements can be safely neglected and the lifetime differences solely stem from the terms in the second line of eq. (1.1).
- (ii) The operators $\tilde{\mathcal{O}}_6$ are four-quark operators which distinguish between e.g. B^+ and B_d^0 , for example $\langle B_d^0 | \bar{b}\gamma_\mu d \bar{d}\gamma^\mu b | B_d^0 \rangle \neq \langle B^+ | \bar{b}\gamma_\mu d \bar{d}\gamma^\mu b | B^+ \rangle$, so that they give a non-zero contribution to the $B^+ - B_d^0$ lifetime difference. The final expression for this quantity will only contain isospin-violating operators and these operators cannot mix into the isospin-conserving operators $\mathcal{O}_n$. In particular, one needs no counterterms proportional to the lower-dimensional operators $\mathcal{O}_3$ or $\mathcal{O}_5$, which are currently show-stoppers to a reliable calculation of the dimension-6 operator matrix elements $\langle \tilde{\mathcal{O}}_6 \rangle$ with lattice QCD.

These points also apply to V-spin symmetry, which corresponds to unitary rotations of the $(u, s)^T$ quark field doublet, with the caveat that V-spin symmetry is broken at a $\sim 30\%$ level due to $m_s - m_u \sim 0.3\Lambda_{\text{QCD}}$. Thus we expect a larger deviation of our prediction of the $D_s^+ - D^0$ lifetime ratio from data than in the cases of isospin doublets.

Theoretical predictions of lifetime ratios are related to calculations of the differences $\Gamma(H_Q) - \Gamma(H'_Q)$ of the total decay rates of the considered hadrons via

$$\frac{\tau(H'_Q)}{\tau(H_Q)} - 1 = \tau^{\text{exp}}(H'_Q) \left(\Gamma(H_Q) - \Gamma(H'_Q) \right). \quad (1.2)$$

Here “exp” means that the experimental value of the lifetime is used in the prefactor. $\Gamma(H_Q)$ is related to the forward matrix element $H_Q \rightarrow H_Q$ through the optical theorem. Specifying the discussion to $\tau(B^+)/\tau(B_d^0)$ now, the Cabibbo-favoured contribution to this lifetime ratio is proportional to $|V_{ud}V_{cb}|^2$. For the computation of $\Gamma(H_b)$ one first integrates out all heavy degrees of freedom with masses of the order of the electroweak scale. This results in the $|\Delta B| = 1$ Hamiltonian $\mathcal{H}_{\text{eff}}^{|\Delta B|=1}$ which describes the weak decays of b -flavoured hadrons in the SM, where B is the beauty quantum number. The low-energy dynamics, related to energy scales of order m_b and below, is described by effective dimension-6 operators. These $|\Delta B| = 1$ operators are multiplied by Wilson coefficients describing the short-distance dynamics, which can be calculated in perturbation theory. The $|\Delta B| = 1$ Wilson coefficients are known up to next-to-next-to-leading order (NNLO) of QCD [6]. The implementation of the HQE for the

calculation of $\tilde{\Gamma}_n$ amounts to the matching of the $\bar{b}q \rightarrow \bar{b}q$ forward matrix element calculated with $\mathcal{H}_{\text{eff}}^{|\Delta B|=1}$ to effective local $\Delta B = 0$ operators, where q is a valence quark of H_b . The perturbative calculation of the $\Delta B = 0$ Wilson coefficients is justified by $m_b \gg \Lambda_{\text{QCD}}$.

In processes like $\bar{b} \rightarrow \bar{c}u\bar{d}$ with four different quark flavours $\mathcal{H}_{\text{eff}}^{|\Delta B|=1}$ comprises only two current-current operators, which describe the W -mediated SM tree diagram including QCD corrections. For decays into a $u\bar{u}$ or $c\bar{c}$ pair one also encounters penguin operators, with much smaller Wilson coefficients. Furthermore, in our calculation of $\tau(B^+)/\tau(B_d^0)$ all penguin effects are CKM-suppressed by two powers of the Wolfenstein parameter $\lambda \sim 0.22$, so that no NNLO precision is currently needed for the latter. The CKM-leading contribution to $\tau(B^+)/\tau(B_d^0)$ has been calculated to NLO in refs. [7, 8]. Judging the accuracy of the result by the dependences on renormalisation scale and scheme, one concludes that an NNLO calculation is needed to confront today's experimental result [9] with theory. This calculation is especially timely, because meanwhile hadronic matrix elements have been calculated with QCD sum rules [10, 11] and also first complete lattice-QCD computations of all matrix elements entering $\tau(D^+)/\tau(D^0)$ and $\tau(D_s^+)/\tau(D^0)$ are available [12, 13].¹

The dominant contribution to $\tau(B^+)/\tau(B_d^0) - 1$ stems from *Weak annihilation (WA)* and the *Pauli interference (PI)* diagrams illustrated in figure 1 at LO. They contribute to neutral and charged B meson decay rates, respectively. The new results of this paper are

- (i) NNLO corrections to the WA and PI contributions, i.e. the CKM-favoured piece of $\tau(B^+)/\tau(B_d^0) - 1$,
- (ii) NLO corrections to the CKM-suppressed WA and PI contributions with two current-current operators, and
- (iii) the application of the results to the D^+-D^0 and $D_s^+-D^0$ lifetime differences.

Conceptually, the calculation is similar to the one of the decay matrix element Γ_{12}^q of $B_q^0-\bar{B}_q^0$ mixing, which determines the width differences $\Delta\Gamma_q$ among the mass eigenstates and the CP asymmetries in flavour-specific decays. Just as in the case of $\tau(B^+)/\tau(B_d^0)$ the starting point was an NLO result for the current-current contribution [18–20] and a systematic calculation of sub-leading contributions [21, 22] up to the complete NNLO prediction [23] lead to a substantial reduction of the scale and scheme dependences of the predicted Γ_{12}^q . Another NNLO calculation in this context has addressed the term Γ_3 in eq. (1.1), which constitutes the universal piece common to all decay rates of b -flavoured hadrons [24, 25].

The remainder of the paper is organised as follows: In section 2, we discuss the effective theories used for our calculation. In particular, we present explicit results for the evanescent operators in both the $|\Delta B| = 1$ and $\Delta B = 0$ theories. Section 3 is devoted to a discussion of certain technical aspects regarding the calculation of the loop integrals. In section 4, we describe the determination of the matching coefficients. The phenomenological results for B and D mesons are presented in sections 5 and 6, respectively. We summarise our results in section 7.

¹First exploratory lattice calculations were presented in refs. [14–16]; for a first determination of bare matrix elements with domain wall fermions see ref. [17].

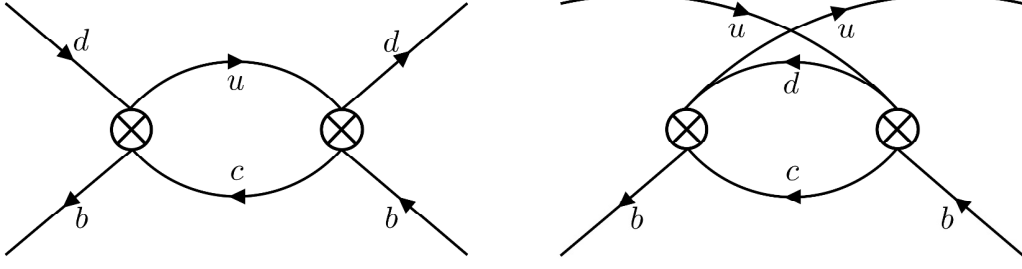


Figure 1. Weak annihilation (left) and Pauli interference (right) diagrams in the leading order of QCD. They contribute to $\Gamma(B_d^0)$ and $\Gamma(B^+)$, respectively. The crosses represent the insertion of $|\Delta B| = 1$ operators from the effective Hamiltonian (2.2). CKM-suppressed contributions are not shown.

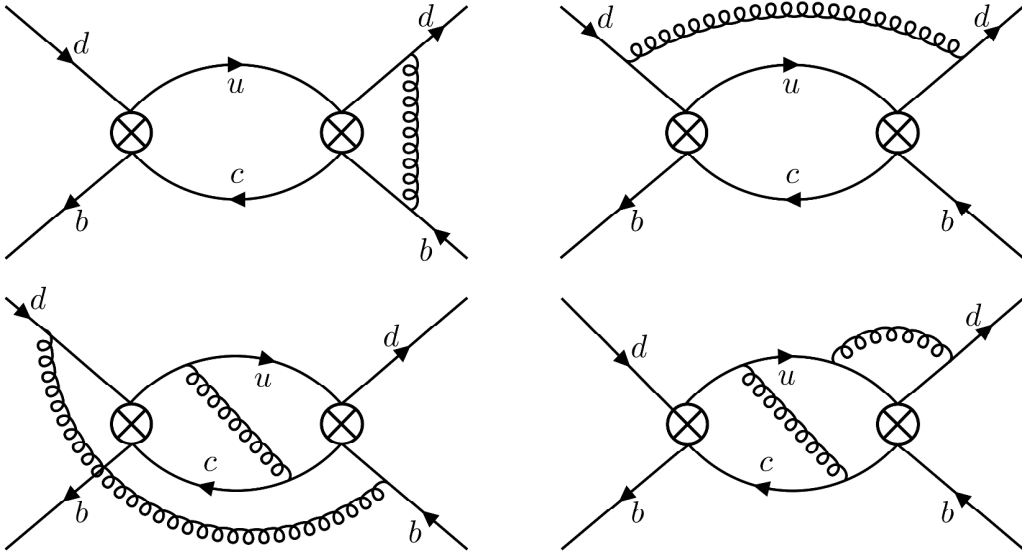


Figure 2. Sample Feynman diagrams contribution to $\Gamma(B_d^0)$ at NLO (top) and NNLO (bottom). Analogue diagrams also exist for the PI contribution.

2 Effective theories

The optical theorem expresses the total decay width of H_b as

$$\Gamma(H_b) = \frac{1}{2M_{H_b}} \langle H_b | \mathcal{T} | H_b \rangle$$

with $\mathcal{T} = \text{Im } i \int d^4x T \left[\mathcal{H}_{\text{eff}}^{|\Delta B|=1}(x) \mathcal{H}_{\text{eff}}^{|\Delta B|=1}(0) \right]$ (2.1)

and “Im” in the definition of transition operator understood to be applied to the matrix element $\langle H_b | \dots | H_b \rangle$.

Sample Feynman diagrams for the decay of a neutral and charge B meson to LO are given in figure 1. For illustration we show in figure 2 Feynman diagrams which contribute to $\Gamma(B_d^0)$ at NLO and NNLO. The following step is to perform a Heavy Quark Expansion (HQE) of the transition operator in powers of $m_b \gg \Lambda_{\text{QCD}}$. This amounts to expressing the

bi-local correlator in eqs. (2.1) and (2.12) in terms of *local* $\Delta B = 0$ operators multiplied by the corresponding matching coefficients. While the matrix elements of such operators need to be evaluated using non-perturbative methods, e.g. lattice QCD or sum rules, the matching coefficients are determined in perturbation theory, and depend on the $|\Delta B| = 1$ Wilson coefficients, the strong coupling constant α_s and the quark masses m_b and m_c . Higher dimensional operators correspond to increasingly suppressed terms in the HQE. The main scope of this work is the evaluation of the matching coefficients to NNLO accuracy in QCD. Effectively, this corresponds to a three-loop calculation on the $|\Delta B| = 1$ side, and a two-loop calculation on the $\Delta B = 0$ side. The methodology of the matching calculation is described in refs. [7, 8, 18, 19, 26–28].

$\mathcal{H}_{\text{eff}}^{|\Delta B|=1}$ will be introduced in section 2.1. The HQE of eq. (1.1) expresses the bilocal matrix element $\langle H_b | \mathcal{T} | H_b \rangle$ in terms of local $\Delta B = 0$ operators, to be discussed in section 2.2.

2.1 $|\Delta B| = 1$ hamiltonian

For convenience, we adopt the traditional operator basis [29, 30] with the $|\Delta B| = 1$ Hamiltonian

$$\mathcal{H}_{\text{eff}}^{|\Delta B|=1} = 4 \frac{G_F}{\sqrt{2}} \sum_{q_3=d,s} \left\{ \sum_{q_{1,2}=u,c} \lambda_{q_1 q_2 q_3} [C_1(\mu_1) Q_1^{q_1 q_2 q_3}(\mu_1) + C_2(\mu_1) Q_2^{q_1 q_2 q_3}(\mu_1)] - \lambda_t^{q_3} \sum_{i=3}^6 C_i(\mu_1) Q_i^{q_3}(\mu_1) \right\} + \text{h.c.}, \quad (2.2)$$

where $\lambda_{q_1 q_2 q_3} = V_{q_1 b}^* V_{q_2 q_3}$ and $\lambda_t^q = V_{tb}^* V_{tq_3}$ comprise the CKM matrix elements and C_i ($i = 1, \dots, 6$) are the Wilson coefficients encoding the short-distance physics associated with energies above the renormalisation scale μ_1 . The current-current operators

$$\begin{aligned} Q_1^{q_1 q_2 q_3} &= (\bar{b}^i \gamma^\mu P_L q_1^j) (\bar{q}_2^j \gamma_\mu P_L q_3^i), \\ Q_2^{q_1 q_2 q_3} &= (\bar{b}^i \gamma^\mu P_L q_1^i) (\bar{q}_2^j \gamma_\mu P_L q_3^j), \end{aligned} \quad (2.3)$$

where the summation over the colour indices i and j is understood, are accompanied with Wilson coefficients C_1 and C_2 of order 1, while those of the four-quark penguin operators in the second line of eq. (2.3) are much smaller. In our NNLO calculation we restrict ourselves to the CKM-favoured contributions, which strictly only involve the CKM factor $|\lambda_{cud}|^2$. In the PI diagrams $|\lambda_{cud}|^2$ trivially combines with the CKM-suppressed contribution proportional to $|\lambda_{cus}|^2$ to an expression found by replacing $V_{ud} \rightarrow 1$ in the CKM-favoured piece, so that we include this piece as well. (The WA contribution does not involve λ_{cus} .)

In addition, we extend the work in ref. [7] and compute the CKM-suppressed contributions proportional to $|\lambda_{ccd}|^2$, λ'_{cud} and $|\lambda_{uud}|^2$ to NLO accuracy. This includes diagrams as in figure 1 with internal uc quarks for the WA case and with internal ud quarks in the PI case, both at one- and two-loop order. Furthermore, we have to consider one-particle reducible diagrams as in figure 3. They lead to CKM factors $|\lambda_{ccd}|^2$ and $|\lambda_{uud}|^2$ but also to

$$\lambda'_{cud} \equiv 2\text{Re}[V_{ub}^* V_{ud} V_{cb} V_{cd}^*] = -2|\lambda_{ccd}|^2 \bar{\rho} \left[1 + \mathcal{O}(\lambda^4)\right], \quad (2.4)$$

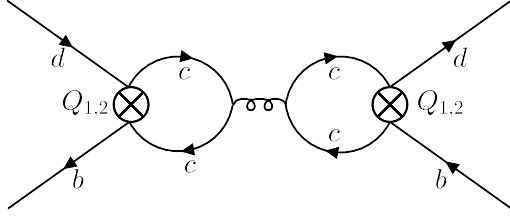


Figure 3. Penguin-like contribution at NLO to the decay width involving an intermediate $c\bar{c}$ pair in the hard loops. These contribution are CKM-suppressed by a factor $|V_{cd}|^2/|V_{ud}|^2$ with respect to the CKM-favoured weak annihilation process. Diagrams with internal u quark are not shown but included in our analysis.

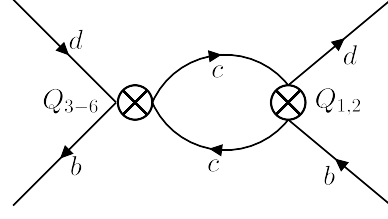


Figure 4. Weak annihilation diagram involving one penguin operator Q_{3-6} . These are needed at NLO as counterterms to the CKM-suppressed part of the decay width. Diagrams with internal uu pairs are not shown but included in our analysis.

which first appear at NLO accuracy. λ'_{cud} comes from diagrams of the form of figure 3, where both a cc and a uu pair appear in the two loops. For the renormalisation of penguin-like contributions we need counterterms proportional to the penguin operators

$$\begin{aligned} Q_3^{q3} &= (\bar{b}^i \gamma^\mu P_L q_3^i) \sum_q (\bar{q}^j \gamma_\mu P_L q^j), \\ Q_4^{q3} &= (\bar{b}^i \gamma^\mu P_L q_3^j) \sum_q (\bar{q}^j \gamma_\mu P_L q^i), \\ Q_5^{q3} &= (\bar{b}^i \gamma^\mu P_L q_3^i) \sum_q (\bar{q}^j \gamma_\mu P_R q^j), \\ Q_6^{q3} &= (\bar{b}^i \gamma^\mu P_L q_3^j) \sum_q (\bar{q}^j \gamma_\mu P_R q^i). \end{aligned} \quad (2.5)$$

They enter LO diagrams as shown in figure 4. Furthermore, we include the small Wilson coefficients C_{3-6} at LO through the diagram of figure 4 as needed to cancel the renormalisation scale dependence. The described penguin effects have been calculated in the context of the Cabibbo-favoured piece of the lifetime ratio $\tau(B_s^0)/\tau(B_d^0)$ in ref. [31], where only the contribution with two charm loops in figure 3 was needed.

The main motivation to work in the traditional basis is that we want to have one b spinor on each external fermion line such that we can apply the equations of motion (EOMs) $\not{p}_b b = m_b b$, where p_b^μ is the external b quark four-momentum, to simplify all the tensor structures. In the original version (cf. figure 1) this is not the case, but we can achieve the desired expression if we replace one $|\Delta B| = 1$ operator in the diagram with its Fierz-transformed version. For e.g. $Q_1^{q_1 q_2 q_3}$ this amounts to

$$Q_1^{q_1 q_2 q_3} = (\bar{b}^i \gamma^\mu P_L q_1^j) (\bar{q}_2^j \gamma_\mu P_L q_3^i) \stackrel{F}{=} (\bar{b}^i \gamma^\mu P_L q_3^i) (\bar{q}_2^j \gamma_\mu P_L q_1^j) = Q_2^{q_3 q_2 q_1}. \quad (2.6)$$

Since the Fierz transformation only holds for $d = 4$, a priori one expects different Wilson coefficients of $Q_j^{q_1 q_2 q_3}$ and its Fierz transform beyond LO. In refs. [24, 30, 32, 33] it has been worked out how this is remedied through an appropriate definition of evanescent operators. As an advantage of the traditional operator basis, the Fierz transform simply interchanges

$Q_1^{q_1 q_2 q_3}$ and $Q_2^{q_3 q_2 q_1}$, which simplifies the calculation. We apply the Fierz transformation to the right operator in our diagrams and obtain for the decay width

$$\tilde{\Gamma}^{q_1 q_2 q_3} = \frac{1}{2M_{H_b}} \sum_{i,j=1,2} \left(4 \frac{G_F}{\sqrt{2}}\right)^2 |\lambda_{q_1 q_2 q_3}|^2 \tilde{C}_i(\mu_1) C_j(\mu_1) \langle H_b | \text{Im } i \int d^4 x e^{iqx} T \left[Q_i^{\dagger q_3 q_2 q_1}(x) Q_j^{q_1 q_2 q_3}(0) \right] | H_b \rangle, \quad (2.7)$$

where $\tilde{C}_i(\mu_1)$ is the Wilson coefficient of the Fierz-transformed operator $Q_i^{q_3 q_2 q_1}$. The forward scattering matrix defined by $\tilde{\Gamma}^{q_3 q_2 q_1}$ leads to amplitudes where a b -spinor is present in each fermion line as illustrated in figure 5, where the effect of the Fierz transformation is illustrated.

For the Wilson coefficients of the current-current operators we have the simple relation

$$\begin{aligned} C_1 &= \tilde{C}_2 \\ C_2 &= \tilde{C}_1, \end{aligned} \quad (2.8)$$

with a judicious choice for the evanescent operators

$$\begin{aligned} E_1^{(1), q_1 q_2 q_3} &= (\bar{b}^i \gamma^{\mu_1 \mu_2 \mu_3} P_L q_1^j) (\bar{q}_2^j \gamma_{\mu_1 \mu_2 \mu_3} P_L q_3^i) - (16 - 4\epsilon + \mathcal{A}_2 \epsilon^2) Q_1^{q_1 q_2 q_3}, \\ E_2^{(1), q_1 q_2 q_3} &= (\bar{b}^i \gamma^{\mu_1 \mu_2 \mu_3} P_L q_1^i) (\bar{q}_2^j \gamma_{\mu_1 \mu_2 \mu_3} P_L q_3^j) - (16 - 4\epsilon + \mathcal{A}_2 \epsilon^2) Q_2^{q_1 q_2 q_3}, \\ E_1^{(1), q_1 q_2 q_3} &= (\bar{b}^i \gamma^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} P_L q_1^j) (\bar{q}_2^j \gamma_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} P_L q_3^i) - (256 - 224\epsilon + \mathcal{B}_1 \epsilon^2) Q_1^{q_1 q_2 q_3}, \\ E_2^{(1), q_1 q_2 q_3} &= (\bar{b}^i \gamma^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} P_L q_1^i) (\bar{q}_2^j \gamma_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} P_L q_3^j) - (256 - 224\epsilon + \mathcal{B}_2 \epsilon^2) Q_2^{q_1 q_2 q_3}, \end{aligned} \quad (2.9)$$

where $\gamma^{\mu_1 \dots \mu_n}$ is a shorthand notation for $\gamma^{\mu_1} \dots \gamma^{\mu_n}$. The coefficients of the ϵ^2 terms first enter the calculation at NNLO and read [24]

$$\mathcal{A}_2 = -4, \quad \mathcal{B}_1 = -\frac{45936}{115}, \quad \mathcal{B}_2 = \frac{115056}{115}. \quad (2.10)$$

For all the related technical details we refer the reader to ref. [24]. The Fierz symmetry of eq. (2.8) only holds for the contributions of current-current diagrams, but not for penguin diagrams like those in figure 3, for which we will not use Fierz-transformed operators.

It is worthwhile to discuss the difference between the result in eq. (2.9) and the corresponding expression in eq. (A.2) of ref. [34], in which the evanescent operators have been fixed to avoid mixing of $Q_+ \equiv Q_2 + Q_1$ into $Q_- \equiv Q_2 - Q_1$ under the renormalisation group evolution, extending the NLO prescription of ref. [30] to NNLO. The non-mixing condition is equivalent to the feature that the *renormalised* two-loop matrix element $\langle Q_{\pm} \rangle^{(2)}$ is proportional to $\langle Q_{\pm} \rangle^{(0)}$, i.e. there is no term involving $\langle Q_{\mp} \rangle^{(0)}$ when the sum of the two-loop diagrams is expressed in terms of the tree-level matrix elements. Since the renormalised matrix element involves counterterm contributions with evanescent operators, the latter feature cannot be expected to hold for the *bare* matrix elements and the counterterm diagrams individually. Now Q_+ is even under Fierz transformation (in the sense $Q_+^{q_1 q_2 q_3} \rightarrow Q_+^{q_3 q_2 q_1}$) while Q_- is odd, thus the procedure of refs. [30, 34] maintains Fierz symmetry for the renormalised matrix elements in the sense that the Fierz parity of $\langle Q_{\pm}(\mu) \rangle$ is equal to ± 1 at all values of the renormalisation scales μ .

In the calculations of this paper and ref. [24] Fierz symmetry is instead implemented in a different way. The Fierz transform of $Q_{1,2}$ is equal to $Q_{2,1}$ up to an exchange of quark flavours (see eq. (2.6)), which is irrelevant as long as no penguin diagrams contribute to $\langle Q_{1,2} \rangle$. The definition in eq. (2.9) ensures that $\langle Q_{1,2} \rangle^{(2)\text{bare}} = \langle Q_{2,1} \rangle^{(2)\text{bare}}$ and the equality of the renormalised matrix elements in the original and fierzed basis follows from the fact that the step from the original to the fierzed basis simply interchanges the evanescent operators associated with Q_1 and Q_2 . The latter feature is absent in ref. [34], because in that paper the CMM basis [35] is used, in which the Fierz transformation neither interchanges the two operators $Q_{1,2}^{\text{CMM}}$ nor their corresponding evanescent operators. Our condition that $\langle Q_{1,2} \rangle$ shall agree with the matrix elements of the corresponding fierzed operators is sufficient to ensure the non-mixing of $Q_{\pm}$ into $Q_{\mp}$.

The validity of eq. (2.8) beyond LO, which is justified by the choice of evanescent operators listed above, thus implies

$$\Gamma^{q_1 q_2 q_3} = \tilde{\Gamma}^{q_2 q_1 q_3} \Big|_{\tilde{C}_1 \rightarrow C_2, \tilde{C}_2 \rightarrow C_1}, \quad (2.11)$$

which we use for the remainder of this paper.

As can be seen from the diagrams in figures 3 and 4, there is no need to use Fierz-arranged operators for the penguin diagrams. It is possible to treat this class of diagrams differently from the current-current diagrams, since it forms a gauge invariant subset. The general expression of the decay width for this subclass of diagrams is the sum of terms of the form

$$\Gamma^{q_1^{(\prime)} q_2^{(\prime)} q_3^{(\prime)}} = \frac{1}{2M_{H_b}} \sum_{i,j=1,2} \left(4 \frac{G_F}{\sqrt{2}}\right)^2 \lambda_{q_1 q_2 q_3} \lambda_{q_1' q_2' q_3'}^* C_i(\mu_1) C_j(\mu_1) \text{Im} \langle H_b | i \int d^4x e^{iqx} T \left[Q_i^{\dagger q_1' q_2' q_3'}(x) Q_j^{q_1 q_2 q_3}(0) \right] | H_b \rangle. \quad (2.12)$$

Here, the notation on the left-hand side reflects the possibility that there may be an insertion of $|\Delta B| = 1$ effective operators with different quark flavours, see figure 3.

As is well-known, the Wilson coefficients C_i that enter the calculation of the decay width mix upon renormalisation. Explicitly,

$$C_{i,B} = Z_{ij} C_j(\mu_1), \quad (2.13)$$

where the index $C_{i,B}$ and $C_i(\mu_1)$ denote the bare and renormalised Wilson coefficients, respectively, and Z_{ij} is the renormalisation matrix, which is obtained by imposing appropriate conditions on the matrix elements of the operators Q_i . In our work, we use the results obtained in the $\overline{\text{MS}}$ renormalisation scheme, which are known in the literature to NNLO accuracy [24, 35–38]. In particular, we use the explicit results given in appendix A of ref. [24], which include the mixing with the evanescent operators in eq. (2.9).

2.2 Expansion in terms of $\Delta B = 0$ operators

As mentioned above, after using the optical theorem, we perform a HQE and expand the bi-local correlator obtained in increasing powers of Λ_{QCD}/m_b

$$\mathcal{T} = [\mathcal{T}_0 + \mathcal{T}_2 + \mathcal{T}_3] \left[1 + \mathcal{O}(\Lambda_{\text{QCD}}^4/m_b^4) \right], \quad (2.14)$$

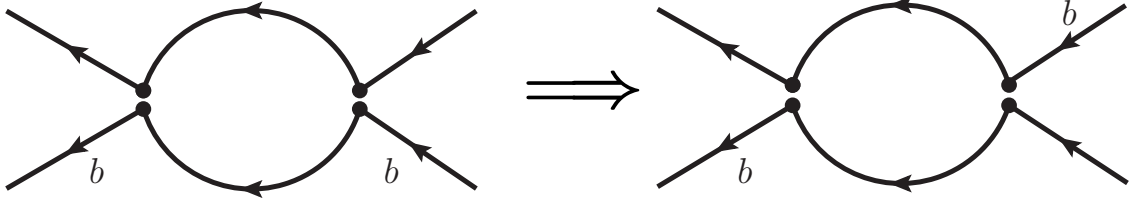


Figure 5. Diagrammatic description of the effect of inserting a Fierz-transformed operator in one of the two vertices, here the vertex on the right. The diagram on the l.h.s. contains a spinor line where b -spinor are absent, such that EOMs cannot be used to simplify the expressions in the calculation. On the other hand, the r.h.s. diagram will generate structures with one b -spinor on each fermion line.

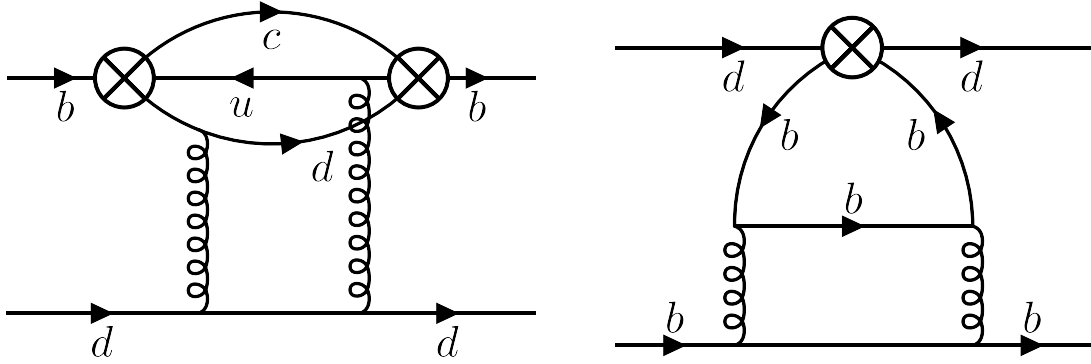


Figure 6. Example of isospin singlet contributions to the decay rate in the $|\Delta B| = 1$ (left) and $\Delta B = 0$ theory (right). These are often referred to as *penguin* or *eye contraction* diagrams. They cancel in the lifetime ratio $\tau(B^+)/\tau(B_d^0)$.

where $\mathcal{T}_n$ denotes the part of the decay width that is suppressed by a factor of $(\Lambda_{\text{QCD}}/m_b)^n$. The leading terms in this expansion, namely $\mathcal{T}_0$ and $\mathcal{T}_2$, are given by the free b quark decay and the chromomagnetic interactions of the final state quarks with the hadronic cloud surrounding the b quark, respectively. $\langle H_b | \mathcal{T}_0 | H_b \rangle$ is known to NNLO accuracy [24, 25], but contributes equally to all decay rates and drops out from lifetime differences. The contributions from $\langle H_b | \mathcal{T}_2 | H_b \rangle$ have negligible effects on the difference in lifetimes of B^+ and B_d^0 , due to the isospin symmetry of QCD.

At $\mathcal{O}(\Lambda_{\text{QCD}}^3/m_b^3)$ we encounter the first corrections to the lifetimes difference. These come from weak interactions between the b quark and the light valence quarks and can be written in the form

$$\mathcal{T}_3 = \mathcal{T}^u + \mathcal{T}^d + \mathcal{T}_{\text{sing}}, \quad (2.15)$$

where the superscript u refers to the pairs of up-type quarks inside the loop diagram in figure 1, while d indicates the internal ud and cd pairs. $\mathcal{T}_{\text{sing}}$ refers to additional isospin singlet corrections, which describe power-suppressed contributions to the free quark decay from strong interactions with the spectator quark (see diagrams in figure 6). These cancel in the difference $\Gamma(B^+) - \Gamma(B_d^0)$.

Contributions to $\mathcal{T}^u$ and $\mathcal{T}^d$ involve the following dimension-6 operators

$$\begin{aligned} Q^q &= (\bar{b}\gamma_\mu P_L q) (\bar{q}\gamma^\mu P_L b) \\ Q_S^q &= (\bar{b}P_L q) (\bar{q}P_R b) \\ T^q &= (\bar{b}\gamma_\mu P_L T^a q) (\bar{q}\gamma^\mu P_L T^a b) \\ T_S^q &= (\bar{b}P_L T^a q) (\bar{q}P_R T^a b). \end{aligned} \quad (2.16)$$

In addition to the physical operators, we define the first-generation evanescent operators as

$$\begin{aligned} E[Q^q] &= (\bar{b}\gamma^{\mu_1} \dots \gamma^{\mu_3} P_L q) (\bar{q}\gamma_{\mu_3} \dots \gamma_{\mu_1} P_L b) - (4 - 8\epsilon + a\epsilon^2)Q^q, \\ E[Q_S^q] &= (\bar{b}\gamma^{\mu_1} \gamma^{\mu_2} P_L q) (\bar{q}\gamma_{\mu_2} \gamma_{\mu_1} P_R b) - (4 - 8\epsilon + a_S\epsilon^2)Q_S^q, \end{aligned} \quad (2.17)$$

the second-generation evanescent operators as

$$\begin{aligned} E[Q^q]^{(2)} &= (\bar{b}\gamma^{\mu_1} \dots \gamma^{\mu_5} P_L q) (\bar{q}\gamma_{\mu_5} \dots \gamma_{\mu_1} P_L b) - (16 + b\epsilon + c\epsilon^2)Q^q, \\ E[Q_S^q]^{(2)} &= (\bar{b}\gamma^{\mu_1} \dots \gamma^{\mu_4} P_L q) (\bar{q}\gamma_{\mu_4} \dots \gamma_{\mu_1} P_R b) - (16 + b_S\epsilon + c_S\epsilon^2)Q_S^q, \end{aligned} \quad (2.18)$$

and finally, the third-generations evanescent operators as

$$\begin{aligned} E[Q^q]^{(3)} &= (\bar{b}\gamma^{\mu_1} \dots \gamma^{\mu_7} P_L q) (\bar{q}\gamma_{\mu_7} \dots \gamma_{\mu_1} P_L b) - (64 + e\epsilon + f\epsilon^2)Q^q, \\ E[Q_S^q]^{(3)} &= (\bar{b}\gamma^{\mu_1} \dots \gamma^{\mu_6} P_L q) (\bar{q}\gamma_{\mu_6} \dots \gamma_{\mu_1} P_R b) - (64 + e_S\epsilon + f_S\epsilon^2)Q_S^q. \end{aligned} \quad (2.19)$$

In our work, we choose the same definitions of evanescent operators related to physical operators with the same Dirac (but different colour) structure, meaning that $E[T^q]^{(i)}$ and $E[T_S^q]^{(i)}$ also involve the coefficients of the ϵ and ϵ^2 terms appearing in eqs. (2.17) to (2.19). Unlike in the case of $|\Delta B| = 1$ operators, there is no preferred choice for the terms of $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$, we only fix the value of 8ϵ in eq. (2.17) to facilitate the comparison with the literature, see e.g. ref. [7]. In the calculation of lifetimes differences among mesons there is no need to maintain Fierz symmetry in the definition of the coefficients of the $\Delta B = 0$ operators, because the quark fields in eq. (2.16) are already arranged in the correct way for the computation of the hadronic matrix elements. Any dependence on the coefficients $a, \dots, f_S$ will cancel as long as the same evanescent operators are used in the calculation of the matrix elements. Specifically, in the gradient-flow method this dependence cancels in the matching to the flowed operators, which are subsequently matched to the operators of lattice QCD [12, 13, 39, 40]. For simplicity, in our analysis we take the coefficients $a, \dots, f_S = 0$.

The situation is different for the calculation of lifetime differences of baryons which involves only two of the four operators in eq. (2.16) at dimension-6 level, while two linear combinations of these operators have power-suppressed baryonic matrix elements after a Fierz transformation [7, 26]. Thus in applications to baryons Fierz symmetry matters.

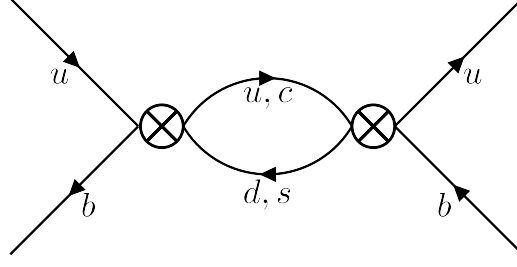


Figure 7. Sample weak annihilation-type Feynman diagrams contribution to the LO decay rate of a B^+ meson.

When we perform the matching between the “full” $|\Delta B| = 1$ side and the effective $\Delta B = 0$ theory, the weak interactions with the valence quarks take the following expression

$$\begin{aligned}
 \mathcal{T}^u &= 4 \frac{G_F^2 m_b^2}{6\pi} \left[|\lambda_{cud}|^2 \left(F^{cu} Q^d + F_S^{cu} Q_S^d + G^{cu} T^d + G_S^{cu} T_S^d \right) \right. \\
 &\quad + |\lambda_{ccd}|^2 \left(F^{cc} Q^d + F_S^{cc} Q_S^d + G^{cc} T^d + G_S^{cc} T_S^d \right) \\
 &\quad + |\lambda_{uud}|^2 \left(F^{uu} Q^d + F_S^{uu} Q_S^d + G^{uu} T^d + G_S^{uu} T_S^d \right) \\
 &\quad \left. + \lambda'_{cud} \left(\tilde{F}^{cu} Q^d + \tilde{F}_S^{cu} Q_S^d + \tilde{G}^{cu} T^d + \tilde{G}_S^{cu} T_S^d \right) \right], \\
 \mathcal{T}^d &= 4 \frac{G_F^2 m_b^2}{6\pi} \left[|V_{cb}|^2 \left(F^{cd} Q^u + F_S^{cd} Q_S^u + G^{cd} T^u + G_S^{cd} T_S^u \right) \right. \\
 &\quad + |V_{ub}|^2 \left(F^{ud} Q^u + F_S^{ud} Q_S^u + G^{ud} T^u + G_S^{ud} T_S^u \right. \\
 &\quad \left. \left. + F^{cs} Q^u + F_S^{cs} Q_S^u + G^{cs} T^u + G_S^{cs} T_S^u \right) \right], \quad (2.20)
 \end{aligned}$$

where the superscripts in the Wilson coefficients $F_{(S)}^{qq'}$ and $G_{(S)}^{qq'}$ refer to the qq' pair in the loop, while the superscripts on the dimension-6 operators refer to the external light spectator quarks, see figure 1. The contributions $\tilde{F}^{cu}, \dots, \tilde{G}_S^{cu}$ come from the CKM-suppressed one-particle-reducible diagrams as in figure 3 where both a uu and a cc pair are considered.

$\mathcal{T}^u$ in eq. (2.20) is easily generalised to the case in which the valence quark is s instead of d , with the appropriate replacement in CKM factors and operators. However, this replacement also interchanges CKM-favoured with CKM-suppressed terms and one should keep in mind that we only quote the CKM-favoured parts of $\mathcal{T}^u$ at NNLO.

For the derivation of $\mathcal{T}^d$, we have used $|V_{ud}|^2 + |V_{us}|^2 \approx 1$, $|V_{cs}|^2 + |V_{cd}|^2 \approx 1$ and $m_d = m_s = 0$, so that $F_{(S)}^{qs} = F_{(S)}^{qd}$ and $G_{(S)}^{qs} = G_{(S)}^{qd}$. Note that the terms proportional to $|V_{ub}|^2$ get contributions from WA and PI. The latter are obtained from the cd terms by taking the limit $m_c \rightarrow 0$. On the other hand, the WA contributions to $\Gamma(B^+)$ originate from Feynman diagrams as shown in figure 7. They contribute to all $|V_{ub}|^2$ -suppressed terms of $\mathcal{T}^d$ and are conveniently obtained from the cu contributions to $\mathcal{T}^u$. This is possible thanks to our choice of evanescent operators in eq. (2.9) which preserves Fierz symmetry, but we stress that in extracting these terms from the cu contributions, we interchange the role of the $|\Delta B| = 1$ Wilson coefficients. Explicitly, we have

$$(F^{cs}, F_S^{cs}, G^{cs}, G_S^{cs}) = (F^{cu}, F_S^{cu}, G^{cu}, G_S^{cu})|_{C_1 \leftrightarrow C_2}, \quad (2.21)$$

with analogous expressions for F^{ud} (where the additional limit $m_c \rightarrow 0$ is taken). We include them up to NLO.

We define the operators in the $\Delta B = 0$ basis at the renormalisation scale μ_0 , which we later take to be $\mu_0 = \bar{m}_b(\bar{m}_b)$ [11]. The dependence of μ_0 in the corresponding matrix elements cancels against the μ_0 dependence of the Wilson coefficients $F_{(S)}^{qq'}$ and $G_{(S)}^{qq'}$. These coefficients depend also on the $|\Delta B| = 1$ renormalisation scale μ_1 , due to the truncation of the perturbative series, and on the ratio $x = m_c/m_b$. The residual dependence on μ_1 is expected to diminish with increasing order of α_s . For this reason it is often used as an estimate of the uncertainty from higher order perturbative corrections, as discussed later in section 5.

As can be seen from eq. (2.20), and as we mentioned earlier, the contributions to the decay rate that contain cc or uu intermediate states, e.g. $F_{(S)}^{cc}$ and $G_{(S)}^{cc}$, are CKM-suppressed with respect to the contributions coming from cu pairs. In ref. [7], the NLO corrections to $\mathcal{T}^u$ have been computed in the approximation $|V_{ub}| = 0$, $|V_{cd}| = 0$ and $|V_{ud}| = 1$, thus neglecting the CKM-suppressed contributions, whose impact was estimated to be at the sub-percent level. We compute $F_{(S)}^{cc}$, $G_{(S)}^{cc}$, $F_{(S)}^{uu}$, $G_{(S)}^{uu}$, $\tilde{F}_{(S)}^{cu}$ and $\tilde{G}_{(S)}^{cu}$ to NLO accuracy. In addition, we include the CKM-suppressed contributions to $\mathcal{T}^d$, that is $F^{ud}, \dots, G_S^{cs}$ to NLO accuracy. We confirm that their contribution is well below the percent level, see section 5. Thus, we extend the approximation of ref. [7] to NNLO. The terms which are neglected introduce an uncertainty of the order of $|V_{cd}|^2 \alpha_s^2 m_c/m_b$, which is below 1%.

The final step to obtain a theoretical prediction for the lifetimes is to compute the hadronic matrix elements of the effective operators appearing in eq. (2.16). They enter the calculation in the isospin breaking contributions and can be parametrised according to [18, 26]

$$\begin{aligned} 4\langle B^+ | (Q^u - Q^d) (\mu_0) | B^+ \rangle &= f_B^2 M_B^2 B_1(\mu_0), \\ 4\langle B^+ | (Q_S^u - Q_S^d) (\mu_0) | B^+ \rangle &= f_B^2 M_B^2 B_2(\mu_0) \\ 4\langle B^+ | (T^u - T^d) (\mu_0) | B^+ \rangle &= f_B^2 M_B^2 \epsilon_1(\mu_0), \\ 4\langle B^+ | (T_S^u - T_S^d) (\mu_0) | B^+ \rangle &= f_B^2 M_B^2 \epsilon_2(\mu_0). \end{aligned} \quad (2.22)$$

Here, f_B is the decay constant of the B meson. $B_{1,2}$ and $\epsilon_{1,2}$ are referred to as *bag parameters* and need to be determined using non-perturbative methods, namely lattice QCD (LQCD) [14–17, 39, 40] and heavy quark effective theory (HQET) sum rules [10, 11, 41].

Using the isospin relation $\langle B^+ | \{Q, T\}^{u,d} | B^+ \rangle = \langle B_d^0 | \{Q, T\}^{d,u} | B_d^0 \rangle$, we can then write the difference of the decay widths $\Gamma(B_d^0)$ and $\Gamma(B^+)$ as

$$\begin{aligned} \Gamma(B_d^0) - \Gamma(B^+) &= \frac{G_F^2 m_b^2}{12\pi} f_B^2 M_B \left[|\lambda_{cud}|^2 \vec{F}^{cu} + |\lambda_{ccd}|^2 \vec{F}^{cc} + |\lambda_{uud}|^2 \vec{F}^{uu} \right. \\ &\quad \left. + \lambda'_{cud} \vec{F}^{cu} - |V_{cb}|^2 \vec{F}^{cd} - |V_{ub}|^2 (\vec{F}^{ud} + \vec{F}^{cs}) \right] \cdot \vec{B}, \end{aligned} \quad (2.23)$$

where we have introduced the shorthand notation

$$\vec{F}^{qq'} = \begin{pmatrix} F^{qq'}(\mu_1, \mu_0, x) \\ F_S^{qq'}(\mu_1, \mu_0, x) \\ G^{qq'}(\mu_1, \mu_0, x) \\ G_S^{qq'}(\mu_1, \mu_0, x) \end{pmatrix}, \quad \vec{B} = \begin{pmatrix} B_1(\mu_0) \\ B_2(\mu_0) \\ \epsilon_1(\mu_0) \\ \epsilon_2(\mu_0) \end{pmatrix}. \quad (2.24)$$

In our analysis of $\tau(B^+)/\tau(B_d)$, we use results for the bag parameters presented in ref. [11], which have been derived from HQET sum rules.

2.3 $\Delta B = 0$ renormalisation constants

The Wilson coefficients $\vec{F}^q$ given in eq. (2.24) mix upon renormalisation and follow a similar relation as in eq. (2.13). Unlike for the renormalisation matrix $\hat{Z}$ needed in the $|\Delta B| = 1$ part of our computations, there are, to our knowledge, no public results of the $\Delta B = 0$ renormalisation constants in a form that can be used in our work. Recently, a comprehensive study has been published [42] where the $\hat{Z}$ matrix for the $\Delta B = 0$ operator basis, and many others, is derived. Yet, in this work, the results are obtained with a fixed scheme for the evanescent operators (see appendix A of ref. [42]). As stated earlier in section 2.2, we want to keep the definition of the evanescent operators general. For this reason, we have computed the renormalisation constants ourselves. We refrain from presenting explicit results for these quantities here, but refer to the supplementary material which can be downloaded from [43], where the $\hat{Z}$ matrix can be found in computer-readable format.

We have cross-checked our results for the Z factors by performing two independent calculations within our collaboration. Furthermore, we have performed the computation in general R_ξ gauge of QCD, and observe that the gauge parameter ξ drops out from the final results. We have also compared our results with an independent calculation performed in refs. [12, 13] and we have found complete agreement.

3 Calculation

In our work, we employ dimensional regularisation with anticommuting γ_5 . For the calculation of the $|\Delta B| = 1$ and $\Delta B = 0$ amplitudes up to three and two loops, respectively, we have used two independent computer codes developed in FORM [44, 45] and Mathematica.

The starting point of our routine is the generation of Feynman diagrams. This is done using **qgraf** [46]. As mentioned in section 2, we neglect isospin singlet contributions (see figure 6) to the total decay rate. At NNLO, in addition, we consider the CKM-favoured processes only, neglecting the CKM-suppressed terms proportional to $|\lambda_{ccd}|^2$, λ'_{cud} and $|\lambda_{uud}|^2$. Effectively, at this order, we set $|V_{ub}| = 0$, $|V_{cd}| = 0$ and $|V_{ud}| = 1$.

We process the Feynman amplitudes using **tapir** [47], which outputs them as FORM code. This step requires some attention, since we deal with four-fermion effective vertices. We find it convenient to separate the four-particle vertices in two three-particle vertices, introducing an auxiliary particle. The corresponding Feynman rules are chosen such that the Lorentz and colour structure of the desired operators is reproduced correctly. This procedure overcomes any possible relative sign problem between different diagrams that could arise in the reconstruction of the spinor chain.

The Feynman amplitudes obtained are then processed using an in-house setup called **calc**, in which we implement a routine that expresses the tensor integrals that appear in the calculations in terms of scalar Feynman integrals via suitable projectors. This routine is detailed in ref. [48]. After this step, the amplitudes are expressed as a linear combination of a basis of spinor structures given in appendix A, with coefficients consisting of rational functions of the dimension d and the masses m_c and m_b , as well as scalar Feynman integrals

up to three-loop order. Since the momenta of the light quarks are set to zero and the bottom quark is on-shell, each Feynman integral is a two-point function with external momentum $q^2 = m_b^2$ and a dependence on the ratio m_c^2/m_b^2 .

Afterwards, we make use of Kira [49–51] to perform an integration-by-parts (IBP) reduction to master integrals. We find that roughly 70% of all the scalar integrals that appear are present already in the IBP tables derived in the computation of the mixing in the neutral B systems [22, 48], while we performed a complementary reduction to cover the remaining 30%. As expected, no new master integrals appear, and we implement the results detailed in the former works. The master integrals are given in terms of a semi-analytic log-polynomial series in $x = m_c/m_b$; explicit expressions can be found in the supplementary files of ref. [48]. We use these results to obtain an expansion of the matching coefficients up to order $(m_c/m_b)^{20}$. We have verified that the final term in the expansion contributes less than 10^{-11} to the central prediction of the lifetime differences. This justifies the use of expansions for the limit $m_c/m_b \rightarrow 0$ in constructing the matching coefficients.

Finally, we perform the renormalisation of the $|\Delta B| = 1$ and $\Delta B = 0$ side separately. We renormalise α_s , the charm quark mass and the matching coefficients of the effective operators in the $\overline{\text{MS}}$ scheme and the bottom quark mass in the pole scheme. The latter is transformed afterwards to more appropriate renormalisation schemes; see section 5 for more details. We refrain from renormalising the wave function of the external quarks since the corresponding Z factor drops out in the matching procedure. Afterwards, we perform the matching to extract the Wilson coefficients $\vec{F}^{qq'}$, which is described in the next section.

4 Matching coefficients to NNLO

In sections 2.1 and 2.2 we have described the calculation of the amplitudes in the $|\Delta B| = 1$ and $\Delta B = 0$ theories. They are ultraviolet finite but still contain infrared poles in ϵ beyond LO. This requires that the matching of the two theories has to be performed to higher order in ϵ . For a further discussion we refer to section 2.7 of ref. [22] where the matching procedure between $|\Delta B| = 1$ and $|\Delta B| = 2$ theories is described in detail including examples demonstrating the role of evanescent operators.

We define the perturbative expansion of the matching coefficients in eq. (2.20) as follows

$$F = F^{(0)} + \frac{\alpha_s}{4\pi} F^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 F^{(2)} + \mathcal{O}(\alpha_s^3) \quad (4.1)$$

where $\alpha_s \equiv \alpha_s(\mu_1)$. We refrain from presenting explicit results for the matching coefficients but refer the supplementary material which contain results for $F^{(0)}$, $F^{(1)}$ and $F^{(2)}$ [43].

There are several checks which are fulfilled by our results:

- In the matching procedure all infrared poles cancel leading to finite matching coefficients.

The cancellation of the poles can even be observed for individual diagrams: We have identified typical three-loops diagrams at the $|\Delta B| = 1$ side and the corresponding two-loop diagrams at the $|\Delta B| = 0$ side (see figure 8 for examples) and have checked that the difference is finite for $\epsilon \rightarrow 0$.

- We have performed the calculation of all parts for general QCD gauge parameter ξ and have checked that in the final result for the matching coefficients ξ drops out. Note that the individual renormalised $|\Delta B| = 0$ and $|\Delta B| = 1$ amplitudes still depend on ξ .
- The calculations performed in sections 2.1 and 2.2 assume a common renormalisation scale μ which is present both in the matching coefficients $F^{qq'}$, $F_S^{qq'}$, $G^{qq'}$ and $G_S^{qq'}$ and in the hadronic matrix elements. We use the anomalous dimensions of the $|\Delta B| = 0$ theory (see section 2.3) in order to separate the scales associated to the two theories, μ_0 and μ_1 . This leads to the form of the matching coefficients used in eq. (2.24). At this point we can use the anomalous dimensions of the $|\Delta B| = 1$ theory to cross-check the μ_1 dependence.
- We have performed a number of internal cross checks. In particular, we have performed the computation of the bare $|\Delta B| = 0$ and $|\Delta B| = 1$ amplitudes, the renormalisation in both theories, and the matching twice using independent computer codes.

For illustration we now provide numerical results for the perturbative expansion of the Wilson coefficients up to NNLO. For the renormalisation scales we choose $\mu_0 = \overline{m}_b(\overline{m}_b) = 4.163$ GeV and $\mu_1 = 4.2$ GeV. The expansions for the cu and cd contributions of F, F_S, G, G_S in terms of $\tilde{\alpha}_s = \alpha_s(4.2 \text{ GeV})/\pi$ read:

$$\begin{aligned}
 F^{cu} &= -0.0114 - 2.7414 \tilde{\alpha}_s + 1.8952 \tilde{\alpha}_s^2, \\
 F_S^{cu} &= 0.0122 + 0.9613 \tilde{\alpha}_s - 10.8561 \tilde{\alpha}_s^2, \\
 G^{cu} &= -2.4254 + 6.9883 \tilde{\alpha}_s + 38.6748 \tilde{\alpha}_s^2, \\
 G_S^{cu} &= 2.6013 - 7.6302 \tilde{\alpha}_s - 47.8491 \tilde{\alpha}_s^2, \\
 F^{cd} &= -0.6952 - 0.6568 \tilde{\alpha}_s + 52.5542 \tilde{\alpha}_s^2, \\
 F_S^{cd} &= 0.7991 \tilde{\alpha}_s + 5.6585 \tilde{\alpha}_s^2, \\
 G^{cd} &= 7.6467 - 8.1621 \tilde{\alpha}_s - 113.5616 \tilde{\alpha}_s^2, \\
 G_S^{cd} &= 2.6913 \tilde{\alpha}_s + 0.5914 \tilde{\alpha}_s^2.
 \end{aligned} \tag{4.2}$$

After inserting $\alpha_s(4.2 \text{ GeV}) = 0.2243$ we obtain

$$\begin{aligned}
 F^{cu} &= -0.0114 (1. + 17.2021_{\text{NLO}} - 0.8491_{\text{NNLO}}), \\
 F_S^{cu} &= 0.0122 (1. + 5.6241_{\text{NLO}} - 4.5350_{\text{NNLO}}), \\
 G^{cu} &= -2.4254 (1. - 0.2057_{\text{NLO}} - 0.0813_{\text{NNLO}}), \\
 G_S^{cu} &= 2.6013 (1. - 0.2094_{\text{NLO}} - 0.0938_{\text{NNLO}}), \\
 F^{cd} &= -0.6952 (1. + 0.0674_{\text{NLO}} - 0.3853_{\text{NNLO}}), \\
 F_S^{cd} &= (0.0571_{\text{NLO}} + 0.0288_{\text{NNLO}}), \\
 G^{cd} &= 7.6467 (1. - 0.0762_{\text{NLO}} - 0.0757_{\text{NNLO}}), \\
 G_S^{cd} &= (0.1922_{\text{NLO}} + 0.0030_{\text{NNLO}}).
 \end{aligned} \tag{4.3}$$

For convenience we normalise the results to the LO coefficient (if available). For some matching coefficients the LO contribution is artificially small, see, e.g., F^{cu} and F_S^{cu} . In

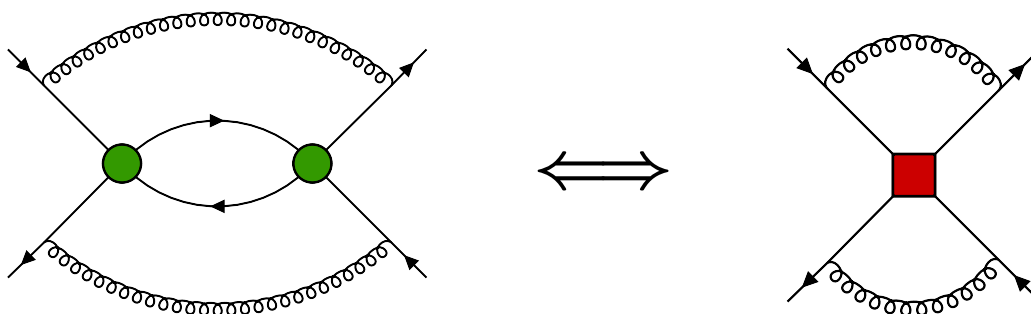


Figure 8. Example of selected diagrams used as a check of correct IR singularities cancellation. A green blob and a red square correspond to the insertion of $|\Delta B| = 1$ and $\Delta B = 0$ operators, respectively. The diagram on the left is IR divergent, but UV finite, while the diagram on the right contains both IR and UV poles. The latter are canceled by appropriate counterterms, while the former cancel in the matching.

such cases, the relative contribution from NLO is large. However, the NNLO term is either significantly smaller or leads to a compensation and ensures the reliability of the perturbative expansion. For F^{cd} the NLO contribution is relatively small, and at the NNLO level we observe a negative correction of 38%. Similarly, for G^{cd} the NLO and NNLO contributions have the same order of magnitude amount to about -7% only. In those cases where the LO contribution is absent (see, e.g., F_S^{cd} and G_S^{cd}) we observe that the correction term at NNLO is smaller than the NLO contribution by at least a factor of two.

5 Phenomenology for B mesons

In this section we discuss the numerical effect of the new NNLO corrections on the lifetime ratio $\tau(B^+)/\tau(B_0)$. We use eq. (2.23) together with the input parameters given in table 1 and the bag parameters of the B mesons obtained from QCD sum rules in ref. [11],

$$\begin{aligned} B_1 &= 1.013_{-0.059}^{+0.066}, \\ B_2 &= 1.004_{-0.081}^{+0.085}, \\ \epsilon_1 &= -0.098_{-0.032}^{+0.026}, \\ \epsilon_2 &= -0.037_{-0.020}^{+0.019}. \end{aligned} \tag{5.1}$$

They are defined at the scale $\mu_0 = \bar{m}_b(\bar{m}_b)$.

Let us briefly discuss our choices of renormalisation schemes. It is convenient to perform the calculation in a first step in a renormalisation scheme where the bottom quark is renormalised in the pole scheme (since we have $q^2 = m_b^2$ for the external momentum) and charm quark mass in the $\overline{\text{MS}}$ scheme. In a next step we transform the bottom quark mass in the quantity $z = m_c^2/m_b^2$ to the $\overline{\text{MS}}$ scheme and introduce separate renormalisation scales μ_c and μ_b for the masses and μ_1 for α_s . For our numerical analysis we identify these scales (i.e. we set $\mu_1 = \mu_c = \mu_b$) and adapt $\mu_1 = 4.2 \text{ GeV}$ as central value. We estimate the uncertainty due to unknown higher order corrections by varying μ_1 between 2.1 GeV and 8.4 GeV. Next we

$\alpha_s(M_Z)$	$= 0.1180(9)$	[52]	G_F	$= 1.166\,378\,7 \times 10^{-5} \text{ GeV}^{-2}$	[52]
M_W	$= 80.3629(133) \text{ GeV}$	[52]	M_Z	$= 91.1880(20) \text{ GeV}$	[52]
m_t^{OS}	$= 172.4(7) \text{ GeV}$	[52]	$\bar{m}_b(\bar{m}_b)$	$= 4.163(16) \text{ GeV}$	[53]
$\bar{m}_c(3 \text{ GeV})$	$= 0.993(8) \text{ GeV}$	[54]	M_B	$= 5279.41(7) \text{ MeV}$	[52]
f_B	$= 0.1900(13) \text{ GeV}$	[55]	$\tau(B^+)$	$= 1.638(4) \text{ ps}$	[52]
$ V_{ud} $	$= 0.974343_{-0.000048}^{+0.000049}$	[56]	$\tau(B^+)/\tau(B)$	$= 1.076(4)$	[9]
$ V_{cb} $	$= 0.04159_{-0.00084}^{+0.00023}$	[56]	$ V_{cd} $	$= 0.22491_{-0.00022}^{+0.00020}$	[56]
$ V_{ub} $	$= 0.003743_{-0.000057}^{+0.000053}$	[56]	$\bar{\rho}$	$= 0.1562_{-0.0045}^{+0.0102}$	[56]

Table 1. Input parameters for the numerical analysis. The quoted m_t^{pole} corresponds to $\bar{m}_t(\bar{m}_t) = (162.6 \pm 0.7) \text{ GeV}$ in the $\overline{\text{MS}}$ scheme.

introduce three choices for the overall factor m_b^2 (see eqs. (2.20) and (2.23)): We either keep it in the pole scheme or we transform it to the $\overline{\text{MS}}$ or potential-subtracted (PS) scheme [57]. In the following we present numerical results in these renormalisation schemes.

In eq. (2.23) we insert NNLO corrections to the leading-CKM matching coefficients $\vec{F}^{cu}$ and $\vec{F}^{cd}$ while only NLO corrections are used for the CKM-suppressed matching coefficient $\vec{F}^{cc}$, $\vec{F}^{uu}$, $\vec{F}^{cu}$, $\vec{F}^{cs}$ and $\vec{F}^{ud}$. Note that the latter only contributes to about 0.005% and 0.06% in the $\overline{\text{MS}}$ and PS scheme, respectively. Also the penguin contribution, which we include to LO are well below the per mille level. Since we do not include $\vec{F}^{cc}$, $\vec{F}^{uu}$, $\vec{F}^{cu}$, $\vec{F}^{cs}$ and $\vec{F}^{ud}$ to NNLO we furthermore use the approximation $|V_{ud}| = 1$, $|V_{ub}| = 0$ and $|V_{cd}| = 0$ at NNLO.

In our analysis we do not include contributions from dimension-7 operators. The LO coefficients are available from refs. [58, 59], but the matrix elements are completely unknown. Confronting our results with experimental data probes the size of these terms.

In the following we present our predictions for $\tau(B^+)/\tau(B_d^0)$ including the uncertainties from the variation of μ_1 (“scale”), from the bag parameters (“bag”), from $|V_{cb}|$ and the remaining input parameters as given in table 1. For the three renormalisation schemes introduced above we obtain

$$\begin{aligned}
 \frac{\tau(B^+)}{\tau(B_d^0)} &= 1.073_{-0.024\text{scale}}^{+0.003} \pm 0.018_{\text{bag}} \pm 0.002_{|V_{cb}|} \pm 0.002_{\text{input}} (\overline{\text{MS}}), \\
 \frac{\tau(B^+)}{\tau(B_d^0)} &= 1.071_{-0.020\text{scale}}^{+0.008} \pm 0.018_{\text{bag}} \pm 0.002_{|V_{cb}|} \pm 0.001_{\text{input}} (\text{PS}), \\
 \frac{\tau(B^+)}{\tau(B_d^0)} &= 1.066_{-0.027\text{scale}}^{+0.013} \pm 0.017_{\text{bag}} \pm 0.002_{|V_{cb}|} \pm 0.001_{\text{input}} (\text{pole}).
 \end{aligned} \tag{5.2}$$

We observe that the central values from the $\overline{\text{MS}}$ and PS schemes are very close with similar uncertainty estimates. The prediction from the pole scheme is slightly smaller and shows a larger uncertainty. The dominant contributions to the uncertainties come from the scale variation and the bag parameters; the parametric uncertainties are much smaller.

In figure 9 we show $\tau(B^+)/\tau(B_d^0) - 1$ as a function of μ_1 between 2 GeV and 10 GeV. For the $\overline{\text{MS}}$ and PS schemes we show LO, NLO and NNLO results as dotted, dashed and solid lines. For comparison also the NNLO predictions in the pole scheme are shown. The

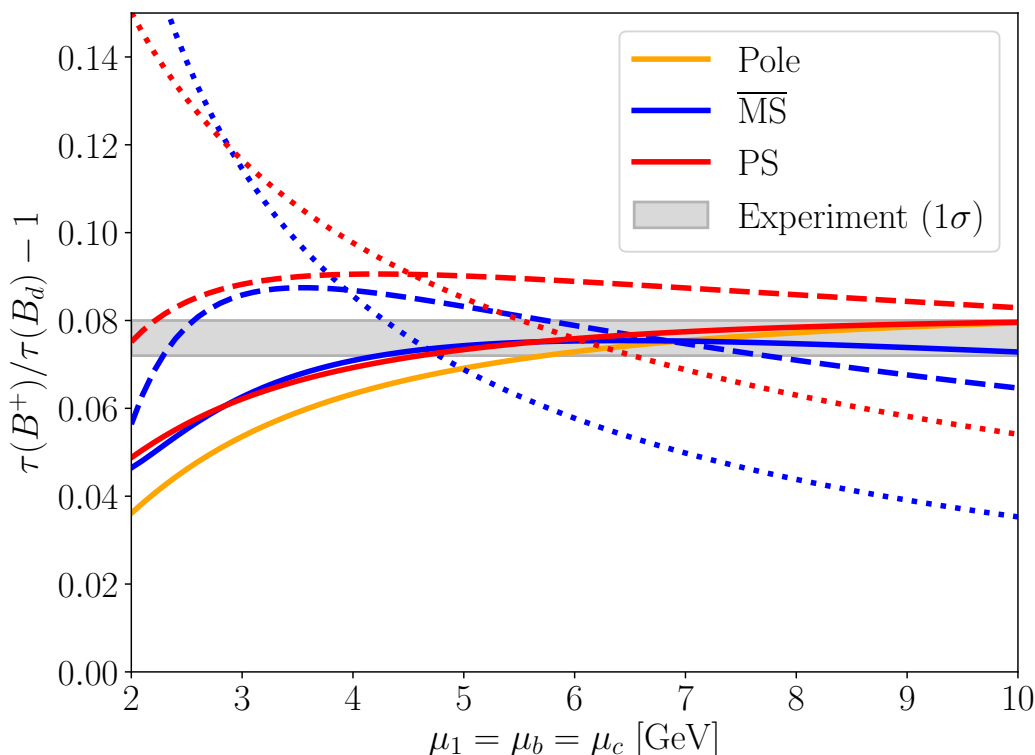


Figure 9. $\tau(B^+)/\tau(B_d^0) - 1$ as a function of μ_1 for different renormalisation schemes, see legend. The current experimental results with a 1σ uncertainty are shown as gray band.

current experimental result is shown as gray band. We observe that the LO result has a strong dependence on μ_1 which is significantly reduced when going to NLO. In the range of $\mu_1 \in [2.1 \text{ GeV}, 8.4 \text{ GeV}]$ the step from NLO to NNLO does not lead to an improvement: There is a reduction from 20% to 19% in the $\overline{MS}$ scheme and even an increase from 7% to 20% in the PS scheme. However, the solid curves are much closer together than the dashed curves which indicates a significant reduction of the renormalisation scheme dependence at NNLO. Also the agreement with the experimental results is improved at NNLO.

For our final prediction we only consider the $\overline{MS}$ and PS schemes. In each scheme we add in quadrature upper and lower uncertainties, symmetrise the result and perform an average across the two schemes. This leads to

$$\left. \frac{\tau(B^+)}{\tau(B_d^0)} \right|_{\text{dim-6}} = 1.072 \pm 0.024, \quad (5.3)$$

which has to be compared to the experimental value [9]

$$\left(\frac{\tau(B^+)}{\tau(B_d^0)} \right)^{\text{exp}} = 1.076 \pm 0.004. \quad (5.4)$$

In eq. (5.3) we have added a label to emphasize that the result does not include the unknown contributions from operators with dimension 7 and beyond. The experimental uncertainty is smaller than the one from the theory prediction by a factor of six. We observe good

agreement, well within the cited uncertainties. Since it is expected that beyond-the-Standard Model effects the lifetime ratio are small, we interpret the good agreement as evidence for the validity of the heavy-quark expansion.

Further reduction of the theory uncertainty can be achieved by improving the predictions of the bag parameters and by computing further higher-order corrections in perturbation theory.

It is interesting to compare our NNLO prediction from eq. (5.3) to previous predictions which were based on NLO calculations. Ref. [59] cites the result

$$\frac{\tau(B^+)}{\tau(B_d^0)} = 1.086(22), \quad (5.5)$$

which is based on an older determination of the non-perturbative matrix elements. It also includes the power-suppressed contributions from dimension-7 operators in the vacuum insertion approximation, which sets the colour-octet matrix elements to zero. Since the colour-singlet operators have small coefficients, similar to the dimension-6 case (see eqs. (5.7) to (5.9) below), this results in a small contribution. We find agreement within the respective uncertainties. The uncertainty given in ref. [59] is smaller than the one given in eq. (5.3) since the scale variation was done in a different way and over a smaller interval.

For convenience we present in the following numerical results for $\tau(B^+)/\tau(B_d^0)$ without inserting numerical results for the bag parameters. This enables us to obtain improved predictions once more precise results for the bag parameters become available. We parametrise $\tau(B^+)/\tau(B_d^0)$ as

$$\left. \frac{\tau(B^+)}{\tau(B_d^0)} \right|_{\text{dim-6}} = 1 + F_{B_1} B_1 + F_{B_2} B_2 + F_{\epsilon_1} \epsilon_1 + F_{\epsilon_2} \epsilon_2 \quad (5.6)$$

and provide results for the coefficients F_i using the same prescription as for eq. (5.2). For the three renormalisation schemes we obtain:

$$\begin{aligned} F_{B_1}^{\overline{\text{MS}}} &= 0.027_{-0.015}^{+0.003}_{\text{scale}} \pm 0.001_{|V_{cb}|} \pm 0.001_{\text{input}}, \\ F_{B_2}^{\overline{\text{MS}}} &= -0.006_{-0.008}^{+0.003}_{\text{scale}} \pm 0.000_{|V_{cb}|} \pm 0.000_{\text{input}}, \\ F_{\epsilon_1}^{\overline{\text{MS}}} &= -0.568_{-0.001}^{+0.028}_{\text{scale}} \pm 0.015_{|V_{cb}|} \pm 0.010_{\text{input}}, \\ F_{\epsilon_2}^{\overline{\text{MS}}} &= 0.116_{-0.003}^{+0.002}_{\text{scale}} \pm 0.003_{|V_{cb}|} \pm 0.002_{\text{input}}, \end{aligned} \quad (5.7)$$

$$\begin{aligned} F_{B_1}^{\text{PS}} &= 0.025_{-0.013}^{+0.007}_{\text{scale}} \pm 0.001_{|V_{cb}|} \pm 0.001_{\text{input}}, \\ F_{B_2}^{\text{PS}} &= -0.006_{-0.006}^{+0.002}_{\text{scale}} \pm 0.000_{|V_{cb}|} \pm 0.000_{\text{input}}, \\ F_{\epsilon_1}^{\text{PS}} &= -0.575_{-0.001}^{+0.014}_{\text{scale}} \pm 0.015_{|V_{cb}|} \pm 0.011_{\text{input}}, \\ F_{\epsilon_2}^{\text{PS}} &= 0.116_{-0.003}^{+0.005}_{\text{scale}} \pm 0.003_{|V_{cb}|} \pm 0.002_{\text{input}}. \end{aligned} \quad (5.8)$$

$$\begin{aligned} F_{B_1}^{\text{pole}} &= 0.022_{-0.017}^{+0.010}_{\text{scale}} \pm 0.000_{|V_{cb}|} \pm 0.001_{\text{input}}, \\ F_{B_2}^{\text{pole}} &= -0.007_{-0.006}^{+0.002}_{\text{scale}} \pm 0.000_{|V_{cb}|} \pm 0.000_{\text{input}}, \\ F_{\epsilon_1}^{\text{pole}} &= -0.554_{-0.012}^{+0.035}_{\text{scale}} \pm 0.014_{|V_{cb}|} \pm 0.010_{\text{input}}, \\ F_{\epsilon_2}^{\text{pole}} &= 0.109_{-0.006}^{+0.008}_{\text{scale}} \pm 0.003_{|V_{cb}|} \pm 0.002_{\text{input}}. \end{aligned} \quad (5.9)$$

Comparing the results across the three schemes, the pole scheme has the largest perturbative uncertainty for all coefficients, which is consistent with the expected poor convergence of the perturbative series in this scheme [60, 61]. For each scheme, the four coefficients differ in the relative size of the perturbative uncertainty. Interestingly, the coefficients of the larger bag parameters B_1 and B_2 also have a larger relative perturbative uncertainty, with that of B_2 exceeding 100%. The coefficient F_{B_2} is small compared to F_{B_1} because the contribution to F_S^{cd} vanishes to LO whereas it gives a sizeable contribution to F^{cd} . Therefore, more accurate results of the lifetime ratio could be achieved by calculating higher orders in perturbation theory of selected matching coefficients.

6 Phenomenology for D mesons

For the numerical evaluation of lifetime ratios in the D meson system we adapt our results obtained for B meson decays. In a first step we set all contributions where charm quarks appear as virtual particles in the gluon propagator to zero. Next we replace m_b by m_c and interpret the quantity $z = m_c^2/m_b^2$ as $z_s = m_s^2/m_c^2$ and adapt the CKM factors such that we end up with matching coefficients expressed in terms of quantities in $n_f = 4$ -flavour QCD, in particular $\alpha_s^{(4)}(\mu_1)$. It is convenient to evaluate the matching coefficients $C_1, C_2, \dots$ in the 5-flavour theory. For this reason one should use the relation between $\alpha_s^{(4)}$ and $\alpha_s^{(5)}$ before the numerical evaluation of the lifetime ratio.

However, since a precision analysis of D meson lifetimes is beyond the scope of this work we neglect all threshold effects between $n_f = 4$ - and $n_f = 5$ -flavour QCD, i.e., we use $m_c^{(5)}$ and $\alpha_s^{(5)}$ in the routines which evaluate the lifetime differences. The numerical effect of this approximation is expected to be very small. Furthermore we must set $z_s = 0$, because effects of $m_s \neq 0$ correspond to higher-dimensions in the HQE.

We consider the isospin doublet (D^+, D^0) and the V-spin doublet (D_s^+, D^0) , and compute the ratios $\tau(D^+)/\tau(D^0)$ and $\tau(D_s^+)/\tau(D^0)$. We stress that the CKM-suppressed and CKM-favoured contributions to $\Gamma(D_s^+)$ are interchanged with respect to $\Gamma(D^+)$ and $\Gamma(B^+)$. The diagrams contributing at LO in QCD to the decay rates of D mesons are displayed in figure 10.

6.1 The lifetimes ratio $\tau(D^+)/\tau(D^0)$

In the case of charmed-hadron lifetime ratios the contributions of the dimension-7 operators in the HQE of the decay width, as in eq. (1.1), are expected to have a larger impact on the theoretical predictions [62] compared to the bottom case, because $\Lambda_{\text{QCD}}/m_c \sim 0.3$ is larger than $\Lambda_{\text{QCD}}/m_b \sim 0.1$. As mentioned above, we deviate from ref. [62] by neglecting these contributions and using the experimental measurement as an estimate of the size of the $1/m_c^4$ terms. Placing such bounds helps to assess the size of the corresponding terms in the b system, whose impact should be three times smaller there.

Following the procedure that leads to eq. (2.23), we obtain [62]

$$\Gamma(D^0) - \Gamma(D^+) = \frac{G_F^2 m_c^2}{12\pi} f_D^2 M_D \left(\vec{F}_D^{s'd'} - |V_{ud}|^2 \vec{F}_D^{s'u} \right) \cdot \vec{B}, \quad (6.1)$$

where $F_D^{qd'} = |V_{ud}|^2 F_D^{qd} + |V_{us}|^2 F_D^{qs}$ and $F_D^{s'q} = |V_{cd}|^2 F_D^{dq} + |V_{cs}|^2 F_D^{sq}$. Here, the indices qq' in the notation of the Wilson coefficients refer to the intermediate qq' quark pair. When

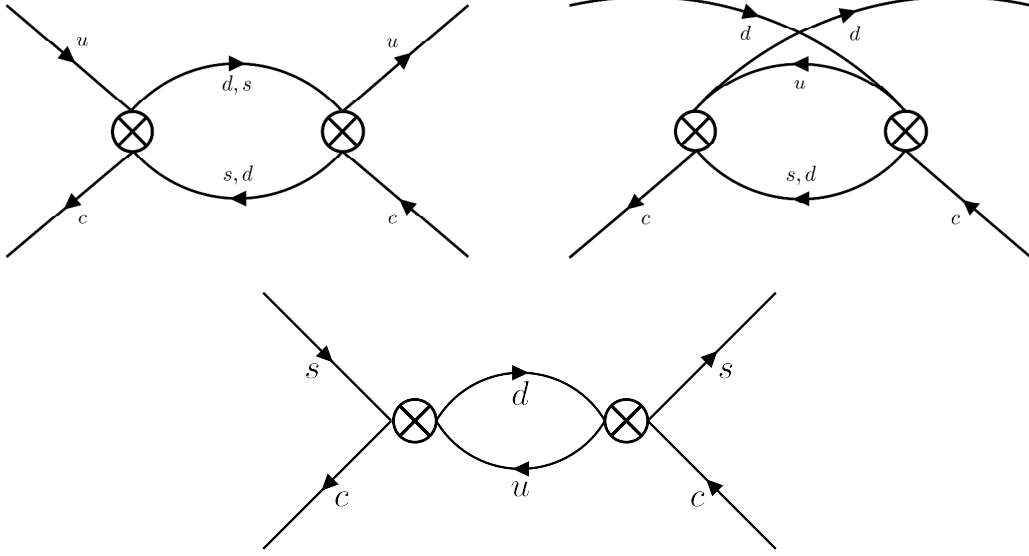


Figure 10. Weak annihilation contribution to $\Gamma(D^0)$ (top left), Pauli interference contribution to $\Gamma(D^+)$ (top right), and weak annihilation contribution to $\Gamma(D_s^+)$ (bottom) in the leading order of QCD. The crosses represent the insertion of $|\Delta C| = 1$ operators from the effective Hamiltonian of weak charm decays. CKM-suppressed contributions are not shown.

$f_D = 0.2120(7) \text{ GeV}$	[55]	$M_D = 1869.66(5) \text{ MeV}$	[52]
$\tau(D^0) = 0.4103(10) \text{ ps}$	[52]	$\tau(D^+) = 1.033(5) \text{ ps}$	[52]
$\tau(D^+)/\tau(D^0) = 2.510(13)(07)$	[63]	$\tau(D_s^+) = 0.5012(22) \text{ ps}$	[52]
$\tau(D_s^+)/\tau(D^0) = 1.222(6)$	[52]	$B(D_s^+ \rightarrow \tau^+ \nu_\tau) = 5.39(9)\%$	[52]
$B(D^0 \rightarrow X e^+ \nu_e) = 6.49(11)\%$	[52]	$B(D_s^+ \rightarrow X e^+ \nu_e) = 6.33(15)\%$	[52]
$f_{D_s} = 0.2499(5) \text{ GeV}$	[55]	$f_{D_s}/f_D = 1.1783(16)$	[55]
$ V_{us} = 0.22504^{+0.00020}_{-0.00022}$	[56]	$ V_{cs} = 0.973497^{+0.000057}_{-0.000054}$	[56]

Table 2. Input parameters for the numerical analysis.

we adapt our results for the B mesons to the D system, we set $x = 0$ and identify $F^{cu} = F_D^{sd} = F_D^{ds} = F_D^{dd} = F_D^{ss}$ and $F^{cd} = F_D^{su} = F_D^{du}$.

For the numerical evaluation we use the input parameters given in table 2 and the bag parameters of the D mesons obtained from QCD sum rules in ref. [11],

$$\begin{aligned}
 B_1^D &= 0.875^{+0.07}_{-0.044}, \\
 B_2^D &= 0.862^{+0.138}_{-0.078}, \\
 \epsilon_1^D &= -0.122^{+0.033}_{-0.042}, \\
 \epsilon_2^D &= 0.0002^{+0.0148}_{-0.0197}.
 \end{aligned} \tag{6.2}$$

They are defined at the scale $\mu_0 = 3 \text{ GeV}$.

For our final prediction we consider only the $\overline{\text{MS}}$ scheme. We adapt $\mu_1 = 3 \text{ GeV}$ as our central value, and let μ_1 vary between 1.5 GeV and 6 GeV to estimate the uncertainty due

to higher order corrections. Explicitly, we find

$$\begin{aligned}
 \left(\frac{\tau(D^+)}{\tau(D^0)} \right)_{\text{dim-6}}^{\text{LO}} &= 2.073_{-0.489}^{+1.179}_{\text{scale}}, \\
 \left(\frac{\tau(D^+)}{\tau(D^0)} \right)_{\text{dim-6}}^{\text{NLO}} &= 1.927_{-0.702}^{+0.002}_{\text{scale}}, \\
 \left(\frac{\tau(D^+)}{\tau(D^0)} \right)_{\text{dim-6}}^{\text{NNLO}} &= 1.737_{-0.182}^{+0.082}_{\text{scale}} \pm 0.185_{\text{bag}} \pm 0.014_{\text{input}}.
 \end{aligned} \tag{6.3}$$

Here and in the following we omit the uncertainties from bag parameters and input in the LO and NLO numbers, because they do not change with the order of the perturbative series. We reiterate that no dimension-7 operators are used in our analysis — neither for the central values nor for the uncertainty estimates, in our spirit to assess the quality of the HQE by comparing the dimension-6 result with data.

Recently the bag parameters for $\tau(D_s^+)/\tau(D^0)$ have been computed with lattice QCD. To apply these to $\tau(D^+)/\tau(D^0)$ one must trust in U-spin symmetry; which in all lattice calculations for $|\Delta B| = 2$ bag parameters is found to hold to an excellent level. The dominant effect of U-spin breaking in hadronic matrix elements is captured by the ratio of the decay constants, so that we assume that D_s^+ matrix elements differ from their D^+ counterparts by a factor of $f_{D_s}^2/f_D^2 \sim 1.4$. In B physics the large U-spin breaking in decay constants has been ascribed to large chiral logarithms which are absent in the bag parameters [64]. Thus we use the same lattice bag parameters for both lifetime ratios [12, 13]:

$$\begin{aligned}
 B_1^D &= 1.103 \pm 0.010, \\
 B_2^D &= 0.9754 \pm 0.0071, \\
 \epsilon_1^D &= -0.2427 \pm 0.0071, \\
 \epsilon_2^D &= -0.0289 \pm 0.0008,
 \end{aligned} \tag{6.4}$$

and obtain

$$\begin{aligned}
 \left(\frac{\tau(D^+)}{\tau(D^0)} \right)_{\text{dim-6}}^{\text{LO}} &= 2.787_{-0.757}^{+1.803}_{\text{scale}} (\overline{\text{MS}}), \\
 \left(\frac{\tau(D^+)}{\tau(D^0)} \right)_{\text{dim-6}}^{\text{NLO}} &= 2.590_{-0.988}^{+0.001}_{\text{scale}} (\overline{\text{MS}}), \\
 \left(\frac{\tau(D^+)}{\tau(D^0)} \right)_{\text{dim-6}}^{\text{NNLO}} &= 2.344_{-0.236}^{+0.094}_{\text{scale}} \pm 0.035_{\text{bag}} \pm 0.026_{\text{input}} (\overline{\text{MS}}).
 \end{aligned} \tag{6.5}$$

Symmetrising the scale uncertainty in eq. (6.5) and adding the three uncertainties in quadrature gives our final prediction

$$\left(\frac{\tau(D^+)}{\tau(D^0)} \right)_{\text{dim-6}}^{\text{NNLO}} = 2.344 \pm 0.170. \tag{6.6}$$

This has to be compared to the experimental value [63]

$$\left(\frac{\tau(D^+)}{\tau(D^0)} \right)^{\text{exp}} = 2.510 \pm 0.015. \tag{6.7}$$

When we use the determination of the bag parameters from ref. [11], the comparison between the central values of our theoretical prediction of the lifetimes ratio and the quoted experimental value implies a magnitude of slightly less than 50% for the $1/m_c^4$ suppressed terms, that is $\tilde{\Gamma}_7$ in eq. (1.1). This is in agreement with the estimate in ref. [62]. The situation, however, changes considerably when we implement in our analysis the results presented recently in refs. [12, 13]. Here, a comparison between the central value in eq. (6.6) and the experimental determination in eq. (6.7) implies a contribution from the dimension-7 operators of slightly less than 10%. This is due to the fact that the bag parameters differ significantly between the two determinations that we implement. In particular, as pointed out in ref. [12], the difference in the value of $\epsilon_1^{\overline{\text{MS}}}$ is the more phenomenologically relevant, given that the Wilson coefficient associated to this matrix element is the largest [58], as can be seen from eq. (5.7).

6.2 The lifetimes ratio $\tau(D_s^+)/\tau(D^0)$

Unlike the decay of the D^+ meson, the leading-CKM contribution to $\Gamma(D_s^+)$ comes from WA diagrams, as shown in figure 10.

(D_s^+, D^0) is a doublet of the V-spin symmetry which rotates u and s quarks into each other. Because of $m_s - m_u \sim 0.3 \Lambda_{\text{QCD}}$ it is broken by 30% from soft QCD dynamics, so that the V-spin singlet terms in $\mathcal{T}_{\text{sing}}$ do not cancel exactly in the lifetimes difference of the two mesons. Furthermore, V-spin breaking implies $\langle D_s^+ | \{Q, T\}^{s,u} | D_s^+ \rangle \neq \langle D^0 | \{Q, T\}^{u,s} | D^0 \rangle$, so that $\tau(D_s^+)/\tau(D^0) - 1$ cannot be solely expressed in terms of matrix elements of differences like $Q^s - Q^u$, cf. eq. (2.22) for the isospin analogue. In view of the remark after eq. (6.3) we incorporate the described V-spin breaking by using $f_{D_s}^2$ in the normalisation of the $\langle D_s^+ | \dots | D_s^+ \rangle$ matrix elements while taking f_D^2 for $\langle D^0 | \dots | D^0 \rangle$.

Other corrections which must be taken into account are described and analysed in detail in ref. [62]: There are CKM-unsuppressed contributions to the WA diagram in figure 10 from a lepton pair in the loop. $D_s^+ \rightarrow \tau^+ \nu_\tau$ defies the HQE because of $m_{D_s} \sim m_\tau$ but this contribution can be taken care of by subtracting $\Gamma(D_s^+ \rightarrow \tau^+ \nu_\tau)$ from Γ_{tot} using data on the branching fraction. We also include the semi-leptonic decays $D_s^+ \rightarrow X \ell \nu_\ell$ and $D^0 \rightarrow X \ell \nu_\ell$ in this way and replace Γ_{tot} by the hadronic rate

$$\Gamma_{\text{had}} \equiv \Gamma_{\text{tot}} - \Gamma_{(\text{semi})\text{lept}}$$

amounting to

$$\begin{aligned} \frac{\tau(D_s^+)}{\tau(D^0)} - 1 &= \tau(D_s^+) \left[\Gamma_{\text{had}}(D^0) - \Gamma_{\text{had}}(D_s^+) - \Gamma(D_s^+ \rightarrow \tau^+ \nu_\tau) + \right. \\ &\quad \left. \sum_{\ell=e,\mu} \left(\Gamma(D^0 \rightarrow X \ell \nu) - \Gamma(D_s^+ \rightarrow X \ell \nu) \right) \right] \\ &= \tau(D_s^+) \left[\Gamma_{\text{had}}(D^0) - \Gamma_{\text{had}}(D_s^+) \right] - B(D_s^+ \rightarrow \tau^+ \nu_\tau) \\ &\quad + \sum_{\ell=e,\mu} \left(\frac{\tau(D_s^+)}{\tau(D^0)} B(D^0 \rightarrow X \ell \nu) - B(D_s^+ \rightarrow X \ell \nu) \right) \\ &= \tau(D_s^+) \left[\Gamma_{\text{had}}(D^0) - \Gamma_{\text{had}}(D_s^+) \right] - 0.0219 \pm 0.0041 \end{aligned} \quad (6.8)$$

where PDG values [52] (assuming equal semi-leptonic branching ratios for e and μ) have been used in the last line. The first term in eq. (6.8) defines $\tau(D_s^+)/\tau(D^0)|_{\text{had}}$ which we calculate in this paper.

Thus we need

$$\Gamma_{\text{had}}(D^0) - \Gamma_{\text{had}}(D_s^+) = \frac{G_F^2 m_c^2}{12\pi} f_D^2 M_D \left[\vec{F}_D^{s'd'} - \frac{f_{D_s}^2}{f_D^2} |V_{cs}|^2 \left(|V_{ud}|^2 \vec{F}_D^{du} + |V_{us}|^2 \vec{F}_D^{su} \right) \right] \cdot \vec{B}, \quad (6.9)$$

where the coefficients $\vec{F}_D^{du}$ are extracted from the coefficients $\vec{F}^{cu}$ in an analogous way to the $\vec{F}^{cs}$ contributions in eq. (2.20) and $F^{cd} = F_D^{su}$. There are no CKM-favoured PI contributions to $\tau(D^+)/\tau(D^0)$; since WA diagrams are much smaller, this lifetime ratio is close to 1. The last term in eq. (6.9) includes the CKM-suppressed PI contribution to $\Gamma(D_s^+)$. We confirm the finding of ref. [62] that this term is numerically smaller than the WA contribution but non-negligible.

For our final prediction we consider only the $\overline{\text{MS}}$ scheme. We adapt $\mu_1 = 3 \text{ GeV}$ as our central value, and let μ_1 vary between 1.5 GeV and 6 GeV to estimate the uncertainty due to higher order corrections. Furthermore, we set $m_s = 0$ at LO, NLO and NNLO. Since we assume exact SU(3) symmetry, we take the values in eq. (6.2) for the bag parameters in eq. (6.9). Explicitly, we find

$$\begin{aligned} \left(\frac{\tau(D_s^+)}{\tau(D^0)} \right)_{\text{had, dim-6}}^{\text{LO}} &= 1.110_{-0.035}^{+0.076}_{\text{scale}} (\overline{\text{MS}}), \\ \left(\frac{\tau(D_s^+)}{\tau(D^0)} \right)_{\text{had, dim-6}}^{\text{NLO}} &= 1.115_{-0.091}^{+0.005}_{\text{scale}} (\overline{\text{MS}}), \\ \left(\frac{\tau(D_s^+)}{\tau(D^0)} \right)_{\text{had, dim-6}}^{\text{NNLO}} &= 1.157_{-0.010}^{+0.074}_{\text{scale}} \pm 0.124_{\text{bag}} \pm 0.004_{\text{input}} (\overline{\text{MS}}). \end{aligned} \quad (6.10)$$

When we use the results in refs. [12, 13] instead, we get

$$\begin{aligned} \left(\frac{\tau(D_s^+)}{\tau(D^0)} \right)_{\text{had, dim-6}}^{\text{LO}} &= 1.288_{-0.085}^{+0.176}_{\text{scale}} (\overline{\text{MS}}), \\ \left(\frac{\tau(D_s^+)}{\tau(D^0)} \right)_{\text{had, dim-6}}^{\text{NLO}} &= 1.275_{-0.135}^{+0.003}_{\text{scale}} (\overline{\text{MS}}), \\ \left(\frac{\tau(D_s^+)}{\tau(D^0)} \right)_{\text{had, dim-6}}^{\text{NNLO}} &= 1.311_{-0.008}^{+0.071}_{\text{scale}} \pm 0.013_{\text{bag}} \pm 0.006_{\text{input}} (\overline{\text{MS}}). \end{aligned} \quad (6.11)$$

In analogy to eq. (6.6) we find our final result for the choice of lattice bag parameters:

$$\left(\frac{\tau(D_s^+)}{\tau(D^0)} \right)_{\text{had, dim-6}}^{\text{NNLO}} = 1.311 \pm 0.042 \quad (6.12)$$

This has to be confronted with the corresponding quantity derived from the experimental value [52]

$$\left(\frac{\tau(D_s^+)}{\tau(D^0)} \right)^{\text{exp}} = 1.222 \pm 0.006. \quad (6.13)$$

Subtracting the contributions from the (semi-)leptonic decays as described around eq. (6.8) one finds the experimental result for the hadronic contribution to the lifetime ratio as

$$\left(\frac{\tau(D_s^+)}{\tau(D^0)}\right)_{\text{had}}^{\text{exp}} = 1.244 \pm 0.007. \quad (6.14)$$

Similarly to what happens in the case of the ratio $\tau(D^+)/\tau(D_0)$, the results obtained with lattice QCD show a better agreement with the experimental value with respect to the results derived using HQET sum rules. Contrary to the case of $\tau(D^+)/\tau(D_0)$, the difference between eq. (6.14) and eqs. (6.10) and (6.11), is not only a measure of the suppressed terms of $\mathcal{O}(1/m_c^4)$ and higher in the HQE and unknown N³LO corrections, but also of the V-spin breaking terms beyond the factorisable contributions accounted for by the factor of $f_{D_s}^2/f_D^2$. Comparing the central values, the experimental value in eq. (6.14) differs from the NNLO results in eqs. (6.10) and (6.11) by 0.087 and -0.067 , respectively, slightly favouring the lattice computation. Yet both values are smaller in magnitude than the experimental value of 0.244 of the calculated valence-quark effects, indicating that the HQE works at the expected level of accuracy.

For completeness we finally add the (semi-)leptonic decay rates of eq. (6.8) to eq. (6.12) to find

$$\left(\frac{\tau(D_s^+)}{\tau(D^0)}\right)_{\text{dim-6}}^{\text{NNLO}} = 1.289 \pm 0.042. \quad (6.15)$$

7 Conclusions

In this paper we have presented a new SM prediction for the lifetime ratio $\tau(B^+)/\tau(B_d^0)$ based on a calculation of next-to-next-leading order (NNLO) QCD corrections to the Wilson coefficients of the dimension-6 operators appearing in the heavy quark expansion (HQE) of this ratio. This is the first such calculation for $\Delta B = 0$ coefficients and constitutes an important step towards theory predictions of lifetime difference among b -flavoured hadrons which match the accuracy of the corresponding measurements. The computation has involved two-scale three-loop integrals which we have determined as deep expansions in the quark mass ratio m_c/m_b . In addition, we have calculated Cabibbo-suppressed contributions to the lifetime ratio at NLO.

We find sizable NNLO corrections, comparable to the NLO contribution, with a significant reduction of the dependence on the renormalisation scheme. The residual renormalisation scale dependence is satisfactory and, most importantly, the NNLO corrections combined with hadronic matrix elements from QCD sum rules [10, 11] bring the calculated $\tau(B^+)/\tau(B_d^0)$ into excellent agreement with the experimental value. This suggest that contributions from yet poorly known dimension-7 terms of the HQE are small, so that the HQE indeed constitutes a valid and useful concept. We have presented our results in a form which permits the implementation of future, more precise computations of the hadronic matrix elements in an easy and straightforward way.

Applying our results to the charm system, we present predictions for the lifetime splittings in the isospin doublet (D^+, D^0) and the V-spin doublet (D_s^+, D^0) . The larger values of $\alpha_s(m_c)$ and Λ_{QCD}/m_c compared to their counterparts in the b system lead to larger uncertainties

from neglected higher-order terms in the HQE. We combine our NNLO coefficients with both the matrix elements calculated with QCD sum rules [11] and the ones computed on the lattice [12, 13]. In both cases our predictions agree with the experimental results within uncertainties. The difference between theoretical and experimental numbers can be viewed as an estimate of the $(\Lambda_{\text{QCD}}/m_c)^4$ terms of the HQE and in the case of $\tau(D_s^+)/\tau(D^0)$ also of unaccounted V-spin breaking terms. For the lattice-based prediction the difference amounts to -11% and 27% of the calculated valence-quark contributions $\tau(D^+)/\tau(D^0) - 1$ and $\tau(D_s^+)/\tau(D^0) - 1$, respectively, if one simply compares the central values of prediction and measurement. This is well below the expected uncertainty of order $\Lambda_{\text{QCD}}/m_c \sim 0.3$ from the next term in the HQE. If one uses the sum-rule results for the matrix elements, this exercise gives larger differences of -51% and -36% for the two quantities.

Our NNLO result supports the HQE as the correct method to calculate lifetime differences between heavy hadrons, for both the beauty and charm systems. Extending predictions to even higher orders of $\alpha_s(m_{b,c})$ and $\Lambda_{\text{QCD}}/m_{b,c}$ in combination with better hadronic matrix elements will reduce the theoretical uncertainties further with the aim to match the small experimental errors. In observables with essentially no sensitivity to physics beyond the SM, like the lifetime ratios studied in this paper, these calculation will permit to assess the final accuracy achievable with the HQE. With justified confidence in the method one can then apply the HQE to observables with high BSM sensitivity, especially those related to the decay width matrices of the $B_{d,s}-\bar{B}_{d,s}$ mixing complexes [23].

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A Basis elements for spinor structures

Here we list all the basis elements of spinor structures that are used to reduce tensor integrals to scalar ones. This basis is derived from the one given in ref. [48] and adapted to the $\Delta B = 0$ basis. We refer to [48] for all the technical details. Explicitly, we have.²

$$\begin{aligned} B_1 &= \mathbb{1} \otimes \mathbb{1} \\ B_2 &= \not{\epsilon}_q \otimes \mathbb{1} \\ B_3 &= \mathbb{1} \otimes \not{\epsilon}_q \\ B_4 &= \not{\epsilon}_q \otimes \not{\epsilon}_q \\ B_5 &= \gamma^{\mu_1} \otimes \gamma_{\mu_1} \\ B_6 &= \not{\epsilon}_q \gamma^{\mu_1} \otimes \gamma_{\mu_1} \end{aligned}$$

²A spinor structure of the form $(\bar{b}\Gamma q)(\bar{q}\Gamma b)$ is understood.

$$\begin{aligned}
B_7 &= \gamma^{\mu_1} \otimes \gamma_{\mu_1} \not\epsilon_q \\
B_8 &= \not\epsilon_q \gamma^{\mu_1} \otimes \gamma_{\mu_1} \not\epsilon_q \\
B_9 &= \gamma^{\mu_1} \gamma^{\mu_2} \otimes \gamma_{\mu_2} \gamma_{\mu_1} \\
B_{10} &= \not\epsilon_q \gamma^{\mu_1} \gamma^{\mu_2} \otimes \gamma_{\mu_2} \gamma_{\mu_1} \\
B_{11} &= \gamma^{\mu_1} \gamma^{\mu_2} \otimes \gamma_{\mu_2} \gamma_{\mu_1} \not\epsilon_q \\
B_{12} &= \not\epsilon_q \gamma^{\mu_1} \gamma^{\mu_2} \otimes \gamma_{\mu_2} \gamma_{\mu_1} \not\epsilon_q \\
B_{13} &= \gamma^{\mu_1} \dots \gamma^{\mu_3} \otimes \gamma_{\mu_3} \dots \gamma_{\mu_1} \\
B_{14} &= \not\epsilon_q \gamma^{\mu_1} \dots \gamma^{\mu_3} \otimes \gamma_{\mu_3} \dots \gamma_{\mu_1} \\
B_{15} &= \gamma^{\mu_1} \dots \gamma^{\mu_3} \otimes \gamma_{\mu_3} \dots \gamma_{\mu_1} \not\epsilon_q \\
B_{16} &= \not\epsilon_q \gamma^{\mu_1} \dots \gamma^{\mu_3} \otimes \gamma_{\mu_3} \dots \gamma_{\mu_1} \not\epsilon_q \\
B_{17} &= \gamma^{\mu_1} \dots \gamma^{\mu_4} \otimes \gamma_{\mu_4} \dots \gamma_{\mu_1} \\
B_{18} &= \not\epsilon_q \gamma^{\mu_1} \dots \gamma^{\mu_4} \otimes \gamma_{\mu_4} \dots \gamma_{\mu_1} \\
B_{19} &= \gamma^{\mu_1} \dots \gamma^{\mu_4} \otimes \gamma_{\mu_4} \dots \gamma_{\mu_1} \not\epsilon_q \\
B_{20} &= \not\epsilon_q \gamma^{\mu_1} \dots \gamma^{\mu_4} \otimes \gamma_{\mu_4} \dots \gamma_{\mu_1} \not\epsilon_q \\
B_{21} &= \gamma^{\mu_1} \dots \gamma^{\mu_5} \otimes \gamma_{\mu_5} \dots \gamma_{\mu_1} \\
B_{22} &= \not\epsilon_q \gamma^{\mu_1} \dots \gamma^{\mu_5} \otimes \gamma_{\mu_5} \dots \gamma_{\mu_1} \\
B_{23} &= \gamma^{\mu_1} \dots \gamma^{\mu_5} \otimes \gamma_{\mu_5} \dots \gamma_{\mu_1} \not\epsilon_q \\
B_{24} &= \not\epsilon_q \gamma^{\mu_1} \dots \gamma^{\mu_5} \otimes \gamma_{\mu_5} \dots \gamma_{\mu_1} \not\epsilon_q \\
B_{25} &= \gamma^{\mu_1} \dots \gamma^{\mu_6} \otimes \gamma_{\mu_6} \dots \gamma_{\mu_1} \\
B_{26} &= \not\epsilon_q \gamma^{\mu_1} \dots \gamma^{\mu_6} \otimes \gamma_{\mu_6} \dots \gamma_{\mu_1} \\
B_{27} &= \gamma^{\mu_1} \dots \gamma^{\mu_6} \otimes \gamma_{\mu_6} \dots \gamma_{\mu_1} \not\epsilon_q \\
B_{28} &= \not\epsilon_q \gamma^{\mu_1} \dots \gamma^{\mu_6} \otimes \gamma_{\mu_6} \dots \gamma_{\mu_1} \not\epsilon_q \\
B_{29} &= \gamma^{\mu_1} \dots \gamma^{\mu_7} \otimes \gamma_{\mu_7} \dots \gamma_{\mu_1} \\
B_{30} &= \not\epsilon_q \gamma^{\mu_1} \dots \gamma^{\mu_7} \otimes \gamma_{\mu_7} \dots \gamma_{\mu_1} \\
B_{31} &= \gamma^{\mu_1} \dots \gamma^{\mu_7} \otimes \gamma_{\mu_7} \dots \gamma_{\mu_1} \not\epsilon_q \\
B_{32} &= \not\epsilon_q \gamma^{\mu_1} \dots \gamma^{\mu_7} \otimes \gamma_{\mu_7} \dots \gamma_{\mu_1} \not\epsilon_q \\
B_{33} &= \gamma^{\mu_1} \dots \gamma^{\mu_8} \otimes \gamma_{\mu_8} \dots \gamma_{\mu_1} \\
B_{34} &= \not\epsilon_q \gamma^{\mu_1} \dots \gamma^{\mu_8} \otimes \gamma_{\mu_8} \dots \gamma_{\mu_1} \\
B_{35} &= \gamma^{\mu_1} \dots \gamma^{\mu_8} \otimes \gamma_{\mu_8} \dots \gamma_{\mu_1} \not\epsilon_q \\
B_{36} &= \not\epsilon_q \gamma^{\mu_1} \dots \gamma^{\mu_8} \otimes \gamma_{\mu_8} \dots \gamma_{\mu_1} \not\epsilon_q \\
B_{37} &= \gamma^{\mu_1} \dots \gamma^{\mu_9} \otimes \gamma_{\mu_9} \dots \gamma_{\mu_1} \\
B_{38} &= \not\epsilon_q \gamma^{\mu_1} \dots \gamma^{\mu_9} \otimes \gamma_{\mu_9} \dots \gamma_{\mu_1} \\
B_{39} &= \gamma^{\mu_1} \dots \gamma^{\mu_9} \otimes \gamma_{\mu_9} \dots \gamma_{\mu_1} \not\epsilon_q \\
B_{40} &= \not\epsilon_q \gamma^{\mu_1} \dots \gamma^{\mu_9} \otimes \gamma_{\mu_9} \dots \gamma_{\mu_1} \not\epsilon_q
\end{aligned}$$

$$\begin{aligned}
B_{41} &= \gamma^{\mu_1} \cdots \gamma^{\mu_{10}} \otimes \gamma_{\mu_{10}} \cdots \gamma_{\mu_1} \\
B_{42} &= \not{q} \gamma^{\mu_1} \cdots \gamma^{\mu_{10}} \otimes \gamma_{\mu_{10}} \cdots \gamma_{\mu_1} \\
B_{43} &= \gamma^{\mu_1} \cdots \gamma^{\mu_{10}} \otimes \gamma_{\mu_{10}} \cdots \gamma_{\mu_1} \not{q} \\
B_{44} &= \not{q} \gamma^{\mu_1} \cdots \gamma^{\mu_{10}} \otimes \gamma_{\mu_{10}} \cdots \gamma_{\mu_1} \not{q}
\end{aligned}$$

where $\not{q} = \not{q}/\sqrt{q^2}$, with q being the external momentum and $q^2 = m_b^2$.

Data Availability Statement. This article has associated data in a data repository. Available at: <https://www.ttp.kit.edu/preprints/2026/ttp26-014/>.

Code Availability Statement. This article has no associated code or the code will not be deposited.

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