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Infimum for General Relations
with Connections to Vector Optimization
and Conjugate Duality

by

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Declaration

I truthfully declare that I have prepared the thesis independently and have fully indicated all aids and sources used, that I have marked the passages taken over verbatim or in terms of content as such, and that I have complied with the KIT statutes on safeguarding good scientific practice.

Date

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Abstract

The concepts of infimum and supremum have been widely used in optimization, but their adoption in vector- and set-optimization has been fragmented. Existing definitions are catered to specific types of assumptions and relations, often characterized by domination structures. This divided landscape inhibits the comparison and transfer of approaches utilizing these concepts.

In this thesis, we propose a framework for defining an infimum operator irrespective of the underlying chosen notion of “smaller” and investigate its properties. We show how this framework is compatible with classical notions of vector optimization and compare it to specific infimum operators from select authors. We introduce the concept of *representativity* of minima and infima, and show how it relates to the concept of external stability of nondominated sets.

Lastly, we apply this framework to investigate the connections between two set-valued conjugate duality approaches, respectively based on supremum and maximum, that could previously not be compared. In this process, we highlight underlying assumptions, including those automatically satisfied by the most commonly used infimum operator by Nieuwenhuis [20].

Introduction and Motivation

During the research on conjugate duality and its subsequent generalizations to topological vector spaces, we repeatedly encountered a focus on weakly efficient solutions and on a generalization of the infimum, which we will call *weak infimum*, by Nieuwenhuis [20]. The advantages of weak infima became clear through the exposition in Löhne [16], but the drawbacks associated to other types of infima were not. A closer reading of Sawaragi et al. [25, Chapt. 6: Conjugate Duality] revealed that duality results were derived for efficient points using minima under some assumptions related to external stability in [25, Sect. 6.1], instead of infima without further assumptions, like for weakly efficient points in the following section [25, Sect. 6.2]. This approach was generalized by Boţ et al. [4, Chapt. 7]. A key observation motivating this work is that, for the usual (single dimensional) infimum and for the weak infimum:

$$\inf \bigcup_{i \in I} \inf Y^i = \inf \bigcup_{i \in I} Y^i,$$

for some collection $(Y^i)_{i \in I}$ of sets. This implies that, for a function $f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$:

$$\begin{aligned} \inf_{(x,y) \in \mathcal{X} \times \mathcal{Y}} f(x,y) &= \inf \bigcup_{(x,y) \in \mathcal{X} \times \mathcal{Y}} \{f(x,y)\} \\ &= \inf \bigcup_{x \in \mathcal{X}} \bigcup_{y \in \mathcal{Y}} \{f(x,y)\} \\ &= \inf \bigcup_{x \in \mathcal{X}} \inf \bigcup_{y \in \mathcal{Y}} \{f(x,y)\} = \inf_{x \in \mathcal{X}} \inf_{y \in \mathcal{Y}} f(x,y). \end{aligned}$$

The above is required to formulate strong conjugate duality for vector-valued, and more broadly set-valued problems. Following these observations, we used [25, Sect. 6.1.6] and [16] as a starting point for generalizing the infimum operator for arbitrary relations, on arbitrary spaces, with the aim of obtaining conditions on the collection $(Y^i)_{i \in I}$ of sets to ensure that

$$\text{Inf}_R \bigcup_{i \in I} \text{Inf}_R Y^i = \text{Inf}_R \bigcup_{i \in I} Y^i.$$

The associated property for the above to hold was named *representativity* (of the infimum, for the lower bounds of Y^i). This then allowed us to apply the same procedure as in Sawaragi et al. [25] (or in a more general setting, Boţ et al. [4]) for deriving conjugate duality results using infima, irrespective of underlying relation. In this context, we were able to investigate how the two conjugate duality approaches relate to each other, and what their underlying assumptions are.

In addition, this work provides a framework in which minima and infima are well defined and still correspond to the traditional literature on the topic, especially in regards to weak infima. By construction, attained infima are always minimal in the proposed generalization. The main studied properties were existence of infimal elements and the possibly new concept of representativity of infima and minima. Additional results under extended calculus rules were also given. Furthermore, general conjugate duality results were formulated for set-valued infimization problems, for which we also proposed two solution concepts.

Further research is specifically needed on the proposed nondominated infimum and to identify conditions under which infima are self-infimal. Regarding applications, we mostly focused on orderings induced by constant conical domination structures. An application to non-constant domination structures constitutes a possible direction for future work. Another potential direction would be to try and formulate a generalized infimum for properly nondominated points. Additionally, generalized conditions ensuring subdifferentiability and stability (in the context of strong duality) with respect to general relations are yet to be established.

Throughout this work, we use the symbols $\leq$ and $\geq$ instead of the usual $\leq$ and $\geq$, as done by several authors, *e.g.* [6], to stay consistent with the formulations typically used in vector optimization. In that regard, using the orderings $<$ and $=$ as used on $\mathbb{R}$, for $a, b \in \mathbb{R}^n$ and $I = \{1, \dots, n\}$, we define:

$$\begin{aligned} a \leq b &: \iff a_i < b_i \vee a_i = b_i, \forall i \in I, \\ a \leq b &: \iff a \leq b \wedge a \neq b, \forall i \in I, \\ a < b &: \iff a_i < b_i, \forall i \in I. \end{aligned}$$

With the above definitions, in the scalar case, $\leq = <$. Moreover, as in [4, 16], the symbol for set inclusion is here $\subseteq$, while strict inclusion is denoted $\subset$. This differs from several authors, especially in convex analysis, *e.g.* [1, 11, 19, 24, 25].

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1 Representing Relations by a Domination Structure

First introduced by Yu [29], the concept of domination structures has been widely adopted to represent relations on vector spaces. Many results in the field of vector optimization have been derived for constant conical domination structures, as a generalization of Pareto optimality. Despite this, some research has been conducted in the general framework (see eg. Eichfelder [7]).

We begin by introducing basic properties of binary relations and their connection to orderings. We then define domination structures and study the induced orderings, before focusing on constant conical domination structures. Finally, we derive results on cones and topological interiors, which will be used in Sect. 3.

1.1 Properties of Relations

Let $\mathcal{Y}$ be a set, not necessarily a vector space, and R a binary relation on $\mathcal{Y}$. The relation R can satisfy different properties on a set $Y \subseteq \mathcal{Y}$ [25]:

- (i) *transitivity* if $yRz \wedge zRw \Rightarrow yRw$,
- (ii) *negative transitivity* if $y\neg R z \wedge z\neg R w \Rightarrow y\neg R w$,
- (iii) *reflexivity* if yRy ,
- (iv) *irreflexivity* if $y\neg Ry$,
- (v) *symmetry* if $yRz \Rightarrow zRy$.
- (vi) *asymmetry* if $yRz \Rightarrow z\neg Ry$,
- (vii) *negative asymmetry* if $y\neg Rz \Rightarrow zRy$,
- (viii) *antisymmetry* if $yRw \wedge wRy \Rightarrow y = w$,

for all $y, z, w \in Y \subseteq \mathcal{Y}$, $y \neq z \neq w$ and $\neg R$ denoting the *binary complement* of R (i.e. “not R ”), itself a binary relation. One may also represent the relation R as a subset of $\mathcal{Y} \times \mathcal{Y}$. In that case, the set of pairs R contains all pairs $(y, z) \in \mathcal{Y} \times \mathcal{Y}$ for which yRz holds [9]. We collect the following results.

1.1 Lemma:

Let R be a binary relation on $Y \subseteq \mathcal{Y}$ and $\bar{R}$ its binary complement. Then:

- (i) R is transitive if and only if $\bar{R}$ is negatively transitive,
- (ii) R is reflexive if and only if $\bar{R}$ is irreflexive,
- (iii) R is asymmetric if and only if $\bar{R}$ is negatively asymmetric,
- (iv) R is symmetric if and only if $\bar{R}$ is symmetric.

Proof. Claims follow directly from the binary complement of $\bar{R}$ being R . $\square$

1.2 Definition:

Let R be a binary relation on $Y \subseteq \mathcal{Y}$. Then R is called

- (i) a partial ordering if it is transitive, reflexive and antisymmetric,
- (ii) a strict partial ordering if it is transitive, irreflexive and asymmetric,
- (iii) an equivalence relation if it is transitive, reflexive and symmetric.

The term *ordering* is sometimes used instead of partial ordering, *e.g.* by N. Bourbaki [19]. A set is called *partially ordered* whenever it is equipped with a partial ordering. One may also consider the *inverse* (or *opposite* [19]) relation $-R$ to R , defined as

$$y(-R)z :\iff zRy.$$

Since it is merely a notational variant of the same relation, any property of R is also a property of $-R$. Lastly, assuming $\mathcal{Y}$ is equipped with addition and scalar multiplication, we may want R to satisfy the following properties, with respect to a set $Y \subseteq \mathcal{Y}$: for any $y, z \in Y$, we say R is

- (i) *translation invariant* if $\forall w \in \mathcal{Y}$ with $w + y, w + z \in Y$:

$$yRz \iff (w + y)R(w + z),$$

- (ii) and *positively responsive* if for any scalar $\alpha > 0$ with $\alpha y, \alpha z \in Y$:

$$yRz \iff (\alpha y)R(\alpha z).$$

The following result is well known (see [19]) and plays a key role in using generalized infima and minima as solution sets of problems, in which R is a preference relation.

1.3 Proposition:

Let R be a binary relation on $Y \subseteq \mathcal{Y}$. If $yRz \wedge zRy, \forall y, z \in Y$, then R is an equivalence relation on Y .

Proof. From $yRz \wedge zRy, \forall y, z \in Y$, including for $y = z$, symmetry and reflexivity immediately hold. Then, since yRz, zRw, yRw hold per assumption for any $y, z, w \in Y$, R is also transitive on Y , and an equivalence relation by Def. 1.2. $\square$

In view of the above proposition, if a binary relation R is representing preferences, a subset on which R is an equivalence relation contains only elements that are equally preferred.

1.2 Domination Structures

Assume now that $\mathcal{Y}$ is a vector space (see Sect. 1.3.1). When dealing with sets, we use Minkowski summation, that is, for $A, B \subseteq \mathcal{Y}$,

$$A + B : \Longleftrightarrow \{a + b : a \in A, b \in B\}.$$

For translations by an element $a \in Y$ of a set $B \subseteq \mathcal{Y}$, we simply write

$$a + B := \{a\} + B.$$

Likewise, scalar multiplication is passed on to the set elements, *i.e.*

$$\alpha A := \{\alpha a : a \in A\}.$$

Binary relations can be characterized (or induced) by sets of rules. In higher dimensional real spaces, Yu [29] proposed such an approach to address structural specificities of competing preferences in multiobjective optimization. The original idea was then expanded and adapted to general vector spaces and has been widely adopted to characterize (weakly) nondominated points. The version we present leans on Sawaragi et al. [25].

Let $Y \subseteq \mathcal{Y}$. The *domination structure* $\mathcal{D}(y)$ on R is given by the set-valued *ordering map*:

$$\mathcal{D} : \mathcal{Y} \rightrightarrows \mathcal{Y}, y \mapsto \mathcal{D}(y) := \{d \in \mathcal{Y} : yR(y+d)\},$$

such that, using Minkowski summation,

$$\begin{aligned} yRz &\iff z \in y + \mathcal{D}(y) &\iff y \in z - \mathcal{D}(y), \\ y\not R z &\iff z \notin y + \mathcal{D}(y) &\iff y \notin z - \mathcal{D}(y). \end{aligned}$$

Essentially, the domination structure $\mathcal{D}(y)$ at y captures the deviations d from y for which $z := y + d$ satisfies yRz . In that case, we say y *dominates* z with respect to $\mathcal{D}$ (and thus to R). If $\mathcal{D}(y)$ is empty, y dominates no element in $\mathcal{Y}$ and if $\mathcal{D}(y) = \mathcal{Y}$, y dominates all elements of $\mathcal{Y}$. Using this alternative representation of R , we can characterize some of its properties.

1.4 Lemma:

Let R be a relation on $Y \subseteq \mathcal{Y}$ characterized by the ordering map $\mathcal{D}$. Then, for any $y \in Y$ and $d, d' \in Y - y$, the relation R is

- (i) *transitive*, iff $d \in \mathcal{D}(y) \wedge d' - d \in \mathcal{D}(y+d) \Rightarrow d' \in \mathcal{D}(y)$,
- (ii) *negatively transitive*, iff $d \notin \mathcal{D}(y) \wedge d' - d \notin \mathcal{D}(y+d) \Rightarrow d' \notin \mathcal{D}(y)$,
- (iii) *reflexive*, iff $0 \in \mathcal{D}(y)$,
- (iv) *irreflexive*, iff $0 \notin \mathcal{D}(y)$,
- (v) *symmetric*, iff $d \in \mathcal{D}(y) \Rightarrow -d \in \mathcal{D}(y+d)$,
- (vi) *asymmetric*, iff $d \in \mathcal{D}(y) \Rightarrow -d \notin \mathcal{D}(y+d)$,
- (vii) *negatively asymmetric*, iff $d \notin \mathcal{D}(y) \Rightarrow -d \in \mathcal{D}(y+d)$,
- (viii) *antisymmetric*, iff $d \in \mathcal{D}(y) \wedge -d \in \mathcal{D}(y+d) \Rightarrow d = 0$.

Proof. Observe that $yRz \Leftrightarrow z \in y + \mathcal{D}(y)$ and $y\not R w \Leftrightarrow w \notin y + \mathcal{D}(y)$, i.e. $z = y + d$ for some $d \in \mathcal{D}(y)$ and $w = y + d'$ with some $d' \notin \mathcal{D}$. Assuming, $d, d' \in Y - y$ guarantees that $z, w \in Y$. In the following, assume $y, z, w \in Y$ with $z := y + d$ and $w := y + d'$.

(i) Equivalence follows from definition $yRz \wedge zRw \Rightarrow yRw$ and

$$\begin{aligned} yRz \wedge zRw &\Longleftrightarrow (z \in y + \mathcal{D}(y)) \wedge (w \in z + \mathcal{D}(z)) \\ &\Longleftrightarrow (y + d \in y + \mathcal{D}(y)) \wedge (y + d' \in y + d + \mathcal{D}(y + d)) \\ &\Longleftrightarrow (d \in \mathcal{D}(y)) \wedge (d' - d \in \mathcal{D}(y + d)), \end{aligned}$$

along with

$$yRw \Longleftrightarrow (w \in y + \mathcal{D}(y)) \Longleftrightarrow (y + d' \in y + \mathcal{D}(y)) \Longleftrightarrow (d' \in \mathcal{D}(y)).$$

(iii) Follows from definition $yRy \Leftrightarrow y \in y + \mathcal{D}(y) \Leftrightarrow 0 \in \mathcal{D}(y)$.

(v) Follows from definition, $yRz \Leftrightarrow z \in y + \mathcal{D}(y) \Leftrightarrow d \in \mathcal{D}(y)$ and $zRy \Leftrightarrow y \in z + \mathcal{D}(z) \Leftrightarrow y \in y + d + \mathcal{D}(y + d) \Leftrightarrow -d \in \mathcal{D}(y + d)$.

(vi) Follows analogously to (v) by replacing zRy with $z\bar{R}y \Leftrightarrow -d \notin \mathcal{D}(y + d)$.

(viii) By the same reformulations as in (v), the assertion follows from $yRz \wedge zRy \Leftrightarrow (d \in \mathcal{D}(y)) \wedge (-d \in \mathcal{D}(y + d))$ and $y = z$ equivalent to $d = 0$.

Cases (ii), (iv) and (vii) then respectively follow from (i), (iii) and (vi) by replacing R with $\bar{R}$ and $\mathcal{D}(\cdot)$ by $(\mathcal{D}(\cdot))^c$. $\square$

The domination structure $\mathcal{D}(y)$ at any point y can be an arbitrary set. However, there are advantages to limiting oneself to less general structures. Consider the inverse relation $-R$. Then,

$$\forall a, b \in \mathcal{Y} : a(-R)b \Longleftrightarrow bRa \Longleftrightarrow a \in b + \mathcal{D}(b) \Longleftrightarrow b \in a + (-\mathcal{D}(b)),$$

which is equivalent to the relation characterized by the ordering map $-\mathcal{D}$ only when $\mathcal{D}(a) = \mathcal{D}(b), \forall a, b \in \mathcal{Y} : a(-R)b$. If we assume that $\mathcal{D}$ is constant with $\mathcal{D}(y) \equiv D, \forall y \in Y$, then $-R$ is immediately characterized by $-\mathcal{D}$. In many applications, one further assumes $\mathcal{D}(y) \equiv D$ is a constant cone. A set $K \subseteq \mathcal{Y}$ is called a *cone* if it is closed under positive scalar multiplication, *i.e.* $\alpha d \in K, \forall d \in K$ and $\alpha > 0$ (see Sect. 1.3.4 on cones). The following result shows the advantages of *constant conical domination structures*.

1.5 Proposition:

A relation R on $\mathcal{Y}$ characterized by the ordering map $\mathcal{D}$ is:

- (i) translation invariant if and only if $\mathcal{D}$ is constant, i.e. $\mathcal{D}(y) \equiv D$,
- (ii) positively responsive if and only if $\alpha\mathcal{D}(y) = \mathcal{D}(\alpha y), \forall y \in \mathcal{Y}, \alpha > 0$.

Both of the above properties are exactly satisfied whenever $\mathcal{D}(y) \equiv D$ is a constant cone, i.e. a constant conical domination structure.

Proof. We proceed to show the first two assertions:

- (i) For the first assertion, note that $yRz \Leftrightarrow z \in y + \mathcal{D}(y)$ and

$$(y+w)R(z+w) \iff (z+w) \in (y+w) + \mathcal{D}(y+w) \iff z \in y + \mathcal{D}(y+w),$$

for any appropriate $w \in \mathcal{Y}$ such that $y+w, z+w \in \mathcal{Y}$. If $\mathcal{D}$ is a constant map, then $\mathcal{D}(y) = \mathcal{D}(y+w) = D$ and R is translation invariant. If $\mathcal{D}$ is not constant, there exists some $y' \in \mathcal{Y}$ with $\mathcal{D}(y) \neq \mathcal{D}(y')$. Set $w := y' - y$ and $z := y + d$ for some $d \in \mathcal{Y}$. Assume there exists some $d \in \mathcal{D}(y) \setminus \mathcal{D}(y')$. Then,

$$z \in y + \mathcal{D}(y) \quad \text{and} \quad z \notin y + \mathcal{D}(y+w),$$

contradicting, by the above, translation invariance. Otherwise, if $\mathcal{D}(y) \setminus \mathcal{D}(y') = \emptyset$, then $\mathcal{D}(y) \subset \mathcal{D}(y')$ and $\mathcal{D}(y') \neq \emptyset$, as otherwise $\mathcal{D}(y) = \mathcal{D}(y') = \emptyset$ would hold, contradicting that the domination structures are different. Thus, we can choose $d \in \mathcal{D}(y') \setminus \mathcal{D}(y)$, which implies

$$z \notin y + \mathcal{D}(y) \quad \text{and} \quad z \in y + \mathcal{D}(y+w),$$

also contradicting translation invariance.

- (ii) Let $z - y = d \in \mathcal{D}(y)$. By assumption, for any $\alpha > 0$:

$$\alpha d = \alpha(z - y) \in \alpha\mathcal{D}(y) = \mathcal{D}(\alpha y) \iff (\alpha y)R(\alpha z),$$

showing positive responsiveness. Assume now that $\alpha\mathcal{D}(y) \neq \mathcal{D}(\alpha y)$ for some $y \in \mathcal{Y}$ and $\alpha > 0$. Assume there is some $d \in \mathcal{D}(y) : \alpha d \notin \mathcal{D}(\alpha y)$. Let $z := y + d$, implying yRz . Then, by

$$\alpha d = \alpha(z - y) \notin \mathcal{D}(\alpha y) \implies (\alpha y) \not R(\alpha z)$$

provides a contradiction to positive responsiveness. Else, if $\alpha\mathcal{D}(y) \subset \mathcal{D}(\alpha y)$, then $\mathcal{D}(\alpha y) \neq \emptyset$ must hold. We can thus choose some

$$d' \in \mathcal{D}(\alpha y) : \quad d' \notin \alpha\mathcal{D}(y) \iff \frac{d'}{\alpha} \notin \mathcal{D}(y),$$

and for $z := y + \frac{d'}{\alpha}$,

$$\alpha z \in \alpha y + \mathcal{D}(\alpha y) \iff (\alpha y)R(\alpha z).$$

However, $y \not R z$ holds from $z = y + \frac{d'}{\alpha} \notin y + \mathcal{D}(y)$ also presents a contradiction to positive responsiveness.

Lastly, whenever $\mathcal{D}(y) \equiv D$ is constant, the condition $\alpha D = D, \forall \alpha > 0$ is equivalent to D being closed under positive scalar multiplication and thus, by definition, a cone. $\square$

These conditions are sufficient but not necessary for R to be translation invariant and positively responsive on proper subsets of $\mathcal{Y}$, as the following example shows.

1.6 Example: Let $\mathcal{Y} = \mathbb{R}^2$ and $Y = \{y^1, y^2\}$ with $y^1 = (1, 0)^\top$ and $y^2 = (0, 1)^\top$. Then, let the domination map $\mathcal{D}$ on Y be defined as follows:

$$\begin{aligned} \mathcal{D}(y^1) &= \{y \in \mathbb{R}^2 : y_1, y_2 \in [0, 1]\}, \\ \mathcal{D}(y^2) &= \{y \in \mathbb{R}^2 : y_1, y_2 \in [-1/2, 1/2]\}. \end{aligned}$$

Since, for R the relation induced by $\mathcal{D}$ on Y ,

$$\begin{aligned} y^2 \notin y^1 + \mathcal{D}(y^1) = [1, 2] \times [0, 1] &\iff y^1 \not R y^2, \\ y^1 \notin y^2 + \mathcal{D}(y^2) = [-1/2, 1/2] \times [1/2, 3/2] &\iff y^2 \not R y^1, \end{aligned}$$

the relation is by definition both translation invariant and positively responsive, despite the domination structures being neither conical, nor constant.

In view of Prop. 1.5, constant conical domination structures, being necessary for inducing translation invariant and positively responsive relations on entire vector spaces, are especially interesting to consider. Furthermore, simple conditions on $\mathcal{D}(y) \equiv D$ induce properties of R on $\mathcal{Y}$ and any of its subsets.

1.7 Lemma:

Let R be a relation on $\mathcal{Y}$ characterized by the constant conical domination structure $\mathcal{D}(y) \equiv D \subseteq \mathcal{Y}$. Then R is

- (i) transitive iff D is convex,
- (ii) negatively transitive iff D^c is convex,
- (iii) reflexive iff $0 \in D$,
- (iv) irreflexive iff $0 \notin D$,
- (v) symmetric iff $D = -D$,
- (vi) asymmetric iff $D \cap (-D) = \emptyset$,
- (vii) negatively asymmetric iff $D^c \cap (-D)^c = \emptyset$,
- (viii) antisymmetric iff $D \cap (-D) = \{0\}$.

Proof. Observe that, for $y, z \in \mathcal{Y}$,

$$yRz \Leftrightarrow z - y \in D \iff y - z \in -D \iff z(-R)y$$

and $y \not R z \Leftrightarrow z - y \notin D$. Let $x, y, w \in \mathcal{Y}$:

- (i) Following Lem. 1.4 (i), let arbitrary $d, d' - d \in D$ be given. Then, by D being a convex cone, it is closed under addition (see Lem. 1.11) and $d + d' - d = d' \in D$, implying transitivity.

It remains to show that convexity of D is necessary. Assume D is not convex. Then $\exists d, d'' \in D$, such that

$$(1 - \alpha) \frac{d}{(1 - \alpha)} + \alpha \frac{d''}{\alpha} = d + d'' =: d' \notin D$$

for some $\alpha \in (0, 1)$ and $\frac{d}{(1 - \alpha)}, \frac{d''}{\alpha} \in D$. For any $y \in \mathcal{Y}$ we have $y + d, y + d'' \in \mathcal{Y}$, whence also $(y + d) + d'' = y + d' \in \mathcal{Y}$. Then, following Lem. 1.4 (i),

$$d \in \underbrace{\mathcal{D}(y)}_{=D} \quad \text{and} \quad \underbrace{d''}_{=d'-d} \in \underbrace{\mathcal{D}(y + d)}_{=D},$$

but not $d' \in \mathcal{D}(y)$, contradicting transitivity.

- (iii) Follows from Lem. 1.4 (iii).
- (v) From Lem. 1.4 (v), for any $d \in D$ we have $-d \in D$, whence D is closed under negative scalar multiplication, and $D = -D$. Assume now $D \neq -D$. Then $D \neq \emptyset$ must hold and since $d \in D \Leftrightarrow -d \in (-D)$, there must exist some $d \in D \setminus (-D)$ with $-d \in (-D) \setminus D$. Otherwise, $(-D) \setminus D$ must also be empty and $D \subseteq -D \subseteq D$ contradicts $D \neq -D$. Thus, for any $y \in \mathcal{Y}$ the point $y + d$ lies in $\mathcal{Y}$ and $-d \notin \mathcal{D}(y + d) = D$, contradicting symmetry.
- (vi) From Lem. 1.4 (vi), $-d \notin D, \forall d \in D$ and since $-d \in -D$, $D \cap (-D) = \emptyset$. The necessity follows similarly to case (v).
- (viii) Following Lem. 1.4 (viii), $-d, d \in D$ implying $d = 0$ is equivalent to $D \cap (-D) = \{0\}$.

Assume now that this does not hold. If $D \cap (-D) = \emptyset$, then asymmetry holds by case (vi), and R cannot be antisymmetric. Thus, assume $\exists d \in D \cap (-D)$ with $d \neq 0$. Then, for any $y \in \mathcal{Y}$ we have $y + d \in \mathcal{Y}$ such that $d \in \mathcal{D}(y) = D$ and $-d \in \mathcal{D}(y + d) = D$, by definition of the domination structure, but contradicting antisymmetry of R .

The remaining cases (ii), (iv) and (vii) respectively follow from (i), (iii) and (vi) by replacing R by $\bar{R}$ and D by D^c . $\square$

A similar result to Lem. 1.7 can be found in Rockafellar and Wets [24], deriving some of the above properties directly for a closed convex ordering cone. That a pointed convex cone containing zero induces an antisymmetric, transitive and reflexive relation is also derived by Göpfert et al. [9, Thm. 2.1.13].

1.8 Remark: In Lem. 1.7 (i) and (ii), neither transitivity nor negative transitivity of R on a proper subset $\emptyset \neq Y \subset \mathcal{Y}$ require D to be convex. This is especially the case when the dimension of D is greater than that of Y . This follows from the fact that elements contradicting the property may not exist in $Y \subset \mathcal{Y}$. The conditions are thus sufficient but not necessary.

If we assume $y + D \subseteq Y$ for any $y \in Y$, the equivalence holds. We proceed to show this for transitivity.

Proof. The first part is covered in Lem. 1.7 (i). It remains to show convexity of D is necessary. Assume D is not convex. Then $\exists d, d'' \in D$, such that $(1 - \alpha)\frac{d}{(1 - \alpha)} + \alpha\frac{d''}{\alpha} = d + d'' =: d' \notin D$ for some $\alpha \in (0, 1)$. For any $y \in Y$ we

have $d, d'' \in Y - y$ from $y + D \subseteq \mathcal{Y}$, such that $y + d, y + d'' \in Y$ whence also $(y + d) + d'' = y + d' \in Y$. Then, following Lem. 1.4 (i),

$$d \in \underbrace{\mathcal{D}(y)}_{=D} \quad \text{and} \quad \underbrace{d''}_{=d'-d} \in \underbrace{\mathcal{D}(y+d)}_{=D},$$

but not $d' \in \mathcal{D}(y)$, contradicting transitivity. $\square$

The analogous holds for negative transitivity. The condition $y + D \subseteq Y, \forall y \in Y$ is especially satisfied for $Y = \mathcal{Y}$. Similar observations can be made on symmetry, asymmetry, negative asymmetry and antisymmetry.

1.9 Proposition:

Let R be a relation on $\mathcal{Y}$ characterized by the constant conical domination structure $D \subseteq \mathcal{Y}$. Then R is

- (i) a partial ordering iff D is convex and $D \cap (-D) = \{0\}$,*
- (ii) a strict partial ordering iff D is convex and $D \cap (-D) = \emptyset$,*
- (iii) an equivalence relation iff D is a linear subspace of $\mathcal{Y}$.*

Proof. The first two claims follow directly from Def. 1.2 and Lem. 1.7. For the last assertion, by definition, D convex, $D = -D$ and $0 \in D$ must hold. The cone is thus closed under addition (being a convex cone), positive and negative scalarization (by $D = -D$) and contains zero, making it a linear subspace. We omit to show necessity. A complete proof can be found in Rockafellar and Wets [24, Prop. 3.8]. $\square$

Note that, in view of Remark 1.8, Prop. 1.9 gives sufficient conditions on D for its induced relation to satisfy the different required properties on any proper subset Y of $\mathcal{Y}$. Indeed, a relation defined on a specific subset of $\mathcal{Y}$ may not require the satisfaction of these conditions (see Prop. 1.3, where a simpler condition is given for an equivalence relation).

1.3 Some Properties of Cones and Topological Vector Spaces

In the following, we give overviews of *vector spaces*, *topological spaces* and their combination in the form of *topological vector spaces*. While these overviews are in no way exhaustive, they serve to derive different results, especially related to *cones*, that are required for Sect. 3.

1.3.1 Vector Spaces

Let $\mathcal{Y}$ be a set and $\mathbb{F}$ a field (*e.g.* the real or complex numbers $\mathbb{R}$ and $\mathbb{C}$). Let $\mathcal{Y}$ be equipped with an addition map $+$: $\mathcal{Y} \times \mathcal{Y} \rightarrow \mathcal{Y}$ and a scalar multiplication map $\cdot$: $\mathbb{F} \times \mathcal{Y} \rightarrow \mathcal{Y}$. Then, $(\mathcal{Y}, +, \cdot)$ is a *vector space* (also sometimes called *linear space* [11] and simply denoted $\mathcal{Y}$) if it satisfies (see *e.g.* [2] or [9, Def. 2.1.9]):

- (i) $(\mathcal{Y}, +)$ is an *abelian group*, *i.e.* $+$ satisfies associativity, commutativity, existence of a zero (or *null*) element and existence of additive inverses on $\mathcal{Y}$.
- (ii) Scalar multiplication is compatible with addition, *i.e.* $\cdot$ satisfies associativity, existence of a one (or *unity*) element and distributivity over both vector and scalar addition, that is:

$$\begin{aligned} \forall a, b \in \mathcal{Y}, \alpha, \beta \in \mathbb{F} : \quad & \alpha \cdot (a + b) = \alpha \cdot a + \alpha \cdot b, \\ & (\alpha + \beta) \cdot a = \alpha \cdot a + \beta \cdot a. \end{aligned}$$

Note that, whenever scalar multiplication is clearly implied, for any $a \in \mathcal{Y}$ and $\alpha \in \mathbb{F}$, we may write αa instead of $\alpha \cdot a$. Moreover, we have no distinct notation for the scalar zero and the additive zero. For most applications one characterizes relations on $\mathcal{Y}$ using cones, whence we must assume the existence of a partial order on $\mathbb{F}$. Otherwise, expressions like $\alpha > 0$ are not defined. For simplicity, we may assume $\mathbb{F} = \mathbb{R}$ and limit ourselves to *real vector spaces*.

1.3.2 Topological Spaces

Let $\mathcal{Y}$ be the underlying set of the vector space $(\mathcal{Y}, +, \cdot)$. A family $\mathcal{T}$ of subsets of $\mathcal{Y}$ is called a *topology* if

- (i) $\emptyset$ and $\mathcal{Y} \in \mathcal{T}$,

- (ii) $\mathcal{T}$ is closed under unions, *i.e.* $\bigcup_{U \in S \subset \mathcal{T}} U \in \mathcal{T}$,
- (iii) $\mathcal{T}$ is closed under finite intersections, *i.e.* $\bigcap_{U \in S \subset \mathcal{T}} U \in \mathcal{T}$ for any finite subset $S \subset \mathcal{T}$.

Then, $(\mathcal{Y}, \mathcal{T})$ forms a *topological space*. Essentially, a topology describes what sets are considered *open* in $\mathcal{Y}$, with respect to $\mathcal{T}$ (although one can also introduce topologies via closed sets, *e.g.* [18, Thm. 17.1]). The complement U^c of an open set $U \in \mathcal{T}$ is called *closed*. A topological space is usually written $(\mathcal{Y}, \mathcal{T})$, to specify the topology used. For simplicity, and since we only use one topology throughout, we simply write $\mathcal{Y}$ and omit $\mathcal{T}$.

Let $A \subseteq \mathcal{Y}$ be an arbitrary set. The *interior* and *closure* of A are respectively defined as the union of all open subsets of A and the intersection of all closed supersets of A , *i.e.*

$$\text{int } A := \bigcup_{O \in \mathcal{T}, O \subseteq A} O \quad \text{and} \quad \text{cl } A := \bigcap_{C^c \in \mathcal{T}, C \supseteq A} C.$$

The *boundary* of A is composed of all elements of the closure of A that are not from the interior of A , *i.e.* $\text{bd } A := \text{cl } A \setminus \text{int } A$. For any set A it holds that $\text{int } A \subseteq A \subseteq \text{cl } A$, with at least one strict inclusion. The set is called *open* whenever $\text{int } A = A$, and *closed* whenever $\text{cl } A = A$.

Let a be an arbitrary element of $\mathcal{Y}$. An open set $U \in \mathcal{T}$ is called an *open neighborhood* (or simply *neighborhood*) of a whenever $a \in U$. Depending on the topology, $\{a\}$ can be a neighborhood of a or not. For example, when the topology is *Hausdorff*, any finite set of elements, including singletons, is closed [18, Thm. 17.8]. Given a set $A \subseteq \mathcal{Y}$, following the above definitions, a lies in the interior of A exactly when there exists a neighborhood U of a with $a \in U \subseteq A$. Similarly, a lies in the closure of A exactly when any neighborhood of a also intersects A (think: “*The point a lies in all closed supersets of A , whence if a lies in an open set U , the complement of U being closed, U^c cannot be a superset of A .*”).

We now introduce the concepts of *nets* (as a generalization of sequences) and *continuity* in topological spaces (see Jahn [11, Def.1.29]).

To represent indices, let I be a nonempty set equipped with a reflexive and transitive relation $\preceq$ (a *preorder*). We call $(I, \preceq)$ (or just I when the ordering is obvious) *directed*, whenever any finite subset of I has an upper bound with respect to $\preceq$. Examples of directed sets and their upper bound for a finite subset S include:

- (i) $(\{1, \dots, 10\}, \leq)$, with $\max\{s : s \in S\}$,
- (ii) $(\mathbb{N} \times \mathbb{N}, \leq)$, with $(\max\{s_1 : s_1 \in S\}, \max\{s_2 : s_2 \in S\})$,
- (iii) Let I be generated by $i^0 \in I$ and $i^{j+1} \subset i^j$. Then $(I, \subseteq)$ is directed.

A *net*, defined as a map from a directed set I to $\mathcal{Y}$ with $I \mapsto (y_i)_{i \in I}$, is said to *converge* to some *limit* $\bar{y} \in \mathcal{Y}$, written $\lim_{i \in I} y_i = \bar{y}$, whenever, for any neighborhood U of $\bar{y}$, there is an index $j \in I$ for which $y_i \in U$ for all $j \preceq i \in I$.

Nets provide an alternative way to characterize closure and boundary points of sets. Let $(a_i)_{i \in I} \subseteq A$ be a net in A (that is, all elements of the net lie in A). Then $\bar{a} \in \text{cl } A$ if and only if it is the limit of a net in A , and lies on the boundary of A if the limit point is not from $\text{int } A$.

Given an additional topological space $(\mathcal{X}, \mathcal{S})$ and some map $f : (\mathcal{X}, \mathcal{S}) \rightarrow (\mathcal{Y}, \mathcal{T})$, we call f *continuous at* $x \in \mathcal{X}$, whenever, for any neighborhood $U \in \mathcal{T}$ of $f(x)$, there exists some neighborhood $V \in \mathcal{S}$ of x satisfying $f(V) \subseteq U$. The map f is called *continuous on* $\mathcal{X}$ whenever it is continuous at any $x \in \mathcal{X}$.

1.3.3 Topological Vector Spaces

If the vector space $(\mathcal{Y}, +, \cdot)$ and the topology $\mathcal{T}$ are such that addition and scalar multiplication are continuous (with respect to the adequate *product topologies*) on $\mathcal{Y} \times \mathcal{Y}$ and $\mathbb{F} \times \mathcal{Y}$, respectively, then $\mathcal{Y}$ is a *topological vector space* (in the following, we shall only focus on $\mathbb{F} = \mathbb{R}$ and *real topological vector spaces*). In that case, the topology is called *linear* and is translation invariant (see Aliprantis and Border [1] or Göpfert et al. [9, Def. 2.2.20]), *i.e.*

$$\forall y \in \mathcal{Y} : \quad U \in \mathcal{T} \iff y + U \in \mathcal{T}.$$

Since for $U \in \mathcal{T}$, whenever $0 \in U$, $y + U$ forms an open neighborhood around y , it makes sense to consider the set $\mathcal{T}_0$ of *open neighborhoods around zero*. An alternative characterization of the interior and closure of a set A is then given by (see Munkres [18, Thm. 17.5]):

$$\begin{aligned} \text{int } A &= \{a \in A : \exists U \in \mathcal{T}_0 \text{ such that } a + U \subseteq A\}, \\ \text{cl } A &= \{a \in A : (a + U) \cap A \neq \emptyset \text{ for any } U \in \mathcal{T}_0\}. \end{aligned}$$

The set of neighborhoods of zero in topological vector spaces has some advantages. Let U be a neighborhood of zero. Then, by continuity of addition

and scalar multiplication, for any $U \in \mathcal{T}_0$, the following holds (see Aliprantis and Border [1, Sect. 5.2]):

- (i) $\lambda U \in \mathcal{T}_0$ for any $\lambda \neq 0$,
- (ii) U is *absorbing*, i.e. for any $y \in \mathcal{Y}$ there exists some $\lambda_0 > 0$ such that $\lambda y \in U$ for any λ with $|\lambda| \leq \lambda_0$,
- (iii) there exists some *balanced* neighborhood of zero $V \in \mathcal{T}_0$ with $V \subseteq U$, i.e. for any $y \in V$ and $|\lambda| \leq 1$, we have $\lambda y \in V$,
- (iv) there exists some neighborhood of zero $W \in \mathcal{T}_0$ with $W \subseteq U$ and $W + W \subseteq U$.

The following result applies the above principles.

1.10 Lemma: [16, Prop. 1.24]

Let $A, B \subseteq \mathcal{Y}$ and $\text{int } B \neq \emptyset$. Then $A + \text{int } B = \text{cl } A + \text{int } B$.

Proof. From $A \subseteq \text{cl } A$, it follows that $A + \text{int } B \subseteq \text{cl } A + \text{int } B$. Let $y \in \text{cl } A + \text{int } B$. Then there exist $a \in \text{cl } A$ and $b \in \text{int } B$ with $y = a + b$.

We show $y \in A + \text{int } B$. Choose an open neighborhood of zero U such that $b + U \subseteq \text{int } B$, which must exist by $b \in \text{int } B$. Then, there exists a balanced neighborhood of zero $V \subseteq U$. From $a \in \text{cl } A$, $a + V$ must intersect A , i.e. $(a + V) \cap A \neq \emptyset$. Choose some $\bar{a} \in (a + V) \cap A$. Then, $\bar{a} - a \in V$ and by V being balanced, $a - \bar{a} \in V$, whence $a \in \bar{a} + V$ and $\bar{a} + V$ forms an open neighborhood of a . Thus,

$$y = a + b \in \bar{a} + b + V \subseteq A + \text{int } B,$$

from $b + V \subseteq b + U \subseteq \text{int } B$, proving the lemma. $\square$

For a (topological) vector space $\mathcal{Y}$, if a partial ordering R is translation invariant and positively responsive on $\mathcal{Y}$, then $(\mathcal{Y}, R)$ forms a *partially ordered (topological) vector spaces* [16, Def. 1.27]. The infimum of a set $Y \subseteq \mathcal{Y}$ is defined as the greatest lower bound of a partially ordered set Y and is unique [19, §1, Def. 6].

1.3.4 Cones in Topological Vector Spaces

A set $K \subseteq \mathcal{Y}$ is called a *cone* if it is closed under positive scalar multiplication, i.e. $\alpha d \in K, \forall d \in K$ and $\alpha > 0$ (some authors assume $\alpha \geq 0$, e.g. [11, 24]). Cones can, e.g. be generated by taking the *conical hull* of an arbitrary set $S \subseteq \mathcal{Y}$, denoted

$$\text{cone } S := \{\alpha s \in \mathcal{Y} : \alpha > 0, s \in S\}.$$

Then, $\text{cone } S$ is the smallest cone containing S (see e.g. [22, Cor. 2.6.3]). A cone K can satisfy two important properties:

- (i) *pointedness* if for any $d \in K \setminus \{0\} : d \notin -K$. Equivalent formulations include $K \cap -K \in \{\emptyset, \{0\}\}$ and $K \setminus \{0\} \cap -K \setminus \{0\} = \emptyset$.
- (ii) *acuteness* if there exists an open halfspace $H^\circ \subset \mathcal{Y}$ such that $\text{cl } K \subseteq H^\circ \cup \{0\}$.

Whenever K is convex, K is acute if and only if $\text{cl } K$ is pointed [22, 25]. Furthermore, convexity of K implies the following:

1.11 Lemma: [11, Lem. 1.11]

If $K \subseteq \mathcal{Y}$ is a convex cone, if and only if $K + K \subseteq K$.

Proof. The expression $K + K \subseteq K$ can be rewritten as $d + d' \in K, \forall d, d' \in K$. Let $K + K \subseteq K$ and $d, d' \in K$. Then for some $\alpha \in (0, 1)$, $\alpha d, (1 - \alpha)d' \in K$ whence $\alpha d + (1 - \alpha)d' \in K$, showing that K is convex. Else, assuming convexity of K , we may write $\frac{1}{2}(d + d') = \frac{1}{2}d + \frac{1}{2}d' \in K$ and by K a cone, $d + d' \in K$. $\square$

The previous lemma is essentially analogous to [22, Thm. 2.6]: K is a convex cone if and only if it is closed under addition and positive scalar multiplication. A set S is called *generating*, whenever $S - S = \mathcal{Y}$. This allows to decompose any $y \in \mathcal{Y}$ into a difference between some $s^1, s^2 \in S : y = s^1 - s^2$.

1.12 Lemma:

If $S \subseteq \mathcal{Y}$ is generating $\mathcal{Y}$, then for any $a, b \in \mathcal{Y} : (a + S) \cap (b + S) \neq \emptyset$.

Proof. Since S is generating, $a - b \in S - S = \mathcal{Y}$ and there exist some $s^1, s^2 \in S$ such that $a - b = s^2 - s^1$. Then, it directly follows that

$$a + s^1 = b + s^2 \in (a + S) \cap (b + S) \neq \emptyset.$$

□

Convex cones are also generating when they have interior points:

1.13 Lemma:

Let $K \subseteq \mathcal{Y}$ be a convex cone with $\text{int } K \neq \emptyset$. Then K is generating $\mathcal{Y}$.

Proof. Choose some $k \in \text{int } K$. Then, there exists an open neighborhood of zero $U \in \mathcal{T}_0$ such that $k + U \subseteq K$. Let $y \in \mathcal{Y}$. Then, there exists some scalar $t > 0$ large enough such that

$$\frac{1}{t}y \in U \implies k + \frac{1}{t}y \in k + U \subseteq K.$$

Since K is a convex cone, by Lem. 1.11 and $t > 0$:

$$tk, y \in K \xrightarrow{\text{Lem. 1.11}} tk + y \in K,$$

whence $y = (y + tk) - tk \in K - K$. By $y \in \mathcal{Y}$ being arbitrarily chosen, $\mathcal{Y} = K - K$ holds and K is generating $\mathcal{Y}$. □

The above essentially states that K is *full dimensional*. Indeed, $K - K$ spans an *affine space*, the *affine hull* of K , i.e. the smallest affine space in which K lies. The assumption of nonempty interior makes it so that the affine hull of K spans the entire space. For more details, see [22].

The following two results are partially analogous to [11, Lem. 1.12].

1.14 Lemma:

Let $K \subseteq \mathcal{Y}$ be a convex cone with $\text{int } K \neq \emptyset$ and $0 \in K$. Then $K + \text{int } K = \text{int } K$.

Proof. Since $0 \in K$, we have $\text{int } K \subseteq K + \text{int } K$. Let $a \in K$ and $b \in \text{int } K$. Then

$$d := a + b \in K + \text{int } K \subseteq K + K.$$

From $b \in \text{int } K$, there exists an open neighborhood of zero $U \in \mathcal{T}_0$ with $b + U \subseteq K$. Then, by Lem. 1.11 and K being a convex cone,

$$d + U = a + (b + U) \subseteq K + K \stackrel{\text{Lem. 1.11}}{\subseteq} K,$$

whence $d \in \text{int } K$ holds. $\square$

1.15 Lemma:

If some $K \subseteq \mathcal{Y}$ is a convex cone, then so is $\text{int } K$. If K is additionally pointed, then $K \setminus \{0\}$ and $\text{int } K$ are also pointed convex cones and do not contain zero.

Proof. We show the three statements separately:

- If $\text{int } K = \emptyset$, then it is trivially a cone, convex and pointed. Let $d^1, d^2 \in \text{int } K$. Then, there exist some open neighborhoods of zero U^1 and U^2 such that

$$d^i + U^i \subseteq K, \quad i = 1, 2.$$

By continuity of scalar multiplication in $\mathcal{Y}$, for $\alpha > 0$, αU^1 and αU^2 are also open neighborhoods of zero. Then, since K is a cone,

$$\alpha d^1 + \alpha U^1 = \alpha(d^1 + U^1) \subseteq \alpha K \subseteq K,$$

whence $\text{int } K$ is also a cone. The same holds for d^2 . Then, by Lem. 1.11, $\alpha d^1 + (1 - \alpha)d^2 \in K + K \subseteq K$ and

$$\begin{aligned} \alpha d^1 + (1 - \alpha)d^2 &\in \alpha(d^1 + U^1) + (1 - \alpha)(d^2 + U^2) \\ &= \alpha d^1 + (1 - \alpha)d^2 + (\alpha U^1 + (1 - \alpha)U^2). \end{aligned}$$

Since, by continuity of scalar multiplication and addition, $(\alpha U^1 + (1 - \alpha)U^2)$ is also an open neighborhood around zero,

$$\alpha d^1 + (1 - \alpha)d^2 \in \text{int } K.$$

This shows $\text{int } K$ is a convex cone.

- If K is also pointed, $K \cap -K \in \{\emptyset, \{0\}\}$. Then, by $\text{int } K \subseteq K$,

$$\text{int } K \cap -\text{int } K \subseteq K \cap -K \in \{\emptyset, \{0\}\},$$

and $\text{int } K$ is pointed. Assume now $0 \in \text{int } K$. Then, there exists an open neighborhood of zero $U \subseteq K$ which contains a balanced and absorbing neighborhood $V \in \mathcal{T}_0$. Choose $0 \neq d \in K$. Then, $\alpha d \in K$ for any $\alpha > 0$ and there exists some $\alpha_0 > 0$ for which $\alpha_0 d \in V$ (by V absorbing) and $-\alpha_0 d \in V \subseteq K$ (by V balanced). Hence,

$$0 \neq \alpha_0 d \in K \cap -K \notin \{\emptyset, \{0\}\},$$

contradicting pointedness of K . Thus $\text{int } K \cap -\text{int } K = \emptyset$ and $0 \notin \text{int } K$.

- If $K \setminus \{0\} = \emptyset$, then it is trivially a cone, convex and pointed. Let $d^1, d^2 \in K \setminus \{0\}$. Naturally, $\alpha d^1 = 0$ with $\alpha > 0$ if and only if $d^1 = 0$. Thus $K \setminus \{0\}$ is a cone. Now consider

$$d_\alpha := \alpha d^1 + (1 - \alpha)d^2 \in K,$$

for $\alpha > 0$. By pointedness of K , $K \setminus \{0\} \cap -K \setminus \{0\} = \emptyset$, making $K \setminus \{0\}$ also pointed. Assume $d_\alpha = 0$ for some $\alpha > 0$, the only point that could violate convexity. Then

$$-\alpha d^1 = (1 - \alpha)d^2 \in K \cap -K \implies d^2 = d^1 = 0,$$

by pointedness of K . Thus, $K \setminus \{0\}$ is a pointed convex cone, not containing zero.

□

That the *algebraic* interior of a convex cone is a cone can be found, (based on a slightly different definition of cones which assumes that zero is necessarily included), in Jahn [11, Lem. 1.12 (a)]. For topological vector spaces, by [11, Lem. 1.32], the algebraic and topological interiors of convex sets with nonempty interiors coincide.

2 Generalization of Minimum and Infimum

In the real scalar case, the *real* infimum of a set $Y \subseteq \mathbb{R}$ is defined as the *greatest lower bound* of Y in $\mathbb{R}$. We say $\hat{y}$ is the infimum of Y if it is:

-
- (i) a lower bound of Y , that is $\hat{y} \leq y, \forall y \in Y$, and
 - (ii) an upper bound to the lower bounds of Y , *i.e.* if for $l \in \mathbb{R} : l \leq y$ holds for all $y \in Y$, then $l \leq \hat{y}$.

When the infimum of a set is part of that set, it is also its minimum. In that case, we say the infimum is *attained*. Hence, $\hat{y}$ is the minimum of Y if it satisfies conditions (i), (ii) and $\hat{y} \in Y$. In $\mathbb{R}$, it can be shown that the infimum (and thus minimum) is unique. Furthermore, extending the space to $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ allows any set $Y \subseteq \mathbb{R}$ to have a well defined infimum. The existence of a minimum is not guaranteed. Analogous results and definitions hold for suprema and maxima.

We now use lower bounds to give an alternative definition of infima and minima. Rather than defining the infimum as a single element, we can view it as the set of *infimal elements* of Y . In the one-dimensional scalar case, it is then only a singleton. Let $A(Y)$ denote the set of lower bounds to Y , *i.e.* all $l \in \mathbb{R}$ satisfying (i), and $B(Y)$ the set of upper bounds to all lower bounds to Y , *i.e.* all $u \in \mathbb{R}$ satisfying (ii). We obtain:

$$\begin{aligned} A(Y) &= \{l \in \mathbb{R} : l \leq y, \forall y \in Y\}, \\ B(Y) &= \{u \in \mathbb{R} : l \leq u, \forall l \in A(Y)\}. \end{aligned}$$

Then, real minimal elements of Y are given by $A(Y) \cap B(Y) \cap Y$ and real infimal elements by $A(Y) \cap B(Y)$. In case there are no real infimal elements, we adopt the following convention:

- If there exist no real lower bounds to Y in $\mathbb{R}$, *i.e.* no $l \in \mathbb{R}$ satisfies condition (i) ($A(Y) = \emptyset$), then $-\infty$ may be considered as the only (thus greatest), non-real, lower bound.
- If there exist lower bounds ($A(Y) \neq \emptyset$) but no real upper bounds to the set of real lower bounds to Y , *i.e.* condition (ii) is not satisfied by any $u \in \mathbb{R}$ ($B(Y) = \emptyset$), we may consider $+\infty$ to be the greatest, non-real, lower bound of Y .

Furthermore, we extend the definitions by allowing $\pm\infty$ as formal elements of sets we take infima and minima over. Without changing the sets $A(Y)$ and $B(Y)$, we obtain an alternative definition of minima and infima of $Y \subseteq \overline{\mathbb{R}}$.

2.1 Definition: Minimum and Infimum

Let $Y \subseteq \overline{\mathbb{R}}$ and $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. The minimum of Y is given by

$$\{\min Y\} = A(Y) \cap B(Y) \cap Y,$$

and the infimum of Y by

$$\{\inf Y\} = \begin{cases} \{-\infty\} & \text{if } A(Y) = \emptyset, \\ \{+\infty\} & \text{if } A(Y) \neq \emptyset \text{ and } B(Y) = \emptyset, \\ A(Y) \cap B(Y) & \text{otherwise.} \end{cases}$$

That $A(Y) \cap B(Y)$ always forms a singleton is shown later in Example 3.13. Note that, despite accepting to take the minimum over $Y \subseteq \overline{\mathbb{R}}$, by $A(Y)$ and $B(Y)$ being subsets of $\mathbb{R}$, only real lower bounds are ever considered. Thus the minimum of Y always remains a subset of $\mathbb{R}$, while the infimum is a subset of the extended space:

$$\forall Y \subseteq \overline{\mathbb{R}} : \quad \min Y \subseteq \mathbb{R}, \quad \inf Y \subseteq \overline{\mathbb{R}}.$$

Other authors have chosen to not allow taking the minimum of a set in $\overline{\mathbb{R}}$ (e.g. [8, 16, 25]). We do so here for ease of use when combining minimum and infimum operators.

The chosen viewpoint on extrapolating the definition of infimum fits with the definition of infimum and minimum on $\mathbb{R}$ (and $\overline{\mathbb{R}}$) given at the start of this section and has several advantages. First, it treats $\pm\infty$ less as elements of a set Y , than as information about the set itself. In this regard, $\min Y$ remains a subset of $\mathbb{R}$ even if it is given some $Y \subseteq \overline{\mathbb{R}}$: $\min Y$ can be considered to extract a real (or more generally *existing*) greatest lower bound of Y , with disregard to the reason as to why it may not exist, while $\inf Y$, mapping to $\overline{\mathbb{R}}$, conveys information as to the reason $\inf Y \neq \min Y$. If $\inf Y = -\infty$, then Y has no lower bounds and is *unbounded from below*. Else, if $\inf Y = +\infty$, then the set is empty. The existence of a real infimum informs of an unattained limit point in Y , and would be closely related to local openness of that set. Secondly, uniquely using $\pm\infty$ in special cases of the infimum reduces the need for addressing arbitrary calculus rules, simplifying generalization. For instance, the conditions of $A(Y)$ and $B(Y)$ are such, that elements of the extended space $\overline{\mathbb{R}}$ are only ever compared to elements of $\mathbb{R}$.

Here, we have chosen the definition of infimum as the greatest lower bound of a set as a starting point, however, some authors have chosen different

approaches. One of these approaches, chosen by Gros [8], is to view the (real) infimum of Y as the minimum taken over some closure of the set Y . Indeed, in $\mathbb{R}$ it holds that:

$$\inf Y \cap \mathbb{R} = \min \text{cl } Y.$$

This property is very useful in many applications, but, in some cases, conflicts with $\min Y = \inf Y \cap Y$, *i.e.* that minimal elements are exactly attained infimal elements. This will be addressed in the case of infimum generalized for nondominated points at the end of Sect. 3.

We also want to address the general definition of infimum by N. Bourbaki [19, §1, Def. 6] on partially ordered sets $(\mathcal{Y}, R)$. Indeed, for some $Y \subseteq \mathcal{Y}$, the infimum of Y is the greatest lower bound of Y in the sense of R . The issue, is that this supposes R forms a partial ordering on $\mathcal{Y}$. Thus, our proposed approach distances itself from the classical definition, but remains compatible.

2.1 A General Approach to Lower Bounds, Infimum and Minimum

Let us generalize the concepts of lower bounds, infima and minima for an arbitrary binary relation R . Let $\bar{\mathcal{Y}} := \mathcal{Y} \cup \{-\infty_R, +\infty_R\}$ be the extension of the space $\mathcal{Y}$ by a globally smallest and a globally largest element with respect to R . That is,

$$\forall y \in \mathcal{Y} : \quad -\infty_R R y, \quad y R +\infty_R,$$

and we assume $\pm\infty_R \notin \mathcal{Y}$.

2.2 Definition: *General Lower Bounds*

Let $Y \subseteq \bar{\mathcal{Y}}$ and R be an arbitrary binary relation on $\mathcal{Y}$. Define the sets

$$\begin{aligned} A_R(Y) &:= \{l \in \mathcal{Y} : l R y, \forall y \in Y\}, \\ B_R(Y) &:= \{u \in \mathcal{Y} : l R u, \forall l \in A_R(Y)\}, \end{aligned}$$

both subsets of $\mathcal{Y}$. Then, $l \in \mathcal{Y}$ is called:

- lower bound of Y with respect to R if $l \in A_R(Y)$,
- infimal element of Y with respect to R if $l \in A_R(Y) \cap B_R(Y)$,
- minimal element of Y with respect to R if $l \in A_R(Y) \cap B_R(Y) \cap Y$.

2.3 Remark: The sets $A_R(Y)$ and $B_R(Y)$ are generalizations of the sets $A(Y)$ and $B(Y)$ from before. $A_R(Y)$ can be thought of as the set of existing lower bounds (i.e. in $\mathcal{Y}$) to Y with respect to R , whilst $B_R(Y)$ defines the set of existing upper bounds to the lower bounds of Y . Hence, an existing infimal element to Y with respect to R is a greatest lower bound in $\mathcal{Y}$ to Y , coherent with the scalar characterization.

2.4 Remark: The condition for some $u \in \mathcal{Y}$ being in $B_R(Y)$ is equivalent to:

$$\text{if } lRy, \forall y \in Y \text{ then } lRu \text{ must hold.}$$

This characterization was used by different authors [16, 20, 25].

Based on the above and Def. 2.1, we introduce the *generalized minimum* and *infimum operators* with respect to the arbitrary relation R .

2.5 Definition: Generalization of Minimum

Let $Y \subseteq \bar{\mathcal{Y}}$. The minimum of Y with respect to R is given by

$$\text{Min}_R Y := A_R(Y) \cap B_R(Y) \cap Y.$$

This definition of minimum is based on minimal elements being attained infimal elements from $\mathcal{Y}$. A more common characterization is given by Prop. 2.16.

2.6 Definition: Generalization of Infimum

Let $Y \subseteq \bar{\mathcal{Y}}$. The infimum of Y with respect to R is given by

$$\text{Inf}_R Y := \begin{cases} \{-\infty_R\} & \text{if } A_R(Y) = \emptyset, \\ \{+\infty_R\} & \text{if } A_R(Y) \neq \emptyset \text{ and } B_R(Y) = \emptyset, \\ A_R(Y) \cap B_R(Y) & \text{otherwise.} \end{cases}$$

2.7 Remark: $\text{Min}_R Y$ is always a subset of $\mathcal{Y}$, whilst $\text{Inf}_R Y$ is a subset of $\bar{\mathcal{Y}}$. This implies that, for instance, given some set $Y \subseteq \bar{\mathcal{Y}}$ with $-\infty_R \in Y$: $\text{Min}_R Y = \emptyset \neq \{-\infty_R\} = \text{Inf}_R Y$.

2.8 Remark: In the usual real scalar case, it can be shown that $\inf Y \neq \emptyset$ for any choice of $Y \subseteq \bar{\mathbb{R}}$ (see Example 3.13). This cannot be assumed generally. For some $Y \subseteq \bar{\mathcal{Y}}$, $\text{Inf}_R Y = \emptyset$ can hold. Let $\mathcal{Y} = \mathbb{R}^2$, $R = \leq$ and $Y = \{(\alpha, \alpha)^\top \in \mathbb{R}^2 : \alpha > 0\}$. Then $A_{\leq}(Y) = \{l \in \mathbb{R}^2 : l \leq 0\}$ and

$B_{\leq}(Y) = \{u \in \mathbb{R}^2 : 0 \leq u\}$. Since neither $A_{\leq}(Y)$ nor $B_{\leq}(Y)$ are empty, $\text{Inf}_{\leq} Y = A_{\leq}(Y) \cap B_{\leq}(Y) = \emptyset$.

The following result shows that any minimal element is also infimal.

2.9 Proposition:

Let $Y \subseteq \overline{\mathcal{Y}}$. Then $\text{Min}_R Y \subseteq \text{Inf}_R Y$.

Proof. If $A_R(Y) = \emptyset$ or if $A_R(Y) \neq \emptyset$ and $B_R(Y) = \emptyset$, $\text{Inf}_R Y = \{\pm\infty_R\}$ and $\text{Min}_R Y = A_R(Y) \cap B_R(Y) \cap Y = \emptyset$, whence, in both cases, $\text{Min}_R Y = \emptyset \subseteq \text{Inf}_R Y$ holds. Otherwise, $\text{Inf}_R Y = A_R(Y) \cap B_R(Y) \supseteq A_R(Y) \cap B_R(Y) \cap Y = \text{Min}_R Y$. $\square$

2.10 Remark: Since any infimal element is both an upper bound and a lower bound to all of the infimal elements, assuming R is reflexive, by $\hat{y}R\hat{y}'$, $\forall \hat{y}, \hat{y}' \in \text{Inf}_R Y$ and Prop. 1.3, R is an equivalence relation on $\text{Inf}_R Y$. The same applies to $\text{Min}_R Y$, since it is a subset of $\text{Inf}_R Y$, by Prop. 2.9.

The set of minimal elements of a set $Y \subseteq \mathcal{Y}$, not from the extended space, coincides with the set of its attained infimal elements. This is a defining property we aimed to keep in the generalization.

2.11 Proposition: Attained Infimal Elements

Let $Y \subseteq \overline{\mathcal{Y}}$ with $\text{Inf}_R Y \neq \{\pm\infty_R\}$ or simply $Y \subseteq \mathcal{Y}$. Then $\text{Min}_R Y = \text{Inf}_R Y \cap Y$.

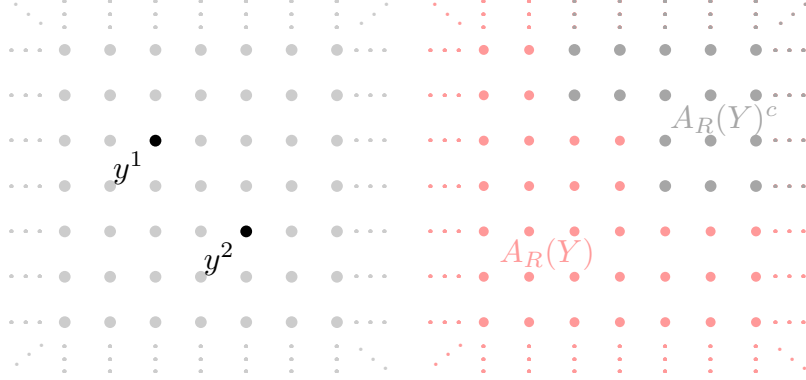
Proof. Whenever $\text{Inf}_R Y = A_R(Y) \cap B_R(Y)$, the result holds directly by Def. 2.5. It remains to show equality for the two other cases of $\text{Inf}_R Y$.

The case $\text{Inf}_R(Y) = \{-\infty_R\}$ only holds when $A_R(Y) = \emptyset$, and the case $\text{Inf}_R(Y) = \{+\infty_R\}$ only holds when $B_R(Y) = \emptyset$ and $A_R(Y) \neq \emptyset$. Both cases imply $A_R(Y) \cap B_R(Y) = \emptyset$ and $\text{Min}_R Y = \emptyset$. Since we have assumed $Y \subseteq \mathcal{Y}$, we obtain

$$\text{Inf}_R Y \cap Y = \{\pm\infty_R\} \cap Y = \emptyset = \text{Min}_R Y.$$

$\square$

2.12 Remark: Prop. 2.11 does not always hold for $Y \subseteq \overline{\mathcal{Y}}$, since $\text{Min}_R Y$ is only the set of all the existing greatest lower bounds of Y with respect to R , that is, lower bounds in $\mathcal{Y}$. See Remark 2.7 for an example.

Figure 2.1: Upper bounds of Y in $\mathbb{Z}^2$ in Example 2.13.

We formulate an example for the discrete space $\mathcal{Y} = \mathbb{Z}^2$. We omit to specify $\pm\infty_R$ and to show that the conditions are satisfied.

2.13 Example: Let $\mathcal{Y} = \mathbb{Z}^2$ and R be defined as follows: aRb holds whenever $a_1 < b_1$ or $a_2 \leq b_2$ or whenever $a = b$. Equivalently, we can use

$$aRb \iff a_1 \geq b_1 \wedge a_2 > b_2, \quad a, b \in \mathbb{Z}^2.$$

Then, given $Y = \{y^1, y^2\}$ with $y^1 = (0, 2)^\top$ and $y^2 = (2, 0)^\top$, we determine the infimum $\text{Inf}_R Y$ (see Fig. 2.1).

- The set $A_R(Y)$ contains all lower bounds of Y , that is

$$A_R(Y) = \{l \in \mathbb{Z}^2 : l_1 < y_1^i \vee l_2 \leq y_2^i \vee l = y^i, i = 1, 2\}.$$

To find this set, consider $A_R(Y)^c$, that is, all elements of $\mathbb{Z}^2$ that are not lower bounds to Y . For y^1 , the set of $v \in \mathbb{Z}^2 : vRy^1$ is given by

$$\{v : v_1 \geq 0 \wedge v_2 > 2\} = \{v : v_1 \geq 0, v_2 \geq 3\}.$$

For y^2 , analogously, the resulting set of $v \in \mathbb{Z}^2 : vRy^2$ is $\{v : v_1 \geq 2, v_2 \geq 1\}$. Thus, altogether, the set of elements in $\mathcal{Y} = \mathbb{Z}^2$ that are not lower bounds to Y is given by:

$$A_R(Y)^c = \{v : v_1 \geq 0, v_2 \geq 3\} \cap \{v : v_1 \geq 2, v_2 \geq 1\}.$$

This is represented in Fig. 2.2.

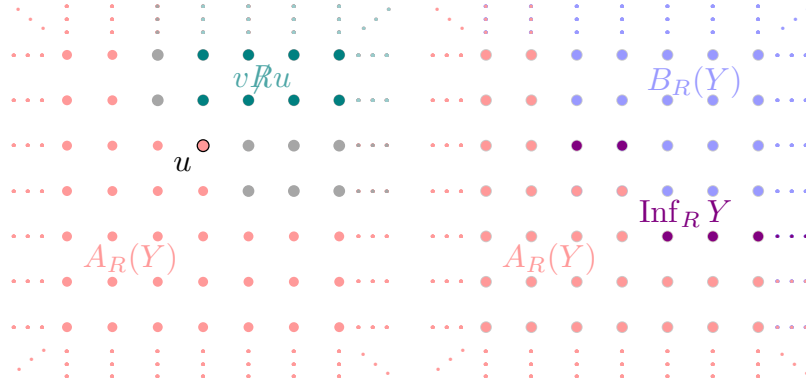


Figure 2.2: Verifying $u = (1, 2)^\top \in B_R(Y)$ and resulting infimum in Example 2.13.

- Proceeding with the upper bounds to $A_R(Y)$, we determine $B_R(Y)$ by looking at all elements u in $\mathcal{Y}$ for which all $v \in \mathbb{Z}^2 : v \not\leq u$ do not lie in $A_R(Y)$. Let us explicitly verify $u = (1, 2)^\top$. The set of v for which $v \not\leq u$ is given by $\{v : v_1 \geq 1, v_2 \geq 3\}$, which is clearly a subset of $A_R(Y)^c$. Since u is never not an upper bound to any $l \in A_R(Y)$, $u \in B_R(Y)$. Repeating this multiple times yields the solution for $B_R(Y)$ as seen in Fig. 2.2.
- Lastly, gathering the elements that lie both in $A_R(Y)$ (neither empty nor $\mathbb{Z}^2$) and $B_R(Y)$, yields the infimum

$$\text{Inf}_R Y = \{y^1, y^2, (1, 2)^\top\} \cup \{(\hat{y}_1, 0)^\top \in \mathbb{Z}^2 : \hat{y}_1 \geq 2\},$$

and the minimum $\text{Min}_R Y = \{y^1, y^2\}$. (See Fig. 2.2)

2.1.1 Reformulations of the Generalized Minimum and Infimum

We introduce two additional conditions on $\pm\infty_R$. First, we assume that $\mathcal{Y}$ does not contain globally smallest and greatest elements. That is,

$$\nexists l^*, u^* \in \mathcal{Y} \text{ such that, } \forall y \in \mathcal{Y} : \quad l^* R y, \quad y R u^*.$$

Assume some $l^* \in \mathcal{Y}$ exists with $l^* R y$ for all $y \in \mathcal{Y}$. Then, for any $Y \subseteq \overline{\mathcal{Y}}$ we have $l^* \in A_R(Y)$ and $\text{Inf}_R Y \neq \{-\infty_R\}$. Thus, there exists no set in the extended space that does not have a lower bound. Something similar happens whenever a globally greatest element u^* exists in $\mathcal{Y}$. Secondly, we

assume that no element of $\mathcal{Y}$ is smaller than $-\infty_R$, nor larger than $+\infty_R$, *i.e.*

$$\forall y \in \mathcal{Y} : \quad y \not R -\infty_R, \quad +\infty_R \not R y.$$

Assume some $\hat{y} \in \mathcal{Y}$ exists with $\hat{y} R +\infty_R$. Then, this would imply that the set $\{-\infty_R\}$ has a lower bound and fails to signal unboundedness from below. We must not assume whether or not $-\infty_R R -\infty_R$ and $+\infty_R R +\infty_R$ hold. Effects of these assumptions are discussed in Remark 2.15.

The next lemma relates $A_R(Y)$ to $B_R(Y)$ and shows that all $y \in Y \setminus \{\pm\infty_R\}$ are upper bounds to lower bounds of Y . It is then used to characterize minima with respect to R , under the assumptions on $\pm\infty_R$ discussed above.

2.14 Lemma:

Let $Y \subseteq \overline{\mathcal{Y}}$. Then

- (i) $Y \cap \mathcal{Y} \subseteq B_R(Y)$,
- (ii) $A_R(Y) = \emptyset \iff B_R(Y) = \mathcal{Y} \iff -\infty_R \in Y$,
- (iii) $A_R(Y) = \mathcal{Y} \iff B_R(Y) = \emptyset \iff Y \in \{\emptyset, \{+\infty_R\}\}$.

Proof. (i) Let some $y \in Y \cap \mathcal{Y}$. Then lRy holds for any $l \in A_R(Y)$, and thus $y \in B_R(Y)$. The same condition is trivially satisfied if $A_R(Y) = \emptyset$.

(ii) Let $A_R(Y) = \emptyset$, then the condition $lRu, \forall l \in A_R(Y)$ in $B_R(Y)$ is trivially satisfied for all $u \in \mathcal{Y}$. Else, if $B_R(Y) = \mathcal{Y}$, then lRu must hold $\forall u \in \mathcal{Y}$. This is by assumption only satisfied for a globally smallest element $-\infty_R$ of $\mathcal{Y}$ with respect to R , which lies not in $\mathcal{Y}$ and thus not in $A_R(Y)$. Hence $A_R(Y) = \emptyset$. Lastly, for $-\infty_R \in Y$, since $lR-\infty_R$ is not satisfied by any $l \in \mathcal{Y}$, $A_R(Y) = \emptyset$ and by the above $B_R(Y) = \mathcal{Y}$.

(iii) We show $A_R(Y) = \mathcal{Y} \Rightarrow B_R(Y) = \emptyset \Rightarrow Y \in \{\emptyset, \{+\infty_R\}\} \Rightarrow A_R(Y) = \mathcal{Y}$.

- Assume $A_R(Y) = \mathcal{Y}$. Then the condition $lRu, \forall l \in A_R(Y)$ of $B_R(Y)$ is only satisfied by $+\infty_R$. Since $B_R(Y) \subseteq \mathcal{Y}$ and $+\infty_R \notin \mathcal{Y}$, $B_R(Y) = \emptyset$.
- Assume $Y \notin \{\emptyset, \{+\infty_R\}\}$. If $-\infty_R \in Y$, (ii) implies $B_R(Y) = \mathcal{Y} \neq \emptyset$. If $-\infty_R \notin Y$, then $\emptyset \neq Y \subseteq \mathcal{Y}$ and $Y \cap \mathcal{Y} \neq \emptyset$. By (i), this

implies $B_R(Y) \neq \emptyset$. Inverting the implication, $B_R(Y) = \emptyset$ implies $Y \notin \{\emptyset, \{+\infty_R\}\}$.

- Let $Y \in \{\emptyset, \{+\infty_R\}\}$. Then the condition $lRy, \forall y \in Y$ of $A_R(Y)$ is trivially satisfied for any $l \in \mathcal{Y}$, as $+\infty_R$ is a globally largest element of $\mathcal{Y}$ with respect to R , whence $A_R(Y) = \mathcal{Y}$.

□

2.15 Remark: Note that both cases (ii) and (iii) of Lem. 2.14 require that $\mathcal{Y}$ does not contain any globally greatest nor smallest element. Furthermore, Lem. 2.14 (ii) requires that no element of Y is smaller than $-\infty_R$ with respect to R . Without these assumptions, one can however still obtain that $A_R(Y) = \mathcal{Y} \Rightarrow B_R(Y) = \emptyset$.

2.16 Proposition: Alternative Definition of Minimum

Let $Y \subseteq \overline{\mathcal{Y}}$. Then $\text{Min}_R Y = A_R(Y) \cap Y$.

Proof. From $A_R(Y) \subseteq \mathcal{Y}$, we have $A_R(Y) = A_R(Y) \cap \mathcal{Y}$ and by Lem. 2.14 (i):

$$\text{Min}_R Y = A_R(Y) \cap \underbrace{B_R(Y) \cap \overbrace{Y \cap \mathcal{Y}}^{\subseteq B_R(Y)}}_{=Y \cap \mathcal{Y}} = A_R(Y) \cap Y.$$

□

The following results give alternative conditions for the occurrence of $\{\pm\infty_R\}$ in $\text{Inf}_R Y$.

2.17 Proposition:

Let $Y \subseteq \overline{\mathcal{Y}}$. Then

$$(i) \text{ Inf}_R Y = \{-\infty_R\} \iff A_R(Y) = \emptyset \iff B_R(Y) = \mathcal{Y},$$

$$(ii) \text{ Inf}_R Y = \{+\infty_R\} \iff A_R(Y) = \mathcal{Y} \iff B_R(Y) = \emptyset.$$

Proof. By Def. 2.2 and Def. 2.6, $\text{Inf}_R Y = \{-\infty_R\}$ is equivalent to $A_R(Y) = \emptyset$ and $\text{Inf}_R Y = \{+\infty_R\}$ is equivalent to $A_R(Y) \neq \emptyset$ and $B_R(Y) = \emptyset$. By Lem. 2.14 (ii), $B_R(Y) = \mathcal{Y}$ if and only if $A_R(Y) = \emptyset$. By Lem. 2.14 (iii), $B_R(Y) = \emptyset$ implies $A_R(Y) = \mathcal{Y}$ and is thus a sufficient condition to verify. □

Prop. 2.17 also shows that infimal elements of Y may exist in $\mathcal{Y}$ if and only if $A_R(Y) \notin \{\emptyset, \mathcal{Y}\}$, or equivalently $B_R(Y) \notin \{\emptyset, \mathcal{Y}\}$. In this case, we may write

$$\text{Inf}_R Y = A_R(Y) \cap B_R(Y) \subseteq \mathcal{Y}.$$

However, there still may not exist any infimal element to Y , *i.e.* $\text{Inf}_R Y = \emptyset$ can still occur.

2.18 Corollary:

Let $Y \subseteq \overline{\mathcal{Y}}$. Then $\text{Inf}_R Y = \{+\infty_R\}$ if and only if $Y \in \{\emptyset, \{+\infty_R\}\}$.

Proof. Follows immediately from Lem. 2.14 (iii) and Prop. 2.17 (ii). $\square$

In order to use Def. 2.6 for a given space $\mathcal{Y}$ and relation R , one needs to be able to characterize the sets $A_R(Y)$ and $B_R(Y)$ for any $Y \in \overline{\mathcal{Y}}$. However, it may be beneficial to tackle $\pm\infty_R \in Y$ specifically.

2.19 Proposition:

Let $Y \subseteq \overline{\mathcal{Y}}$. Then

$$\begin{aligned} A_R(Y) &= \begin{cases} \emptyset & \text{if } -\infty_R \in Y \\ A_R(Y \setminus \{+\infty_R\}) & \text{otherwise,} \end{cases} \\ B_R(Y) &= \begin{cases} \mathcal{Y} & \text{if } -\infty_R \in Y \\ B_R(Y \setminus \{+\infty_R\}) & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. Let $-\infty_R \in Y$. Then, by Lem. 2.14 (ii), $A_R(Y) = \emptyset$ and $B_R(Y) = \mathcal{Y}$. Now let $-\infty_R \notin Y$. If $Y \subseteq \mathcal{Y}$ the assertions trivially hold. Thus, let $Y = Y' \cup \{+\infty_R\}$ for some $Y' \subseteq \mathcal{Y}$. Then some l lies in $A_R(Y)$ if it satisfies $lRy, \forall y \in Y'$ and $lR+\infty_R$. Since the latter is always true, it is sufficient to verify the former condition on Y' , whence $A_R(Y) = A_R(Y' \setminus \{+\infty_R\})$. The analogous holds immediately for $B_R(Y)$ by replacing $A_R(Y)$ by $A_R(Y' \setminus \{+\infty_R\})$ in its definition. $\square$

2.20 Corollary:

Let $Y \subseteq \overline{\mathcal{Y}}$. Then, for $Y' := Y \cap \mathcal{Y}$,

$$\text{Min}_R Y = \begin{cases} \emptyset & \text{if } -\infty_R \in Y, \\ \text{Min}_R Y' & \text{otherwise.} \end{cases}$$

Proof. From Prop. 2.16 and Prop. 2.19 we immediately have $\text{Min}_R Y = \emptyset$ if $-\infty_R \in Y$. Otherwise, for $-\infty_R \notin Y$,

$$\begin{aligned} \text{Min}_R Y &\stackrel{\text{Prop. 2.16}}{=} A_R(Y) \cap Y \\ &\stackrel{\text{Prop. 2.19}}{=} A_R(Y \setminus \{+\infty_R\}) \cap Y \\ &\stackrel{A_R(Y \setminus \{+\infty_R\}) \subseteq \mathcal{Y}}{=} A_R(Y \setminus \{+\infty_R\}) \cap (Y \setminus \{+\infty_R\}) \\ &\stackrel{\text{Prop. 2.16}}{=} \text{Min}_R(Y \setminus \{+\infty_R\}) = \text{Min}_R Y'. \end{aligned}$$

□

Next, we show an important property of the sets $A_R(Y)$ and $B_R(Y)$ that allows us to reuse the Min_R operator in the characterization of Inf_R . The following lemma implies that, for some set $Y \subseteq \overline{\mathcal{Y}}$, the lower bounds of the upper bounds to the lower bounds of Y are exactly the lower bounds of Y .

2.21 Lemma:

Let $Y \subseteq \overline{\mathcal{Y}}$. Then $A_R(B_R(Y)) = A_R(Y)$ and $B_R(B_R(Y)) = B_R(Y)$.

Proof. We first show $A_R(B_R(Y)) = A_R(Y)$. From Def. 2.2 we have $A_R(B_R(Y)) = \{l \in \mathcal{Y} : lRy, \forall y \in B_R(Y)\}$. Following Prop. 2.19, we may assume $+\infty_R \notin Y$. By Lem. 2.14 (ii), if $A_R(Y) = \emptyset$, then $B_R(Y) = \mathcal{Y}$. Since $A_R(\mathcal{Y}) = \emptyset$, the assertion holds for $A_R(Y) = \emptyset$. This case occurs especially when $-\infty_R \in Y$.

We may now assume $Y \subseteq \mathcal{Y}$ and $A_R(Y) \neq \emptyset$. By Lem. 2.14 (i), $Y \subseteq B_R(Y)$ such that, for any $l \in \mathcal{Y}$ with

$$lRy, \forall y \in B_R(Y) \implies lRy, \forall y \in Y,$$

and as such $A_R(B_R(Y)) \subseteq A_R(Y)$. For the reverse, let $l^* \in A_R(Y)$, i.e. $l^*Ry, \forall y \in Y$. Assume $l^* \notin A_R(B_R(Y))$. Then l^*Ru does not hold for some $u \in B_R(Y)$. Yet, since $u \in B_R(Y)$, lRu holds for all $l \in A_R(Y)$, including

l^* , contradicting the assumption. The second assertion follows from the first using Def. 2.2:

$$\begin{aligned} B_R(Y) &= \{u \in \mathcal{Y} : lRu, \forall l \in A_R(Y)\} \\ &\stackrel{A_R(B_R(Y))=A_R(Y)}{=} \{u \in \mathcal{Y} : lRu, \forall l \in A_R(B_R(Y))\} \\ &= B_R(B_R(Y)). \end{aligned}$$

□

Finally, we can give two alternative characterizations of the generalized infimum with respect to the arbitrary relation R , with Cor. 2.23 addressing $\pm\infty_R \in Y$ explicitly.

2.22 Proposition:

Let $Y \subseteq \bar{\mathcal{Y}}$. Then

$$\text{Inf}_R Y = \text{Inf}_R B_R(Y) = \begin{cases} \{-\infty_R\} & \text{if } B_R(Y) = \mathcal{Y}, \\ \{+\infty_R\} & \text{if } B_R(Y) = \emptyset, \\ \text{Min}_R B_R(Y) & \text{otherwise.} \end{cases}$$

Proof. From Def. 2.6 and Prop. 2.17, it immediately follows that

$$\text{Inf}_R Y = \begin{cases} \{-\infty_R\} & \text{if } B_R(Y) = \mathcal{Y}, \\ \{+\infty_R\} & \text{if } B_R(Y) = \emptyset, \\ A_R(Y) \cap B_R(Y) & \text{otherwise.} \end{cases} \quad (1)$$

By Lem. 2.21, $A_R(B_R(Y)) = A_R(Y)$ and $B_R(B_R(Y)) = B_R(Y)$ holds, that is, both Y and $B_R(Y)$ have the same lower bounds and upper bounds to lower bounds. Thus, we may substitute Y by $B_R(Y)$ in (1) and obtain $\text{Inf}_R Y = \text{Inf}_R B_R(Y)$.

It remains to show that $A_R(Y) \cap B_R(Y) = \text{Min}_R B_R(Y)$, whenever $\emptyset \neq B_R(Y) \neq \mathcal{Y}$. Using the characterization of Min_R from Prop. 2.16 and Lem. 2.21, for any $Y \subseteq \bar{\mathcal{Y}}$,

$$\begin{aligned} A_R(Y) \cap B_R(Y) &\stackrel{\text{Lem. 2.21}}{=} A_R(B_R(Y)) \cap B_R(Y) \\ &\stackrel{\text{Prop. 2.16}}{=} \text{Min}_R B_R(Y). \end{aligned}$$

□

2.23 Corollary:

Let $Y \subseteq \overline{\mathcal{Y}}$. Then, for $Y' := Y \cap \mathcal{Y}$,

$$\text{Inf}_R Y = \begin{cases} \{-\infty_R\} & \text{if } B_R(Y') = \mathcal{Y} \text{ or } -\infty_R \in Y, \\ \{+\infty_R\} & \text{if } Y' = \emptyset \text{ and } -\infty_R \notin Y, \\ \text{Min}_R B_R(Y') & \text{otherwise.} \end{cases}$$

Proof. We start from Prop. 2.22. By Prop. 2.19, $B_R(Y) = \mathcal{Y}$ if $-\infty_R \in Y$ and $B_R(Y) = B_R(Y')$ otherwise. In the latter case, $\text{Min}_R B_R(Y) = \text{Min}_R B_R(Y')$ holds and $\text{Inf}_R Y = \text{Inf}_R Y'$. Lastly, by Cor. 2.18, $\text{Inf}_R Y = \{+\infty_R\}$ exactly when $Y' = \emptyset$ and $-\infty_R \notin Y$. This completes the proof. $\square$

2.1.2 Properties of the Generalized Minimum and Infimum

Recall that the scalar infimum satisfies many desirable properties. For any $Y \subseteq \mathbb{R}$, the infimum always exists and is unique, *i.e.* $\text{Inf}_{\leq} Y = \{\inf Y\} \neq \emptyset$ with $\inf Y \in \overline{\mathbb{R}}$. Furthermore, both the minimum and infimum of any set Y are *self-minimal*, *i.e.* $\min \inf Y = \inf Y$ and $\min \min Y = \min Y$, and the infimum is even always *self-infimal*, *i.e.* $\inf \inf Y = \inf Y$.

We investigate how properties of the relation R and of the set Y , affect the generalized infimum $\text{Inf}_R Y$ with regards to existence and uniqueness of infimal and minimal elements, as well as self-minimality of infima and minima. Self-infimality will be addressed in the context of Sect. 2.2. We first gather a few results which will be useful for later proofs:

2.24 Lemma:

Let I be an index set, $Y, Y', Y_i \subseteq \overline{\mathcal{Y}}, \forall i \in I$ and $Y \subseteq Y'$. Then, the following holds:

- (i) $A_R(Y) \supseteq A_R(Y')$,
- (ii) $A_R(\bigcup_{i \in I} Y^i) = \bigcap_{i \in I} A_R(Y^i)$,
- (iii) $A_R(Y) \subseteq A_R(\text{Inf}_R Y) \subseteq A_R(\text{Min}_R Y)$,
- (iv) $B_R(Y) \subseteq B_R(Y')$,
- (v) $B_R(Y) \supseteq B_R(\text{Inf}_R Y) \supseteq B_R(\text{Min}_R Y)$,
- (vi) $A_R(Y) = A_R(Y')$ if and only if $B_R(Y) = B_R(Y')$.

Proof. (i) Let $l \in A_R(Y')$. Then $lRy, \forall y \in Y' \supseteq Y$ implies $lRy, \forall y \in Y$ and $l \in A_R(Y)$.

(ii) Follows directly from reformulation:

$$\begin{aligned} A_R(\bigcup_{i \in I} Y^i) &= \{l \in \mathcal{Y} : lRy, \forall y \in \bigcup_{i \in I} Y^i\} \\ &= \{l \in \mathcal{Y} : lRy, \forall y \in Y^i, \forall i \in I\} \\ &= \bigcap_{i \in I} \{l \in \mathcal{Y} : lRy, \forall y \in Y^i\} = \bigcap_{i \in I} A_R(Y^i). \end{aligned}$$

(iii) Since from Prop. 2.9 $\text{Min}_R Y \subseteq \text{Inf}_R Y$, case (i) yields $A_R(\text{Inf}_R Y) \subseteq A_R(\text{Min}_R Y)$. It remains to show that $A_R(Y) \subseteq A_R(\text{Inf}_R Y)$.

- If $\text{Inf}_R Y = \{-\infty_R\}$, by definition $A_R(Y) = \emptyset$ and by Prop. 2.19, $A_R(\text{Inf}_R Y) = \emptyset$.
- If $\text{Inf}_R Y = \{+\infty_R\}$ or $\text{Inf}_R Y = \emptyset$, then by Lem. 2.14 (iii) and Prop. 2.19, $A_R(\text{Inf}_R Y) = \mathcal{Y}$, which is a superset to any $A_R(Y)$.
- We may now assume $\text{Inf}_R Y = A_R(Y) \cap B_R(Y) \neq \emptyset$. Let $l^* \in A_R(Y)$, i.e. $l^*Ry, \forall y \in Y$. Assume $l^* \notin A_R(\text{Inf}_R Y)$, that is $\exists \hat{y} \in \text{Inf}_R Y : l^* \not R \hat{y}$. Yet, $\hat{y} \in A_R(Y) \cap B_R(Y)$, and especially in $B_R(Y)$, whence $lR\hat{y}, \forall l \in A_R(Y)$, including l^* and contradicting the assumption of $l^* \notin A_R(\text{Inf}_R Y)$.

This proves $A_R(Y) \subseteq A_R(\text{Inf}_R Y)$. With the first part, the inclusion chain is shown.

- (iv) Let $u \in B_R(Y)$. Then $lRu, \forall l \in A_R(Y) \supseteq A_R(Y')$ by (i) and implies $lRu, \forall l \in A_R(Y')$ and $u \in B_R(Y')$.
- (v) Follows from the same argument as the proof of (iv) utilizing case (iii).
- (vi) If $B_R(Y) = B_R(Y')$, by Lem. 2.21,

$$A_R(Y) = A_R(B_R(Y)) = A_R(B_R(Y')) = A_R(Y'),$$

and if $A_R(Y) = A_R(Y')$, $B_R(Y) = B_R(Y')$ holds by definition.

□

2.25 Proposition: *Effect of Irreflexivity*

If R is irreflexive on $\mathcal{Y}$, $\text{Min}_R Y = \emptyset$ for any $Y \subseteq \overline{\mathcal{Y}}$.

Proof. From Prop. 2.16, $\text{Min}_R Y = A_R(Y) \cap Y$. If $A_R(Y) = \emptyset$, then $\text{Min}_R Y = \emptyset$. Let $l \in A_R(Y)$. Then, lRy holds for all $y \in Y$ and by irreflexivity of R , $y \not R y$ for any $y \in Y$, whence $y \notin A_R(Y)$ and $\text{Min}_R Y = \emptyset$. □

Prop. 2.25 implies that, if R is irreflexive on $\mathcal{Y}$, for any set $Y \subseteq \overline{\mathcal{Y}}$, the minimum of Y with respect to R is empty. Moreover, since $B_R(Y) \subseteq \mathcal{Y}$, its minimum is also empty and by Prop. 2.22, the infimum of Y is either empty or $\{\pm\infty_R\}$.

It is important to note that reflexivity of R on $\mathcal{Y}$ is not sufficient to guarantee $\text{Inf}_R Y \neq \emptyset$ for any $Y \subseteq \overline{\mathcal{Y}}$. This will be prominent in specific formulations of the generalized infimum on a vector space $\mathcal{Y}$ in Sect. 3. In general, not much can be said about the necessary existence of minimal elements.

2.26 Proposition: *Effect of Antisymmetry*

If R is antisymmetric and $Y \subseteq \overline{\mathcal{Y}}$, $\text{Inf}_R Y$ is either empty or a singleton.

Proof. The cases $\text{Inf}_R Y \in \{\emptyset, \{+\infty_R\}, \{-\infty_R\}\}$ trivially satisfy the assertion. Hence, we must only show that whenever $\exists \hat{y} \in \text{Inf}_R Y = A_R(Y) \cap B_R(Y) : \text{Inf}_R Y = \{\hat{y}\}$. Assume $\hat{y}, \hat{y}' \in \text{Inf}_R Y$. Then, similarly to Remark 2.10, $\hat{y}R\hat{y}'$ from $\hat{y} \in A_R(Y), \hat{y}' \in B_R(Y)$ and vice versa. Hence

$$\hat{y}R\hat{y}' \wedge \hat{y}'R\hat{y} \xrightarrow{R \text{ antisymmetric}} \hat{y} = \hat{y}',$$

proving that $\text{Inf}_R Y$ is a singleton. □

Note that, by Prop. 2.9, a unique infimum implies that its corresponding minimum is either exactly a singleton of the attained infimal element or empty. Thus, Prop. 2.26 implies that under antisymmetry, infima are unique, provided they exist, and so are minima, provided their corresponding infimum is attained.

We now address self-minimality and -infimality.

2.27 Definition: Self-Minimality and -Infimality

A set $Y \subseteq \overline{\mathcal{Y}}$ is called self-minimal (with respect to R) if $\text{Min}_R Y = Y$, and self-infimal (with respect to R) if $\text{Inf}_R Y = Y$.

By Prop. 2.11, whenever $Y \subseteq \mathcal{Y}$, self-infimality implies self-minimality: the infimal elements of Y are exactly Y and are all attained. We may show how minima are always self-minimal.

2.28 Proposition: Self-Minimality of Minima

For any $Y \subseteq \overline{\mathcal{Y}}$, $\text{Min}_R Y$ is self-minimal with respect to R .

Proof. We must show $\text{Min}_R \text{Min}_R Y = \text{Min}_R Y$. By $A_R(Y) \subseteq A_R(\text{Min}_R Y)$ from Lem. 2.24 (iii) and Prop. 2.16 we obtain

$$\text{Min}_R \text{Min}_R Y = \underbrace{A_R(\text{Min}_R Y)}_{\supseteq A_R(Y)} \cap \underbrace{\text{Min}_R Y}_{= A_R(Y) \cap Y} = A_R(Y) \cap Y = \text{Min}_R Y.$$

□

The *restricted infimum* $\text{Inf}_R Y \cap \mathcal{Y} = A_R(Y) \cap B_R(Y)$ is also self-minimal. Whenever $\text{Inf}_R Y \neq \{\pm\infty_R\}$, $\text{Inf}_R Y \cap \mathcal{Y} = \text{Inf}_R Y$. Then, self-minimality follows from Prop. 2.22 and self minimality of the minimum:

$$\begin{aligned} \text{Min}_R \text{Inf}_R Y &\stackrel{\text{Prop. 2.22}}{=} \text{Min}_R \text{Min}_R B_R(Y) \\ &\stackrel{\text{Prop. 2.28}}{=} \text{Min}_R B_R(Y) \stackrel{\text{Prop. 2.22}}{=} \text{Inf}_R Y. \end{aligned}$$

Self-minimality does not hold in general for the infimum. The assumption $\text{Inf}_R Y \neq \{\pm\infty_R\}$ is required to compensate for the minimum operator ignoring elements from the extended space $\overline{\mathcal{Y}}$. Indeed, given $\text{Inf}_R Y = \{-\infty_R\}$ for some $Y \subseteq \overline{\mathcal{Y}}$:

$$\text{Min}_R \text{Inf}_R Y \stackrel{\text{Cor. 2.20}}{=} \emptyset \neq \{-\infty_R\} = \text{Inf}_R Y.$$

The same also holds for $\text{Inf}_R Y = \{+\infty_R\}$.

2.2 On Representativity

While infima gather the “greatest lower bounds” of a given set, one may be interested in knowing how much information about the lower bounds is captured. The question may be:

“With regards to the lower bounds of Y , is it sufficient to consider only the lower bounds of $\text{Inf}_R Y$?”

Taking inspiration from Komuro [14], we define an equivalence class for sets that have the same lower bounds with respect to R .

2.29 Definition: *Representativity*

For any $Y \subseteq \overline{\mathcal{Y}}$, the class of representative sets of (lower bounds of) $\mathcal{Y}$ (with respect to R) is given by

$$[Y] := \{Y' \subseteq \overline{\mathcal{Y}} : A_R(Y') = A_R(Y)\}.$$

Any set $Y' \in [Y]$ is called *representative* of $[Y]$ (with respect to R).

Whenever it is clear with respect to what relation and set Y representativity is meant, it is sufficient to call $Y' \in [Y]$ simply *representative*. In this sense, whenever $\text{Min}_R Y \in [Y]$ or $\text{Inf}_R Y \in [Y]$, we may call the respective sets simply *representative*. We immediately observe the following:

2.30 Lemma:

Let $Y^1, Y^2 \subseteq \overline{\mathcal{Y}}$. Then, $B_R(Y^1) \in [Y^1]$ always holds and the following are equivalent:

- (i) $Y^1 \in [Y^2]$,
- (ii) $Y^2 \in [Y^1]$,
- (iii) $A_R(Y^1) = A_R(Y^2)$,
- (iv) $B_R(Y^1) = B_R(Y^2)$.

Moreover, any of the above implies $\text{Inf}_R Y^1 = \text{Inf}_R Y^2$.

Proof. $B_R(Y^1) \in [Y^1]$ is equivalent to $A_R(B_R(Y^1)) = A_R(Y^1)$, which holds by Lem. 2.21. Cases (i), (ii) and (iii) are all equivalent by Def. 2.29. By

Lem. 2.24 (vi), they are also equivalent to (iv). Lastly, whenever Y^1 and Y^2 have the same sets of lower bounds (case (iii)) and of upper bounds to their lower bounds (case (iv)), by definition, they share the same infimum $\text{Inf}_R Y^1 = \text{Inf}_R Y^2$. $\square$

Lem. 2.30 shows how all representative sets of Y share the same lower bounds, upper bounds to lower bounds and infimum. Representative sets do however not always share the same minimum. It is also easy to see that, whenever an infimum is representative, it is also self-infimal. Representativity is however only a sufficient condition for self-infimality. Since we may obtain the same infimum from different sets of lower bounds, if $\text{Inf}_R Y$ is representative for some $[Y']$ where $\text{Inf}_R Y = \text{Inf}_R Y'$, self infimality holds despite $\text{Inf}_R Y$ not being representative for Y (see Example 3.3). Thus, self-infimality of $\text{Inf}_R Y$ can be reduced to asking, whether or not there exists such a set Y' with $\text{Inf}_R Y = \text{Inf}_R Y' \in [Y']$. This existence question appears difficult to answer in this general setting.

We now focus on representative minima and infima. Consider a representative infimum of some $Y \subseteq \overline{\mathcal{Y}}$, i.e. $\text{Inf}_R Y \in [Y]$. Then, $\text{Inf}_R Y$ being representative of Y in the sense of its lower bounds (i.e. $A_R(Y) = A_R(\text{Inf}_R Y)$) can be understood as the infimum encapsulating all relevant lower bounds of Y , such that the infimum of Y is sufficient to characterize all lower bounds of Y . The analogous holds for a representative minimum $\text{Min}_R Y \in [Y]$. Using Lem. 2.24 (iii), it is easy to see how representativity of a minimum implies representativity of the associated infimum.

2.31 Proposition:

For any $Y \subseteq \overline{\mathcal{Y}}$, if $\text{Min}_R Y \in [Y]$, then $\text{Inf}_R Y \in [Y]$.

Proof. Using Lem. 2.24 (iii) and $\text{Min}_R Y \in [Y]$ we obtain

$$A_R(Y) \subseteq A_R(\text{Inf}_R Y) \subseteq A_R(\text{Min}_R Y) = A_R(Y),$$

implying $A_R(\text{Inf}_R Y) = A_R(Y)$, i.e. $\text{Inf}_R Y \in [Y]$. $\square$

2.32 Remark: *Note how infimal elements must not be attained for the minimum to be representative. In Example 2.13, the set Y is self-minimal, whence $\text{Min}_R Y$ is representative of Y , even though not all infimal points are attained.*

In the following subsections, we investigate conditions for representativity of minima and infima and show applicability for taking minima and infima over unions of sets.

2.2.1 Sufficient and Necessary Conditions for Representativity

Minima and infima are not always representative. In fact, empty infima are never representative.

2.33 Lemma: *Representativity of Empty Infima*

For any $Y \subseteq \overline{\mathcal{Y}}$, if $\text{Inf}_R Y = \emptyset$, then $\text{Inf}_R Y \notin [Y]$.

Proof. If $\text{Inf}_R Y = \emptyset$, by Lem. 2.14 (iii), $A_R(\text{Inf}_R Y) = \mathcal{Y} \neq A_R(Y)$, as $A_R(Y) = \mathcal{Y}$ implies $\text{Inf}_R Y = \{+\infty_R\}$ by Prop. 2.17 (ii), contradicting the assumption. This disproves representativity. $\square$

Regarding the absence of minimal elements, we can make the following observation:

2.34 Lemma: *Representativity of Empty Minima*

For any $Y \subseteq \overline{\mathcal{Y}}$, $\text{Min}_R Y = \emptyset \in [Y]$ holds if and only if $Y \in \{\emptyset, \{+\infty_R\}\}$.

Proof. The case $\text{Min}_R Y = \emptyset \in [Y]$ is, by Lem. 2.14 (iii), equivalent to

$$A_R(Y) = A_R(\text{Min}_R Y) = A_R(\emptyset) \stackrel{\text{Lem. 2.14 (iii)}}{=} \mathcal{Y},$$

which holds exactly for $Y \in \{\emptyset, \{+\infty_R\}\}$, proving the assertion. $\square$

We derive conditions on the relation R and the set Y for $\text{Inf}_R Y$ and $\text{Min}_R Y$ to be representative. The following result assumes only conditions on Y .

2.35 Proposition:

For $Y \subseteq \overline{\mathcal{Y}}$, $\text{Inf}_R Y \in [Y]$ if and only if $\text{Inf}_R Y = \{\pm\infty_R\}$ or Y satisfies

$$A_R(A_R(Y) \cap B_R(Y)) = A_R(Y). \quad (2)$$

Proof. We go through all the possible cases of $\text{Inf}_R Y$ for an arbitrary $Y \subseteq \overline{\mathcal{Y}}$ and show $A_R(\text{Inf}_R Y) = A_R(Y)$:

- Let $\text{Inf}_R Y = \{-\infty_R\}$. Then, following Def. 2.6, $A_R(Y) = \emptyset$ and by Prop. 2.19 $A_R(\text{Inf}_R Y) = \emptyset$.
- Let $\text{Inf}_R Y = \{+\infty_R\}$. Then, by Cor. 2.18, $Y \in \{\emptyset, \{+\infty_R\}\}$ such that, by Lem. 2.14 (iii), $A_R(\text{Inf}_R Y) = A_R(Y) = \mathcal{Y}$.
- We may now assume $\text{Inf}_R Y = A_R(Y) \cap B_R(Y)$. Then, by (2), $A_R(\text{Inf}_R Y) = A_R(A_R(Y) \cap B_R(Y)) = A_R(Y)$.

For the reverse, assume $\text{Inf}_R Y \subseteq \mathcal{Y}$ and $A_R(A_R(Y) \cap B_R(Y)) \neq A_R(Y)$. Then, by $\text{Inf}_R Y = A_R(Y) \cap B_R(Y)$, $A_R(\text{Inf}_R Y) \neq A_R(Y)$ and $\text{Inf}_R Y \notin [Y]$. This completes the proof. $\square$

Prop. 2.35 shows how representativity of the infimum fails whenever Y has lower bounds that are not lower bounds of $\text{Inf}_R Y$. Under additional assumptions on the relation R , one can derive specific necessary and sufficient conditions for representativity to hold.

2.36 Theorem: *Representativity under Transitivity*

Let R be transitive and $Y \subseteq \overline{\mathcal{Y}}$. Then

- (i) $\text{Inf}_R Y \in [Y]$ if and only if $\text{Inf}_R Y \neq \emptyset$,
- (ii) $\text{Min}_R Y \in [Y]$ if and only if $\text{Min}_R Y \neq \emptyset$ or $Y \in \{\emptyset, \{+\infty_R\}\}$.

Proof. Let $Y \subseteq \overline{\mathcal{Y}}$ and assume R is transitive on $\mathcal{Y}$. Whenever $\text{Inf}_R Y = \{\pm\infty_R\}$, $\text{Inf}_R Y \in [Y]$ by Prop. 2.35 and by Lem. 2.33, whenever $\text{Inf}_R Y = \emptyset$, the infimum is not representative. Moreover, if $\text{Min}_R Y = \emptyset$, by Lem. 2.34, an empty minimum is representative if and only if $Y \in \{\emptyset, \{+\infty_R\}\}$. It remains to show respective representativity for $\emptyset \neq \text{Inf}_R Y \subseteq \mathcal{Y}$ and $\text{Min}_R Y \neq \emptyset$.

In both cases, we may further assume, by Prop. 2.17, $A_R(Y), B_R(Y) \notin \{\emptyset, \mathcal{Y}\}$ and, by Lem. 2.14 (ii) and (iii), $-\infty_R \notin Y \notin \{\emptyset, \{+\infty_R\}\}$.

- (i) Assume $\emptyset \neq \text{Inf}_R Y \subseteq \mathcal{Y}$ whence, by definition, $\text{Inf}_R Y = A_R(Y) \cap B_R(Y) \neq \emptyset$. We show $A_R(\text{Inf}_R Y) = A_R(Y)$. As $A_R(\text{Inf}_R Y) \supseteq A_R(Y) \neq \emptyset$ holds from Lem. 2.24 (iii), it remains to show $A_R(\text{Inf}_R Y) \subseteq A_R(Y)$. Let $l^* \in A_R(\text{Inf}_R Y)$, i.e. $l^* R \hat{y}, \forall \hat{y} \in \text{Inf}_R Y$. Choose some

$\widehat{y} \in \text{Inf}_R Y$, existing by assumption. Then $\widehat{y} \in A_R(Y)$ such that $\widehat{y} R y, \forall y \in Y$. Then, by transitivity of R ,

$$l^* R \widehat{y} \wedge \widehat{y} R y \xrightarrow{R \text{ transitive}} l^* R y, \quad \forall y \in Y,$$

whence $l^* \in A_R(Y)$, showing $A_R(Y) = A_R(\text{Inf}_R Y)$, *i.e.* $\text{Inf}_R Y \in [Y]$.

- (ii) Let $\text{Min}_R Y := A_R(Y) \cap B_R(Y) \cap Y \neq \emptyset$, following Def. 2.5. As $A_R(\text{Min}_R Y) \supseteq A_R(Y) \neq \emptyset$ holds from Lem. 2.24 (iii), it remains to show that $A_R(\text{Min}_R Y) \subseteq A_R(Y)$. Let $l^* \in A_R(\text{Min}_R Y)$, *i.e.* $l^* R \widehat{y}, \forall \widehat{y} \in \text{Min}_R Y$. Choose some $\widehat{y} \in \text{Min}_R Y$, existing by assumption. Then $\widehat{y} \in A_R(Y)$ such that $\widehat{y} R y, \forall y \in Y$. Then, by transitivity of R ,

$$l^* R \widehat{y} \wedge \widehat{y} R y \xrightarrow{R \text{ transitive}} l^* R y, \quad \forall y \in Y,$$

whence $l^* \in A_R(Y)$, showing $A_R(Y) = A_R(\text{Min}_R Y)$, *i.e.* $\text{Min}_R Y \in [Y]$.

This concludes the proof. $\square$

The key assumption of Thm. 2.36 (i), besides transitivity of R , is the nonempty-ness of the infimum. An example of empty infimum for a transitive and reflexive relation on $\mathbb{R}^3$ is given in Example 3.15.

2.37 Theorem: *Representativity under Negative Transitivity*

Let R be negatively transitive and $Y \subseteq \overline{\mathcal{Y}}$. Then

- (i) $\text{Inf}_R Y \in [Y]$ if and only if $\text{Inf}_R Y = \{\pm\infty_R\}$ or

$$\forall y \in Y \setminus \text{Inf}_R Y : \exists \widehat{y} \in \text{Inf}_R Y : y \not R \widehat{y}, \quad (3)$$

- (ii) $\text{Min}_R Y \in [Y]$ if and only if $Y \in \{\emptyset, \{+\infty_R\}\}$ or

$$\forall y \in Y \setminus \text{Min}_R Y : \exists \widehat{y} \in \text{Min}_R Y : y \not R \widehat{y}. \quad (4)$$

Proof. Let $Y \subseteq \overline{\mathcal{Y}}$ and assume R is negatively transitive on $\mathcal{Y}$. Whenever $\text{Inf}_R Y = \{\pm\infty_R\}$, then $\text{Inf}_R Y \in [Y]$ by Prop. 2.35 and if $Y \in \{\emptyset, \{+\infty_R\}\}$, then $\text{Min}_R Y \in [Y]$ by Lem. 2.34. In both cases, we may then further assume, by Prop. 2.17, $A_R(Y), B_R(Y) \notin \{\emptyset, \mathcal{Y}\}$ and, by Lem. 2.14 (ii) and (iii), $-\infty_R \notin Y \notin \{\emptyset, \{+\infty_R\}\}$.

- (i) We show $A_R(\text{Inf}_R Y) = A_R(Y)$. As $A_R(\text{Inf}_R Y) \supseteq A_R(Y) \neq \emptyset$ holds from Lem. 2.24 (iii), it remains to show that $A_R(\text{Inf}_R Y) \subseteq A_R(Y)$. Assume condition (3) is satisfied. Let $l^* \in A_R(\text{Inf}_R Y)$, *i.e.* $l^* R \hat{y}, \forall \hat{y} \in \text{Inf}_R Y$. Assume $l^* \notin A_R(Y)$. Then $\exists y^* \in Y \setminus \text{Inf}_R Y : l^* \not R y^*$. By (3), $\exists \hat{y} \in \text{Inf}_R Y : y^* \not R \hat{y}$, whence we obtain

$$l^* \not R y^* \wedge y^* \not R \hat{y} \xrightarrow{R \text{ neg. transitive}} l^* \not R \hat{y},$$

contradicting $l^* \in A_R(\text{Inf}_R Y)$. Thus $A_R(\text{Inf}_R Y) = A_R(Y)$.

For the reverse, assume now that (3) is not satisfied. That is, there exists some $y \in Y \setminus \text{Inf}_R Y$ with $y R \hat{y}, \forall \hat{y} \in \text{Inf}_R Y$. This implies $y \in A_R(\text{Inf}_R Y)$. However, if $y \in A_R(Y)$, since by Lem. 2.14 (i) $y \in Y \subseteq B_R(Y)$, $y \in A_R(Y) \cap B_R(Y) = \text{Inf}_R Y$, contradicting the assumption that $y \in Y \setminus \text{Inf}_R Y$. Thus $y \notin A_R(Y)$, showing $A_R(\text{Inf}_R Y) \neq A_R(Y)$ and disproving representativity.

- (ii) Assume condition (4) is satisfied. From Lem. 2.24 (iii) and existing lower bounds of Y , we already know $A_R(Y) \subseteq A_R(\text{Min}_R Y) \neq \emptyset$, whence we must only show $A_R(Y) \supseteq A_R(\text{Min}_R Y)$. Let $l^* \in A_R(\text{Min}_R Y)$, *i.e.* $l^* R \hat{y}, \forall \hat{y} \in \text{Min}_R Y$. Assume $l^* \notin A_R(Y)$. Then $\exists y^* \in Y \setminus \text{Min}_R Y : l^* \not R y^*$. By condition (4), $\exists \hat{y} \in \text{Min}_R Y : y^* \not R \hat{y}$, whence we obtain

$$l^* \not R y^* \wedge y^* \not R \hat{y} \xrightarrow{R \text{ neg. transitive}} l^* \not R \hat{y},$$

contradicting $l^* \in A_R(\text{Min}_R Y)$. Thus $A_R(\text{Min}_R Y) = A_R(Y)$, *i.e.* $\text{Min}_R Y \in [Y]$, holds.

Assume now that condition (4) is violated. Then, $\exists y \in Y \setminus \text{Min}_R Y$ with $y R \hat{y}$ for all $\hat{y} \in \text{Min}_R Y$. This implies $y \in A_R(\text{Min}_R Y)$. However, if $y \in A_R(Y)$, then $y \in A_R(Y) \cap Y = \text{Min}_R Y$ by Prop. 2.16, contradicting the assumption that $y \in Y \setminus \text{Min}_R Y$. Thus $y \notin A_R(Y)$, showing $A_R(Y) \neq A_R(\text{Min}_R Y)$ and disproving representativity.

□

Notice how condition (4) is a direct generalization of the domination property of $\text{Min}_R Y$, as mostly used in the context of sets of *nondominated points* (with respect to an ordering cone D) when choosing $R = \not\prec_D$ (see Sect. 3.4.1). This is especially important with regards to Thm. 2.40, for which the condition for satisfaction by equality is in the literature traditionally linked to external stability of $\text{Min}_R Y$. The property (3) can be viewed as an expansion of that concept to the infimum $\text{Inf}_R Y$. This is further addressed in Sect. 3.4.1 and in Thm. 3.23.

2.2.2 Effects of Representative Minima and Infima

Representativity is of particular interest when considering unions of sets. Indeed, the lower bounds of a union of sets may be sufficiently characterized by the union of the respective infima or minima. The following theorems are generalizations using representativity of well-known results specific to nondominated and weakly nondominated infima (*e.g.* [20, Thm. I-28], [4, Cor. 7.1.9, Cor. 7.4.5] or in a slightly different setting [16, Thm. 1.54]).

2.38 Lemma:

Let I be an index set and $Y^i, Z^i \subseteq \bar{\mathcal{Y}}$ for all $i \in I$. If $Z^i \in [Y^i]$, $\forall i \in I$, then $\bigcup_{i \in I} Z^i \in [\bigcup_{i \in I} Y^i]$.

Proof. From Def. 2.29, $Z^i \in [Y^i]$ for all $i \in I$ is equivalent to

$$A_R(Z^i) = A_R(Y^i), \quad \forall i \in I, \quad (5)$$

Thus, by Lem. 2.24,

$$\begin{aligned} A_R\left(\bigcup_{i \in I} Z^i\right) &\stackrel{\text{Lem. 2.24 (ii)}}{=} \bigcap_{i \in I} A_R(Z^i) \\ &\stackrel{(5)}{=} \bigcap_{i \in I} A_R(Y^i) \stackrel{\text{Lem. 2.24 (ii)}}{=} A_R\left(\bigcup_{i \in I} Y^i\right), \end{aligned}$$

showing $\bigcup_{i \in I} Z^i \in [\bigcup_{i \in I} Y^i]$. $\square$

2.39 Theorem:

Let I be an index set and $Y^i \subseteq \bar{\mathcal{Y}}$ for all $i \in I$. If $\text{Inf}_R Y^i \in [Y^i]$, $\forall i \in I$, then

$$\text{Inf}_R \bigcup_{i \in I} \text{Inf}_R Y^i = \text{Inf}_R \bigcup_{i \in I} Y^i.$$

Proof. Setting $Z^i = \text{Inf}_R Y^i$ in Lem. 2.38 yields

$$\bigcup_{i \in I} \text{Inf}_R Y^i \in \left[\bigcup_{i \in I} Y^i\right].$$

Then, by Lem. 2.30, $\text{Inf}_R \bigcup_{i \in I} \text{Inf}_R Y^i = \text{Inf}_R \bigcup_{i \in I} Y^i$ holds. $\square$

It is important to note that Thm. 2.39 does not guarantee that infimal elements will exist for $\text{Inf}_R \bigcup_{i \in I} Y^i$. The following result parallels Thm. 2.39, but replaces infima with minima.

2.40 Theorem:

Let I be an index set and $Y^i \subseteq \bar{\mathcal{Y}}$ for all $i \in I$. Then

$$\text{Min}_R \left(\bigcup_{i \in I} Y^i \right) \subseteq \text{Min}_R \left(\bigcup_{i \in I} \text{Min}_R Y^i \right).$$

Furthermore, if $\text{Min}_R Y^i \in [Y^i], \forall i \in I$, the reverse inclusion holds.

Proof. Throughout this proof, we use $\text{Min}_R Y = A_R(Y) \cap Y$ for any $Y \subseteq \bar{\mathcal{Y}}$ by Prop. 2.16. By Lem. 2.24 (ii) and $\bigcup_{i \in I} \text{Min}_R Y^i \subseteq \bigcup_{i \in I} Y^i$ we obtain the inclusion

$$A_R \left(\bigcup_{i \in I} \text{Min}_R Y^i \right) \supseteq A_R \left(\bigcup_{i \in I} Y^i \right). \quad (6)$$

Starting from the right-hand side of the assertion,

$$\begin{aligned} \text{Min}_R \left(\bigcup_{i \in I} \text{Min}_R Y^i \right) &\stackrel{\text{Prop. 2.16}}{=} A_R \left(\bigcup_{i \in I} \text{Min}_R Y^i \right) \cap \left(\bigcup_{i \in I} \text{Min}_R Y^i \right) \\ &\stackrel{(6)}{\supseteq} A_R \left(\bigcup_{i \in I} Y^i \right) \cap \left(\bigcup_{i \in I} \text{Min}_R Y^i \right) \\ &= \bigcup_{j \in I} \left(A_R \left(\bigcup_{i \in I} Y^i \right) \cap \text{Min}_R Y^j \right). \end{aligned} \quad (7)$$

We may use, by Lem. 2.24 (ii) and $j \in I$,

$$A_R \left(\bigcup_{i \in I} Y^i \right) = \bigcap_{i \in I} A_R(Y^i) \subseteq A_R(Y^j). \quad (8)$$

By Prop. 2.16 and (8), we write, for any $j \in I$:

$$\begin{aligned} A_R \left(\bigcup_{i \in I} Y^i \right) \cap \text{Min}_R Y^j &= \bigcap_{i \in I} A_R(Y^i) \cap A_R(Y^j) \cap Y^j \\ &= \bigcap_{i \in I} A_R(Y^i) \cap Y^j \\ &= A_R \left(\bigcup_{i \in I} Y^i \right) \cap Y^j. \end{aligned} \quad (9)$$

Altogether, we get

$$\begin{aligned} \text{Min}_R \left(\bigcup_{i \in I} \text{Min}_R Y^i \right) &\stackrel{(7)}{\supseteq} \bigcup_{j \in I} \left(A_R \left(\bigcup_{i \in I} Y^i \right) \cap \text{Min}_R Y^j \right) \\ &\stackrel{(9)}{=} \bigcup_{j \in I} \left(A_R \left(\bigcup_{i \in I} Y^i \right) \cap Y^j \right) \\ &= A_R \left(\bigcup_{i \in I} Y^i \right) \cap \left(\bigcup_{i \in I} Y^i \right) \\ &\stackrel{\text{Prop. 2.16}}{=} \text{Min}_R \left(\bigcup_{i \in I} Y^i \right). \end{aligned} \quad (10)$$

This concludes the first part of the proof. For the second part, if the minima over Y^i are representative for all $i \in I$, then

$$A_R \left(\bigcup_{i \in I} \text{Min}_R Y^i \right) = A_R \left(\bigcup_{i \in I} Y^i \right)$$

holds by setting $Z^i = \text{Min}_R Y^i$ in Lem. 2.38, whence (6) is satisfied with equality, implying the same for (7) and (10), completing the proof. $\square$

2.41 Remark: In view of Thm. 2.39, Min_R cannot be replaced with Inf_R in Thm. 2.40: in general the inclusion using infima does not hold, i.e.

$$\text{Inf}_R \bigcup_{i \in I} \text{Inf}_R Y^i \not\subseteq \text{Inf}_R \bigcup_{i \in I} Y^i.$$

Using the (nondominated) infimum characterized later in Thm. 3.20, we may define for $R = \not\leq_D$, $D = \mathbb{R}_{\geq}^2$ and $\mathcal{Y} = \mathbb{R}^2$ the two sets

$$Y^1 = (0, 2] \times [0, 2] \cup (0, 2), \quad Y^2 = [0, 2] \times (0, 2] \cup (2, 0).$$

See Sect. 3.1 and Sect. 3.4.1 for details and a computation procedure. Then, taking the infima of Y^1 and Y^2 (see Fig. 2.3) yields

$$\text{Inf}_{\not\leq_D} Y^1 = \{(0, 2)\}, \quad \text{Inf}_{\not\leq_D} Y^2 = \{(2, 0)\}.$$

Clearly, the infima are not representative for their respective sets.

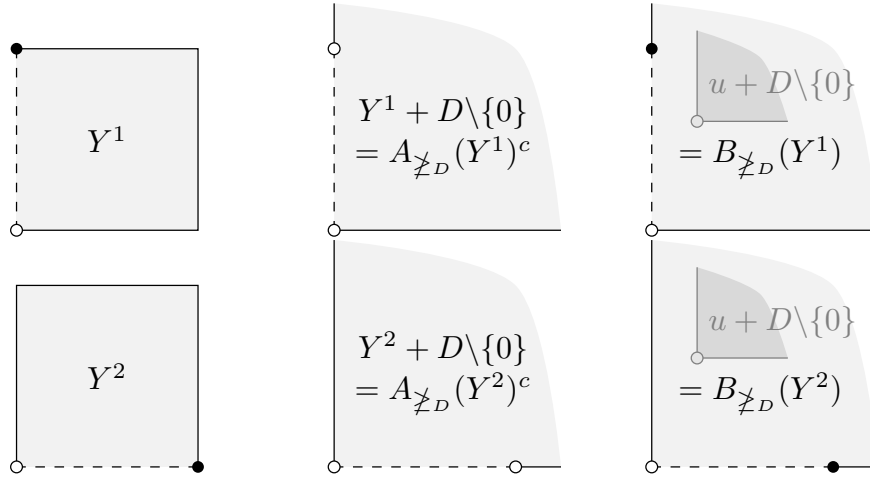
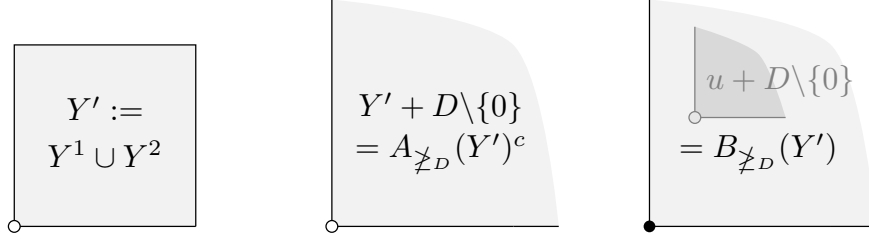
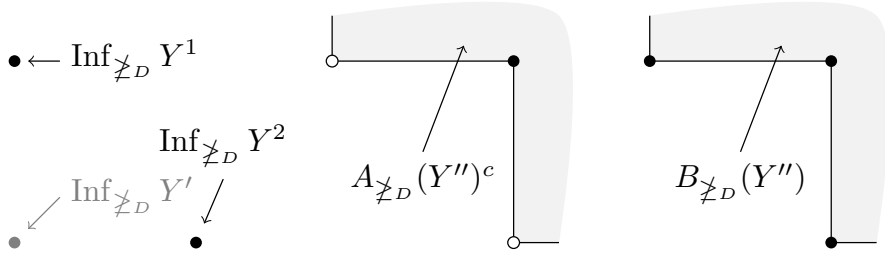


Figure 2.3: Computing nondominated infima of Y^1 and Y^2 .

Let $Y' = Y^1 \cup Y^2 = [0, 2] \times [0, 2] \setminus \{(0, 0)\}$, by Fig. 2.4, and $Y'' := \text{Inf}_{\not\leq_D} Y^1 \cup \text{Inf}_{\not\leq_D} Y^2 = \{(0, 2), (2, 0)\}$, by Fig. 2.5. We observe

$$\text{Inf}_{\not\leq_D} Y' = \{(0, 0)\} \not\subseteq \{(0, 2), (2, 0)\} = \text{Inf}_{\not\leq_D} Y''.$$

Figure 2.4: Computing nondominated infimum of $Y' = Y^1 \cup Y^2$.Figure 2.5: Computing nondominated infimum of $Y'' = \text{Inf}_{\not\leq_D} Y^1 \cup \text{Inf}_{\not\leq_D} Y^2$.

Regarding Thm. 2.40, we may observe that the infima of Y^1, Y^2 and Y'' are attained, hence minimal, while the infimum of Y' is not. In fact, $\text{Min}_{\not\leq_D} Y' = \emptyset$. Thus, substituting Y' and Y'' again,

$$\text{Min}_{\not\leq_D}(Y^1 \cup Y^2) = \emptyset \subseteq Y'' = \text{Min}_{\not\leq_D}(\text{Min}_{\not\leq_D} Y^1 \cup \text{Min}_{\not\leq_D} Y^2),$$

which is coherent with Thm. 2.40.

We may now compare Thm. 2.39 and Thm. 2.40. It is clear that, whenever all infimal elements are attained, both results are equivalent. There is an inherent trade-off between the two results. Whenever, minima are not representative but the infima are, Thm. 2.39 applies with equality, compared to the inclusion of Thm. 2.40. Furthermore, as soon as minima are representative, independently of attainment, by Prop. 2.31, Thm. 2.39 also applies with equality. In short: given representativity of infima and/or of minima, the infimum equation is more robust. However, whenever infima are not representative, by Prop. 2.31, neither are the corresponding minima. Then, Thm. 2.40 yields an inclusion, whereas Thm. 2.39 is simply not applicable, as demonstrated by Rem. 2.41. In a sense, the equation using infima requires milder, easier satisfied, assumptions, but the inclusion using minima is always true, irrespective of representativity.

2.3 Defining a Generalized Maximum and Supremum

Let R be a binary relation on $\mathcal{Y}$. Using the inverse relation $-R$, we can define a generalized maximum and supremum with the same approach as for the generalized infimum and minimum. In this case, $A_{-R}(Y)$ are the *upper bounds* and $B_{-R}(Y)$ the *lower bounds to the upper bounds* of Y with respect to R . We obtain, by $a(-R)b :\Leftrightarrow bRa, \forall a, b \in \mathcal{Y}$,

$$\begin{aligned} A_{-R}(Y) &= \{l \in \mathcal{Y} : yRl, \forall y \in Y\}, \\ B_{-R}(Y) &= \{u \in \mathcal{Y} : uRl, \forall l \in A_{-R}(Y)\}. \end{aligned}$$

Furthermore, the roles of $-\infty_R$ and $+\infty_R$ are exchanged: since $-\infty_R$ is the globally smallest element of $\mathcal{Y}$ with respect to R , it is automatically the largest element of $\mathcal{Y}$ with respect to $-R$. This reasoning is analogously applicable to $+\infty_R$. We may thus write $-\infty_{-R} = +\infty_R$ and $+\infty_{-R} = -\infty_R$, and the extended space $\bar{\mathcal{Y}} = \{\pm\infty_R\}$ with respect to R is also the extended space of $\mathcal{Y}$ with respect to $-R$.

Thus, in an analogous way to Def. 2.5 and Def. 2.6, the *maximum* and *supremum* of $Y \subseteq \bar{\mathcal{Y}}$ are given by

$$\text{Max}_R Y := \text{Min}_{-R} Y = A_{-R}(Y) \cap B_{-R}(Y) \cap Y, \quad (11)$$

$$\text{Sup}_R Y := \text{Inf}_{-R} Y = \begin{cases} \{+\infty_R\} & \text{if } A_{-R}(Y) = \emptyset, \\ \{-\infty_R\} & \text{if } B_{-R}(Y) = \emptyset, \\ A_{-R}(Y) \cap B_{-R}(Y) & \text{otherwise,} \end{cases} \quad (12)$$

All previous results on minima and infima can also be applied to maxima and suprema. This includes for example Prop. 2.16, whence

$$\text{Max}_R Y = A_{-R}(Y) \cap Y.$$

Further research is required to fully characterize the relationship between infima and suprema. The following result shows how the supremum over the lower bounds of a set provides the infimum of that set.

2.42 Proposition:

For any $Y \subseteq \bar{\mathcal{Y}}$:

- (i) $A_{-R}(A_R(Y)) = B_R(Y)$ and $B_{-R}(A_R(Y)) = A_R(Y)$,
- (ii) $\text{Sup}_R(A_R(Y)) = \text{Inf}_R Y$.
- (iii) $\text{Max}_R(A_R(Y)) = \text{Inf}_R Y \cap \mathcal{Y}$.

Proof. Using the definitions and $a(-R)b = bRa$,

$$A_{-R}(A_R(Y)) = \{l' \in \mathcal{Y} : lRl', \forall l \in A_R(Y)\} = B_R(Y) \quad (13)$$

$$\begin{aligned} B_{-R}(A_R(Y)) &= \{l' \in \mathcal{Y} : lRl', \forall l \in A_{-R}(Y)\} \stackrel{(13)}{=} A_R(B_R(Y)) \\ &\stackrel{\text{Lem. 2.21}}{=} A_R(Y). \end{aligned} \quad (14)$$

We go through the possible cases of $\text{Inf}_R Y$ for some $Y \subseteq \bar{\mathcal{Y}}$:

- If $\text{Inf}_R Y = \{+\infty_R\}$, by Prop. 2.22, $B_R(Y) = \emptyset$ and, by (13), $A_{-R}(A_R(Y)) = \emptyset$, implying $\text{Sup}_R A_R(Y) = \{+\infty_R\}$ by (12), and $\text{Max}_R A_R(Y) = \emptyset$.
- If $\text{Inf}_R Y = \{-\infty_R\}$, by Def. 2.6, $A_R(Y) = \emptyset$ and, by (14), $B_{-R}(A_R(Y)) = \emptyset$, implying $\text{Sup}_R A_R(Y) = \{-\infty_R\}$ by (12), and $\text{Max}_R A_R(Y) = \emptyset$.

Otherwise, if $\text{Inf}_R Y \neq \{\pm\infty_R\}$, by the above arguments,

$$\begin{aligned} \text{Sup}_R A_R(Y) &= A_{-R}(A_R(Y)) \cap B_{-R}(A_R(Y)) \\ &\stackrel{(14)}{=} A_{-R}(A_R(Y)) \cap A_R(Y) \stackrel{\text{Prop. 2.16}}{=} \text{Max}_R A_R(Y) \\ &\stackrel{(13)}{=} B_R(Y) \cap A_R(Y) = \text{Inf}_R Y. \end{aligned}$$

This completes the proof. □

Moreover, if, for some $Y \subseteq \mathcal{Y}$, the infimum $\text{Inf}_R Y$ is such that $A_{-R}(\text{Inf}_R Y) = B_R(Y)$, then $\text{Sup}_R \text{Inf}_R Y = \text{Inf}_R Y$. This is notably a property of *weak infima* (see Sect. 3.4.2), as stated in [25, Lem. 6.2.1].

2.4 Properties under Translation Invariance and Positive Responsiveness

We observe how some properties of the relation R on $\mathcal{Y}$ are expanded for the extended space $\bar{\mathcal{Y}} = \mathcal{Y} \cup \{\pm\infty_R\}$. Assume $\mathcal{Y}$ is a vector space. Let $-\infty_R$ and $+\infty_R$ be globally smallest and greatest elements of $\mathcal{Y}$ with respect to R , respectively. Furthermore, as in Sect. 2.1, $\pm\infty_R \notin \mathcal{Y}$ and for any $y \in \mathcal{Y}$:

$$-\infty_R R y, \quad y R +\infty_R, \quad y \not R -\infty_R, \quad -\infty_R \not R y.$$

We may assume the following calculus rules:

$$v + (-\infty_R) = -\infty_R, \quad v + (+\infty_R) = +\infty_R, \quad (\text{C1})$$

$$-(-\infty_R) = +\infty_R, \quad -(+\infty_R) = -\infty_R, \quad (\text{C2})$$

$$\alpha(-\infty_R) = -\infty_R, \quad \alpha(+\infty_R) = +\infty_R, \quad (\text{C3})$$

for any $v \in \mathcal{Y}$ and $\alpha > 0$. Note that, together, (C2) and (C3) imply

$$-\alpha(-\infty_R) = +\infty_R, \quad -\alpha(+\infty_R) = -\infty_R.$$

Additional calculus rules may be formulated (see *e.g.* Löhne [16]) to tackle cases like $\alpha = 0$ or $-\infty_R + (+\infty_R)$, but are not strictly necessary in this section. Throughout Sect. 2, we have assured that the relation R is only ever applied to pairs of elements where one is from the extended space $\overline{\mathcal{Y}}$ and the other from $\mathcal{Y}$. This becomes especially apparent in the following:

2.43 Lemma: Relations in Extended Vector Spaces

Let R be a relation on $\mathcal{Y}$ and $\overline{\mathcal{Y}}$ be an extension of $\mathcal{Y}$ with respect to R . Then, for any $(a, b) \in (\mathcal{Y} \times \overline{\mathcal{Y}}) \cup (\overline{\mathcal{Y}} \times \mathcal{Y})$, $v \in \mathcal{Y}$ and $\alpha > 0$:

(i) if R is translation invariant on $\mathcal{Y}$ and (C1) holds,

$$aRb \iff (v + a)R(v + b),$$

(ii) if R is translation invariant on $\mathcal{Y}$ and (C2) holds,

$$aRb \iff (-b)R(-a),$$

(iii) if R is positively responsive on $\mathcal{Y}$ and (C3) holds,

$$aRb \iff (\alpha a)R(\alpha b).$$

Proof. Let $a \in \mathcal{Y}$ and $b \in \overline{\mathcal{Y}} = \mathcal{Y} \cup \{\pm\infty_R\}$:

(i) By $\mathcal{Y}$ closed under addition, for any $a, b \in \mathcal{Y}$, we have $v + a, v + b \in \mathcal{Y}$, whence, for R translation invariant,

$$aRb \xrightarrow{R \text{ trans. inv.}} (v + a)R(v + b).$$

Furthermore, under the calculus rule (C1), if $b = -\infty_R$, then $v+b = -\infty_R$ and $aRb \Leftrightarrow (v+a)R(v+b)$. Otherwise if $b = +\infty_R$, then $v+b = +\infty_R$ and $aRb \Leftrightarrow (v+a)R(v+b)$.

- (ii) By $\mathcal{Y}$ closed under addition, $v := b - a$, $-a, -b \in \mathcal{Y}$ given $a, b \in \mathcal{Y}$, and for R translation invariant,

$$aRb \xLeftrightarrow[R \text{ trans. inv.}] (v+a)R(v+b) \xLeftrightarrow[v=b-a] (-b)R(-a).$$

Furthermore, under the calculus rule (C2), if $b = -\infty_R$, then $-b = +\infty_R$ and $aRb \Leftrightarrow (-b)R(-a)$. Otherwise if $b = +\infty_R$, then $-b = -\infty_R$ and $aRb \Leftrightarrow (-b)R(-a)$.

- (iii) By $\mathcal{Y}$ closed under scalar multiplication, for any $a, b \in \mathcal{Y}$, we have $\alpha a, \alpha b \in \mathcal{Y}$, whence, for R positively responsive,

$$aRb \xLeftrightarrow[R \text{ pos. hom.}] (\alpha a)R(\alpha b).$$

Furthermore, under the calculus rule (C3), if $b = -\infty_R$, then $\alpha b = -\infty_R$ and $aRb \Leftrightarrow (\alpha a)R(\alpha b)$. Otherwise if $b = +\infty_R$, then $\alpha b = +\infty_R$ and $aRb \Leftrightarrow (\alpha a)R(\alpha b)$.

Lastly, the same argument can be made by using $a \in \bar{\mathcal{Y}}$ and $b \in \mathcal{Y}$, thus proving the lemma for any pair $(a, b) \in (\mathcal{Y} \times \bar{\mathcal{Y}}) \cup (\bar{\mathcal{Y}} \times \mathcal{Y})$. $\square$

It is easy to see that the above also applies to $-R$, since $\bar{\mathcal{Y}}$ is also an extension of $\mathcal{Y}$ with respect to $-R$ and both translation invariance and positive responsiveness apply simultaneously to $-R$ and R . We derive the effects of the calculus rules (C1) to (C3) on concepts introduced throughout Sect. 2.

2.44 Proposition: *Effects of Calculus Rules*

For any $Y \subseteq \bar{\mathcal{Y}}$, $v \in \mathcal{Y}$ and $\alpha > 0$:

(i) if R is translation invariant on $\mathcal{Y}$ and (C1) holds,

$$\begin{aligned} v + A_R(Y) &= A_R(v + Y), & v + B_R(Y) &= B_R(v + Y), \\ v + \text{Min}_R Y &= \text{Min}_R(v + Y), & v + \text{Max}_R Y &= \text{Max}_R(v + Y), \\ v + \text{Inf}_R Y &= \text{Inf}_R(v + Y), & v + \text{Sup}_R Y &= \text{Sup}_R(v + Y), \end{aligned}$$

(ii) if R is translation invariant on $\mathcal{Y}$ and (C2) holds,

$$\begin{aligned} A_{-R}(Y) &= -A_R(-Y), & B_{-R}(Y) &= -B_R(-Y), \\ \text{Max}_R Y &= -\text{Min}_R(-Y), & \text{Sup}_R Y &= -\text{Inf}_R(-Y), \end{aligned}$$

(iii) if R is positively responsive on $\mathcal{Y}$ and (C3) holds,

$$\begin{aligned} \alpha A_R(Y) &= A_R(\alpha Y), & \alpha B_R(Y) &= B_R(\alpha Y), \\ \alpha \text{Min}_R Y &= \text{Min}_R(\alpha Y), & \alpha \text{Max}_R Y &= \text{Max}_R(\alpha Y), \\ \alpha \text{Inf}_R Y &= \text{Inf}_R(\alpha Y), & \alpha \text{Sup}_R Y &= \text{Sup}_R(\alpha Y). \end{aligned}$$

Proof. (i) For any $Y \subseteq \bar{\mathcal{Y}}$ and $v \in \mathcal{Y}$, using Lem. 2.43 (i),

$$\forall l, u \in \mathcal{Y}, y \in Y : \quad \begin{aligned} lRy &\iff (v + l)R(v + y), \\ lRu &\iff (v + l)R(v + u). \end{aligned} \quad (15)$$

Then, by Def. 2.2,

$$\begin{aligned} v + A_R(Y) &= v + \{l \in \mathcal{Y} : lRy, \forall y \in Y\} \\ &\stackrel{(15)}{=} \{(v + l) \in \mathcal{Y} : (v + l)R(v + y), \forall (v + y) \in v + Y\} \\ &= A_R(v + Y), \end{aligned} \quad (16)$$

$$\begin{aligned} v + B_R(Y) &= v + \{u \in \mathcal{Y} : lRu, \forall l \in A_R(Y)\} \\ &\stackrel{(15)}{=} \{v + u \in \mathcal{Y} : (v + l)R(v + u), \forall v + l \in v + A_R(Y)\} \\ &\stackrel{(16)}{=} B_R(v + Y). \end{aligned} \quad (17)$$

Then, by Prop. 2.16,

$$\begin{aligned} v + \text{Min}_R Y &= v + (A_R(Y) \cap Y) \\ &\stackrel{(16)}{=} A_R(v + Y) \cap (v + Y) = \text{Min}_R(v + Y), \end{aligned} \quad (18)$$

and by Prop. 2.22, using the calculus rule (C1),

$$\begin{aligned} v + \text{Inf}_R Y &= \begin{cases} v + \{-\infty_R\} & \text{if } B_R(Y) = \mathcal{Y}, \\ v + \{+\infty_R\} & \text{if } B_R(Y) = \emptyset, \\ v + \text{Min}_R B_R(Y) & \text{otherwise,} \end{cases} \\ &\stackrel{(C1), (17), (18)}{=} \begin{cases} \{-\infty_R\} & \text{if } B_R(v + Y) = \mathcal{Y}, \\ \{+\infty_R\} & \text{if } B_R(v + Y) = \emptyset, \\ \text{Min}_R B_R(v + Y) & \text{otherwise,} \end{cases} \\ &= \text{Inf}_R(v + Y). \end{aligned} \quad (19)$$

Lastly, using Lem. 2.43 (i) on $-R$, we get analogous results to (16), (17), (18) and (19):

$$\begin{aligned} v + A_{-R}(Y) &= A_{-R}(v + Y), & v + B_{-R}(Y) &= B_{-R}(v + Y), \\ v + \text{Min}_{-R} Y &= \text{Min}_{-R}(v + Y), & v + \text{Inf}_{-R} Y &= \text{Inf}_{-R}(v + Y). \end{aligned}$$

Then, by the above and $\text{Max}_R := \text{Min}_{-R}$ and $\text{Sup}_R := \text{Inf}_{-R}$,

$$v + \text{Max}_R Y = \text{Max}_R(v + Y), \quad v + \text{Sup}_R Y = \text{Sup}_R(v + Y).$$

(ii) For any $Y \subseteq \overline{\mathcal{Y}}$, using Lem. 2.43 (ii),

$$\forall l, u \in \mathcal{Y}, y \in Y : \quad \begin{aligned} l(-R)y &\iff (-l)R(-y), \\ l(-R)u &\iff (-l)R(-u). \end{aligned} \quad (20)$$

Thus, by Def. 2.2, we obtain

$$\begin{aligned} A_{-R}(Y) &= \{l \in \mathcal{Y} : l(-R)y, \forall y \in Y\} \\ &\stackrel{(20)}{=} -\{-l \in \mathcal{Y} : (-l)R(-y), \forall -y \in -Y\} \\ &= -A_R(-Y) \end{aligned} \quad (21)$$

$$\begin{aligned} B_{-R}(Y) &= \{u \in \mathcal{Y} : l(-R)u, \forall l \in A_{-R}(Y)\} \\ &\stackrel{(20)}{=} -\{-u \in \mathcal{Y} : (-l)R(-u), \forall -l \in -A_{-R}(Y)\} \\ &\stackrel{(21)}{=} -B_R(-Y). \end{aligned} \quad (22)$$

Then, by (11), $\text{Max}_R Y := \text{Min}_{-R} Y$ with

$$\begin{aligned} \text{Min}_{-R} Y &= A_{-R}(Y) \cap B_{-R}(Y) \cap Y \\ &\stackrel{(21),(22)}{=} -(A_R(-Y) \cap B_R(-Y) \cap (-Y)) \\ &= -\text{Min}_R(-Y), \end{aligned}$$

and, using the calculus rule (C2), by (12), $\text{Sup}_R Y := \text{Inf}_{-R} Y$ with

$$\begin{aligned} \text{Inf}_{-R} Y &\stackrel{(12)}{=} \begin{cases} \{+\infty_R\} & \text{if } A_{-R}(Y) = \emptyset, \\ \{-\infty_R\} & \text{if } B_{-R}(Y) = \emptyset, \\ A_{-R}(Y) \cap B_{-R}(Y) & \text{otherwise,} \end{cases} \\ &\stackrel{(C2),(21),(22)}{=} \begin{cases} -\{-\infty_R\} & \text{if } A_R(-Y) = \emptyset, \\ -\{+\infty_R\} & \text{if } B_R(-Y) = \emptyset, \\ -(A_R(-Y) \cap B_R(-Y)) & \text{otherwise,} \end{cases} \\ &= -\text{Inf}_R(-Y). \end{aligned}$$

(iii) For any $Y \subseteq \overline{\mathcal{Y}}$ and $\alpha > 0$, using Lem. 2.43 (iii),

$$\forall l, u \in \mathcal{Y}, y \in Y : \quad \begin{aligned} lRy &\iff (\alpha l)R(\alpha y), \\ lRu &\iff (\alpha l)R(\alpha u). \end{aligned} \quad (23)$$

Then, by Def. 2.2,

$$\begin{aligned} \alpha A_R(Y) &= \alpha \{l \in \mathcal{Y} : lRy, \forall y \in Y\} \\ &\stackrel{(23)}{=} \{(\alpha l) \in \mathcal{Y} : (\alpha l)R(\alpha y), \forall (\alpha y) \in \alpha Y\} \\ &= A_R(\alpha Y), \end{aligned} \quad (24)$$

$$\begin{aligned} \alpha B_R(Y) &= \alpha \{u \in \mathcal{Y} : lRu, \forall l \in A_R(Y)\} \\ &\stackrel{(23)}{=} \{\alpha u \in \mathcal{Y} : (\alpha l)R(\alpha u), \forall \alpha l \in \alpha A_R(Y)\} \\ &\stackrel{(24)}{=} B_R(\alpha Y). \end{aligned} \quad (25)$$

Then, by Prop. 2.16,

$$\begin{aligned} \alpha \text{Min}_R Y &= \alpha (A_R(Y) \cap Y) \\ &\stackrel{(24)}{=} (\alpha A_R(Y)) \cap (\alpha Y) = \text{Min}_R(\alpha Y), \end{aligned} \quad (26)$$

and by Prop. 2.22, using the calculus rule (C3),

$$\begin{aligned}
\alpha \operatorname{Inf}_R Y &= \begin{cases} \alpha\{-\infty_R\} & \text{if } B_R(Y) = \mathcal{Y}, \\ \alpha\{+\infty_R\} & \text{if } B_R(Y) = \emptyset, \\ \alpha \operatorname{Min}_R B_R(Y) & \text{otherwise,} \end{cases} \\
&\stackrel{(C3),(25),(26)}{=} \begin{cases} \{-\infty_R\} & \text{if } B_R(\alpha Y) = \mathcal{Y}, \\ \{+\infty_R\} & \text{if } B_R(\alpha Y) = \emptyset, \\ \operatorname{Min}_R B_R(\alpha Y) & \text{otherwise,} \end{cases} \\
&= \operatorname{Inf}_R(\alpha Y). \tag{27}
\end{aligned}$$

Now, using $-R$, we also get the analogous results to (24),(25)(26) and (27):

$$\begin{aligned}
\alpha A_{-R}(Y) &= A_{-R}(\alpha Y), & \alpha B_{-R}(Y) &= B_{-R}(\alpha Y), \\
\alpha \operatorname{Min}_{-R} Y &= \operatorname{Min}_{-R}(\alpha Y), & \alpha \operatorname{Inf}_{-R} Y &= \operatorname{Inf}_{-R}(\alpha Y),
\end{aligned}$$

which, by the above and $\operatorname{Max}_R := \operatorname{Min}_{-R}$ and $\operatorname{Sup}_R := \operatorname{Inf}_{-R}$,

$$\alpha \operatorname{Max}_R Y = \operatorname{Max}_R(\alpha Y), \quad \alpha \operatorname{Sup}_R Y = \operatorname{Sup}_R(\alpha Y).$$

□

2.4.1 Equivalence Class for Upper Bounds

For any $Y \subseteq \overline{\mathcal{Y}}$, we may define the equivalence class for the upper bounds of Y (with respect to R) as

$$[Y]^- := \{Y' \in \overline{\mathcal{Y}} : A_{-R}(Y') = A_{-R}(Y)\}.$$

Note that, in general, $[Y]^- \neq [Y]$, as $[Y]$ is characterized by $A_R(Y)$, not by $A_{-R}(Y)$. We call $\operatorname{Max}_R Y$ (resp. $\operatorname{Sup}_R Y$) *representative* (implicitly referring to Y and $-R$) whenever the set is representative of Y with respect to $-R$, *i.e.*

$$\begin{aligned}
\operatorname{Max}_R Y \in [Y]^- &\iff A_{-R}(\operatorname{Max}_R Y) = A_{-R}(Y), \\
\operatorname{Sup}_R Y \in [Y]^- &\iff A_{-R}(\operatorname{Sup}_R Y) = A_{-R}(Y).
\end{aligned}$$

In this regard, by $\operatorname{Max}_R = \operatorname{Min}_{-R}$ and $\operatorname{Sup}_R = \operatorname{Inf}_{-R}$, we may apply results formulated around the representativity of minima and infima with maxima and suprema. The following result shows the effect of calculus rules on the equivalence class $[\cdot]^-$. The same is applicable on $[\cdot]^+$.

2.45 Corollary:

For any $Y, Y' \subseteq \overline{\mathcal{Y}}$, $v \in \mathcal{Y}$ and $\alpha > 0$:

(i) if R is translation invariant on $\mathcal{Y}$ and (C1) holds,

$$Y' \in [Y] \iff v + Y' \in [v + Y],$$

(ii) if R is translation invariant on $\mathcal{Y}$ and (C2) holds,

$$Y' \in [Y]^- \iff -Y' \in [-Y],$$

(iii) if R is positively responsive on $\mathcal{Y}$ and (C3) holds,

$$Y' \in [Y] \iff \alpha Y' \in [\alpha Y].$$

Proof. (i) For any $v \in \mathcal{Y}$, by Prop. 2.44 (i),

$$\begin{aligned} A_R(Y') = A_R(Y) &\iff v + A_R(Y') = v + A_R(Y) \\ &\stackrel{\text{Prop. 2.44 (i)}}{\iff} A_R(v + Y') = A_R(v + Y), \end{aligned}$$

whence $Y' \in [Y] \iff v + Y' \in [v + Y]$.

(ii) From Prop. 2.44 (ii),

$$\begin{aligned} A_{-R}(Y') = A_{-R}(Y) &\stackrel{\text{Prop. 2.44 (ii)}}{\iff} -A_R(-Y') = -A_R(-Y) \\ &\iff A_R(-Y') = A_R(-Y), \end{aligned}$$

whence $Y' \in [Y]^- \iff -Y' \in [-Y]$.

(iii) For any $\alpha > 0$, by Prop. 2.44 (iii),

$$\begin{aligned} A_R(Y') = A_R(Y) &\iff \alpha A_R(Y') = \alpha A_R(Y) \\ &\stackrel{\text{Prop. 2.44 (iii)}}{\iff} A_R(\alpha Y') = A_R(\alpha Y), \end{aligned}$$

whence $Y' \in [Y] \iff \alpha Y' \in [\alpha Y]$.

□

2.4.2 Infima and Minima over Sums of Sets

Under (C1), we obtain the following results:

2.46 Corollary:

Let R be translation invariant on $\mathcal{Y}$ and (C1) hold. For any index set I and collections of sets $S^i \subseteq \mathcal{Y}$ and $Y^i \subseteq \overline{\mathcal{Y}}$, $i \in I$:

$$\text{Min}_R \bigcup_{i \in I} (S^i + Y^i) \subseteq \text{Min}_R \bigcup_{i \in I} (S^i + \text{Min}_R Y^i).$$

Moreover, if $\text{Min}_R Y^i \in [Y^i]$, $\forall i \in I$, the above inclusion is satisfied with equality.

Proof. By Minkowski addition and calculus rule (C1), from $S^i \subseteq \mathcal{Y}$,

$$S^i + Y^i = \bigcup_{s^i \in S^i} (s^i + Y^i) \in \overline{\mathcal{Y}} \quad (28)$$

holds for any $i \in I$. Using Prop. 2.44 (i), by Thm. 2.40:

$$\begin{aligned} \text{Min}_R \bigcup_{i \in I} (S^i + Y^i) &\stackrel{(28)}{=} \text{Min}_R \bigcup_{i \in I, s^i \in S^i} (s^i + Y^i) \\ &\stackrel{\text{Thm. 2.40}}{\subseteq} \text{Min}_R \bigcup_{i \in I, s^i \in S^i} \text{Min}_R (s^i + Y^i) \\ &\stackrel{\text{Prop. 2.44 (i)}}{=} \text{Min}_R \bigcup_{i \in I, s^i \in S^i} (s^i + \text{Min}_R Y^i) \\ &= \text{Min}_R \bigcup_{i \in I} (Y^1 + \text{Min}_R Y^2). \end{aligned} \quad (29)$$

Futhermore, assuming $\text{Min}_R Y^i \in [Y^i]$, $\forall i \in I$, by Cor. 2.45 (i),

$$\forall i \in I, s^i \in Y^i : s^i + \text{Min}_R Y^i \in [s^i + Y^i].$$

Lastly, by Prop. 2.44 (i) and $S^i \subseteq \mathcal{Y}$, $\text{Min}_R (s^i + Y^i) = s^i + \text{Min}_R (Y^i)$, whence (29) is satisfied with equality, by Thm. 2.40, completing the proof. $\square$

2.47 Corollary:

Let R be translation invariant on $\mathcal{Y}$ and (C1) hold. For any index set I and collections of sets $S^i \subseteq \mathcal{Y}$ and $Y^i \subseteq \overline{\mathcal{Y}}$, $i \in I$ with $\text{Inf}_R Y^i \in [Y^i]$ for every $i \in I$:

$$\text{Inf}_R \bigcup_{i \in I} (S^i + Y^i) = \text{Inf}_R \bigcup_{i \in I} (S^i + \text{Inf}_R Y^i).$$

Proof. By Minkowski addition and calculus rule (C1), from $S^i \subseteq \mathcal{Y}$,

$$S^i + Y^i = \bigcup_{s^i \in S^i} (s^i + Y^i) \in \overline{\mathcal{Y}} \quad (30)$$

holds for any $i \in I$. By Cor. 2.45 (i), for any $i \in I$ and any $s^i \in S^i$,

$$\text{Inf}_R Y^i \in [Y^i] \iff s^i + \text{Inf}_R Y^i \in [s^i + Y^i],$$

and, by Prop. 2.44 (i) and $S^i \subseteq \mathcal{Y}$,

$$\text{Inf}_R (s^i + Y^i) \in [s^i + Y^i], \quad \forall i \in I, s^i \in S^i. \quad (31)$$

Thus, we may use Thm. 2.39 to obtain:

$$\begin{aligned} \text{Inf}_R \bigcup_{i \in I} (S^i + Y^i) &\stackrel{(30)}{=} \text{Inf}_R \bigcup_{i \in I, s^i \in S^i} (s^i + Y^i) \\ &\stackrel{\text{Thm. 2.39, (31)}}{=} \text{Inf}_R \bigcup_{i \in I, s^i \in S^i} \text{Inf}_R (s^i + Y^i) \\ &\stackrel{\text{Prop. 2.44 (i)}}{=} \text{Inf}_R \bigcup_{i \in I, s^i \in S^i} (s^i + \text{Inf}_R Y^i) \\ &= \text{Inf}_R \bigcup_{i \in I} (S^i + \text{Inf}_R Y^i). \end{aligned}$$

□

Analogously, whenever R satisfies translation invariance and (C1), so does $-R$. Thus, Cor. 2.46 implies

$$\text{Max}_R(Y^1 + Y^2) \subseteq \text{Max}_R(Y^1 + \text{Max}_R Y^2)$$

for any $Y^1 \subseteq \mathcal{Y}$ and $Y^2 \subseteq \overline{\mathcal{Y}}$. If additionally $\text{Max}_R Y^2$ is representative (*i.e.* $\text{Max}_R Y^2 \in [Y^2]^-$), then the above holds with equality and implies representativity of $\text{Sup}_R Y^2$, by Prop. 2.31. Whenever $\text{Sup}_R Y^2 \in [Y^2]^-$, by Cor. 2.47,

$$\text{Sup}_R(Y^1 + Y^2) = \text{Sup}_R(Y^1 + \text{Sup}_R Y^2).$$

Lastly, echoing [16, Cor. 1.49(ii)], we can also show *inf-additivity* under representativity.

2.48 Corollary: *Inf-Additivity*

Let R be translation invariant on $\mathcal{Y}$ and (C1) hold. For any two sets $Y^1 \subseteq \mathcal{Y}$ and $Y^2 \subseteq \overline{\mathcal{Y}}$ with $\text{Inf}_R Y^i \in [Y^i]$ for $i = 1, 2$:

$$\text{Inf}_R(Y^1 + Y^2) = \text{Inf}_R(\text{Inf}_R Y^1 + \text{Inf}_R Y^2),$$

provided $\text{Inf}_R Y^i \neq \{\pm\infty_R\}$ for some $i \in \{1, 2\}$.

Proof. By $Y^1 \subseteq \mathcal{Y}$, Cor. 2.47 immediately yields

$$\text{Inf}_R(Y^1 + Y^2) = \text{Inf}_R(Y^1 + \text{Inf}_R Y^2). \quad (32)$$

Provided $\text{Inf}_R Y^2 \subseteq \mathcal{Y}$, by the same result,

$$\text{Inf}_R(Y^1 + \text{Inf}_R Y^2) = \text{Inf}_R(\text{Inf}_R Y^1 + \text{Inf}_R Y^2),$$

showing the assertion. Otherwise, $\text{Inf}_R Y^1 \subseteq \mathcal{Y}$ and $\text{Inf}_R Y^2 = \{\pm\infty_R\}$. If $\text{Inf}_R Y^2 = \{-\infty_R\}$, by the calculus rules (C1),

$$-\infty_R \in (Y^1 + \text{Inf}_R Y^2) \cap (\text{Inf}_R Y^1 + \text{Inf}_R Y^2),$$

such that, by Prop. 2.22 and (32),

$$\text{Inf}_R(Y^1 + Y^2) = \text{Inf}_R(\text{Inf}_R Y^1 + \text{Inf}_R Y^2) = \{-\infty_R\}.$$

On the other hand, if $\text{Inf}_R Y^2 = \{+\infty_R\}$, again by the calculus rules (C1),

$$\{+\infty_R\} = (Y^1 + \text{Inf}_R Y^2) = (\text{Inf}_R Y^1 + \text{Inf}_R Y^2),$$

implying, by Prop. 2.22 and (32),

$$\text{Inf}_R(Y^1 + Y^2) = \text{Inf}_R(\text{Inf}_R Y^1 + \text{Inf}_R Y^2) = \{+\infty_R\}.$$

This completes the proof. □

3 Applications of the Generalized Infimum on Vector Spaces

In this section, we apply the generalized infimum framework introduced in Sect. 2 to relations defined on vector spaces, and use their ordering map characterizations, as addressed in Sect. 1. We focus on constant ordering maps, before specifically addressing constant conical domination structures and their application for the three most common orderings used in vector optimization. We derive associated properties for each and, when applicable, compare our approach to other authors.

Note that the term *ordering* is used more loosely than in traditional literature, where an ordering is necessarily at least a partial ordering (*i.e.* reflexive, transitive and antisymmetric) [19]. In this section, we call ordering any partial, strict partial ordering *and* their complements (which are necessarily negatively transitive).

3.1 Generalized Infimum under Constant Domination Structures

Consider that a binary relation R on a topological vector space $\mathcal{Y}$ is represented by an ordering map $\mathcal{D} : \mathcal{Y} \rightrightarrows \mathcal{Y}$. Assuming R is both translation invariant and positively responsive on the entire space, by Lem. 1.1, the ordering map is constant and a cone, *i.e.*

$$aRb \iff b - a \in \mathcal{D}(a) \equiv \mathcal{D},$$

with constant conical domination structure $\mathcal{D} \subseteq \mathcal{Y}$. If R forms a partial ordering, then $(\mathcal{Y}, R)$ forms a partially ordered topological vector space. This is the viewpoint chosen in many applications and allows to easily construct many examples for the previous sections. Thus, if we assume the calculus rules (C1), (C2) and (C3), we can use the properties derived in Sect. 2.4. Moreover, we can relate maxima and suprema operators to infima and minima operators using Prop. 2.44 (ii). In this regard, we shall here only focus on infima and minima. The following result combines domination structures with the sets of lower bounds and of upper bounds to lower bounds from Def. 2.2.

3.1 Proposition:

Let $\mathcal{D} : \mathcal{Y} \rightrightarrows \mathcal{Y}$ be an ordering map characterizing R on $\mathcal{Y}$ and let $Y \subseteq \mathcal{Y}$. Then $l \in A_R(Y)$ if and only if

$$l \notin Y - \mathcal{D}(l)^c \iff Y \cap (l + \mathcal{D}(l)^c) = \emptyset.$$

If the ordering map $\mathcal{D}$ is constant on $\mathcal{Y}$ with a constant domination structure $\mathcal{D} \subseteq \mathcal{Y}$, then

$$\begin{aligned} A_R(Y) &= \bigcap_{y \in Y} (y - \mathcal{D}) \\ &= (Y - \mathcal{D}^c)^c \\ &= \{l \in \mathcal{Y} : Y \cap (l + \mathcal{D}^c) = \emptyset\}, \\ B_R(Y) &= \{u \in \mathcal{Y} : (u - \mathcal{D}^c) \subseteq (Y - \mathcal{D}^c)\}. \end{aligned}$$

Proof. Using the definition of domination structures, $yRz \Leftrightarrow z \in y + \mathcal{D}(y) \Leftrightarrow y \in z - \mathcal{D}(y)$ and $(z - \mathcal{D}(y))^c = z - \mathcal{D}(y)^c$ for any $y, z \in \mathcal{Y}$, whence we can write:

$$\begin{aligned} A_R(Y) &= \{l \in \mathcal{Y} : l \in y - \mathcal{D}(l), \forall y \in Y\} \\ &= \{l \in \mathcal{Y} : l \in \bigcap_{y \in Y} (y - \mathcal{D}(l))\} \stackrel{\mathcal{D}(l)=\mathcal{D}}{=} \bigcap_{y \in Y} (y - \mathcal{D}) \\ &= \{l \in \mathcal{Y} : l \notin \bigcap_{y \in Y} (y - \mathcal{D}(l))^c\}^c \\ &= \{l \in \mathcal{Y} : l \in \bigcup_{y \in Y} y - \mathcal{D}(l)^c\}^c \\ &= \{l \in \mathcal{Y} : l \in Y - \mathcal{D}(l)^c\}^c \stackrel{\mathcal{D}(l)=\mathcal{D}}{=} (Y - \mathcal{D}^c)^c. \end{aligned}$$

Reformulating differently, we obtain, by $(y + \mathcal{D}(y))^c = y + \mathcal{D}(y)^c$ for any $y \in \mathcal{Y}$,

$$\begin{aligned} A_R(Y) &= \{l \in \mathcal{Y} : y \in l + \mathcal{D}(l), \forall y \in Y\} \\ &= \{l \in \mathcal{Y} : y \notin l + \mathcal{D}(l)^c, \forall y \in Y\} \\ &= \{l \in \mathcal{Y} : Y \cap (l + \mathcal{D}(l)^c) = \emptyset\} \\ &\stackrel{\mathcal{D}(l)=\mathcal{D}}{=} \{l \in \mathcal{Y} : Y \cap (l + \mathcal{D}^c) = \emptyset\}. \end{aligned}$$

Lastly, given a constant domination structure, from $\mathcal{D}(l) = \mathcal{D}$ and $A_R(Y) =$

$(Y - \mathcal{D}^c)^c$, and the definition of $B_R(Y)$:

$$\begin{aligned} B_R(Y) &= \{u \in \mathcal{Y} : l \in u - \mathcal{D}(l), \forall l \in A_R(Y)\} \\ &\stackrel{\mathcal{D}(l) = \mathcal{D}}{=} \{u \in \mathcal{Y} : (Y - \mathcal{D}^c)^c \subseteq (u - \mathcal{D})\} \\ &= \{u \in \mathcal{Y} : (u - \mathcal{D}^c) \subseteq (Y - \mathcal{D}^c)\}. \end{aligned}$$

□

The characterizations of $A_R(Y)$ and $B_R(Y)$ from Prop. 3.1 can be directly used in Cor. 2.20 and Cor. 2.23 to obtain minima and infima with respect to R .

3.2 Remark: *Prop. 3.1 only characterizes $A_R(Y)$ and $B_R(Y)$ for $Y \subseteq \mathcal{Y}$. To account for sets Y from the extended space $\bar{\mathcal{Y}}$, following Prop. 2.19, we may write:*

$$\begin{aligned} A_R(Y) &= \begin{cases} \emptyset & \text{if } -\infty_R \in Y \\ A_R(Y \setminus \{+\infty_R\}) & \text{otherwise,} \end{cases} \\ B_R(Y) &= \begin{cases} \mathcal{Y} & \text{if } -\infty_R \in Y \\ B_R(Y \setminus \{+\infty_R\}) & \text{otherwise.} \end{cases} \end{aligned}$$

Then, all characterizations of the operators Min_R and Inf_R from Sect. 2 are applicable. For instance, by Prop. 2.16 and Prop. 2.22:

$$\begin{aligned} \text{Min}_R Y &= A_R(Y) \cap Y, \\ \text{Inf}_R Y &= \begin{cases} \{-\infty_R\} & \text{if } B_R(Y) = \mathcal{Y}, \\ \{+\infty_R\} & \text{if } B_R(Y) = \emptyset, \\ \text{Min}_R B_R(Y) & \text{otherwise.} \end{cases} \end{aligned}$$

The proof of Prop. 3.1 shows the difficulties associated with reformulating the conditions in $B_R(Y)$ without considering specific constant domination structures. In contrast, necessary and sufficient conditions for $l \in A_R(Y)$ are relatively easy to obtain.

We use Prop. 3.1 along with the constant domination structure $\mathcal{D}$ to construct examples by first determining the complement of the lower bounds $A_R(Y)^c = (Y - \mathcal{D}^c)$, then gathering all points $u \in \mathcal{Y}$ that satisfy $(u - \mathcal{D}^c) \subseteq A_R(Y)^c$. The infimum, when existing, is then comprised of exactly the points that lie in $B_R(Y)$ but not in $A_R(Y)^c$. This is, in essence, analogous to the procedure described in Example 2.13. The following example applies this procedure for $\mathcal{Y} = \mathbb{R}^2$.

3.3 Example: Let $\mathcal{Y} = \mathbb{R}^2$ and $R = \not\leq$. The ordering map $\mathcal{D} : \mathcal{Y} \rightrightarrows \mathcal{Y}$ is constant with a conical domination structure $\mathcal{D} = (-D \setminus \{0\})^c$ and $D = \mathbb{R}_{\geq}^2$ (see Sect. 3.2). For the set

$$Y = \{(0, 2] \times (0, 2]\} \cup \{[1, 2] \times \{0\}\} \cup \{\{0\} \times (1, 2]\} \subset \mathbb{R}^2,$$

as shown in Fig. 3.1, we obtain

$$\text{Inf}_R Y = \{(1, 0)^\top, (0, 1)^\top\} \quad \text{and} \quad \text{Min}_R Y = \{(1, 0)^\top\}.$$

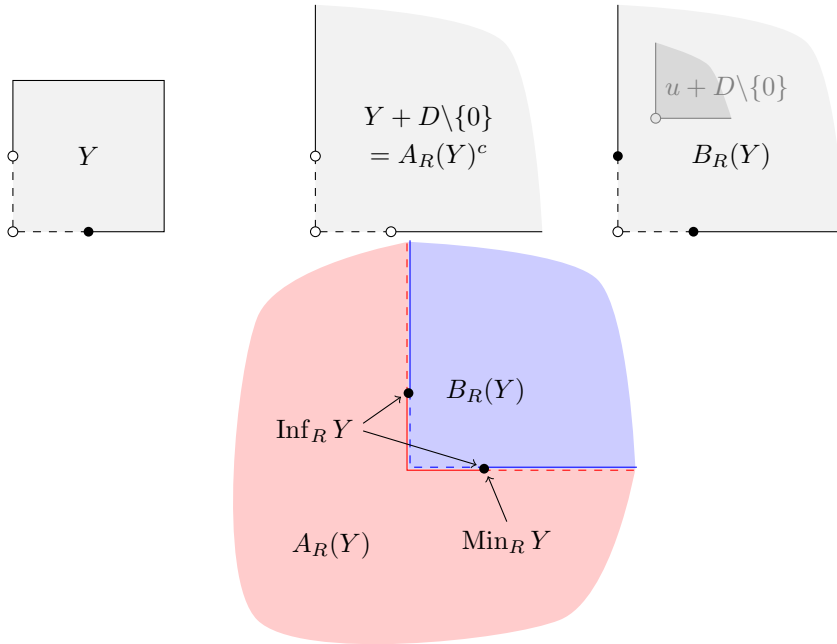
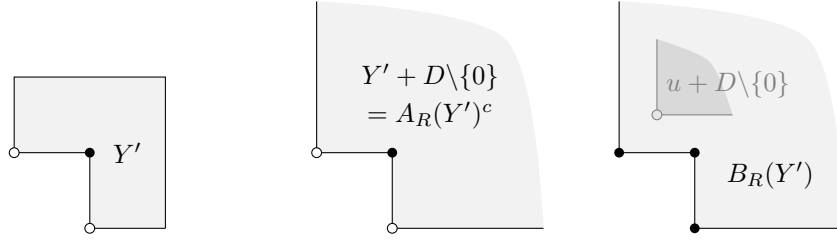


Figure 3.1: Sets for Y from Example 3.3.

The infimum is not representative for $[Y]$. However, for a different set

$$Y' = \{y \in \mathbb{R}^2 : y_1, y_2 \in [0, 2], \max\{y_1, y_2\} \geq 1\} \setminus \{(0, 1)^\top, (1, 0)^\top\},$$

the infimum is the same as for Y , and $\text{Inf}_R Y' \in [Y']$ (see Fig. 3.2). Thus, despite $\text{Inf}_R Y$ not being representative, by existence of a class for which it is, self-infimality holds. Furthermore, Y' is also an example for the absence of minimum and existing infimum with respect to R .

Figure 3.2: Sets for Y' from Example 3.3.

3.2 Orderings with Respect to a Common Ordering Cone

Let $\mathcal{Y}$ be a vector space. Whenever $\mathcal{Y}$ is a real finite dimensional vector space, one can apply the well defined orderings $\leq$, $\leq$ and $<$. Using $\mathcal{Y} = \mathbb{R}^n$ for some $n \in \mathbb{N}$, we can write:

$$\begin{aligned} a \leq b &\iff b - a \in \mathbb{R}_{\geq}^n, \\ a \leq b &\iff b - a \in \mathbb{R}_{\geq}^n = \mathbb{R}_{\geq}^n \setminus \{0\}, \\ a < b &\iff b - a \in \mathbb{R}_{>}^n = \text{int } \mathbb{R}_{\geq}^n. \end{aligned}$$

By Sect. 1.2, $\mathbb{R}_{\geq}^n$, $\mathbb{R}_{\geq}^n$ and $\mathbb{R}_{>}^n$ can be viewed as the constant (conical) domination structures of their respective orderings. These orderings are commonly generalized by replacing $\mathbb{R}_{\geq}^n$ by an arbitrary pointed, nonempty and convex *ordering cone* $D \subseteq \mathbb{R}^n$ containing zero (or assumed closed) [4, 11, 16, 25]. Here, the explicit inclusion of zero replaces the assumption of D being closed, often assumed in the literature due to diverging definitions of cones. We then obtain the following relations with respect to $D \subseteq \mathcal{Y}$, for an arbitrary topological vector space $\mathcal{Y}$:

$$\begin{aligned} a \leq_D b &\iff b - a \in D, \\ a \leq_D b &\iff b - a \in D \setminus \{0\}, \\ a <_D b &\iff b - a \in \text{int } D. \end{aligned}$$

The properties of the above relations can be derived from their domination structures. Lem. 1.15 shows that, if D is a pointed convex cone, then so are $D \setminus \{0\}$ and $\text{int } D$. The, we may show the following:

3.4 Proposition:

Let $D \subset \mathcal{Y}$ be a pointed convex cone containing zero. Then

- (i) $\leq_D$ forms a partial ordering on $\mathcal{Y}$,
- (ii) $\leq_D$ and $<_D$ form strict partial orderings on $\mathcal{Y}$.

Proof. From D being pointed and $0 \in D$, the condition $D \cap -D = \{0\}$ holds. Then, the first assertion follows directly from D being convex and Prop. 1.9. Similarly, the domination structures $D \setminus \{0\}$ and $\text{int } D$ are both pointed convex cones not containing zero by Lem. 1.15. Thus $D \setminus \{0\} \cap -D \setminus \{0\} = \text{int } D \cap -\text{int } D = \emptyset$ and $\leq_D$ and $<_D$ both form strict partial orderings by Prop. 1.9. $\square$

Since $\leq_D$ and $<_D$ form strict partial orderings on $\mathcal{Y}$, they are especially irreflexive. Then, by Prop. 2.25 and subsequent discussion, the respective minima of $R \in \{\leq_D, <_D\}$ are always empty, and their respective infima are either empty or $\{\pm\infty_R\}$. Both orderings are therefore not meaningfully suitable for our generalized infimum framework.

3.2.1 Types of Lower Bounds

Consider the complements of the inverse relations to $\leq_D$ and $<_D$, *i.e.*

$$\begin{aligned} a \not\leq_D b &\iff b - a \notin (-D \setminus \{0\}), \\ a \not<_D b &\iff b - a \notin (-\text{int } D). \end{aligned}$$

In the real scalar case $\mathcal{Y} = \mathbb{R}$, the orderings $\leq$, $\not\leq$ and $\not<$ are all equivalent. That is,

$$\forall a, b \in \mathbb{R} : a \leq b \iff a \not\leq b \iff a \not< b.$$

In $\mathbb{R}$, any nontrivial pointed convex ordering cones $D \subseteq \mathbb{R}$ containing zero is either $\mathbb{R}_{\geq} = [0, +\infty)$ or $\mathbb{R}_{\leq} = (-\infty, 0]$. Assuming D generalizes $\mathbb{R}_{\geq}$, then

$$D = (-D \setminus \{0\})^c = (-\text{int } D)^c = \mathbb{R}_{\geq},$$

such that a being *smaller or equal* ($\leq$) to b is the same as a being *not strictly greater* ($\not\leq$ and $\not<$) than b , for any choice of D . This equivalence does not hold, in general, for higher dimensions. Therefore, we may obtain three different ways to generalize lower bounds, infima and minima from differing notions of

“smaller” induced by a common ordering cone D . Depending on the choice of interpretation of $\leq$ in the scalar definition of lower bounds with respect to a conical ordering, a higher dimensional lower bound l can be defined as:

- (i) a *dominant point*, that is, l dominates all $y \in Y$,
- (ii) a *nondominated point*, that is, l is not dominated by any $y \in Y$,
- (iii) or a *weakly nondominated point*, that is, l is not strictly dominated by any $y \in Y$.

In Jahn [11], these correspond to *strongly minimal*, *minimal* and *weakly minimal* elements of Y . These approaches applied to minimization of a functional, yield, respectively, *strongly efficient solutions*, *efficient solutions* and *weakly efficient solutions*. In this regard, we will also investigate *strong infima*, *(nondominated) infima* and *weak infima* operators (with respect to D), respectively denoted

$$\text{s-Inf}_D := \text{Inf}_{\leq_D}, \quad \text{Inf}_D := \text{Inf}_{\not\leq_D}, \quad \text{w-Inf}_D := \text{Inf}_{\not\leq_D}.$$

3.2.2 Properties of Conically Induced Orderings

We investigate the properties of the orderings $\leq_D$, $\not\leq_D$ and $\not\leq_D$ characterized by D on the space $\mathcal{Y}$. Following Sect. 1.2, the respective constant conical domination structures $\mathcal{D}$ are, respectively, D , $(-D \setminus \{0\})^c$ and $(-\text{int } D)^c$, whence we may write:

$$\begin{aligned} a \leq_D b &\iff b - a \in \mathcal{D} = D, \\ a \not\leq_D b &\iff b - a \in \mathcal{D} = (-D \setminus \{0\})^c, \\ a \not\leq_D b &\iff b - a \in \mathcal{D} = (-\text{int } D)^c. \end{aligned}$$

3.5 Proposition:

Let $D \subset \mathcal{Y}$ be a pointed convex cone with $0 \in D$. Then

- (i) $\leq_D$ satisfies reflexivity, transitivity and antisymmetry on $\mathcal{Y}$,
- (ii) $\not\leq_D$ and $\not\leq_D$ satisfy reflexivity, negative transitivity and negative asymmetry on $\mathcal{Y}$.

Proof. From Prop. 3.4, $\leq_D$ forms a partial ordering, showing the first assertion by Def. 1.2, and $\leq_D$ and $<_D$ strict partial orderings on $\mathcal{Y}$. Then, both inverse relations $\geq_D$ and $>_D$ form strict partial orderings on $\mathcal{Y}$, by definition irreflexive, transitive and asymmetric. By Lem. 1.1 (i),(ii) and (iii), $\not\leq_D$ and $\not>_D$ then satisfy reflexivity, negative transitivity and negative asymmetry. This completes the proof. $\square$

Note that whenever $\text{int } D = \emptyset$, the relation $>_D$ has an empty domination structure such that, $\forall a, b \in \mathcal{Y} : a \not>_D b$ always holds. Therefore, for $\not>_D$, we will want to assume that D has a nonempty interior. For the other two orderings, we assume D is neither empty, nor $\{0\}$.

3.2.3 Common Globally Smallest and Greatest Elements

In order to introduce infima with respect to the different orderings induced by D , we must first define a globally smallest and a globally greatest element for $\mathcal{Y}$, with respect the different orderings, that satisfy the conditions established in Sect. 2.1. We use the following relation:

3.6 Proposition:

Let $D \subset \mathcal{Y}$ be a pointed convex cone with $0 \in D$. Then

$$\forall a, b \in \mathcal{Y} : a \leq_D b \implies a \not\leq_D b \implies a \not>_D b,$$

$$\text{and } a >_D b \implies a \geq_D b \implies a \not\leq_D b.$$

Proof. By D being pointed, convex and $0 \in D$, we obtain $D \cap -D = \{0\}$, whence $D \cap (-D \setminus \{0\}) = \emptyset$. Then,

$$D \subseteq (-D \setminus \{0\})^c \subseteq (-\text{int } D)^c,$$

from $-D \setminus \{0\} \supseteq -\text{int } D$. The result then follows from

$$\begin{aligned} a \leq_D b &\iff b - a \in D, \\ a \not\leq_D b &\iff b - a \in (-D \setminus \{0\})^c, \\ a \not>_D b &\iff b - a \in (-\text{int } D)^c. \end{aligned}$$

The second assertion is the contrapositive statement of the first. $\square$

In view of Prop. 3.6, defining $+\infty_D$ and $-\infty_D$ as the respectively globally smallest and globally greatest elements for $\mathcal{Y}$ with respect to $<_D$ automatically makes them respectively globally smallest and globally greatest elements for $\mathcal{Y}$ with respect to $\leq_D$, $\not\leq_D$ and $\not\geq_D$.

3.7 Proposition: *Common Globally Smallest and Greatest Elements*

Let $D \subset \mathcal{Y}$ be a pointed convex cone with $0 \in D$ and $\bar{\mathcal{Y}}$ be the extension of $\mathcal{Y}$ by $-\infty_D$ and $+\infty_D$, chosen such that

$$\forall y \in \mathcal{Y} : \quad -\infty_D <_D y <_D +\infty_D.$$

Then, for any $R \in \{\leq_D, \not\leq_D, \not\geq_D\}$,

$$\forall y \in \mathcal{Y} : \quad -\infty_D R y R +\infty_D \wedge +\infty_D R y R -\infty_D.$$

Proof. From Prop. 3.6, we can infer that:

$$\begin{aligned} -\infty_D <_D y <_D +\infty_D &\implies -\infty_D \leq_D y \leq_D +\infty_D \\ &\implies -\infty_D \not\leq_D y \not\leq_D +\infty_D, \end{aligned}$$

and by $<_D$ asymmetric, as a strict partial ordering by Prop. 3.4,

$$\begin{aligned} -\infty_D \not\geq_D y \not\geq_D +\infty_D &\implies -\infty_D \not\leq_D y \not\leq_D +\infty_D \\ &\implies -\infty_D \leq_D y \leq_D +\infty_D. \end{aligned}$$

This completes the proof. $\square$

Constructing $\pm\infty_D$ following Prop. 3.7 satisfies part of the assumptions made on $\pm\infty_R$ in Sect. 2.1 for $R \in \{\leq_D, \not\leq_D, \not\geq_D\}$. It remains to guarantee that $\mathcal{Y}$ does not already contain any globally greatest nor least element with respect to any of these orderings. By irreflexivity of $\leq_D$ and $<_D$, this is already satisfied for $\not\leq_D$ and $\not\geq_D$, as no element of $\mathcal{Y}$ can be smaller (*e.g.* in the sense of $\not\leq_D$) and simultaneously not larger (*e.g.* in the sense of $\leq_D$) than itself. Assuming that

$$\nexists l^*, u^* \in \mathcal{Y} : l^* \leq_D y \leq_D u^*, \forall y \in \mathcal{Y},$$

assures that $\mathcal{Y}$ contains neither a globally smallest nor globally greatest element with respect to $\leq_D$. Essentially, the above guarantees a sort of unboundedness of $\mathcal{Y}$ with respect to $\leq_D$. The importance of these assumptions

are addressed in Remark 2.15. In the following, we will consider the extended space $\bar{\mathcal{Y}} := \mathcal{Y} \cup \{\pm\infty_D\}$. Note that complements are still taken with respect to $\mathcal{Y}$, not $\bar{\mathcal{Y}}$.

3.3 Infimum for Dominant Points

The first case we consider is that of dominant points on the extended topological vector space $\bar{\mathcal{Y}}$ with $R = \leq_D$. Let $D \subseteq \mathcal{Y}$ be a convex, pointed cone containing zero. We define the *strong minimum* and *strong infimum* operators with respect to the ordering cone D by

$$\text{s-}\text{Min}_D := \text{Min}_{\leq_D} \quad \text{and} \quad \text{s-}\text{Inf}_D := \text{Inf}_{\leq_D}.$$

From Prop. 3.4, $\leq_D$ forms a partial ordering on $\mathcal{Y}$. By Prop. 1.5 and Prop. 1.9, the infimum of any partial ordering that is translation invariant, positively responsive and represented by a domination structure, is a strong infimum with respect to its constant domination structure $\mathcal{D}(y) \equiv D, \forall y \in \mathcal{Y}$.

3.8 Definition: *Dominating Lower Bounds*

Let $Y \subseteq \bar{\mathcal{Y}}$. Define the sets

$$\begin{aligned} A_{\leq_D}(Y) &:= \{l \in \mathcal{Y} : l \leq_D y, \forall y \in Y\}, \\ B_{\leq_D}(Y) &:= \{u \in \mathcal{Y} : l \leq_D u, \forall l \in A_{\leq_D}(Y)\}. \end{aligned}$$

Then, $l \in \mathcal{Y}$ is called:

- dominating Y if $l \in A_{\leq_D}(Y)$,
- strong infimal point of Y if $l \in A_{\leq_D}(Y) \cap B_{\leq_D}(Y)$,
- dominant point of Y if $l \in A_{\leq_D}(Y) \cap B_{\leq_D}(Y) \cap Y$.

Dominating lower bounds harbor some very useful properties. The following lemma characterizes the existence and uniqueness of strong infimal points.

3.9 Lemma:

For any $Y \subseteq \overline{\mathcal{Y}}$,

$$A_{\leq D}(Y) \cap B_{\leq D}(Y) = \begin{cases} \{\hat{y}\} & \text{if } \exists \hat{y} \in \mathcal{Y} : A_{\leq D}(Y) = \hat{y} - D, \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. We break the proof down in three states of $A_{\leq D}(Y)$ and show equivalence each time:

- Whenever $A_{\leq D}(Y) = \emptyset$, by $0 \in D$, any $\hat{y} \in \mathcal{Y}$ chosen such that $A_{\leq D}(Y) = \hat{y} - D$ would need to lie in $A_{\leq D}(Y)$, yielding a contradiction, whence the assertion trivially holds.
- Else, for $A_{\leq D}(Y) = \mathcal{Y} \Leftrightarrow B_{\leq D}(Y) = \emptyset$ by Lem. 2.14 (iii), $A_{\leq D}(Y) \cap B_{\leq D}(Y) = \emptyset$. Assume $\exists \hat{y} \in \mathcal{Y} : A_{\leq D}(Y) = \mathcal{Y} = \hat{y} - D$. Then, $\hat{y} \geq_D y, \forall y \in \mathcal{Y}$ and $\hat{y}$ is a globally greatest element of $\mathcal{Y}$ with respect to $\leq_D$. However, by assumption, no globally greatest element of $\mathcal{Y}$ can lie in $\mathcal{Y}$, disproving the existence of $\hat{y}$.
- Assume now $\emptyset \neq A_{\leq D}(Y) \neq \mathcal{Y}$. If $A_{\leq D}(Y) = \hat{y} - D$ for some $\hat{y} \in \mathcal{Y}$, then, by $0 \in D$, $\hat{y} \in A_{\leq D}(Y)$ and by $l \in \hat{y} - D$, we have $l \leq_D \hat{y}, \forall l \in A_{\leq D}(Y)$, whence $\hat{y} \in B_{\leq D}(Y)$ and

$$\hat{y} \in A_{\leq D}(Y) \cap B_{\leq D}(Y).$$

Choose some other $\hat{y}' \in A_{\leq D}(Y) \cap B_{\leq D}(Y)$. Then, $\hat{y} \leq_D \hat{y}'$ and $\hat{y}' \leq_D \hat{y}$ must hold, whence, by antisymmetry of $\leq_D$,

$$\hat{y}' = \hat{y} \implies A_{\leq D}(Y) \cap B_{\leq D}(Y) = \{\hat{y}\}.$$

Now assume $A_{\leq D}(Y)$ is not a translate of $-D$. Then

$$\nexists \hat{y} \in A_{\leq D}(Y) : l \leq_D \hat{y}, \forall l \in A_{\leq D}(Y),$$

i.e. $A_{\leq D}(Y)$ contains no element that is also an upper bound to the lower bounds, whence $A_{\leq D}(Y) \cap B_{\leq D}(Y) = \emptyset$.

This concludes the proof. □

We reformulate $A_{\leq_D}(Y)$ and $B_{\leq_D}(Y)$ using Prop. 3.1, and proceed to characterize strong minima and strong infima.

3.10 Proposition:

For any $Y \subseteq \mathcal{Y}$,

$$\begin{aligned} A_{\leq_D}(Y) &= \bigcap_{y \in Y} (y - D), \\ B_{\leq_D}(Y) &= \{u \in \mathcal{Y} : \bigcap_{y \in Y} (y - D) \subseteq (u - D)\}. \end{aligned}$$

Proof. Follows from $R = \leq_D$, $\mathcal{D} = D$ and Prop. 3.1. $\square$

Observe that the above characterizations do not accept sets from the extended space $\overline{\mathcal{Y}}$. This is due to domination structures being only defined on $\mathcal{Y}$ and not $\overline{\mathcal{Y}}$. One may refer to Rem. 3.2 to circumvent this restriction. The following characterization of strong minima and infima avoids this constraint.

3.11 Theorem:

Let $Y \subseteq \overline{\mathcal{Y}}$ and $Y' = Y \cap \mathcal{Y}$. Then the strong minimum of Y with respect to D is given by

$$\text{s-Min}_D Y = \begin{cases} \{\hat{y}\} & \text{if } -\infty_D \notin Y \text{ and} \\ & \exists \hat{y} \in Y' : \bigcap_{y \in Y'} (y - D) = \hat{y} - D, \\ \emptyset & \text{otherwise.} \end{cases}$$

and the strong infimum of Y with respect to D by

$$\text{s-Inf}_D Y = \begin{cases} \{-\infty_D\} & \text{if } \bigcap_{y \in Y'} (y - D) = \emptyset \text{ or } -\infty_D \in Y, \\ \{+\infty_D\} & \text{if } Y' = \emptyset \text{ and } -\infty_D \notin Y, \\ \{\hat{y}\} & \text{if } -\infty_D \notin Y \text{ and} \\ & \exists \hat{y} \in \mathcal{Y} : \bigcap_{y \in Y'} (y - D) = \hat{y} - D, \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. For the strong infimum, we use Cor. 2.23 and $R = \leq_D$ as a starting

point, *i.e.*

$$\text{s-Inf}_D Y = \begin{cases} \{-\infty_D\} & \text{if } B_{\leq_D}(Y') = \mathcal{Y} \text{ or } -\infty_D \in Y, \\ \{+\infty_D\} & \text{if } Y' = \emptyset \text{ and } -\infty_D \notin Y, \\ \text{s-Min}_D B_{\leq_D}(Y') & \text{otherwise.} \end{cases}$$

In the first case, $B_{\leq_D}(Y') = \mathcal{Y} \Leftrightarrow A_{\leq_D}(Y') = \emptyset$ by Lem. 2.14 (ii). In the last case, $\text{s-Min}_D B_{\leq_D}(Y') = A_{\leq_D}(Y') \cap B_{\leq_D}(Y')$, as shown in the proof of Prop. 2.22. We explicitly add that $-\infty_D \notin Y$ must hold, such that, by Prop. 2.19, $A_{\leq_D}(Y') = A_{\leq_D}(Y)$ and $B_{\leq_D}(Y') = B_{\leq_D}(Y)$. Lastly, we can apply Lem. 3.9 to obtain the assertion.

For the characterization of strong minima, we apply Def. 2.5 and Cor. 2.20. By explicit assumption that $-\infty_D \notin Y$, we can reuse the above argument:

$$\text{s-Min}_D Y' = A_{\leq_D}(Y') \cap B_{\leq_D}(Y') \cap Y' = \{\hat{y}\} \cap Y',$$

whenever $\exists \hat{y} \in \mathcal{Y} : \bigcap_{y \in Y'} (y - D) = \hat{y} - D$. This concludes the proof. $\square$

Lem. 3.9 is also used to derive uniqueness and representativity of strong infima, upon existence. The following result gathers the most important properties of strong infima.

3.12 Theorem:

Let $Y \subseteq \overline{\mathcal{Y}}$. Then, the following are equivalent:

- (i) $\text{s-Inf}_D Y \neq \emptyset$,
- (ii) $\text{s-Inf}_D Y \in [Y]$,
- (iii) $\text{s-Inf}_D Y$ is a singleton.

Additionally, $\text{s-Inf}_D Y = \{\hat{y}\}$ for some $\hat{y} \in \mathcal{Y}$ if and only if $A_{\leq_D} = \hat{y} - D$.

Proof. The relation $R = \leq_D$ forms a partial ordering on $\mathcal{Y}$, hence antisymmetric and transitive. From Prop. 2.26 and Thm. 2.36 (i), the strong infimum is not empty, if and only if it is a singleton and if and only if it is representative. This implies that

$$\text{s-Inf}_D Y \in \{\{-\infty_D\}, \{+\infty_D\}, \{\hat{y}\}\}$$

for some $\hat{y} \in \mathcal{Y}$. For the last assertion, we may thus assume

$$\text{s-Inf}_D = A_{\leq_D}(Y) \cap B_{\leq_D}(Y),$$

which, by Lem. 3.9, is equal to $\{\hat{y}\}$ for some $\hat{y} \in \mathcal{Y}$ exactly when $A_{\leq_D} = \hat{y} - D$. $\square$

Thm. 3.12 yields sufficient and necessary conditions for the existence and representativity of strong infima with respect to a D . It immediately follows that only nonempty infima are self-infimal. Different choices of ordering cones may impact this existence of infimal points. However, if the choice of ordering cone can be made such that $A_{\leq_D}(Y)$ is always translate of $-D$, then not only is the existence of $\text{s-Inf}_D Y$ guaranteed, but also its representativity and uniqueness. The geometry of the ordering cone is thus key to the existence of strong infimal points. We proceed with two examples where the ordering cone for $\leq$, respectively on $\mathbb{R}^n$ with $n = 1$ and $n > 1$, displays such a compatible geometry.

3.13 Example: We use Thm. 3.12 to show that for any $Y \subseteq \overline{\mathbb{R}}$ there always exists a unique infimum. We will use, $\mathcal{Y} = \mathbb{R}$ and $R = \leq$, which can be represented by the constant conical domination map

$$\mathcal{D} \equiv D = \mathbb{R}_{\geq} = [0, +\infty).$$

Since D is a convex pointed cone containing zero, we can use $\text{Inf}_{\leq} Y = \text{s-Inf}_D Y$. Then

$$A_{\leq}(Y) = \bigcap_{y \in Y} (y - D) = \bigcap_{y \in Y} (-\infty, y],$$

as per Prop. 3.10, and a closed interval as the intersection of closed intervals. Thus $A_{\leq}(Y)$ is either empty, $\mathbb{R}$ or an interval of the form $(-\infty, \hat{y}]$ for some $\hat{y} \in \mathbb{R}$. By Thm. 3.12, $\text{s-Inf}_D Y = \{\hat{y}\} =: \{\inf Y\}$. Additionally, this also proves representativity and existence of the single dimensional real infimum.

For higher dimensional ordering cones, it may be difficult to guarantee that all intersections of a potentially infinite number of translates of $-D$ form a translate of the same cone. The following example relies on the existence of the one dimensional real infimum of any set in $\overline{\mathbb{R}}$, as shown in Example 3.13, to show existence, uniqueness and representativity of infima on $\mathbb{R}^n$ with respect to $\leq$.

3.14 Example: Let $\mathcal{Y} = \mathbb{R}^n$ and $R = \leq$, which can be represented by the constant conical domination map $\mathcal{D} \equiv D = \mathbb{R}_{\geq}^n$, for $\mathbb{R}_{\geq}^n$ the non-negative orthant in $\mathbb{R}^n$. Then, for any $a, b \in \mathbb{R}^n : a \leq b \Leftrightarrow a_i \leq b_i, \forall i = 1, \dots, n$. As in Example 3.13, by $D = \mathbb{R}_{\geq}^n$ being a pointed convex cone including zero, $\leq$ forms a partial ordering on $\overline{\mathbb{R}}^n$.

The space $\mathbb{R}^n$ is extended by $\pm\infty_D$ to $\overline{\mathbb{R}}^n := \mathbb{R}^n \cup \{\pm\infty_D\}$. In this context, $\overline{\mathbb{R}}^n \neq \mathbb{R} \times \dots \times \mathbb{R}$, as a point like $(-\infty, 0, \dots, 0)^\top \notin \overline{\mathbb{R}}^n$, since it is not globally

smallest with respect to $\leq$ on $\mathbb{R}^n$. The analogous holds for $+\infty_D$. For this reason, we can write

$$-\infty_D := (-\infty, \dots, -\infty)^\top \quad \text{and} \quad +\infty_D := (+\infty, \dots, +\infty)^\top.$$

Let $Y \subseteq \overline{\mathbb{R}}^n$. Let Y_i be the projections of Y onto the n dimensions, that is $Y_i := \{y_i \in \overline{\mathbb{R}} : y \in Y\}, i = 1, \dots, n$. We proceed to show that:

$$\text{s-Inf}_D Y = \begin{cases} \{-\infty_D\} & \text{if } \exists i \in I : \inf Y_i = -\infty, \\ \{+\infty_D\} & \text{if } Y = \emptyset \text{ or } \exists i \in I : \inf Y_i = +\infty, \\ \{(\inf Y_1, \dots, \inf Y_n)^\top\} & \text{otherwise.} \end{cases}$$

with $(\inf Y_1, \dots, \inf Y_n)^\top$ the componentwise infimum of Y . Assume $Y \neq \emptyset$:

- If $\inf Y_i = -\infty$, then $\nexists l_i \leq y_i, \forall y_i \in Y_i$ and thus $A_{\leq}(Y) = \emptyset$ and $\text{s-Inf}_D Y = \{-\infty_D\}$.
- If $\inf Y_i = +\infty$, then Y_i can only be containing $+\infty$, as we have assumed Y is nonempty. Thus, $y_i = +\infty$ for any $y \in Y$. By $Y \subseteq \overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\pm\infty_D\}$, $y = +\infty_D$ must hold and $\text{s-Inf}_D Y = \{+\infty_D\}$.
- We may now assume $\inf Y_i \in \mathbb{R}, \forall i \in \{1, \dots, n\}$. Then, $(\inf Y_1, \dots, \inf Y_n)^\top \in A_{\leq}(Y)$, since it is a componentwise lower bound. The componentwise infimum lies also in $B_{\leq}(Y)$, as any $l \in A_{\leq}(Y)$ must satisfy $l_i \leq y_i, \forall y_i \in Y_i \Rightarrow l_i \leq \inf Y_i$, by definition. Hence, the componentwise infimum is an infimum of Y with respect to R , and by Thm. 3.12, uniquely so.

Altogether, the infimum with respect to $\leq$ in $\mathbb{R}^n$ always uniquely exists and is representative, by Thm. 3.12.

This happens to be exactly the generalized infimum introduced by Brumelle [5], defined directly as the componentwise infimum. However, the definition is not transposable to infima that are not decomposable.

Thm. 3.11 and Thm. 3.12 can especially be used to construct examples, where, for given sets $Y \subset \mathcal{Y}$, the strong infimum with respect to D does not exist. For this, we use two points $y^1, y^2 \in \mathcal{Y}$ placed in space such that the intersection $(y^1 - D) \cap (y^2 - D)$ is not a translate of D , violating the condition for the existence of strong infimal elements. The following geometric example uses a closed, pointed, convex ordering cone with square cross-sections.

3.15 Example: Let $R = \leq_D$ be characterized by the closed convex pointed ordering cone (with square cross-section)

$$D = \text{cone}([-1, 1] \times \{2\} \times [-1, 1]) \cup \{0\},$$

and let $Y = \{y^1, y^2\}$ with $y^1 = (2, 2, 1)^\top$ and $y^2 = (2, 2, 2)^\top$. Then,

$$(y^1 - D) \cap (y^2 - D) = ([3/2, 5/2] \times \{1\} \times \{3/2\}) - D,$$

such that, by Thm. 3.12, $\text{s-Inf}_D Y = \emptyset$. The set $B_{\leq_D}(Y)$ can then also be constructed by using Prop. 3.10, as displayed in Fig. 3.3.

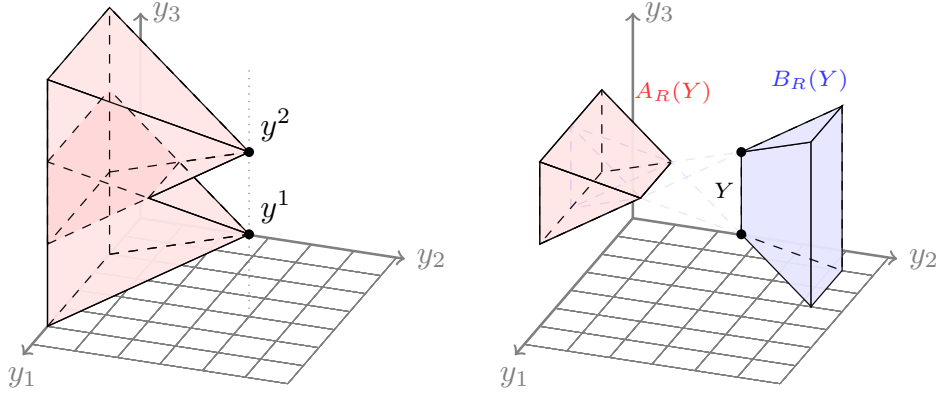


Figure 3.3: Empty Strong infimum.

3.3.1 Comparison to Komuro's Infimum

Different approaches, specifically tailored to infima and suprema with respect to translation invariant and positively responsive partial orderings, can be derived to circumvent inexistent strong infima, like in Example 3.15.

Komuro [14] introduced a generalization of infimum and supremum for real vector spaces partially ordered by an ordering cone D and investigated their properties. Although the focus of the original paper was put on supremum, we present its equivalent infimum for better comparison. Similarly to our approach, the infimal elements of a set Y are the greatest lower bounds to that set. Komuro [14] defined these as follows:

$$\text{Inf}_{\leq_D}^K Y := \{\hat{y} \in \mathcal{Y} : \hat{y} \leq_D l, l \in A_{\leq_D}(Y) \Rightarrow \hat{y} = l\},$$

for some $Y \subseteq \mathbb{R}^k$, with D some pointed convex cone containing zero and a nonempty interior. Note that no notion of $\pm\infty_D$ are addressed. Based on the above, we can assume the following generalization for an arbitrary binary relation R for some $Y \subseteq \mathcal{Y}$:

$$\text{Inf}_R^K Y := \{\hat{y} \in \mathcal{Y} : \hat{y} R l, l \in A_R(Y) \Rightarrow \hat{y} = l\}.$$

However related, Komuro's definition is not equivalent to our definition of infimal elements: the interpretation of *greatest* in “greatest lower bounds” differs. While we assumed a greatest lower bound $\hat{y}$ with respect to R satisfies $l R \hat{y}, \forall l \in A_R(Y)$ and $\hat{y} \in A_R(Y)$, Komuro assumes $\hat{y} R l, \forall l \in A_R(Y) \setminus \{\hat{y}\}$ to be sufficient. We can interpret this as $\hat{y}$ being a lower bound only smaller to itself. As such, it is not an upper bound to the set of lower bounds in the sense of the ordering.

We show the equivalence of Komuro's approach with our restricted infimum in the special case of R being antisymmetric (given, for example, when it forms a partial ordering on $\mathcal{Y}$, as is the case with $\leq_D$) and there exists an infimal element in $\mathcal{Y}$. We do not extend Komuro's infimum to map into the extended space $\bar{\mathcal{Y}}$, whence we'll assume, for comparison, $\text{Inf}_R Y \neq \{\pm\infty_R\}$.

3.16 Proposition:

Let R be antisymmetric and $Y \in \bar{\mathcal{Y}}$. Then, if $A_R(Y) \cap B_R(Y) \neq \emptyset$,

$$\text{Inf}_R^K Y = \text{Inf}_R Y = \{\hat{y}^*\},$$

for some $\hat{y}^* \in \mathcal{Y}$ and $\text{Inf}_R^K Y := \{\hat{y} \in \mathcal{Y} : \hat{y} R l, l \in A_R(Y) \Rightarrow \hat{y} = l\}$.

Proof. Since we assume $\exists \hat{y}^* \in A_R(Y) \cap B_R(Y)$, we know $\emptyset \neq B_R(Y) \neq \mathcal{Y}$, whence $\text{Inf}_R Y \neq \{\pm\infty_R\}$, by Prop. 2.17. Then, by R antisymmetric and Prop. 2.26, $\text{Inf}_R Y = \{\hat{y}^*\}$. We wish to show:

$$\{\hat{y}^*\} = A_R(Y) \cap B_R(Y) = \{\hat{y} \in \mathcal{Y} : \hat{y} R l, l \in A_R(Y) \Rightarrow \hat{y} = l\} =: \text{Inf}_R^K Y.$$

We show that $\hat{y}^* \in \text{Inf}_R Y$ is an infimal element of the Komuro infimum. We already know $\hat{y}^* \in A_R(Y)$ and by $\hat{y}^* \in B_R(Y)$, $l R \hat{y}^*, \forall l \in A_R(Y)$. Thus, whenever $\hat{y}^* R l$ for some $l \in A_R(Y)$, by antisymmetry of R and existence of $\hat{y}^*$:

$$\hat{y}^* R l \wedge l R \hat{y}^* \implies \hat{y}^* = l.$$

Thus $\text{Inf}_R Y \subseteq \text{Inf}_R^K Y$. Let now $\hat{y} \in \text{Inf}_R^K Y$, which, by the above, exists as soon as $\text{Inf}_R Y \neq \emptyset$. Since $\hat{y} \in A_R(Y)$ always holds, assume $\hat{y} \notin B_R(Y)$. Then, $\hat{y}^* \not R \hat{y}$, since $\hat{y}^* \in A_R(Y)$. However, by $\hat{y}^* \in B_R(Y)$, $\hat{y} R \hat{y}^*$ holds, and by $\hat{y} \in \text{Inf}_R^K Y$:

$$\hat{y} R \hat{y}^* \wedge \hat{y}^* \in A_R(Y) \implies \hat{y} = \hat{y}^*,$$

contradicting $\hat{y} \notin B_R(Y)$. Thus, $\hat{y} = \hat{y}^*$, concluding the proof. $\square$

Prop. 3.16 shows how our infimum is the same as Komuro's, given $\text{Inf}_R Y \subseteq \mathcal{Y}$. Differences arise when $\text{Inf}_R Y = \emptyset$. In this regard, Komuro's infimum is more lenient. However, in these cases, representativity and all properties related to representativity, become an issue. The following example shows how Komuro's infimum fails at self-infimality.

3.17 Example: Let $\mathcal{Y} = \mathbb{R}^3$, $Y = \{(2, 2, 1)^\top, (2, 2, 2)^\top\}$ and $R = \leq_D$, characterized by the same ordering cone as in Example 3.15. Then,

$$\begin{aligned} \text{s-Inf}_D Y &= \emptyset, \\ \text{Inf}_{\leq_D}^K Y &= \{y \in [3/2, 5/2] \times \{1\} \times \{3/2\}\}, \\ \text{Inf}_{\leq_D}^K \text{Inf}_{\leq_D}^K Y &= \{y \in \{2\} \times \{0\} \times [1, 2]\}, \end{aligned}$$

showing especially that $\text{Inf}_{\leq_D}^K Y \neq \text{Inf}_{\leq_D}^K \text{Inf}_{\leq_D}^K Y$. (See Fig. 3.4)

3.4 Infimum for Nondominated and Weakly Nondominated Points

Let $R \in \{\not\leq_D, \not\prec_D\}$ for the pointed convex ordering cone D . Then, R^c forms a strict partial ordering on $\mathcal{Y}$ (Prop. 3.4) and R is reflexive, negatively transitive and negatively asymmetric (Prop. 3.5). Since R is negatively transitive, Thm. 2.37 applies as a necessary and sufficient condition for representativity. That is, specifically for the infimum, $\text{Inf}_R Y \in [Y]$ and $\text{Inf}_R Y \subseteq \mathcal{Y}$ if and only if

$$\forall y \in Y, \exists \hat{y} \in \text{Inf}_R Y : \hat{y} \notin y + \mathcal{D},$$

which simplifies to:

$$Y \subseteq \text{Inf}_R Y - \mathcal{D}^c.$$

The analogous holds for minima. Throughout this section, we may assume $\text{int } D \neq \emptyset$, as else $\not\prec_D$ is constantly true. The ordering $\not\prec_D$ is also a special case of $\not\leq_D$: both become equivalent as soon as D is open.

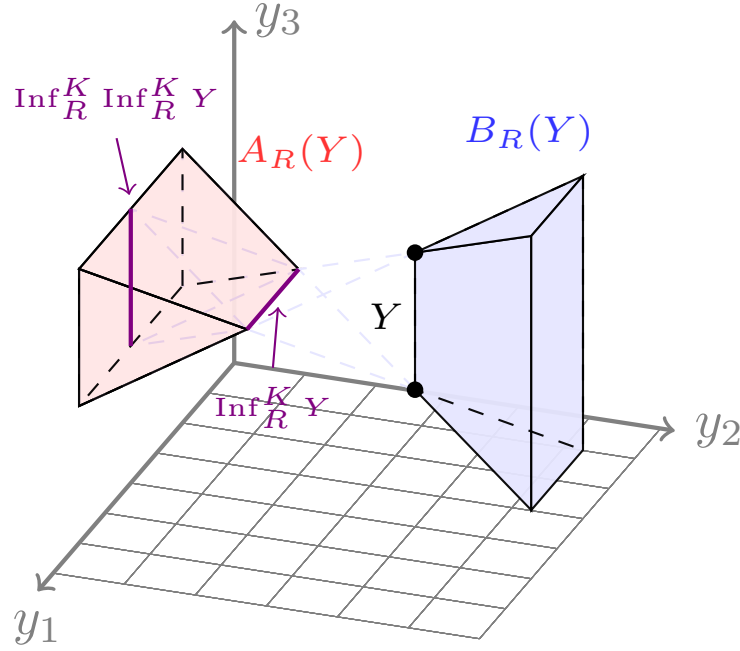


Figure 3.4: Komuro's Infimum and Absence of Self-Infimality.

We will show basic reformulations for the (nondominated) infimum with respect to D and use examples to show its weaknesses. Then, we will examine how the weak infimum is especially useful, in that it is always representative of the set it is taken over, making it the prime choice for conjugate vector duality. Lastly, we will go back to the nondominated infimum, and compare it to a different approach by Gros [8], which displays great similarities to some characterization of the weak infimum.

3.4.1 Infimum for Nondominated Points

Consider nondominated points on the extended topological vector space $\overline{\mathcal{Y}}$. Let $R = \not\prec_D$, with $D \subseteq \mathcal{Y}$ a pointed convex cone. Then, $\not\prec_D$ is characterized by the constant conical domination structure $\mathcal{D} = (-D \setminus \{0\})^c$ and satisfies reflexivity, negative transitivity and negative asymmetry by Prop. 3.5. The *minimum* and *infimum* operators with respect to D are defined as:

$$\text{Min}_D := \text{Min}_{\not\prec_D} \quad \text{and} \quad \text{Inf}_D := \text{Inf}_{\not\prec_D}.$$

This notation is consistent with [25]. To avoid confusion, we may explicitly refer to Min_D and Inf_D as the *nondominated minimum* and *nondominated*

infimum, respectively. We use Def. 2.2 to introduce *nondominated lower bounds*.

3.18 Definition: Nondominated Lower Bounds

Let $Y \subseteq \overline{\mathcal{Y}}$. Define the sets

$$\begin{aligned} A_{\not\leq_D}(Y) &:= \{l \in \mathcal{Y} : l \not\leq_D y, \forall y \in Y\}, \\ B_{\not\leq_D}(Y) &:= \{u \in \mathcal{Y} : l \not\leq_D u, \forall l \in A_{\not\leq_D}(Y)\}. \end{aligned}$$

Then, $l \in \mathcal{Y}$ is called:

- nondominated by Y if $l \in A_{\not\leq_D}(Y)$,
- (nondominated) infimal point of Y if $l \in A_{\not\leq_D}(Y) \cap B_{\not\leq_D}(Y)$,
- (nondominated) minimal or nondominated point of Y if $l \in A_{\not\leq_D}(Y) \cap B_{\not\leq_D}(Y) \cap Y$.

Notice the difference between a point being nondominated *by* Y and a point being a nondominated point *of* Y . The naming is coherent since, by Prop. 2.16, the set of nondominated points of Y is $\text{Min}_{\not\leq_D} Y = \text{Min}_D Y = A_{\not\leq_D}(Y) \cap Y$, that is the points nondominated by and elements of Y .

3.19 Proposition:

For any $Y \subseteq \mathcal{Y}$,

$$\begin{aligned} A_{\not\leq_D}(Y) &= (Y + D \setminus \{0\})^c \\ &= \{l \in \mathcal{Y} : Y \cap (l - D \setminus \{0\}) = \emptyset\}, \\ B_{\not\leq_D}(Y) &= \{u \in \mathcal{Y} : (u + D \setminus \{0\}) \subseteq (Y + D \setminus \{0\})\}. \end{aligned}$$

Proof. The domination structure for $\not\leq_D$ being $\mathcal{D} = (-D \setminus \{0\})^c$, the assertions follow immediately from Prop. 3.1 with $-\mathcal{D}^c = D \setminus \{0\}$. $\square$

Observe how, using Prop. 3.19, some $\hat{y} \in Y$ is a minimal point of Y if and only if

$$Y \cap (\hat{y} - D \setminus \{0\}) = \emptyset,$$

which is a common characterization of nondominated points, *e.g.* [11, Def. 4.1 ff.]. We characterize the (nondominated) minima and infima.

3.20 Theorem:

Let $Y \subseteq \overline{\mathcal{Y}}$ and $Y' = Y \cap \mathcal{Y}$. Then the minimum of Y with respect to D is given by

$$\text{Min}_D Y = \begin{cases} \emptyset & \text{if } -\infty_D \in Y, \\ \{y \in Y' : Y' \cap (y - D \setminus \{0\}) = \emptyset\} & \text{otherwise,} \end{cases}$$

and the infimum of Y with respect to D by

$$\text{Inf}_D Y = \begin{cases} \{-\infty_D\} & \text{if } B_{\not\leq D}(Y') = \mathcal{Y} \text{ or } -\infty_D \in Y, \\ \{+\infty_D\} & \text{if } Y' = \emptyset \text{ and } -\infty_D \notin Y, \\ \text{Min}_D B_{\not\leq D}(Y') & \text{otherwise,} \end{cases}$$

with $B_{\not\leq D}(Y') = \{y \in \mathcal{Y} : (y + D \setminus \{0\}) \subseteq (Y' + D \setminus \{0\})\}$.

Proof. The first part follows directly from Cor. 2.20 with $\text{Min}_{\not\leq D}(Y') = A_{\not\leq D}(Y') \cap Y'$ from Prop. 2.16 and using the reformulation of $A_{\not\leq D}(Y')$ from Prop. 3.19. The second part follows from Cor. 2.23 combined with Prop. 3.19. $\square$

3.21 Remark: The characterization of Min_D and Inf_D in Thm. 3.20 only uses $A_{\not\leq D}(Y')$ and $B_{\not\leq D}(Y')$ for $Y' \subseteq \mathcal{Y}$. Naturally, the two sets can also be formulated directly for $Y \subseteq \overline{\mathcal{Y}}$, akin to Remark 3.2:

$$\begin{aligned} A_{\not\leq D}(Y) &= \begin{cases} \emptyset & \text{if } -\infty_D \in Y \\ A_{\not\leq D}(Y \setminus \{+\infty_D\}) & \text{otherwise,} \end{cases} \\ B_{\not\leq D}(Y) &= \begin{cases} \mathcal{Y} & \text{if } -\infty_D \in Y \\ B_{\not\leq D}(Y \setminus \{+\infty_D\}) & \text{otherwise.} \end{cases} \end{aligned}$$

Then, all characterizations of the operators Min_R and Inf_R are usable for $R = \not\leq_D$. For instance, by Prop. 2.16 and Prop. 2.22:

$$\begin{aligned} \text{Min}_D Y &= A_{\not\leq D}(Y) \cap Y, \\ \text{Inf}_D Y &= \begin{cases} \{-\infty_D\} & \text{if } B_{\not\leq D}(Y) = \mathcal{Y}, \\ \{+\infty_D\} & \text{if } B_{\not\leq D}(Y) = \emptyset, \\ \text{Min}_D B_{\not\leq D}(Y) & \text{otherwise.} \end{cases} \end{aligned}$$

Next, we investigate representativity of $\text{Inf}_D Y$. As already seen in Example 3.3 for $\mathcal{Y} = \mathbb{R}^2$ and $D = \mathbb{R}_{\geq}^2$, there exist sets $Y \subseteq \mathcal{Y}$ that have

non-representative infima $\text{Inf}_D Y \notin [Y]$. Furthermore, the infimum with respect to D may not exist even for simple cases, as the following example shows:

3.22 Example: Let $\mathcal{Y} = \mathbb{R}^2$ and $D = \mathbb{R}_{\geq}^2$. For the open set

$$Y = \text{int}(\text{cone}([1, 2] \times [1, 2])),$$

$A_{\not\geq D}(Y)^c = B_{\not\geq D}(Y) = \text{int } D$ and as such $\text{Inf}_D Y = \text{Min}_D Y = \emptyset$. (See Fig. 3.5)

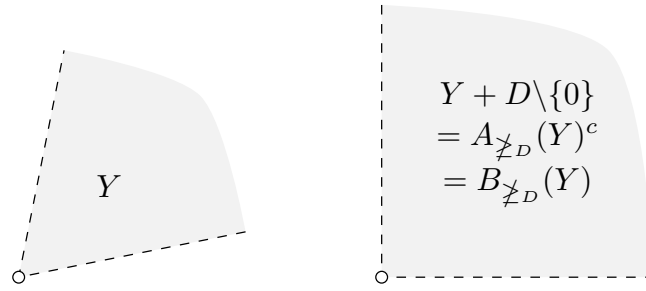


Figure 3.5: Absence of infimum with respect to D in Example 3.22.

In the context of vector optimization, given a feature space $\mathcal{X}$ and functional

$$f : \mathcal{X} \rightarrow \mathcal{Y}, \quad x \mapsto f(x),$$

with $f(X) \subseteq \mathcal{Y}$ the image of f over some $X \subseteq \mathcal{X}$, the set of *efficient* solutions (with respect to the ordering cone D) is given by

$$\mathcal{E}(f, X) := \{x \in \mathcal{X} : f(x) \in \text{Min}_D f(X)\}.$$

Hence, results regarding the existence of nondominated points (and, by extension, of efficient solutions), are directly applicable. See [4, 6, 11, 25] among others.

One such result is the so-called *external stability* [25] (or *domination property* [3, 10, 17] or *nondominance boundedness* [29]) of the set of nondominated points of the image $Y = f(X)$. In Sawaragi et al. [25, Def. 3.2.6], we call a subset $S \subseteq Y$ *externally stable* if

$$\forall y \in Y \setminus S : \exists \hat{y} \in S : \quad y \in \hat{y} + D \iff y \geq_D \hat{y},$$

which corresponds to condition (4) in Thm. 2.37 (ii). In that case, for a collection of sets $(Y^i)_{i \in I}$, slightly modifying Cor. 6.1.3 from Sawaragi et al. [25], like in Thm. 2.40:

$$\text{Min}_D \bigcup_{i \in I} Y^i \subseteq \text{Min}_D \bigcup_{i \in I} \text{Min}_D Y^i. \quad (33)$$

Sawaragi et al. [25, Sect. 6.1] uses (33) to derive conjugate duality for vector and set optimization problems with regards to efficiency. If we drop the condition $S \subseteq Y$ for the domination property and use $S = \text{Inf}_D Y$, we obtain condition (3) from Thm. 2.37 (i), characterizing representative infima for negatively transitive relations like $\not\geq_D$.

Furthermore, in case of existing infimal elements, we can use $\text{Inf}_D Y = \text{Min}_D B_{\not\geq_D}(Y)$ to show that representativity of $\text{Inf}_D Y$ holds if and only if the set of (nondominated) minimal points of $B_{\not\geq_D}(Y)$ is externally stable.

3.23 Theorem:

Let $Y \subseteq \bar{\mathcal{Y}}$ and $\emptyset \neq \text{Inf}_D Y \subseteq \mathcal{Y}$. Then $\text{Inf}_D Y \in [Y]$ if and only if $\text{Min}_D B_{\not\geq_D}(Y)$ is externally stable.

Proof. From Thm. 3.20, since $\emptyset \neq \text{Inf}_D Y \subseteq \mathcal{Y}$:

$$\text{Inf}_D Y = \text{Min}_D B_{\not\geq_D}(Y) = \text{Min}_D B_{\not\geq_D}(Y') = \text{Inf}_D Y',$$

with $Y' = Y \cap \mathcal{Y}$. In the following we directly assume $Y = Y'$. By $\not\geq_D$ negatively transitive from Prop. 3.5, and Thm. 2.37, $\text{Inf}_D Y \in [Y]$ if and only if condition (3) is satisfied, *i.e.*

$$\forall y \in Y \setminus \text{Inf}_D Y : \exists \hat{y} \in \text{Inf}_D Y : y \geq_D \hat{y}.$$

If $\text{Min}_D B_{\not\geq_D}(Y)$ is externally stable,

$$\forall y \in B_{\not\geq_D}(Y) \setminus \text{Inf}_D Y : \exists \hat{y} \in \text{Inf}_D Y : y \geq_D \hat{y}, \quad (34)$$

must be satisfied. By $Y \subseteq \mathcal{Y}$ and Lem. 2.14 (i), $Y \subseteq B_{\not\geq_D}(Y)$, whence external stability implies condition (3) and representativity. Now if $\text{Inf}_D Y$ is representative, so is $B_{\not\geq_D}(Y)$. Then, $\text{Inf}_D Y = \text{Inf}_D B_{\not\geq_D}(Y) \in [Y]$ and condition (3) corresponds exactly to (34), showing external stability. This completes the proof. $\square$

Thus, representativity of $\text{Inf}_D Y \subseteq \mathcal{Y}$ is equivalent to the external stability of $\text{Min}_D B_{\not\prec_D}(Y)$ and, by Prop. 2.31, is a milder assumption than the external stability, *i.e.* representativity, of $\text{Min}_D Y$. One can apply existence results for nondominated points of the set $B_{\not\prec_D}(Y)$. However, for lack of an easier characterization of $B_{\not\prec_D}(Y)$, usability remains limited. More research is to be conducted on this topic.

3.4.2 Infimum for Weakly Nondominated Points

Consider weakly nondominated points on the extended topological vector space $\overline{\mathcal{Y}}$. Let $R = \not\prec_D$, with $D \subseteq \mathcal{Y}$ a pointed convex cone with nonempty interior (as else $\not\prec_D$ is trivially true) and $0 \in D$ (for direct application of previous results). Then, $\not\prec_D$ is characterized by the constant conical domination structure $\mathcal{D} = (-\text{int } D)^c$ and satisfies reflexivity, negative transitivity and negative asymmetry by Prop. 3.5. The *weak minimum* and *weak infimum* operators with respect to D are defined as:

$$\text{w-Min}_D := \text{Min}_{\not\prec_D} \quad \text{and} \quad \text{w-Inf}_D := \text{Inf}_{\not\prec_D}.$$

This notation is consistent with [25], but this section predominantly uses the notation from Löhne [16]. We use Def. 2.2 to introduce *weakly nondominated lower bounds*.

3.24 Definition: *Weakly Nondominated Lower Bounds*

Let $Y \subseteq \overline{\mathcal{Y}}$. Define the sets

$$\begin{aligned} A_{\not\prec_D}(Y) &:= \{l \in \mathcal{Y} : l \not\prec_D y, \forall y \in Y\}, \\ B_{\not\prec_D}(Y) &:= \{u \in \mathcal{Y} : l \not\prec_D u, \forall l \in A_{\not\prec_D}(Y)\}. \end{aligned}$$

Then, $l \in \mathcal{Y}$ is called:

- weakly nondominated by Y if $l \in A_{\not\prec_D}(Y)$,
- weak infimal point of Y if $l \in A_{\not\prec_D}(Y) \cap B_{\not\prec_D}(Y)$,
- weak minimal or weakly nondominated point of Y if $l \in A_{\not\prec_D}(Y) \cap B_{\not\prec_D}(Y) \cap Y$.

Again, notice the difference between a point being weakly nondominated *by* Y and a point being a weakly nondominated point *of* Y . The naming is coherent since, by Prop. 2.16, the set of weakly nondominated points of some

set Y is $w\text{-Min}_D Y := \text{Min}_{\not\prec_D} Y = A_{\not\prec_D}(Y) \cap Y$, that is the points weakly nondominated by and elements of Y .

3.25 Proposition:

For any $Y \subseteq \mathcal{Y}$,

$$\begin{aligned} A_{\not\prec_D}(Y) &= (Y + \text{int } D)^c \\ &= \{l \in \mathcal{Y} : Y \cap (l - \text{int } D) = \emptyset\}, \\ B_{\not\prec_D}(Y) &= \{u \in \mathcal{Y} : (u + \text{int } D) \subseteq (Y + \text{int } D)\}. \end{aligned}$$

Proof. The domination cone for $\not\prec_D$ is given by $\mathcal{D} = (-\text{int } D)^c$. The assertions follow immediately from Prop. 3.1 with $-\mathcal{D}^c = \text{int } D$. $\square$

The weak infimum presents several advantages compared to the infimum with respect to D . Following the reformulation by Löhne [16] of the original definition by Nieuwenhuis [20], we show the equivalence $B_{\not\prec_D}(Y)$ with the *upper closure* of a set $Y \subseteq \bar{\mathcal{Y}}$. In turn, this allows us to retrieve the closure property that got lost with the Min_D operator: the weak infimum of Y is the weak minimum over the upper closure of Y . For the upper closure, we use the notation from [16].

3.26 Definition: Upper Closure [16, Prop. 1.40]

Let $Y \subseteq \bar{\mathcal{Y}}$. The upper closure of $Y \subseteq \mathcal{Y}$ (with respect to D) is given by

$$\text{Cl}_+ Y := \begin{cases} \mathcal{Y} & \text{if } -\infty_D \in Y, \\ \emptyset & \text{if } Y = \{+\infty_D\}, \\ \text{cl}(Y \setminus \{+\infty_D\} + D) & \text{otherwise.} \end{cases}$$

3.27 Remark: The first definition of upper closure [16, Def. 1.39] is closely related to the characterization we obtained for $B_{\not\prec_D}(Y)$ and is immediately followed by the characterization made in [16, Prop. 1.40] which we take as definition. We show the equivalence in Prop. 3.28.

3.28 Proposition: [16, Def. 1.39, Prop. 1.40]

Let $Y \subseteq \bar{\mathcal{Y}}$. Then $B_{\not\prec_D}(Y) = \text{Cl}_+ Y$.

Proof. First, recall that by Prop. 2.19 and Prop. 3.25:

$$B_{\not\prec_D}(Y) = \begin{cases} \mathcal{Y} & \text{if } -\infty_D \in Y, \\ \{u \in \mathcal{Y} : (u + \text{int } D) \subseteq (Y' + \text{int } D)\} & \text{otherwise,} \end{cases}$$

for any $Y \subseteq \overline{\mathcal{Y}}$ and $Y' = Y \cap \mathcal{Y}$. In case $-\infty_D \in Y$ or $Y \in \{\emptyset, \{+\infty_D\}\}$, by Lem. 2.14 and Def. 3.26, $B_{\not\prec_D}(Y) = \text{Cl}_+ Y \in \{\mathcal{Y}, \emptyset\}$, respectively. Otherwise, $\text{Cl}_+ Y = \text{Cl}_+ Y'$ and it remains to show

$$B_{\not\prec_D}(Y') = \{u \in \mathcal{Y} : (u + \text{int } D) \subseteq (Y' + \text{int } D)\} = \text{cl}(Y' + D) = \text{Cl}_+ Y',$$

for $Y' \neq \emptyset$. From Lem. 2.14 (iii) we may assume $B_{\not\prec_D}(Y') \neq \emptyset$. Let $u \in B_{\not\prec_D}(Y')$ and $d \in \text{int } D$. Since D is a cone, $\forall n \in \mathbb{N} : \frac{1}{n}d \in \text{int } D$. From, $u + \text{int } D \subseteq Y' + \text{int } D$, $u + \frac{1}{n}d \in Y' + \text{int } D$ for all $n \in \mathbb{N}$. It follows that $\lim_{n \rightarrow \infty} u + \frac{1}{n}d = u$ and thus, $u \in \text{cl}(Y' + D)$. Now let $u \in \text{Cl}_+ Y' = \text{cl}(Y' + D)$. Then, by Lem. 1.10 and Lem. 1.14,

$$\begin{aligned} u + \text{int } D &\subseteq \text{cl}(Y' + D) + \text{int } D \\ &\stackrel{\text{Lem. 1.10}}{=} Y' + D + \text{int } D \\ &\stackrel{\text{Lem. 1.14}}{=} Y' + \text{int } D. \end{aligned}$$

This concludes the proof. $\square$

An alternative proof for Prop. 3.28 can also be found in [20, Thm. I-17], when substituting Y_+ for $-D$.

3.29 Theorem:

Let $Y \subseteq \overline{\mathcal{Y}}$. Then $A_{\not\prec_D}(Y) \cap B_{\not\prec_D}(Y) = \text{bd } \text{Cl}_+ Y$.

Proof. Note that whenever $\text{Cl}_+ Y \in \{\emptyset, \mathcal{Y}\}$, the boundary $\text{bd } \text{Cl}_+ Y$ is empty. Thus, in view of Lem. 2.14 and Def. 3.26, whenever $Y \in \{\emptyset, \{+\infty_D\}\}$ or $-\infty_D \in Y$, we obtain

$$A_{\not\prec_D}(Y) \cap B_{\not\prec_D}(Y) = \text{bd } \text{Cl}_+ Y = \emptyset.$$

By $\text{Cl}_+ Y = \text{Cl}_+ Y \setminus \{+\infty_D\}$ and Prop. 2.19, we now assume $\emptyset \neq Y \subseteq \mathcal{Y}$.

First, we show that $B_{\not\prec_D}(Y) + D \subseteq B_{\not\prec_D}(Y)$. Let $u \in B_{\not\prec_D}(Y)$. By Prop. 3.25, $u \in B_{\not\prec_D}(Y)$ is equivalent to $u + \text{int } D \subseteq Y + \text{int } D$. Then, by Lem. 1.14,

$$u + D + \text{int } D \stackrel{\text{Lem. 1.14}}{=} u + \text{int } D \stackrel{u \in B_{\not\prec_D}(Y)}{\subseteq} Y + \text{int } D,$$

and thus $B_{\not\prec_D}(Y) + D \subseteq B_{\not\prec_D}(Y)$.

Next, we show that $(A_{\not\prec_D}(Y))^c = \text{int } B_{\not\prec_D}(Y)$. Let $\hat{y} \notin A_{\not\prec_D}(Y)$. Then, by Prop. 3.25, $\hat{y} \in (Y + \text{int } D)$ and there must exist some $y \in Y \subseteq B_{\not\prec_D}(Y)$ (from Lem. 2.14 (i)) with $\hat{y} \in y + \text{int } D$. The set $y + \text{int } D$ forms an open neighborhood around $\hat{y}$ and by the first part,

$$y + \text{int } D \subseteq B_{\not\prec_D}(Y).$$

Thus, there is an open neighborhood around $\hat{y}$ entirely in $B_{\not\prec_D}(Y)$, showing $\hat{y} \in \text{int } B_{\not\prec_D}(Y)$. Let now $\hat{y} \in \text{int } B_{\not\prec_D}(Y)$. Then, there exists some open neighborhood $U \in \mathcal{T}_0$ such that $\hat{y} + U \subseteq B_{\not\prec_D}(Y)$. Choose some $d \in \text{int } D$. Since U is absorbing,

$$\exists \alpha_0 > 0 : \quad -\alpha d \in U, \text{ for any } |\alpha| \leq \alpha_0,$$

and by $\text{int } D$ a cone, $\alpha_0 d \in \text{int } D$. Then,

$$y' := \hat{y} - \alpha_0 d \in (\hat{y} - \text{int } D) \cap (\hat{y} + U)$$

implies $y' <_D \hat{y}$, i.e. $\hat{y}$ is not smaller than y' in the sense of $\not\prec_D$, and $y' \in \hat{y} + U \subseteq B_{\not\prec_D}(Y)$. Thus,

$$\hat{y} \notin A_{\not\prec_D}(B_{\not\prec_D}(Y)) \stackrel{\text{Lem. 2.21}}{=} A_{\not\prec_D}(Y).$$

Altogether, this shows $(A_{\not\prec_D}(Y))^c = \text{int } B_{\not\prec_D}(Y)$.

Lastly, for any $\hat{y} \in \text{bd } B_{\not\prec_D}(Y)$, by the above, $\hat{y} \notin \text{int } B_{\not\prec_D}(Y) \iff \hat{y} \in A_{\not\prec_D}(Y)$, and by $B_{\not\prec_D}(Y)$ closed, $\hat{y} \in A_{\not\prec_D}(Y) \cap B_{\not\prec_D}(Y)$. The opposite inclusion follows analogously. Whenever $\hat{y} \in A_{\not\prec_D}(Y) \cap B_{\not\prec_D}(Y)$, then $\hat{y} \in B_{\not\prec_D}(Y) \setminus \text{int } B_{\not\prec_D}(Y) = \text{bd } B_{\not\prec_D}(Y)$. Using $B_{\not\prec_D}(Y) = \text{Cl}_+ Y$ from Prop. 3.28, we obtain the assertion. $\square$

The core result of Thm. 3.29 has been shown in different settings, *e.g.* [16, 20, 26]. Furthermore, we also obtain some useful properties given the infimum of some $Y \subset \mathcal{Y}$ lies in $\mathcal{Y}$:

- $A_{\not\prec_D}(Y) = (\text{int } (Y + D))^c = (Y + \text{int } D)^c$,
- $A_{\not\prec_D}(Y) \cup B_{\not\prec_D}(Y) = \mathcal{Y}$
- $(\text{Cl}_+ Y)^c \cup (\text{int } \text{Cl}_+ Y) \cup (\text{bd } \text{Cl}_+ Y) = \mathcal{Y}$,
- $(\text{Cl}_+ Y)^c, \text{int } \text{Cl}_+ Y, \text{bd } \text{Cl}_+ Y$ are pairwise disjoint.

These and more results can be found in [16, Cor. 1.48]. We proceed with a characterization of the weak minimum and infimum.

3.30 Theorem:

Let $Y \subseteq \overline{\mathcal{Y}}$ and $Y' = Y \cap \mathcal{Y}$. Then the weak minimum of Y with respect to D is given by

$$\text{w-}\min_D Y = \begin{cases} \emptyset & \text{if } -\infty_D \in Y, \\ \{y \in Y' : Y' \cap (y - \text{int } D) = \emptyset\} & \text{otherwise,} \end{cases}$$

and the weak infimum of Y with respect to D by

$$\text{w-}\inf_D Y = \begin{cases} \{-\infty_D\} & \text{if } \text{Cl}_+ Y' = \mathcal{Y} \text{ or } -\infty_D \in Y, \\ \{+\infty_D\} & \text{if } Y' = \emptyset \text{ and } -\infty_D \notin Y, \\ \text{bd } \text{Cl}_+ Y' & \text{otherwise,} \end{cases}$$

with $\text{Cl}_+ Y' = \text{cl}(Y' + D)$.

Proof. The first part follows directly from Cor. 2.20 with $\text{Min}_{\not\prec_D} Y' = A_{\not\prec_D}(Y') \cap Y'$ from Prop. 2.16 and using the reformulation of $A_{\not\prec_D}(Y')$ from Prop. 3.25. For the second part, we use $B_{\not\prec_D}(Y') = \text{Cl}_+ Y' = \text{cl}(Y' + D)$ from Def. 3.26 and Prop. 3.28 to obtain from Cor. 2.23:

$$\text{Min}_{\not\prec_D} B_{\not\prec_D}(Y') = A_{\not\prec_D}(Y') \cap B_{\not\prec_D}(Y') = \text{bd } \text{Cl}_+ Y'.$$

□

Again, following Prop. 2.22 with Prop. 3.28, an equivalent formulation of $\text{w-}\inf_D Y$ is given, as it is defined in [16], by

$$\text{w-}\inf_D Y = \begin{cases} \{-\infty_D\} & \text{if } \text{Cl}_+ Y = \mathcal{Y}, \\ \{+\infty_D\} & \text{if } \text{Cl}_+ Y = \emptyset, \\ \text{Min}_D \text{Cl}_+ Y & \text{otherwise.} \end{cases}$$

We show two important properties of the weak infimum: for any set $Y \subseteq \overline{\mathcal{Y}}$ the weak infimum $\text{w-}\inf_D Y$ is representative of $[Y]$ and nonempty. This is a stronger statement than self-infimality of infima [20, Thm. I-21]. We proceed to show that $\text{w-}\inf_D Y$ is always representative, which makes weak infima especially suitable for duality results. We make use of $\mathcal{D}^c = -\text{int } D$ being a convex cone with nonempty interior (see Lem. 1.15) to apply Lem. 1.12 and Lem. 1.13. The underlying argument of the following lemma is essentially the

same as in [16, Thm. 1.26]. We show that the line segment between any pair of lower bound and upper bound to lower bounds contains a weakly infimal element.

3.31 Lemma:

For any $Y \subseteq \bar{\mathcal{Y}}$ and any pair $(l, u) \in A_{\nearrow_D}(Y) \times B_{\nearrow_D}(Y)$,

$$\exists \bar{\lambda} \in [0, 1] : \quad \hat{y} := \bar{\lambda}l + (1 - \bar{\lambda})u \in \text{w-Inf}_D Y.$$

Proof. Whenever $A_{\nearrow_D}(Y)$ or $B_{\nearrow_D}(Y)$ is empty, the assertion trivially holds. Let $l \in A_{\nearrow_D}(Y)$ and $u \in B_{\nearrow_D}(Y) = \text{Cl}_+ Y$, by Prop. 3.28. Then, by Prop. 2.17 and Thm. 3.30, the weak infimum of Y is exactly $\text{bd Cl}_+ Y$. Construct some

$$y_\lambda := \lambda l + (1 - \lambda)u, \quad \lambda \in [0, 1].$$

Consider all λ for which $y_\lambda \in \text{Cl}_+ Y$ and let

$$\bar{\lambda} := \sup\{\lambda \in [0, 1] : y_\lambda \in \text{Cl}_+ Y\}.$$

Then, there exists a sequence $(\lambda_n)_{n \in \mathbb{N}} \rightarrow \bar{\lambda}$ with $y_{\lambda_n} \in \text{Cl}_+ Y$, $\forall n \in \mathbb{N}$ and

$$\hat{y} := \lim_{n \in \mathbb{N}} y_{\lambda_n} = y_{\bar{\lambda}} \in \text{cl Cl}_+ Y.$$

Next, consider all $\mu \in (\bar{\lambda}, 1]$ for which $y_\mu \notin \text{Cl}_+ Y$. We observe that $y_\lambda \notin \text{Cl}_+ Y$, $\forall \lambda \in (\bar{\lambda}, 1]$, as otherwise $\bar{\lambda}$ would not be an upper bound to $\{\lambda \in [0, 1] : y_\lambda \in \text{Cl}_+ Y\}$. It follows that

$$\inf\{\mu \in (\bar{\lambda}, 1] : y_\mu \notin \text{Cl}_+ Y\} = \inf(\bar{\lambda}, 1] = \bar{\lambda}.$$

Then, there exists a sequence $(\mu_n)_{n \in \mathbb{N}} \rightarrow \bar{\lambda}$ with $y_{\mu_n} \in (\text{Cl}_+ Y)^c$, $\forall n \in \mathbb{N}$ and

$$\lim_{n \in \mathbb{N}} y_{\mu_n} = \hat{y} \in \text{cl}(\text{Cl}_+ Y)^c \implies \hat{y} \notin \text{int Cl}_+ Y.$$

Altogether, $\hat{y} \in \text{cl Cl}_+ Y \setminus \text{int Cl}_+ Y = \text{bd Cl}_+ Y$, and is thus weakly infimal, by Thm. 3.30. $\square$

3.32 Theorem: Representativity of the weak Infimum

Let $Y \subseteq \bar{\mathcal{Y}}$. Then $\text{w-Inf}_D Y \in [Y]$.

Proof. In view of Thm. 2.37, it is sufficient to verify that property (3) is satisfied whenever $\text{w-Inf}_D Y \neq \{\pm\infty_D\}$. That is, we aim to show that, for nonempty $A_{\not\prec_D}(Y)$ and $B_{\not\prec_D}(Y)$,

$$\forall y \in Y \setminus \text{w-Inf}_D Y : \exists \hat{y} \in \text{w-Inf}_D Y : \hat{y} <_D y \iff \hat{y} \in y - \text{int } D.$$

We use $B_{\not\prec_D}(Y) = \text{Cl}_+ Y$ by Prop. 3.28 and apply Thm. 3.30. From Lem. 2.14, we may assume $\emptyset \neq Y \subset \mathcal{Y}$ and choose some $y \in Y \setminus \text{w-Inf}_D Y$ and $l^0 \in A_{\not\prec_D}(Y)$.

We first construct a lower bound $l \in A_{\not\prec_D}(Y)$ with $l <_D y$. By $\mathcal{D}^c = -\text{int } D$ being a convex cone with nonempty interior, it is generating by Lem. 1.13, and by Lem. 1.12,

$$\exists l \in (y - \text{int } D) \cap (l^0 - \text{int } D),$$

i.e. $l <_D y$ and $l <_D l^0$, such that l dominates y and l^0 . Since $\text{int } D$ is closed under addition (see Lem. 1.11), from $l^0 - l \in \text{int } D$ we obtain

$$l^0 - l + d \in \text{int } D \implies l - d \in l^0 - \text{int } D,$$

for any $d \in \text{int } D$. Thus, $l - \text{int } D \subset l^0 - \text{int } D$ and by l^0 being a lower bound of Y , by Prop. 3.25:

$$Y \cap (l^0 - \text{int } D) = \emptyset \implies Y \cap (l - \text{int } D) = \emptyset,$$

implying $l \in A_{\not\prec_D}(Y)$. We now use $l \in (y - \text{int } D)$. Since $y \in Y \setminus \text{w-Inf}_D Y$, we have

$$y \notin \text{w-Inf}_D Y = \text{bd } \text{Cl}_+ Y = \text{w-Min}_D \text{Cl}_+ Y$$

by the proof of Thm. 3.30, whence by the same result and $-\infty_D \notin Y$,

$$\exists y' \in \text{Cl}_+ Y \cap (y - \text{int } D).$$

By Lem. 3.31, there exists some weakly infimal point

$$\hat{y} \in \text{conv}\{l, y'\} \cap \text{w-Inf}_D Y.$$

Then, by $l, y' \in y - \text{int } D$ a convex set, $\hat{y} \in (y - \text{int } D)$, dominating y , and satisfying property (3). By Thm. 2.37, $\text{w-Inf}_D Y$ is thus representative for any $Y \subseteq \overline{\mathcal{Y}}$.

□

Thm. 3.32 can also be shown using [16, Thm. 1.26, Lem. 1.42, Lem. 1.43] and with $B := (y - \text{int } D)$ as the chosen open convex set. Existence of weak infima is well known (see *e.g.* [16, Cor. 1.44]) and follows here directly from representativity.

3.33 Corollary: *Nonemptiness of the weak Infimum*

Let $Y \subseteq \overline{\mathcal{Y}}$. Then $\text{w-Inf}_D Y \neq \emptyset$.

Proof. From Lem. 2.33, for any $Y \in \overline{\mathcal{Y}}$ if $\text{w-Inf}_D Y = \emptyset$, then $\text{w-Inf}_D Y \notin [Y]$, contradicting Thm. 3.32. Thus $\text{w-Inf}_D Y \neq \emptyset$. $\square$

Conversely, the above results also imply representativity and existence of all weak suprema:

$$\emptyset \neq \text{w-Sup}_D Y \in [Y]^-. \quad (35)$$

The result in Thm. 3.32 can most likely also be shown by using the notions of *cone-closedness*, *-boundedness*, *-compactness* and *-semicompactness*, as compiled and used, *e.g.* by Sawaragi et al. [25] for results on the existence of efficient solutions. Indeed, the above proof essentially shows how $B_{\not\prec_D}(Y)$ is D -semicompact and thus satisfies the conditions from Thm. 2.37. The difficulty of such an approach resides in showing the cone semicompactness, which relies on finite subcovers of sets. We refer the reader to Sawaragi et al. [25, Sect. 3.2].

As a last addition, in a similar approach to Thm. 3.32, we may show that the lower bounds to the upper bounds of an infimum are exactly its lower bounds, and vice versa. This allows us to show that any weak infimum is self-supremal.

3.34 Lemma:

For any $Y \subseteq \overline{\mathcal{Y}}$,

$$\begin{aligned} A_{\not\prec_D}(\text{w-Inf}_D Y) &= B_{\not\prec_D}(\text{w-Inf}_D Y), \\ A_{\not\prec_D}(\text{w-Inf}_D Y) &= B_{\not\prec_D}(\text{w-Inf}_D Y). \end{aligned}$$

Proof. For comparison with general results, bear in mind $R = \not\prec_D$ and $-R = \not\prec_D$. We first show $A_{\not\prec_D}(\text{w-Inf}_D Y) = B_{\not\prec_D}(\text{w-Inf}_D Y)$. The equation is satisfied whenever $\text{w-Inf}_D Y = \{\pm\infty_D\}$ (see Sect. 2.3 and Lem. 2.14).

Moreover, by Cor. 3.33, we have $\text{w-Inf}_D Y \neq \emptyset$. In the following, we shall assume $\emptyset \neq \text{w-Inf}_D Y \subseteq \mathcal{Y}$.

By Def. 2.6, $\text{w-Inf}_D Y \subseteq A_{\not\prec_D}(Y)$. Thus, by Lem. 2.24 (i), Prop. 2.42 (i) and representativity of weak infima by Thm. 3.32,

$$\begin{aligned} A_{\not\prec_D}(\text{w-Inf}_D Y) &\stackrel{\text{Lem. 2.24 (i)}}{\supseteq} A_{\not\prec_D}(A_{\not\prec_D}(Y)) \\ &\stackrel{\text{Prop. 2.42 (i)}}{=} B_{\not\prec_D}(Y) \stackrel{\text{w-Inf}_D Y \in [Y]}{=} B_{\not\prec_D}(\text{w-Inf}_D Y). \end{aligned}$$

It remains to show $A_{\not\prec_D}(\text{w-Inf}_D Y) \subseteq B_{\not\prec_D}(\text{w-Inf}_D Y)$. Assume there exists some $\bar{u} \in A_{\not\prec_D}(\text{w-Inf}_D Y) \setminus B_{\not\prec_D}(\text{w-Inf}_D Y)$. By $\bar{u} \notin B_{\not\prec_D}(\text{w-Inf}_D Y)$, there exists some $l \in A_{\not\prec_D}(\text{w-Inf}_D Y)$ with $l >_D \bar{u}$, *i.e.*

$$l \in \bar{u} + \text{int } D.$$

We construct a point u dominated by l . Choose an arbitrary $\hat{y} \in \text{w-Inf}_D Y$. By $\text{int } D$ being a nonempty convex cone, it is generating by Lem. 1.13 and by Lem. 1.12

$$\exists u \in (l + \text{int } D) \cap (\hat{y} + \text{int } D),$$

implying $l <_D u$ and $\hat{y} <_D u$. Since

$$u + \text{int } D \subseteq \hat{y} + \text{int } D \subseteq (\text{w-Inf}_D Y + \text{int } D),$$

we have $u \in B_{\not\prec_D}(\text{w-Inf}_D Y)$ by Prop. 3.25. Furthermore, we also observe that $\bar{u} <_D l <_D u$ (see the proof of Thm. 3.32 or use transitivity of $<_D$ by Prop. 3.4), *i.e.*

$$u \in l + \text{int } D \subset \bar{u} + \text{int } D.$$

Then, both l and u lie in the convex set $\bar{u} + \text{int } D$, and by Lem. 3.31, there exists some $\hat{y}' \in \text{w-Inf}_D Y$ in the convex hull of l and u , implying $\hat{y}' \in \bar{u} + \text{int } D$. Thus, $\bar{u} <_D \hat{y}'$ holds, disproving that $\bar{u}$ is an upper bound (in the sense of $\not\prec_D$) to the weak infimum of Y , and contradicting $\bar{u} \in A_{\not\prec_D}(\text{w-Inf}_D Y)$. This shows the first assertion.

We obtain the second assertion by using the first, Lem. 2.21 and Prop. 2.42 (i): from $A_{\not\prec_D}(\text{w-Inf}_D Y) = B_{\not\prec_D}(\text{w-Inf}_D Y)$, taking the lower bounds on both sides yields

$$\underbrace{A_{\not\prec_D}(A_{\not\prec_D}(\text{w-Inf}_D Y))}_{\stackrel{\text{Prop. 2.42 (i)}}{=} B_{\not\prec_D}(\text{w-Inf}_D Y)} = \underbrace{A_{\not\prec_D}(B_{\not\prec_D}(\text{w-Inf}_D Y))}_{\stackrel{\text{Lem. 2.21}}{=} A_{\not\prec_D}(\text{w-Inf}_D Y)}.$$

This completes the proof. □

Conversely to Lem. 3.34, by exchanging the roles of $\not\prec_D$ and $\prec_D$, we obtain

$$\begin{aligned} A_{\not\prec_D}(\text{w-Sup}_D Y) &= B_{\prec_D}(\text{w-Sup}_D Y), \\ A_{\prec_D}(\text{w-Sup}_D Y) &= B_{\not\prec_D}(\text{w-Sup}_D Y). \end{aligned} \quad (36)$$

The following shows self-supremality of infima and self-infimality of suprema, as evoked in [25, Lem. 6.2.1] and originally shown in slightly different terms by Kawasaki [12, Lem. 2.3]. This property does not hold for most other types of infimum and supremum.

3.35 Proposition: [25, Lem. 6.2.1]

For any $Y \subseteq \bar{\mathcal{Y}}$,

$$\begin{aligned} \text{w-Sup}_D \text{w-Inf}_D Y &= \text{w-Inf}_D Y, \\ \text{w-Inf}_D \text{w-Sup}_D Y &= \text{w-Sup}_D Y. \end{aligned}$$

Proof. By Prop. 2.22 applied to the supremum,

$$\text{w-Sup}_D \text{w-Inf}_D Y = \text{w-Sup}_D B_{\prec_D}(\text{w-Inf}_D Y). \quad (37)$$

Since, by Thm. 3.32, any infimum is representative, by Lem. 3.34, we also know

$$A_{\not\prec_D}(Y) = A_{\not\prec_D}(\text{w-Inf}_D Y) = B_{\prec_D}(\text{w-Inf}_D Y). \quad (38)$$

Thus, by Prop. 2.42 (ii),

$$\begin{aligned} \text{w-Sup}_D \text{w-Inf}_D Y &\stackrel{(37)}{=} \text{w-Sup}_D B_{\prec_D}(\text{w-Inf}_D Y) \\ &\stackrel{(38)}{=} \text{w-Sup}_D A_{\not\prec_D}(Y) \stackrel{\text{Prop. 2.42 (ii)}}{=} \text{w-Inf}_D Y. \end{aligned} \quad (39)$$

The second assertion is obtained analogously by applying Prop. 2.42 (iii):

$$\begin{aligned} \text{w-Inf}_D \text{w-Sup}_D Y &\stackrel{\text{Prop. 2.22}}{=} \text{w-Inf}_D B_{\not\prec_D}(\text{w-Sup}_D Y) \\ &\stackrel{(36)}{=} \text{w-Inf}_D A_{\prec_D}(\text{w-Sup}_D Y) \\ &\stackrel{(35)}{=} \text{w-Inf}_D A_{\prec_D}(Y) \\ &\stackrel{\text{Prop. 2.42 (iii)}}{=} \text{w-Sup}_D Y. \end{aligned}$$

□

3.4.3 On the Approach by Gros

Attempts to circumvent the drawbacks of the nondominated infimum and obtain similar benefits to the weak infimum (*e.g.* existence and representativity) were made. Gros [8] defined an infimum on $\mathbb{R}^k$ with Pareto optimal solutions in mind. Using our notation, the author defined

$$\begin{aligned}\text{Inf}_D^G Y &:= A_{\not\leq D}(\text{cl}(Y + D \setminus \{0\})) \cap \text{cl}(Y + D \setminus \{0\}) \\ &= \text{Min}_D \text{cl}(Y + D \setminus \{0\}) \subseteq \mathcal{Y}, \quad Y \subseteq \mathcal{Y},\end{aligned}$$

originally for $\mathcal{Y} = \mathbb{R}^k$ and $D = \mathbb{R}_{\leq}^k$. The structure is very similar to that of the infimum on weakly efficient solutions by Nieuwenhuis [20], applies the analogous systematic approach as Kawasaki [12], but formulated with a different ordering cone (and without extending the space). By replacing $D \setminus \{0\}$ by $\text{int } D$ and $\not\leq_D$ by $\not\triangleright_D$, by Lem. 2.21 and Prop. 3.28, we obtain

$$B_{\not\triangleright_D}(Y) \cap A_{\not\triangleright_D}(B_{\not\triangleright_D}(Y)) = B_{\not\triangleright_D}(Y) \cap A_{\not\triangleright_D}(Y),$$

which is exactly the restricted weak infimum. This definition was motivated by the property that

$$\begin{aligned}\text{Inf}_D^G Y &= \text{Inf}_D^G Y(\text{cl } Y) \\ &= \text{Inf}_D^G Y(\text{cl}(Y + D \setminus \{0\})) \\ &= \text{Inf}_D^G Y(Y + D \setminus \{0\}),\end{aligned}$$

as commented below the original definition (see [8, Def. 2.2 ff.]). The main issue with this approach, is that a minimal element may not be infimal. This makes it so that Prop. 2.11 no longer holds: infimal point may not be minimal.

3.36 Example: *This example is inspired by Sawaragi et al. [25, Example 6.1.1]. Let $\mathcal{Y} = \mathbb{R}^2$ and $D = \mathbb{R}_{\leq}^2$. Assume $Y = \{y \in \mathbb{R}^2 : y_1 \in (0, 1], y_2 > 0\} \cup \{0\}$. Then $\text{Min}_D Y = \text{Inf}_D Y = \{(1, 0)^\top\}$ whilst $\text{Inf}_D^G Y = \{0\}$. (See Fig. 3.6)*

Ponstein [21] expanded the definition on the extended space $(\overline{\mathbb{R}})^n = \overline{\mathbb{R}} \times \dots \times \overline{\mathbb{R}}$. This is different to the extension we would need (see Example 3.14). We refer the reader to an example by Sawaragi et al. [25, Example 6.1.1] showing an existing minimum not being part of the infimum by Ponstein.

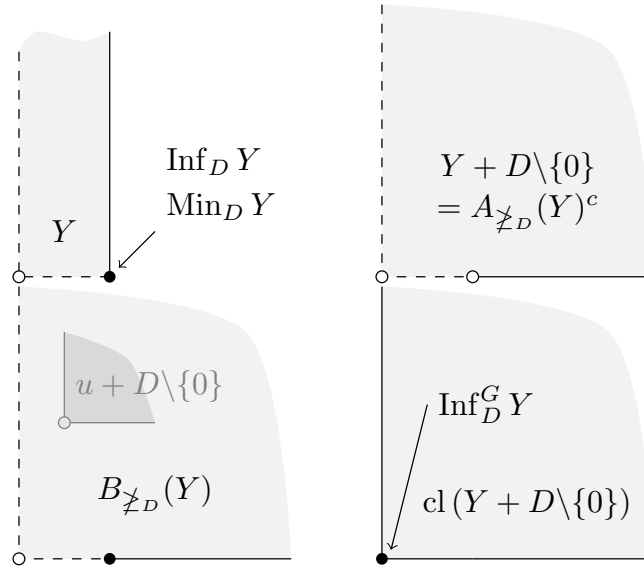


Figure 3.6: Infimum by Gros not attaining nondominated points.

3.5 Notes on the Literature

Nieuwenhuis [20] generalized the concept of supremum in the context of vector optimization for Pareto optimal and weak-maximal points, which respectively parallel efficient and weakly efficient solutions when minimization is considered. The author followed a similar approach to ours, defining supremal points as smallest upper bounds to a set, and derived results that coincide with ours, in the special case of $R = \not\leq_D$ (eg. see [20, Thm. I-28]).

The concept of weak infimum was expanded to the extended space by Kawasaki [12, 13], Tanino [26] and Löhne [16]. The presented version is taken from Löhne [16], where the author expanded the results from Nieuwenhuis [20] and Tanino [26], focusing on complete lattices induced by self-infimal sets and a subsequent generalization of conjugate and Lagrange duality. While Kawasaki [12, 13] introduced the weak infimum of a set as the minimum over the upper closure of that set, Nieuwenhuis [20], Tanino [26] and Löhne [16] used the characterization of infimal points as greatest lower bounds. Some minor differences remain between authors, *e.g.* regarding the upper closure always containing $+\infty_D$.

A form of strong infimum was introduced by Brumelle [5], as a decomposition in one-dimensional infima on real vector spaces for the ordering $\leq$, and gener-

alized in [26]. The given definition is however not immediately suitable for generalization with regards to different ordering cones, as it assumes that the infimum is a singleton. The reformulation by Tanino [26] does coincide with our definition, up to the assumption of uniqueness.

Komuro and Yoshimura [15] generalized the approach from considering weak-maximal and supremal solutions to maximal solutions with respect to a convex cone P on a linear space over $\mathbb{R}$ [14, 15]. The authors focused on monotone order completeness in the induced partially ordered topological vector space. The generalized suprema and infima were not defined on the extended space and the results seem to only be applicable to transitive orderings. The notion of greatest in *greatest lower bound* differs from ours. Generalizations of conjugates were given.

Many results and applications require that the cone D be pointed and convex with nonempty interior. We investigated where most of these assumptions individually need to be applied. We also bridged the gap between definitions of infimum relying on the set of lower bounds (like [14, 15] and Def. 2.6) and its upper closure (like [16, 20] and Prop. 2.22).

A different approach was taken by Gros [8] and extended by Ponstein [21] with regards to efficient solutions, using an analogous point of view to Kawasaki [12]. This proved to lack some properties making it unsuitable for applying a generalization of conjugate duality as used on weak infima in Sawaragi et al. [25] and Boř et al. [4]. Nevertheless, both Gros [8] and Ponstein [21] derived conjugate duality results.

Tanino [26] analyzed the connections between suprema defined via upper and lower bounds against those defined via closures (like Kawasaki [12] or Gros [8]). In this context, a definition of the (nondominated) infimum Inf_D with respect to a cone D , that is analogous to our definition, was introduced, but not further investigated. Tanino [26] emphasized that the weak infimum provides the most suitable generalization of the classical infimum in higher dimensions when compared to the strong and nondominated infima, since it can be characterized both via lower and upper bounds (ensuring that attained infima are minimal) and via taking a (weak) minimum over a closure (ensuring existence), *i.e.* for an adequate set Y ,

$$\text{w-Inf}_D Y = A_{\not\leq_D}(Y) \cap B_{\not\leq_D}(Y) = \text{w-Min cl}(Y + D).$$

4 Conjugate Duality for Set-Valued Mappings

Sect. 3 has shown that the infimum framework developed in Sect. 2 is compatible with the traditional notions around dominating, nondominated and weakly nondominated points, whilst providing a generalized definition of infimum in line with the commonly accepted definition of weak infima. As such, this framework may be used to investigate the assumptions and implications of methods and approaches that have been formulated only for a specific type of relation, *e.g.* for nondominated or weakly nondominated points, for general relations.

This section will focus on comparing two duality approaches for set-valued optimization problems that were formulated for efficiency and weak efficiency, that is, based on nondominated and weakly nondominated points, respectively. The first approach is due to Tanino and Sawaragi [28] and relies on nondominated maximization of the dual objective mapping to find efficient solutions. An expansion to topological vector spaces and set-valued mappings was proposed by Boř et al. [4]. The second approach relies on weak supremization of the dual objective mapping to find weakly efficient solutions. It can be traced back to Kawasaki [12, 13], with later expansion to topological vector spaces by Tanino [27]. Both approaches can be seen as a generalization of the conjugate duality theory formulated by Rockafellar [23], which notably encapsulates Fenchel- and Lagrange-Duality. The theory is based on embedding the primal problem into a family of perturbed problems by means of a so-called *perturbation function*, and using conjugates of that perturbation function to approach the primal optimal value “from below”, *i.e.* through supremization (or maximization) of a dual objective function.

The similarities between the two approaches were addressed by both Sawaragi et al. [25] and Boř et al. [4], but a direct comparison remained impossible for lack of a common framework. For instance, the notion of nondominated infimum has found no consensus in the literature, rendering the transfer of the supremum-based theory from weak efficiency to efficiency hazardous. Furthermore, as we have repeatedly seen throughout Sect. 3.4.2, weak infima present several advantages and features that are not shared by nondominated infima (*e.g.* nonemptiness, representativity, separation of $\mathcal{Y}$), leading to assumptions, which are necessary to expand the approach, becoming hidden in the specificity of weak infimization.

Based on the work of Boř et al. [4, Chapt. 7], we shall generalize both conjugate duality approaches for topological vector spaces, making them applicable for

any relation R , whilst understanding their underlying assumptions and limitations. For most of the considerations, we will assume R to be translation invariant, but not necessarily positively responsive.

We will start by introducing some terminology around set-valued mappings and propose an approach to generalize affine minorants. We show how affine minorants support the already established generalizations of subdifferentiability and conjugates from [4] and give the reader some geometrically motivated intuitions for these concepts. We then propose two solution concepts for set-valued infimization problems, respectively based on minimization and epigraphical-minimization. From there, we introduce a perturbation approach, leading to weak and strong duality results for both maximum- and supremum-based dual problems, and investigate reflexive duality conditions. To conclude, we show the specificities of the supremum-based approach for weak nondomination, circling back to the existing results from Boţ et al. [4], and highlighting to which extend the displayed properties are exceptional to this type of relation.

4.1 Infimization and Minimization over Set-Valued Mappings

We begin by establishing common notions for set-valued mappings. Let $\mathcal{X}$ and $\mathcal{Y}$ be sets. Following Rockafellar and Wets [24, Chapt. 5], we consider mappings $x \mapsto F(x)$ where $x \in \mathcal{X}$ and $F(x) \subseteq \mathcal{Y}$ is a set. For these *set-valued mappings* (also sometimes referred to as *point-to-set maps*, *multifunctions* or *correspondences*) we write

$$F : \mathcal{X} \rightrightarrows \mathcal{Y}, \quad x \mapsto F(x) \subseteq \mathcal{Y}.$$

Whenever $F(x)$ is a singleton, F is called *single-valued* at $x \in \mathcal{X}$. If F is single-valued at every $x \in \mathcal{X}$, then F is uniquely identified by the *vector-valued function*

$$f : \mathcal{X} \rightarrow \mathcal{Y}, \quad x \mapsto f(x), \quad \text{with } F(x) = \{f(x)\}, \forall x \in \mathcal{X}.$$

If $\mathcal{Y} = \mathbb{R}$, f is simply called a *function*. In this sense, set-valued mappings can be seen as a generalization of both functions and vector-valued functions.

Some terminology of functions can be used for set-valued mappings. Let $F : \mathcal{X} \rightrightarrows \mathcal{Y}$. For any $X \subseteq \mathcal{X}$, the *image* of F under X is given by

$$F(X) := \bigcup_{x \in X} F(x).$$

The *range* of F is its image over the entire space $\mathcal{X}$, written $\text{rge } F := F(\mathcal{X})$. The *inverse* of F is the set-valued mapping $F^{-1} : \mathcal{Y} \rightrightarrows \mathcal{X}$ with

$$F^{-1}(y) := \{x \in \mathcal{X} \mid y \in F(x)\}.$$

Then, for any $Y \subseteq \mathcal{Y}$, the *inverse image* of F under Y is given by

$$F^{-1}(Y) := \bigcup_{y \in Y} F^{-1}(y) = \{x \in \mathcal{X} \mid F(x) \cap Y \neq \emptyset\}.$$

The *domain* of F is its inverse image over the entire space $\mathcal{Y}$, written $\text{dom } F := F^{-1}(\mathcal{Y})$. The *graph* of F is given by

$$\text{gph } F := \{(x, y) \in \mathcal{X} \times \mathcal{Y} : y \in F(x)\},$$

and fully characterizes both F and its inverse [24].

Assume now F is a set-valued mapping from $\mathcal{X}$ to the extended space $\bar{\mathcal{Y}} = \mathcal{Y} \cup \{\pm\infty_R\}$, according to an underlying relation R (see Sect. 2.1). In this case, the above definitions still apply, but the domain of F is considered to be the inverse image of F over $\mathcal{Y}$, instead of over $\bar{\mathcal{Y}}$, *i.e.*

$$\text{dom } F = F^{-1}(\mathcal{Y}) = \{x \in \mathcal{X} \mid F(x) \cap \mathcal{Y} \neq \emptyset\}.$$

As a result, if for some $\bar{x} \in \mathcal{X}$, $F(\bar{x}) \in \{\{-\infty_R\}, \{+\infty_R\}\}$ holds, then $\bar{x} \notin \text{dom } F$. The set-valued mapping F is called *proper* (with respect to R) whenever $\text{dom } F \neq \emptyset$ and $-\infty_R \notin \text{rge } F$. In other words, the mapping F maps to at least one element from $\mathcal{Y}$ and never to $-\infty_R$, *i.e.* $\exists x \in \mathcal{X} : F(x) \cap \mathcal{Y} \neq \emptyset$ and $-\infty_R \notin F(x)$, $\forall x \in \mathcal{X}$.

Since $\bar{\mathcal{Y}}$ is equipped with a relation, we can also propose a generalization of the epigraphical value function for F . The *epigraphical set mapping* of F with respect to R is given by

$$E_F : \mathcal{X} \rightrightarrows \mathcal{Y}, x \mapsto E_F(x) := B_R(F(x)).$$

As such, $E_F(x)$ is the set of elements that are upper bounds to the lower bounds of $F(x)$, for any given $x \in \mathcal{X}$. By Prop. 2.22, it immediately follows that

$$\text{Inf}_R F(x) = \text{Inf}_R B_R(F(x)) = \text{Inf}_R E_F(x),$$

which coincides with $\text{Min}_R E_F(x)$, provided $\emptyset \neq E_F(x) \neq \mathcal{Y}$. The *epigraph* of F refers to the graph of the epigraphical set mapping and is given by $\text{epi } F := \text{gph } E_F$. The same approach can be used to generalize hypographical value functions and hypographs. Note that, in the literature, for a common

ordering cone D , a different definitions of epigraph has been formulated with regards to nondomination ($R = \not\leq_D$) and weak nondomination ($R = \not\leq_D$). The proposed definition defines a form of closure over those epigraphs, with the goal of capturing infimal points though minimization.

Using this terminology, we may discuss infimization and minimization over set-valued mappings. We are taking the infimum and minimum of the image of F over a set. The parallel can be drawn to functions: the infimum of a function f over a set X is exactly the infimum over the image $f(X) \subseteq \mathbb{R}$ of f over X . Thus, for some $F : \mathcal{X} \rightrightarrows \mathcal{Y}$ and $X \subseteq \mathcal{X}$, the set

$$\text{Inf}_R F := \text{Inf}_R \bigcup_{x \in X} F(x) = \text{Inf}_R F(X)$$

is called the *infimum of F (over X)* and any point from that set is called an *infimal point of F (over X)*. Analogously, the set

$$\text{Min}_R F := \text{Min}_R \bigcup_{x \in X} F(x) = \text{Min}_R F(X)$$

is called the *minimum of F (over X)* and any point from that set is called a *minimal point of F (over X)*. An analogous terminology can be established for supremization and maximization of F . In this sense, infimization and minimization of a set-valued mapping is equivalent to taking the infimum and minimum of an image set, which can be expressed as a union of sets. Considerations for the points in $\mathcal{X}$ contributing to the infimum and minimum of F are postponed for now.

For functions, the infimum of some $f : \mathcal{X} \rightarrow \mathbb{R}$ over X is also the infimum of the projection of the epigraph $\text{epi } f$ onto $\mathbb{R}$, which compounds to an interval. The following results relate the epigraphical set mapping E_F to F under minimization and infimization.

4.1 Lemma:

For any $F : \mathcal{X} \rightrightarrows \mathcal{Y}$ and $X \subseteq \mathcal{X}$:

$$(i) \bigcup_{x \in X} B_R(F(x)) \subseteq B_R \left(\bigcup_{x \in X} F(x) \right),$$

$$(ii) A_R \left(\bigcup_{x \in X} B_R(F(x)) \right) = A_R \left(B_R \left(\bigcup_{x \in X} F(x) \right) \right) = A_R \left(\bigcup_{x \in X} F(x) \right).$$

Proof. (i) Let $u \in B_R(F(\bar{x}))$ for some $\bar{x} \in X$. Then lRu holds for all $l \in A_R(F(\bar{x})) \supseteq A_R \left(\bigcup_{x \in X} F(x) \right)$, by Lem. 2.24 (i), implying $u \in B_R \left(\bigcup_{x \in X} F(x) \right)$.

(ii) By Lem. 2.21, we immediately have

$$A_R \left(\bigcup_{x \in X} F(x) \right) = A_R \left(B_R \left(\bigcup_{x \in X} F(x) \right) \right).$$

Using Lem. 2.24 (ii), the second assertion then follows from

$$\begin{aligned} A_R \left(\bigcup_{x \in X} B_R(F(x)) \right) &\stackrel{\text{Lem. 2.24 (ii)}}{=} \bigcap_{x \in X} A_R(B_R(F(x))) \\ &\stackrel{\text{Lem. 2.21}}{=} \bigcap_{x \in X} A_R(F(x)) \stackrel{\text{Lem. 2.24 (ii)}}{=} A_R \left(\bigcup_{x \in X} F(x) \right). \end{aligned}$$

□

4.2 Theorem: (Set-Valued and Epigraphical Set Mappings)

For any $F : \mathcal{X} \rightrightarrows \overline{\mathcal{Y}}$ and $X \subseteq \mathcal{X}$:

$$\inf_{x \in X} F(x) = \inf_{x \in X} E_F(x) \supseteq \min_{x \in X} E_F(x) \supseteq \min_{x \in X} F(x).$$

Proof. Throughout, we use $E_F(x) := B_R(F(x))$ for any $x \in X$. By Lem. 4.1 (ii), $\bigcup_{x \in X} B_R(F(x)) \in [\bigcup_{x \in X} F(x)]$, whence by definition of E_F and Lem. 2.30,

$$\inf_R \bigcup_{x \in X} F(x) = \inf_R \bigcup_{x \in X} E_F(x).$$

Next, decomposing the minimum operator using Prop. 2.16 and using Lem. 4.1 (ii), we obtain

$$\begin{aligned} \min_R \bigcup_{x \in X} E_F(x) &= A_R \left(\bigcup_{x \in X} B_R(F(x)) \right) \cap \bigcup_{x \in X} B_R(F(x)), \\ &= A_R \left(\bigcup_{x \in X} F(x) \right) \cap \bigcup_{x \in X} B_R(F(x)). \end{aligned} \quad (40)$$

By Lem. 2.14 (i), $B_R(F(x)) \supseteq \mathcal{Y} \cap F(x)$, $\forall x \in X$, whence

$$\bigcup_{x \in X} B_R(F(x)) \supseteq \mathcal{Y} \cap \bigcup_{x \in X} F(x). \quad (41)$$

Thus, assuming $\pm\infty_R \notin \bigcup_{x \in X} F(x)$,

$$\text{Min}_R \bigcup_{x \in X} E_F(x) \stackrel{(40),(41)}{\supseteq} A_R \left(\bigcup_{x \in X} F(x) \right) \cap \bigcup_{x \in X} F(x) = \text{Min}_R \bigcup_{x \in X} F(x).$$

Otherwise, by Cor. 2.20, the occurrence of $+\infty_R \in \bigcup_{x \in X} F(x)$ can be ignored and if $-\infty_R \in \bigcup_{x \in X} F(x)$, both minima are empty by the same corollary and (40). Thus $\text{Min}_R \bigcup_{x \in X} E_F(x) \supseteq \text{Min}_R \bigcup_{x \in X} F(x)$ always holds. The remaining inclusion

$$\text{Inf}_R E_F(x) \supseteq \text{Min}_R E_F(x)$$

follows from Prop. 2.9. □

The above theorem shows how the epigraphical set mapping cannot always be used to determine the infimal points of F over X through minimization alone. This is coherent with infimization over functions. However, contrary to what holds for functions, it doesn't hold, in general, that the minimum of the epigraphical set mapping over X corresponds to the minimum of F over X .

4.1.1 Extended Set Relations

We introduce a notation, similar to Kawasaki [12], to express sets in $\overline{\mathcal{Y}}$ exclusively containing lower (resp. upper) bounds to one-another. For any two sets $S^1, S^2 \subseteq \overline{\mathcal{Y}}$, we define the binary set relation $\ll_R$ by

$$S^1 \ll_R S^2 \quad :\Longleftrightarrow \quad \begin{cases} s^1 R s^2, \forall (s^1, s^2) \in S^1 \times (S^2 \setminus \{\pm\infty_R\}), \\ s^1 R s^2, \forall (s^1, s^2) \in (S^1 \setminus \{\pm\infty_R\}) \times S^2, \\ (+\infty_R, -\infty_R) \notin S^1 \times S^2. \end{cases}$$

The above notation circumvents the restriction, that the relation R is assumed to be defined only on $(\mathcal{Y} \times \overline{\mathcal{Y}}) \cup (\overline{\mathcal{Y}} \times \mathcal{Y})$, which was prominent in Sect. 2.4. Indeed, $\ll_R$ allows us to consider *extended* lower and upper bounds, *e.g.*

$$-\infty_R \ll_R -\infty_R, \quad -\infty_R \ll_R +\infty_R, \quad +\infty_R \ll_R +\infty_R,$$

as well as to compare any pair of sets from $\overline{\mathcal{Y}} \times \overline{\mathcal{Y}}$. We can make the following observations for $S^1 \ll_R S^2$:

4.3 Lemma:

For any $S^1, S^2 \subseteq \bar{\mathcal{Y}}$:

- (i) If $+\infty_R \notin S^1$, $S^1 \ll_R S^2$ if and only if $S^1 \cap \mathcal{Y} \subseteq A_R(S^2)$,
- (ii) If $-\infty_R \notin S^2$, $S^1 \ll_R S^2$ if and only if $S^2 \cap \mathcal{Y} \subseteq A_{-R}(S^1)$,
- (iii) If $+\infty_R \in S^1$ and $S^1 \ll_R S^2$, then $S^2 \in \{\emptyset, \{+\infty_R\}\}$,
- (iv) If $-\infty_R \in S^2$ and $S^1 \ll_R S^2$, then $S^1 \in \{\emptyset, \{-\infty_R\}\}$.

Proof. We begin by reformulating the conditions for $S^1 \ll_R S^2$. Recall that $A_R(S^2)$ refers to the lower bounds of S^2 , with respect to R , that lie in $\mathcal{Y}$ (see Def. 2.2). Thus, we can write

$$s^1 R s^2, \forall (s^1, s^2) \in (S^1 \cap \mathcal{Y}) \times S^2 \iff S^1 \cap \mathcal{Y} \subseteq A_R(S^2),$$

and analogously for $s^1 R s^2, \forall (s^1, s^2) \in S^1 \times (S^2 \cap \mathcal{Y})$. We obtain

$$S^1 \ll_R S^2 \iff \begin{cases} S^1 \cap \mathcal{Y} \subseteq A_R(S^2), & (42a) \\ S^2 \cap \mathcal{Y} \subseteq A_{-R}(S^1), & (42b) \\ (+\infty_R, -\infty_R) \notin S^1 \times S^2. & (42c) \end{cases}$$

Assume $+\infty_R \in S^1$ and $S^1 \ll_R S^2$. Then, $A_{-R}(S^1) = \emptyset$ by Lem. 2.14 (ii), such that $S^2 \cap \mathcal{Y} = \emptyset$ by (42b). Since $-\infty_R \notin S^2$ by (42c), we obtain $S^2 \in \{\emptyset, \{+\infty_R\}\}$, showing (iii). The proof for (iv) is analogous.

For (i), assume now $+\infty_R \notin S^1$ and $S^1 \cap \mathcal{Y} \subseteq A_R(S^2)$. Immediately, (42a) and (42c) hold. For (42b), consider the following two cases:

- If $-\infty_R \in S^2$, by case (iv), $S^1 \in \{\emptyset, \{-\infty_R\}\}$ whence, by Lem. 2.14 (iii), $A_{-R}(S^1) = \mathcal{Y}$. Condition (42b) is then automatically satisfied.
- If now $-\infty_R \notin S^2$, by Prop. 2.19, $A_R(S^2) = A_R(S^2 \cap \mathcal{Y})$, and as $+\infty_R \notin S^1$, by the same result, $A_{-R}(S^1) = A_{-R}(S^1 \cap \mathcal{Y})$. Thus, by (42a),

$$\begin{aligned} S^1 \cap \mathcal{Y} \subseteq A_R(S^2 \cap \mathcal{Y}) &\iff \forall s^1 \in S^1 \cap \mathcal{Y}, s^2 \in S^2 \cap \mathcal{Y} : s^1 R s^2 \\ &\iff S^2 \cap \mathcal{Y} \subseteq A_{-R}(S^1 \cap \mathcal{Y}) = A_{-R}(S^1), \end{aligned}$$

implying condition (42b).

For the reverse, by (42a), $S^1 \ll_R S^2$ directly implies $S^1 \cap \mathcal{Y} \subseteq A_R(S^2)$. This shows case (i). Case (ii) follows analogously. $\square$

Under (C1), by R assumed translation invariant and Lem. 2.43 (i), one can easily verify that

$$S^1 \ll S^2 \iff v + S^1 \ll v + S^2$$

holds for any $v \in \mathcal{Y}$ and $S^1, S^2 \subseteq \overline{\mathcal{Y}}$. It is also noteworthy to mention, that the relation remains true for subsets:

$$\forall V^1 \subseteq S^1, V^2 \subseteq S^2 : \quad S^1 \ll_R S^2 \implies V^1 \ll_R V^2.$$

The following lemma shows, how the infimal and supremal sets of S^1 and S^2 relate to one another for $S^1 \ll_R S^2$. It also highlights, that if S^1 and S^2 intersect, the intersection is composed of attained supremal points of S^1 and attained infimal points of S^2 .

4.4 Lemma:

Let $S^1, S^2 \subseteq \overline{\mathcal{Y}}$ and $S^1 \ll_R S^2$ hold. Then

$$\left. \begin{array}{c} S^1 \\ \text{Max}_R S^1 \\ \text{Sup}_R S^1 \end{array} \right\} \ll_R \left\{ \begin{array}{c} S^2, \\ \text{Min}_R S^2, \\ \text{Inf}_R S^2. \end{array} \right.$$

and $S^1 \cap S^2 = (\text{Sup}_R S^1 \cap S^1) \cap (\text{Inf}_R S^2 \cap S^2)$.

Proof. Observe that, for any set $S \subseteq \overline{\mathcal{Y}}$, $-\infty_R \ll_R S \ll_R +\infty_R$ holds. Furthermore, as $\text{Min}_R S \subseteq \text{Inf}_R S$ and $\text{Max}_R S \subseteq \text{Sup}_R S$ from Prop. 2.9, if $S' \ll_R \text{Inf}_R S$ then $S' \ll_R \text{Min}_R S$ and if $\text{Sup}_R S \ll_R S'$ then $\text{Max}_R S \ll_R S'$. Thus, it suffices to show

$$\left. \begin{array}{c} S^1 \\ \text{Sup}_R S^1 \end{array} \right\} \ll_R \left\{ \begin{array}{c} S^2, \\ \text{Inf}_R S^2. \end{array} \right.$$

We specifically address the cases $+\infty_R \in S^1$ and $-\infty_R \in S^2$.

- Let $+\infty_R \in S^1$. Then, by Lem. 4.3 (iii), $S^2 \in \{\emptyset, \{+\infty_R\}\}$ holds. By Cor. 2.23 we obtain both $\text{Sup}_R S^1 = \{+\infty_R\}$ and $\text{Inf}_R S^2 = \{+\infty_R\}$, whence

$$\text{Sup}_R S^1 \ll_R S^2, \quad \text{Sup}_R S^1 \ll_R \text{Inf}_R S^2, \quad S^1 \ll_R \text{Inf}_R S^2,$$

as any set is smaller in terms of $\ll_R$ to the empty set and to $\{+\infty_R\}$. Furthermore, if $S^2 = \{+\infty_R\}$, then $S^1 \cap S^2 = \{+\infty_R\}$ and $\text{Inf}_R S^2 \cap S^2 = \{+\infty_R\}$, and if $S^2 = \emptyset = S^1 \cap S^2$, then $\text{Inf}_R S^2 \cap S^2 = \emptyset$. In both cases, the second assertion $S^1 \cap S^2 = (\text{Sup}_R S^1 \cap S^1) \cap (\text{Inf}_R S^2 \cap S^2)$ holds.

- Let $-\infty_R \in S^2$. Then, by Lem. 4.3 (iv), $S^1 \in \{\emptyset, \{-\infty_R\}\}$ holds. By Cor. 2.23 we obtain both $\text{Inf}_R S^2 = \{-\infty_R\}$ and $\text{Sup}_R S^1 = \{-\infty_R\}$, whence

$$\text{Sup}_R S^1 \ll_R S^2, \quad \text{Sup}_R S^1 \ll_R \text{Inf}_R S^2, \quad S^1 \ll_R \text{Inf}_R S^2,$$

as the empty set and $\{+\infty_R\}$ is smaller in terms of $\ll_R$ to any other set. Furthermore, if $S^1 = \{-\infty_R\}$, then $S^1 \cap S^2 = \{-\infty_R\}$ and $\text{Sup}_R S^1 \cap S^1 = \{-\infty_R\}$, and if $S^1 = \emptyset = S^1 \cap S^2$, then $\text{Sup}_R S^1 \cap S^1 = \emptyset$. In both cases, the second assertion $S^1 \cap S^2 = (\text{Sup}_R S^1 \cap S^1) \cap (\text{Inf}_R S^2 \cap S^2)$ holds.

Assume now $+\infty_R \notin S^1$ and $-\infty_R \notin S^2$. Then by Lem. 2.24 (iii) and Lem. 4.3 (i) and (ii),

$$S^1 \cap \mathcal{Y} \subseteq A_R(S^2) \subseteq A_R(\text{Inf}_R S^2) \implies S^1 \ll_R \text{Inf}_R S^2, \quad (43)$$

$$S^2 \cap \mathcal{Y} \subseteq A_{-R}(S^1) \subseteq A_{-R}(\text{Sup}_R S^1) \implies \text{Sup}_R S^1 \ll_R S^2. \quad (44)$$

Lastly, by the same results,

$$\text{Sup}_R S^1 \cap \mathcal{Y} \subseteq A_R(S^2) \subseteq A_R(\text{Inf}_R S^2) \implies \text{Sup}_R S^1 \ll_R \text{Inf}_R S^2.$$

This show the first assertion. For the second assertion, we have $\pm\infty_R \notin S^1 \cap S^2 \subseteq \mathcal{Y}$. Then, by (43) and (44):

$$\begin{aligned} S^1 \cap S^2 &\stackrel{(43),(44)}{\subseteq} A_{-R}(S^1) \cap S^1 \cap A_R(S^2) \cap S^2 \\ &\stackrel{\text{Prop. 2.16}}{=} \text{Max}_R S^1 \cap \text{Min}_R S^2 \cap S^1 \cap S^2 \\ &\stackrel{\text{Prop. 2.9}}{\subseteq} (\text{Sup}_R S^1 \cap S^1) \cap (\text{Inf}_R S^2 \cap S^2) \subseteq S^1 \cap S^2, \end{aligned}$$

proving the equation and completing the proof. $\square$

4.1.2 Affine Minorants

For the rest of this section, we assume $\mathcal{X}$ and $\mathcal{Y}$ to be topological vector spaces and R to be a translation invariant relation on $\mathcal{Y}$, and equipped with the calculus rules (C1) and (C2) from Sect. 2.4.

Consider some (single-valued) mapping $T : \mathcal{X} \rightarrow \mathcal{Y}$, $x \mapsto Tx$. We call T *linear* whenever, for any scalar α and any $x^1, x^2 \in \mathcal{X}$:

$$T(\alpha x^1) = \alpha T(x^1) \quad \text{and} \quad T(x^1 + x^2) = Tx^1 + Tx^2.$$

The set of all linear mappings that are continuous on $\mathcal{Y}$ is denoted by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$.

Let the space $\mathcal{Y}$ be equipped with a relation R and extended to $\overline{\mathcal{Y}}$, following Sect. 2.1. The (single-valued) mapping $x \mapsto Tx + c$ is called *affine* (or *affine linear*) with *slope* T and *shift* $c \in \overline{\mathcal{Y}}$, whenever T is linear. In the following, we shall consider affine mappings with continuous linear component. Rather than defining affine set-valued mappings as the translation of linear set-valued mapping by some constant $c \in \overline{\mathcal{Y}}$, we instead call *affine set-valued* any mapping expressed as the translation of a linear single-valued mapping T by some (*set-valued*) *shift* $C \subseteq \overline{\mathcal{Y}}$. A *continuous affine set-valued mapping* may thus be expressed as

$$x \mapsto Tx + C, \quad \text{with slope } T \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \text{ and shift } C \subseteq \overline{\mathcal{Y}}.$$

Let $F : \mathcal{X} \rightrightarrows \overline{\mathcal{Y}}$, $x \mapsto F(x)$ be a set-valued mapping from $\mathcal{X}$ to $\overline{\mathcal{Y}}$. A set-valued mapping $G : \mathcal{X} \rightrightarrows \overline{\mathcal{Y}}$, $x \mapsto G(x)$ is called a *minorant* of F (with respect to R), whenever

$$\forall x \in \mathcal{X} : \quad G(x) \ll_R F(x)$$

Moreover, G is called a (*continuous*) *affine minorant* of F with slope $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and shift $C \subseteq \overline{\mathcal{Y}}$, if

$$\forall x \in \mathcal{X} : \quad G(x) := Tx + C \ll_R F(x).$$

We specifically call any single-valued affine minorant $x \mapsto Tx + c$ of F with shift $c \in \overline{\mathcal{Y}}$ an *affine minorant ray* of F . By

$$Tx + C \ll_R F(x) \quad \Longleftrightarrow \quad Tx + c \ll_R F(x), \quad \forall c \in C,$$

any affine minorant can then be thought of as a collection of affine minorant rays of the same slope, in the sense that $x \mapsto Tx + C$ is an affine minorant of F if and only if $x \mapsto Tx + c$ is an affine minorant ray of F for every $c \in C$.

4.5 Remark: *Provided $F(x) \notin \{\emptyset, \{+\infty_R\}\}$ for every $x \in \mathcal{X}$, by the calculus rule (C1) and $Tx \in \mathcal{Y}$ for any $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $c = +\infty_R$ is the shift of no affine minorant ray and belongs to no set-valued shift. Similarly, by the same calculus rule, for any slope T , the shift $c = -\infty_R$ characterizes an affine minorant ray.*

4.1.3 Supporting Affine Minorants

The graph of some affine minorant ray of F may intersect with the graph of F . We call the affine minorant ray $x \mapsto Tx + c$ *supporting* in $(\hat{x}, \hat{y}) \in \text{gph } F$ if

$$T\hat{x} + c = \hat{y} \in F(\hat{x}) \cap \mathcal{Y},$$

in which case we call the mapping a *supporting affine minorant ray* of F (at $(\hat{x}, \hat{y})$). Notice how graph points like $(\hat{x}, -\infty_R)$ or $(\hat{x}, +\infty_R)$ are not considered as potential supports. Otherwise, upon existence of some $\hat{x} \in \mathcal{X}$ with $-\infty_R \in F(\hat{x})$ (resp. $+\infty_R \in F(\hat{x})$), the family of affine minorant rays with shift $c = -\infty_R$ (resp. $+\infty_R$) would be supporting in $(\hat{x}, -\infty_R)$ (resp. $(\hat{x}, +\infty_R)$). One can think of a supporting affine minorant ray as “reaching the graph of F ” at $(\hat{x}, \hat{y})$, while it otherwise maps to points “below” the graph. We say an affine minorant with slope T and shift C is a *supporting affine minorant* of F , if it is a collection of supporting affine minorant rays, i.e. if for every $c \in C$, $x \mapsto Tx + c$ is a supporting affine minorant ray of F .

4.6 Proposition:

Let $F : \mathcal{X} \rightrightarrows \overline{\mathcal{Y}}$ with $\text{dom } F \neq \emptyset$ for some $x \in \mathcal{X}$. An affine mapping $x \mapsto Tx + C$, with slope $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and shift $C \subseteq \overline{\mathcal{Y}}$, is

(i) An affine minorant of F if and only if

$$C \subseteq A_R \left(\bigcup_{x \in \mathcal{X}} F(x) - Tx \right) \cup \{-\infty_R\},$$

(ii) A supporting affine minorant ray of F at $(\hat{x}, \hat{y}) \in \text{gph } F$ if and only if $C = \{c\}$ with

$$c = \hat{y} - T\hat{x} \in A_R \left(\bigcup_{x \in \mathcal{X}} F(x) - Tx \right).$$

(iii) A supporting affine minorant of F if and only if

$$C \subseteq \text{Min}_R \bigcup_{x \in \mathcal{X}} (F(x) - Tx).$$

Proof. For a given slope T and set-valued shift $C \subseteq \overline{\mathcal{Y}}$, any affine minorant of F satisfies, by definition,

$$\forall x \in \mathcal{X} : \quad Tx + C \ll_R F(x) \iff C \ll_R F(x) - Tx, \quad (45)$$

due to $Tx \in \mathcal{Y}$ by $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and translation invariance of R .

Assume $+\infty_R \in C$. By $Tx \in \mathcal{Y}$ and (C1), $+\infty_R \in Tx + C$ holds, and by Lem. 4.3 (ii) applied to (45), $F(x) \in \{\emptyset, \{+\infty_R\}\}$ must hold for every $x \in \mathcal{X}$, contradicting that F maps to points in $\mathcal{Y}$. We may therefor assume $+\infty_R \notin C$. Then, for any $x \in \mathcal{X}$, by Lem. 4.3 (i), $C \ll_R F(x) - Tx$ is equivalent to

$$C \setminus \{-\infty_R\} \subseteq A_R(F(x) - Tx).$$

Thus, $x \mapsto Tx + C$ is an affine minorant of F if and only if

$$C \setminus \{-\infty_R\} \subseteq \bigcap_{x \in \mathcal{X}} A_R(F(x) - Tx) \stackrel{\text{Lem. 2.24 (ii)}}{=} A_R\left(\bigcup_{x \in \mathcal{X}} F(x) - Tx\right).$$

This demonstrates the first assertion. Any affine minorant ray $x \mapsto Tx + c$ is supporting F at $(\hat{x}, \hat{y}) \in \text{gph } F$, if $\hat{y} \in \mathcal{Y}$ and its shift is $c = \hat{y} - T\hat{x}$. This implies $c \neq \pm\infty_R$ and by case (i),

$$c = \hat{y} - T\hat{x} \in A_R\left(\bigcup_{x \in \mathcal{X}} F(x) - Tx\right). \quad (46)$$

For the converse, it is easy to verify, that any affine minorant ray whose single valued shift c satisfies the above is, by (i), an affine minorant ray, and by $c = \hat{y} - T\hat{x} \in \mathcal{Y}$, necessarily supporting at $(\hat{x}, \hat{y})$ with $\hat{y} \in F(\hat{x}) \cap \mathcal{Y}$. For the last assertion, we may observe that

$$c \in F(\hat{x}) - T\hat{x} \subseteq \bigcup_{x \in \mathcal{X}} (F(x) - Tx). \quad (47)$$

Therefor, by Prop. 2.16, (46) and (47), the shift of any supporting affine minorant ray with slope T lies in $\text{Min}_R \bigcup_{x \in \mathcal{X}} (F(x) - Tx)$, and any shift c which does not, violates either (46) or (47). Then, by definition, C is the shift of a supporting affine minorant of F with slope T if and only if, for every $c \in C$, $x \mapsto Tx + c$ is a supporting affine minorant ray of F , which is equivalent to

$$C \subseteq \text{Min}_R \bigcup_{x \in \mathcal{X}} (F(x) - Tx).$$

This completes the proof. □

4.1.4 Subgradients

We have seen how supporting affine minorants gather multiple supporting affine minorant rays of the same slope. One may also look at the

affine minorant rays that support F at a common graph point $(\hat{x}, \hat{y}) \in \text{gph } F$.

The slope T of a supporting affine minorant ray of F at $(\hat{x}, \hat{y})$ is called a *subgradient* of F at $\hat{x}$ for $\hat{y}$ (with $\hat{y} \in F(\hat{x}) \cap \mathcal{Y}$). The set of subgradients of F at $\hat{x}$ for $\hat{y}$, denoted $\partial F(\hat{x} \mid \hat{y})$, is called the *subdifferential* of F at $\hat{x}$ for $\hat{y}$. By Prop. 4.6 (ii), it is characterized by

$$\partial F(\hat{x} \mid \hat{y}) := \{T \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \mid \hat{y} - T\hat{x} \in A_R(\bigcup_{x \in \mathcal{X}} F(x) - Tx)\}.$$

F is called *subdifferentiable* at $(\hat{x}, \hat{y}) \in \text{gph } F$ if $\partial F(\hat{x} \mid \hat{y}) \neq \emptyset$, *subdifferentiable at $\hat{x}$* if $\partial F(\hat{x} \mid \hat{y}) \neq \emptyset, \forall \hat{y} \in F(\hat{x})$, and *subdifferentiable (everywhere)* if it is subdifferentiable at every $\hat{x} \in \text{dom } F$. In this sense, all graph points $(\hat{x}, \hat{y}) \in \text{gph } F$, for which F is subdifferentiable at $\hat{x}$ for $\hat{y}$ with subgradient $T \in \partial F(\hat{x} \mid \hat{y})$, characterize the shift $\hat{y} - T\hat{x} \in \mathcal{Y}$ of an affine minorant ray of slope T , that is supporting in $(\hat{x}, \hat{y})$.

Note that, by construction, F cannot be subdifferentiable at $\hat{x}$ for $\pm\infty_R \in F(\hat{x})$, as affine minorants can only support F in graph points from $\mathcal{X} \times \mathcal{Y}$.

4.7 Remark: *There is a discussion to be had regarding the global view of the definition of subgradients. Indeed, some authors like Rockafellar and Wets [24] define subgradients at $\hat{x}$ for $\hat{y}$ locally around the graph point. In that case, one can relate subgradients to tangential and normal cones of the epigraph. The authors could not expand subgradients to vector-valued functions, as the considered image space was not equipped with a relation, thus preventing the construction of epigraphs. We here base our definitions on classical convex analysis results, under the constraints of which local subgradients are global, as can be found in Sawaragi et al. [25] or Boř et al. [4].*

Note further, that the generalization of differentiability proposed by Rockafellar and Wets [24] is incompatible with the present characterization of subgradients, as it relies on tangential hyperplanes to the graph of F , which, for higher dimensional spaces, do not accurately describe the variations of $F(\hat{x})$ for varying x around $\hat{x}$. This discrepancy is again essentially due to the absence of underlying relation on the image space and the use of scalar products, which are akin to using $T \in \mathcal{L}(\mathcal{X}, \mathbb{R})$.

4.1.5 Conjugate Set-Valued Mappings

Related to both affine minorants and subgradients, we introduce two types of *conjugate set-valued mappings* (or simply *conjugates*) of F . We differentiate

between (general) conjugates and attained conjugates.

4.8 Definition: *Conjugates*

Let $F : \mathcal{X} \rightrightarrows \overline{\mathcal{Y}}$.

(i) The (general) conjugate of F is given by

$$F^* : \mathcal{L}(\mathcal{X}, \mathcal{Y}) \rightrightarrows \overline{\mathcal{Y}}, F^*(T) := \sup_{x \in \mathcal{X}} (Tx - F(x)).$$

The (general) biconjugate of F is the conjugate of F^* , that is

$$F^{**} : \mathcal{X} \rightrightarrows \overline{\mathcal{Y}}, F^{**}(x) := (F^*)^* = \sup_{T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})} (Tx - F^*(T)).$$

(ii) The attained conjugate of F is given by

$$F^\# : \mathcal{L}(\mathcal{X}, \mathcal{Y}) \rightrightarrows \mathcal{Y}, F^\#(T) := \max_{x \in \mathcal{X}} (Tx - F(x)).$$

The attained biconjugate of F is the attained conjugate of $F^\#$, that is

$$F^{\#\#} : \mathcal{X} \rightrightarrows \mathcal{Y}, F^{\#\#}(x) := (F^\#)^\# = \max_{T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})} (Tx - F^\#(T)).$$

By Prop. 4.6 (iii) and $-F^\#(T) = \min_{x \in \mathcal{X}} Tx - F(x)$, the attained conjugate of F at T is the negative shift of the supporting affine minorant $x \mapsto Tx - F^\#(T)$. As such, we may view the negative attained conjugate $-F^\#(T)$ as gathering the single-valued shifts of every supporting affine minorant ray of slope T . For comparison, we may view the subdifferential of F at x for y as gathering the slopes of every supporting affine minorant rays of F supporting in $(x, y) \in \text{gph } F$.

Later on, we shall consider *mixed biconjugates* which combine supremization and maximization. For an arbitrary mapping or $F : \mathcal{X} \rightrightarrows \overline{\mathcal{Y}}$, we denote

$$F^{*\#} : \mathcal{X} \rightrightarrows \mathcal{Y}, F^{*\#}(x) := \max_{T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})} (Tx - F^*(T)),$$

hence maximizing over the general conjugate F^* of F . It is easy to see that $F^{*\#}(x) \subseteq F^{**}(x)$, for any $x \in \mathcal{X}$.

The conjugation scheme can also be applied to the epigraphical set mapping E_F . The next result indirectly shows how, for a given slope, there are more supporting affine minorant rays to the epigraph than to the graph of F .

4.9 Proposition: *Epi-Conjugates*

For $F : \mathcal{X} \rightrightarrows \overline{\mathcal{Y}}$ and $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$,

$$E_F^*(T) = F^*(T) \supseteq E_F^\#(T) \supseteq F^\#(T).$$

Proof. By rewriting the conjugation scheme for any $F : \mathcal{X} \rightrightarrows \overline{\mathcal{Y}}$ using infimization, we may write

$$-F^*(T) = -\sup_R \bigcup_{x \in \mathcal{X}} (Tx - F(x)) \stackrel{\text{Prop. 2.44 (i)}}{=} \inf_R \bigcup_{x \in \mathcal{X}} (F(x) - Tx).$$

and similarly, for the epigraphical set mapping E_F of F ,

$$-E_F^*(T) = \inf_R \bigcup_{x \in \mathcal{X}} (E_F(x) - Tx),$$

The analogous holds for $F^\#$ and $E_F^\#$, expressed through minimization. Combined with $Tx \in \mathcal{Y}$, $\forall T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, we may further use that the epigraphical set mapping of $x \mapsto F(x) - Tx$ is given by

$$x \mapsto E_{F-T}(x) := B_R(F(x) - Tx) \stackrel{\text{Prop. 2.44 (i)}}{=} E_F(x) - Tx.$$

Then, Thm. 4.2 immediately applies and yields the assertion. $\square$

The next result is central to the theory of conjugates:

4.10 Proposition: *Generalized Young-Fenchel Inequality*

Let $F : \mathcal{X} \rightrightarrows \overline{\mathcal{Y}}$. Then, for any $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $x \in \mathcal{X}$:

$$Tx - \begin{Bmatrix} F^*(T) \\ F^\#(T) \\ E_F^*(T) \\ E_F^\#(T) \end{Bmatrix} \ll_R \begin{Bmatrix} F(x) \\ E_F(x) \end{Bmatrix}.$$

If $\text{dom } F \neq \emptyset$, then the left-hand side of the relation does not contain $+\infty_R$.

Proof. Let some $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. For the conjugate mapping F^* , by definition,

$$-F^*(T) \stackrel{\text{Prop. 2.44 (i)}}{=} \text{Inf}_R \bigcup_{x \in \mathcal{X}} (F(x) - Tx).$$

By Cor. 2.18 and Prop. 4.9, $-F^*(T) = -E_F^*(T) = \{+\infty_R\}$ if and only if

$$\bigcup_{x \in \mathcal{X}} (F(x) - Tx) \in \{\emptyset, \{+\infty_R\}\},$$

implying $\text{dom } F = \emptyset$. Any subset $S \subseteq \overline{\mathcal{Y}}$ then satisfies $S \ll_R F(x) \in \{\emptyset, \{+\infty_R\}\}$. The same holds for the epigraphical set mapping since, for any $x \in \mathcal{X}$, $F(x) \in \{\emptyset, \{+\infty_R\}\}$ is equivalent to $E_F(x) = \emptyset$ by Lem. 2.14 (iii). Assume now $\text{dom } F \neq \emptyset$ and thus $-F^*(T) \neq \{+\infty_R\}$. By the definition of infimum, we obtain

$$-F^*(T) \subseteq A_R \left(\bigcup_{x \in \mathcal{X}} F(x) - Tx \right) \cup \{-\infty_R\}, \quad (48)$$

and the inclusion chain from Prop. 4.9 yields

$$+\infty_R \notin -F^*(T) = -E_F^*(T) \supseteq -E_F^\#(T) \supseteq -F^\#(T). \quad (49)$$

It then follows from Prop. 4.6 that $F^\#(T)$, $-E_F^\#(T)$, $-E_F^*(T)$ and $-F^*(T)$ are shifts of affine minorants with slope T . By T chosen arbitrarily, for every $x \in \mathcal{X}$ and $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$:

$$+\infty_R \notin Tx - \begin{Bmatrix} F^*(T) \\ F^\#(T) \\ E_F^*(T) \\ E_F^\#(T) \end{Bmatrix} \ll_R F(x).$$

For the remaining assertions, by Lem. 4.1 (ii) and Prop. 2.44,

$$\begin{aligned} A_R \left(\bigcup_{x \in \mathcal{X}} F(x) - Tx \right) &\stackrel{\text{Lem. 4.1 (ii)}}{=} A_R \left(\bigcup_{x \in \mathcal{X}} B_R(F(x) - Tx) \right) \\ &\stackrel{\text{Prop. 2.44 (i)}}{=} A_R \left(\bigcup_{x \in \mathcal{X}} E_F(x) - Tx \right). \end{aligned}$$

Thus, by (48), (49) and Prop. 4.6, we additionally obtain:

$$+\infty_R \notin Tx - \begin{Bmatrix} F^*(T) \\ F^\#(T) \\ E_F^*(T) \\ E_F^\#(T) \end{Bmatrix} \ll_R E_F(x),$$

completing the proof. □

The generalized Young-Fenchel Inequality allows to see why we may want to consider conjugates in the first place. Additionally, it allows to easily show how any biconjugate is, as the pointwise supremum (resp. maximum) over affine minorants of F , a form of best approximation “from below” of the pointwise infimum (and minimum) of F , and minorants of F . By Prop. 4.10, for any $x \in \mathcal{X}$,

$$\begin{aligned} \forall T \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) : \quad Tx - \left\{ \begin{array}{c} F^*(T) \\ F^\#(T) \end{array} \right\} &\ll_R F(x) \\ \implies \bigcup_{T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})} Tx - \left\{ \begin{array}{c} F^*(T) \\ F^\#(T) \end{array} \right\} &\ll_R F(x), \end{aligned}$$

and by Lem. 4.4, as the above extended relation is kept under maximization and supremization of the the left-hand side, and the minimization and supremization of the right-hand side:

$$\left. \begin{array}{l} \sup_{T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})} Tx - F^*(T) = F^{**}(x) \\ \max_{T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})} Tx - F^*(T) = F^{*\#}(x) \\ \max_{T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})} Tx - F^\#(T) = F^{\#\#}(x) \end{array} \right\} \ll_R \left\{ \begin{array}{l} F(x), \\ \min_R F(x), \\ \inf_R F(x). \end{array} \right.$$

Conditions may be added to F to try and strengthen the above. For functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $R = \leq$, whenever f is proper, lower semicontinuous and convex, $f = f^{**}$ holds [24, Sect. 11.A]. One of the challenges in applying conjugacy to set-valued mappings is finding appropriate conditions on F to “close the gap” between F and its biconjugates.

4.1.6 Properties of Subgradients and Conjugates

We provide a few helpful results relating subgradients and conjugates. The next proposition is a commonly known characterization of subgradients using attained conjugates.

4.11 Proposition: *Alternative Characterization of Subgradients*

Let $F : \mathcal{X} \rightrightarrows \overline{\mathcal{Y}}$. A continuous linear map $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a subgradient of F at $\hat{x}$ for $\hat{y} \in F(\hat{x}) \cap \mathcal{Y}$ if and only if $T\hat{x} - \hat{y}$ is the shift of a supporting affine minorant ray of F at $(\hat{x}, \hat{y})$, i.e.

$$T \in \partial F(\hat{x} \mid \hat{y}) \iff T\hat{x} - \hat{y} \in F^\#(T).$$

Proof. By rewriting the attained conjugation scheme for any $F : \mathcal{X} \rightrightarrows \overline{\mathcal{Y}}$ using minimization, we may write

$$-F^\#(T) \stackrel{\text{Prop. 2.44 (ii)}}{=} \text{Min}_R \bigcup_{x \in \mathcal{X}} (F(x) - Tx). \quad (50)$$

Then, $T \in \partial F(\hat{x} \mid \hat{y})$ if and only if

$$T\hat{x} - \hat{y} \in A_R \left(\bigcup_{x \in \mathcal{X}} F(x) - Tx \right) \stackrel{(50)}{\supseteq} -F^\#(T).$$

Assume now there exists some $T \in \partial F(\hat{x} \mid \hat{y})$ with $T\hat{x} - \hat{y} \notin F^\#(T)$. The existence of a subgradient of F at some $\hat{x}$ for $\hat{y}$ immediately implies $\hat{y} \in F(\hat{x}) \cap \mathcal{Y}$. Then, by Prop. 4.6 (iii), $T\hat{x} - \hat{y}$ is not the shift of an affine minorant, thus not of an affine minorant ray, contradicting that T is a subgradient of F at $(\hat{x}, \hat{y})$. $\square$

In view of the above, $-F^\#(T)$ contains the shifts of all supporting affine minorant rays of slope T , and these shifts take the form of $\hat{y} - T\hat{x}$ for an affine minorant ray supporting F in $(\hat{x}, \hat{y}) \in \text{gph } F$, in which case $T \in \partial F(\hat{x}, \hat{y})$. A connection can also be drawn to general conjugates:

4.12 Lemma:

Let $F : \mathcal{X} \rightrightarrows \overline{\mathcal{Y}}$ with $\text{dom } F \neq \emptyset$. Then, for any $(\hat{x}, \hat{y}) \in \text{gph } F$ with $\hat{y} \neq -\infty_R$ and $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$,

$$T\hat{x} - \hat{y} \in F^*(T) \iff T\hat{x} - \hat{y} \in F^\#(T).$$

Proof. Assume $\hat{y} = +\infty_R$. Then, by Prop. 2.44, for any $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $T\hat{x} - \hat{y} = -\infty_R$. By Prop. 4.10, as F has a nonempty domain, $-\infty_R \notin F^*(T)$ must hold, and $T\hat{x} - \hat{y} \notin F^*(T) \supseteq F^\#(T)$. The equivalence assertion is thus always true. Assume now $\hat{y} \in \mathcal{Y}$. The first implication follows from $F^\#(\hat{x}) \subseteq F^*(\hat{x})$. It remains to show that, $T\hat{x} - \hat{y} \in F^*(T) \subseteq F^\#(\hat{x})$ specifically for $\hat{y} \in F(\hat{x}) \cap \mathcal{Y}$. By the existence of a supremal point in $F^*(T)$, by definition,

$$\begin{aligned} T\hat{x} - \hat{y} \in F^*(T) &= \text{Sup}_R \bigcup_{x \in \mathcal{X}} Tx - F(x) \neq \{\pm\infty_R\} \\ \implies F^*(T) &\subseteq A_{-R} \left(\bigcup_{x \in \mathcal{X}} Tx - F(x) \right). \end{aligned}$$

Then, since $\hat{y} \in F(\hat{x})$, it follows that $T\hat{x} - \hat{y} \in \bigcup_{x \in \mathcal{X}} Tx - F(x)$, and altogether, by Prop. 2.16,

$$T\hat{x} - \hat{y} \in \text{Max}_R \bigcup_{x \in \mathcal{X}} Tx - F(x) = F^\#(T),$$

completing the proof. $\square$

If we do not assume that $\hat{y} \neq -\infty_R$, we could obtain $T\hat{x} - \hat{y} = +\infty_R \in F^*(T)$, in which case $T\hat{x} - \hat{y}$ cannot lie in $F^\#(T)$, due to attained conjugates being defined by maximization. Whenever $\hat{y} \in F(\hat{x}) \cap \mathcal{Y}$, Lem. 4.12 can be easily combined with Prop. 4.11.

The following result echoes Fermat's rule, without relying on a notion of differentiability.

4.13 Proposition: Minimality and Infimality

Let $F : \mathcal{X} \rightrightarrows \overline{\mathcal{Y}}$.

- (i) $\hat{y} \in \text{Min}_R \bigcup_{x \in \mathcal{X}} F(x) \iff -\hat{y} \in F^\#(0) \iff 0 \in \partial F(\hat{x} \mid \hat{y}),$
- (ii) $\hat{y} \in \text{Inf}_R \bigcup_{x \in \mathcal{X}} F(x) \iff -\hat{y} \in F^*(0).$

Proof. That $0 \in \partial F(\hat{x} \mid \hat{y})$ is equivalent to $-\hat{y} \in F^\#(0)$ immediately follows from Prop. 4.11. Using the calculus rules from Prop. 2.44 (i),

$$\begin{aligned} \hat{y} \in \text{Min}_R \bigcup_{x \in \mathcal{X}} F(x) &\iff -\hat{y} \in \text{Max}_R \bigcup_{x \in \mathcal{X}} (0x - F(x)) = F^\#(0), \\ \hat{y} \in \text{Inf}_R \bigcup_{x \in \mathcal{X}} F(x) &\iff -\hat{y} \in \text{Sup}_R \bigcup_{x \in \mathcal{X}} (0x - F(x)) = F^*(0), \end{aligned}$$

with “ $0x$ ” referring to the value of the constant linear mapping $0 \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ evaluated at $x \in \mathcal{X}$. $\square$

We discuss further properties of conjugates with regards to subdifferentiability and connections to biconjugates.

4.14 Proposition: Effects of Subdifferentiability

Let $F : \mathcal{X} \rightrightarrows \mathcal{Y}$, $\hat{x} \in \mathcal{X}$ and $\hat{y} \in \mathcal{Y}$. The following are equivalent:

- (i) $\partial F(\hat{x} \mid \hat{y}) \neq \emptyset$,
- (ii) $\hat{y} \in F(\hat{x}) \cap F^{\#\#}(\hat{x})$,
- (iii) $\hat{y} \in F(\hat{x}) \cap F^{*\#}(\hat{x})$.

Moreover, any of the above implies $\hat{y} \in \text{Min}_R F(\hat{x})$ and

$$\hat{y} \in F(\hat{x}) \cap F^{**}(\hat{x}).$$

Proof. Let some $T \in \partial F(\hat{x} \mid \hat{y})$ exist. By definition, $\hat{y} \in F(\hat{x}) \cap \mathcal{Y}$. We first show that $\hat{y}$ is a minimal element of $F(\hat{x})$. By definition of subgradients, Lem. 2.24 (i) and Prop. 2.44, we know that

$$\begin{aligned} T\hat{x} - \hat{y} &\in A_{-R} \left(\bigcup_{x \in \mathcal{X}} Tx - F(x) \right) \stackrel{\text{Lem. 2.24 (i)}}{\subseteq} A_{-R} (T\hat{x} - F(\hat{x})) \\ &\stackrel{\text{Prop. 2.44 (ii)}}{\iff} \hat{y} \in T\hat{x} + A_R (F(\hat{x}) - T\hat{x}) \stackrel{\text{Prop. 2.44 (i)}}{=} A_R (F(\hat{x})), \end{aligned}$$

such that, by Prop. 2.16, $\hat{y} \in \text{Min}_R F(\hat{x})$. We proceed to show that $\hat{y}$ belongs to the biconjugates $F^{\#\#}(T)$, $F^{*\#}(T)$ and $F^{**}(T)$. Using the alternative characterization of subgradients from Prop. 4.11 and $F^\#(T) \subseteq F^*(T)$,

$$\begin{aligned} T\hat{x} - \hat{y} &\in F^\#(T) && \subseteq F^*(T) \\ \iff \hat{y} &\in T\hat{x} - F^\#(T) && \subseteq T\hat{x} - F^*(T) \\ \implies \hat{y} &\in \bigcup_{T' \in \mathcal{L}(\mathcal{X}, \mathcal{Y})} T'\hat{x} - F^\#(T') && \subseteq \bigcup_{T' \in \mathcal{L}(\mathcal{X}, \mathcal{Y})} T'\hat{x} - F^*(T'). \end{aligned} \quad (51)$$

Moreover, by Prop. 4.10 and Prop. 4.3 (ii), for any $\hat{y} \in F(\hat{x}) \cap \mathcal{Y}$:

$$T\hat{x} - \left\{ \begin{array}{c} F^*(T) \\ F^\#(T) \end{array} \right\} \ll_R \{\hat{y}\} \iff \{\hat{y}\} \subseteq \left\{ \begin{array}{c} A_{-R} (T\hat{x} - F^*(T)) \\ A_{-R} (T\hat{x} - F^\#(T)) \end{array} \right\},$$

hold for every $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. We then obtain:

$$\hat{y} \in A_{-R} \left(\bigcup_{T' \in \mathcal{L}(\mathcal{X}, \mathcal{Y})} T'\hat{x} - F^*(T') \right) \subseteq A_{-R} \left(\bigcup_{T' \in \mathcal{L}(\mathcal{X}, \mathcal{Y})} T'\hat{x} - F^\#(T') \right). \quad (52)$$

Thus, by Prop. 2.16, (51) and (52):

$$\hat{y} \in \begin{cases} \text{Max}_R \bigcup_{T' \in \mathcal{L}(\mathcal{X}, \mathcal{Y})} T' \hat{x} - F^*(T') = F^{*\#}(\hat{x}) \subseteq F^{**}(\hat{x}), \\ \text{Max}_R \bigcup_{T' \in \mathcal{L}(\mathcal{X}, \mathcal{Y})} T' \hat{x} - F^\#(T') = F^{\#\#}(\hat{x}). \end{cases}$$

We have shown how (i) implies (ii) and (iii). We now address the two remaining implications:

- (ii) $\Rightarrow$ (i): Let $\hat{y} \in F^{\#\#}(\hat{x}) \cap F(\hat{x})$. As $\hat{y}$ is part of a maximal set, $\hat{y} \in \mathcal{Y}$ holds. Moreover, $\hat{y} \in \bigcup_{T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})} T\hat{x} - F^\#(T)$ and there must specifically exist some $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ for which $\hat{y} \in T\hat{x} - F^\#(\hat{x})$ holds. Then, $T\hat{x} - \hat{y} \in F^\#(\hat{x})$ and by $\hat{y} \in F(\hat{x}) \cap \mathcal{Y}$ and Prop. 4.11, we obtain $T \in \partial F(\hat{x} \mid \hat{y})$.
- (iii) $\Rightarrow$ (ii): Let $\hat{y} \in F^{*\#}(\hat{x}) \cap F(\hat{x})$. As $\hat{y}$ is part of a maximal set, $\hat{y} \in \mathcal{Y}$ holds. Moreover, $\hat{y} \in \bigcup_{T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})} T\hat{x} - F^*(T)$ and there must specifically exist some $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ for which $\hat{y} \in T\hat{x} - F^*(\hat{x})$. Then, by $T\hat{x} - \hat{y} \in F^*(T)$ and $\hat{y} \in F(\hat{x}) \cap \mathcal{Y}$, by Lem. 4.12, $T\hat{x} - \hat{y} \in F^\#(\hat{x})$ holds.

This completes the proof. $\square$

4.15 Remark: The above highlights, that F can only be subdifferentiable at $\hat{x}$ for $\hat{y}$ if $\hat{y}$ is a minimal point of $F(\hat{x})$, in which case $(\hat{x}, \hat{y})$ is also a graph point of the biconjugates $F^{\#\#}$ and $F^{*\#}$.

Note also, that $\hat{y} \in F^{**}(\hat{x}) \cap F(\hat{x})$ does not imply subdifferentiability of F at $(\hat{x}, \hat{y})$ as we can only assume that

$$\hat{y} \in B_{-R} \left(\bigcup_{T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})} T\hat{x} - F^*(T) \right) \supseteq \bigcup_{T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})} T\hat{x} - F^*(T),$$

which proves to be insufficient to guarantee the existence of a subgradient of F at $\hat{x}$ for $\hat{y}$.

4.16 Corollary:

Some set-valued mapping $F : \mathcal{X} \rightrightarrows \overline{\mathcal{Y}}$ is subdifferentiable at $\hat{x} \in \mathcal{X}$ if and only if $F(\hat{x}) \subseteq F^{\#\#}(\hat{x})$. The same also holds when replacing $F^{\#\#}$ with $F^{*\#}$. Any of the above implies $F(\hat{x}) = \text{Min}_R F(\hat{x})$.

Proof. The mapping F is called subdifferentiable at $\hat{x}$ whenever $\partial F(\hat{x} \mid \hat{y}) \neq \emptyset$ for every $\hat{y} \in F(\hat{x})$. By Prop. 4.14, subdifferentiability of F at $\hat{x}$ for $\hat{y}$ is equivalent to

$$\hat{y} \in F(\hat{x}) \cap F^{\#\#}(\hat{x}) \quad (\text{resp. } \hat{y} \in F(\hat{x}) \cap F^{**}(\hat{x})),$$

and whenever this holds for every $\hat{y} \in F(\hat{x})$, we obtain

$$F(\hat{x}) \subseteq F^{\#\#}(\hat{x}) \quad (\text{resp. } F(\hat{x}) \subseteq F^{**}(\hat{x})).$$

By the same proposition, any of the above conditions implies that every $\hat{y}$ must be a minimal point of $F(\hat{x})$. Thus, $F(\hat{x})$ can only be self-minimal, i.e. $F(\hat{x}) = \text{Min}_R F(\hat{x})$. $\square$

4.17 Remark: For any set-valued mapping $F : \mathcal{X} \rightrightarrows \overline{\mathcal{Y}}$, if $\hat{y} \in F(x) \cap F^{**}(x)$, then $\hat{y} \in \text{Inf}_R F(x)$. To show this we use that, by Lem. 4.4 and Prop. 4.10,

$$\forall x \in \mathcal{X} : \quad F^{**}(x) \ll_R F(x).$$

Assume $\hat{y} \in F(x) \cap F^{**}(x)$. Then:

- If $\hat{y} = +\infty_R$, by $F^{**}(x) \ll_R F(x)$ Lem. 4.3 (iii), $F(x) \in \{\emptyset, \{+\infty_R\}\}$ and by Cor. 2.18, $\hat{y} \in \text{Inf}_R F(x) = \{+\infty_R\}$.
- If $\hat{y} = -\infty_R$, by $-\infty_R \in F(x)$ and Cor. 2.23, $\hat{y} \in \text{Inf}_R F(x) = \{-\infty_R\}$.
- If $\hat{y} \in \mathcal{Y}$, then $+\infty_R \notin F^{**}(x) \subseteq \mathcal{Y}$ as the biconjugate is characterized by a supremum and contains $\hat{y}$. Thus, by $F^{**}(x) \ll_R F(x)$ and Lem. 4.3 (ii),

$$\hat{y} \in F^{**}(x) \subseteq A_R(F(x)),$$

and, by Prop. 2.9 and Prop. 2.16,

$$\hat{y} \in F(x) \cap A_R(F(x)) = \text{Min}_R F(x) \subseteq \text{Inf}_R F(x).$$

This highlights the fact that, whenever $F(x) \subseteq F^{**}(x)$ (or even $F = F^{**}$), $F \subseteq \text{Inf}_R F(x)$ must hold, i.e. F can only map to infimal points. For functions, this is automatically satisfied.

4.2 Conjugate Duality for Set-Valued Problems

Let the set-valued mapping $F : \mathcal{X} \rightrightarrows \bar{\mathcal{Y}}$, $x \mapsto F(x)$ be the (*primal*) *objective mapping* of the following optimization problem:

$$(P) \quad \inf_{x \in \mathcal{X}} F(x).$$

The problem is considered an *unconstrained problem*. If one wishes to consider infimization of F over a specific set $S \subseteq \mathcal{X}$, then the constraint can be embedded in a set-valued mapping $\tilde{F} : \mathcal{X} \rightrightarrows \bar{\mathcal{Y}}$ with $\tilde{F}(x) = F(x)$ for every $x \in S$, and $\tilde{F}(x) = \{+\infty_R\}$ or $\emptyset$ otherwise. We differentiate between the following:

- $\hat{y} \in \bar{\mathcal{Y}}$ is an *infimal point* of (P) if $\hat{y} \in \text{Inf}_R(P) := \text{Inf}_R \bigcup_{x \in \mathcal{X}} F(x)$,
- $\hat{y} \in \mathcal{Y}$ is a *minimal point* of (P) if $\hat{y} \in \text{Min}_R(P) := \text{Min}_R \bigcup_{x \in \mathcal{X}} F(x)$,
- $(\hat{x}, \hat{y}) \in \text{gph } F$ is a *minimal pair* of (P) if $\hat{y}$ is a *minimal point* of (P) with $\hat{y} \in F(\hat{x})$.

The sets of infimal and minimal points are the counterparts to the infimal and minimal values in scalar minimization. By Prop. 2.9, $\text{Min}_R(P) \subseteq \text{Inf}_R(P)$. For any minimal point $\hat{y} \in \text{Min}_R(P)$, there exists some $\hat{x} \in \text{dom } F$ with $(\hat{x}, \hat{y})$ a minimal pair of (P) . Infimal points that are not minimal cannot easily be associated with an inverse image set. For $\hat{y}$ an infimal point of (P) , there may not exist any $\hat{x}$ with $\hat{y} \in F(\hat{x})$. Similar notions and notation also apply to maximization and supremization problems.

For the problem (P) , we call $\text{Inf}_R(P)$ the *optimal set* of (P) , as a counterpart for the optimal value in scalar optimization. We further call a problem (P') a subproblem of (P) whenever the optimal set of (P') is a subset of the optimal set of (P) . At some occasions we shall consider the subproblem

$$(P_{=}) \quad \min_{x \in \mathcal{X}} F(x),$$

with optimal set $\text{Min}_R(P_{=})$ coinciding with the set of minimal points of (P) . To introduce a counterpart to minimal pairs, we may consider the *epi-primal problem*

$$(P_{\text{epi}}) \quad \inf_{x \in \mathcal{X}} E_F(x).$$

In view of Thm. 4.2, (P_{epi}) is a subproblem of (P) . Using the epigraph of F , we shall say $(\hat{x}, \hat{y}) \in \text{epi } F$ is an (*epi*-) *infimal pair* of (P) with *epi-infimal*

point $\hat{y}$, if it is a minimal pair of (P_{epi}) , i.e. $\hat{y} \in E_F(\hat{x}) = B_R(F(\hat{x}))$. The set of epi-infimal points is thus denoted by $\text{Min}_R(P_{\text{epi}})$. Still, in general, not every infimal point belongs to an infimal pair.

4.2.1 Solution Concepts for Set-Valued Infimization

There is a discussion to be had on what constitutes a *complete* solution to the problem (P) . Depending on application, a subset of that complete solution may be satisfactory. Consider the scalar optimization problem

$$(P_f) \quad \inf_{x \in \mathcal{X}} f(x),$$

with $f : \mathcal{X} \rightarrow \overline{\mathbb{R}}$ and infimal value $v := \inf_{x \in \mathcal{X}} f(x)$. Some set $\overline{X} \subseteq \mathcal{X}$ is the *complete solution* of (P_f) if for any $\bar{x} \in \mathcal{X}$ with $f(\bar{x}) = v$ implies $\bar{x} \in \overline{X}$. The infimal value v of (P_f) is unique and exists even if there is no solution to (P_f) . If a solution to (P_f) exists, then any $\bar{x} \in \overline{X}$ is associated with v by $f(\bar{x}) = v$.

For vector- and set-valued problems, the infimal set may not be a singleton. As such, let a *complete solution* S of (P) be a collection of pairs $(x, y) \in \mathcal{X} \times \overline{\mathcal{Y}}$, such that x is associated with y . Pairs give us information regarding which pre-image points $\bar{x} \in \mathcal{X}$ contribute to the occurrence of an infimal point $\bar{y} \in \text{Inf}_R(P)$.

We may consider two different solution concepts. The first solution concept is based on *attained infimal points*, directly transferred from the scalar case. Consider the collection of pairs

$$\overline{\mathcal{S}} := \{(\bar{x}, \bar{y}) \mid \bar{y} \in \text{Inf}_R(P) \cap F(\bar{x})\}.$$

For any $(\bar{x}, \bar{y}) \in \overline{\mathcal{S}}$, $\bar{y}$ is an infimal point of (P) , attained at $\bar{x}$. If F is proper, then $-\infty_R \notin \text{dom } F$ and $\text{Inf}_R(P) \neq \{+\infty_R\}$, such that we may reformulate $\overline{\mathcal{S}}$ to:

$$\begin{aligned} \overline{\mathcal{S}} &= \{(\bar{x}, \bar{y}) \mid \bar{y} \in \text{Min}_R(P) \cap F(\bar{x})\}, \\ &= \{(\bar{x}, \bar{y}) \mid \text{is a minimal pair of } (P)\}. \end{aligned}$$

As such, solving (P) becomes synonymous with finding all of its minimal pairs. $\overline{\mathcal{S}}$ can thus be considered a solution of both (P) and (P_-) .

There is however another solution concept we can use. In scalar optimization, $f(x)$ is self-infimal for any $x \in \mathcal{X}$. As such, for $\bar{x}$ to be a part of the set

of solutions, $\bar{x}$ is also associated to v via $\inf\{f(\bar{x})\} = v$. We consider the collection of pairs

$$\tilde{\mathcal{S}} := \{(\bar{x}, \bar{y}) \mid \bar{y} \in \inf_R(P) \cap \inf_R F(\bar{x})\}.$$

For any $(\bar{x}, \bar{y}) \in \tilde{\mathcal{S}}$, $\bar{y}$ is an infimal point of (P) and is associated with $\bar{x}$ through $\bar{y} \in \inf_R F(\bar{x})$. In the scalar case, $\bar{\mathcal{S}}$ and $\tilde{\mathcal{S}}$ coincide.

Assuming F is proper and $\inf_R(P) \neq \{-\infty_R\}$, we can show that $\emptyset \neq E_F(x) \neq \mathcal{Y}$, and thus $\inf_R F(x) = \min_R E_F(x)$ for every $x \in \mathcal{X}$. Using the epigraphical problem (P_{epi}) , we also obtain $\inf_R(P) = \inf_R(P_{\text{epi}}) \subseteq \mathcal{Y}$, and as such

$$\inf_R(P) = \inf_R(P_{\text{epi}}) = A_R \left(\bigcup_{x \in \mathcal{X}} E_F(x) \right) \cap B_R \left(\bigcup_{x \in \mathcal{X}} E_F(x) \right).$$

Using Lem. 2.24 and Lem. 4.1, we get

$$\begin{aligned} & \inf_R(P_{\text{epi}}) \cap \min_R E_F(\bar{x}) \\ &= A_R \left(\bigcup_{x \in \mathcal{X}} E_F(x) \right) \cap A_R(E_F(\bar{x})) \cap B_R \left(\bigcup_{x \in \mathcal{X}} E_F(x) \right) \cap E_F(\bar{x}) \\ & \quad \underbrace{\qquad\qquad\qquad}_{=A_R(\bigcup_{x \in \mathcal{X}} E_F(x))} \quad \underbrace{\qquad\qquad\qquad}_{\supseteq \bigcup_{x \in \mathcal{X}} E_F(x) \supseteq E_F(\bar{x})} \\ &= A_R \left(\bigcup_{x \in \mathcal{X}} E_F(x) \right) \cap \left(\bigcup_{x \in \mathcal{X}} E_F(x) \right) \cap E_F(\bar{x}) \\ &= \min_R(P_{\text{epi}}) \cap E_F(\bar{x}). \end{aligned}$$

We may therefor reformulate $\tilde{\mathcal{S}}$ to:

$$\begin{aligned} \tilde{\mathcal{S}} &= \{(\bar{x}, \bar{y}) \mid \bar{y} \in \min_R(P_{\text{epi}}) \cap E_F(\bar{x})\}, \\ &= \{(\bar{x}, \bar{y}) \mid \text{is an epi-infimal pair of } (P)\}. \end{aligned}$$

As such, solving (P) becomes synonymous with finding all of its epi-infimal pairs, which coincide with the minimal pairs of (P_{epi}) . In view of both of the above solution concepts, it is clear how central minimal pairs are to solving (P) . Further considerations regarding how representative of $F(\mathcal{X})$ these solution concepts are may be investigated. For an investigation of solution concepts in vector optimization in the setting of weak efficiency, refer to [16, Chapt. 2].

4.18 Example: Consider the problem (P) for the set valued mapping $F : \mathbb{R} \rightrightarrows \mathbb{R}$ with

$$F(x) := \left(1 + \cos x, 3 + \sin \frac{x}{2}\right].$$

The infimum of F over $\mathbb{R}$ is 0, but it is never attained. Thus, there exist no minimal pairs for F over $\mathbb{R}$, but the infimal pairs

$$\tilde{\mathcal{S}} = \{(x, y) \mid (2\pi i + \pi, 0), i \in \mathbb{Z}\}$$

communicate where the mapping approaches its infimum value the most.

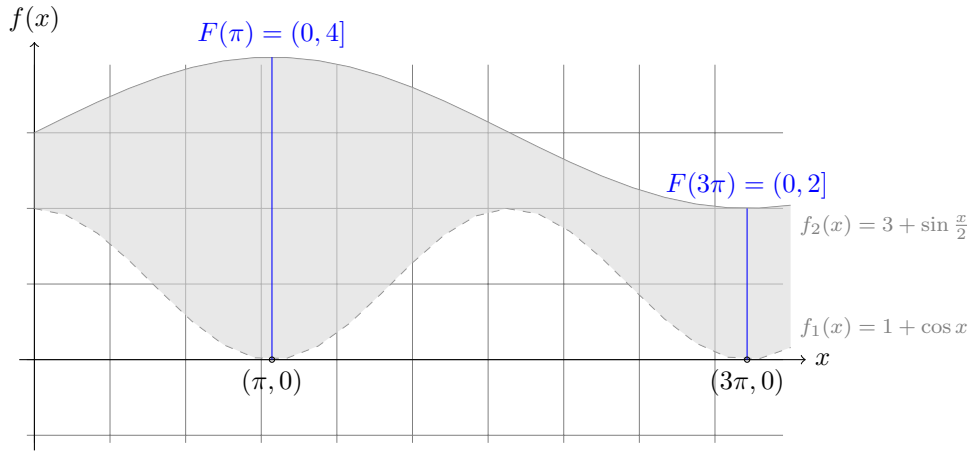


Figure 4.1: Infimal pairs of an interval valued mapping.

4.2.2 Perturbation Approach

We now address a set-valued perturbation approach for the primal problem (P) . Let $\mathcal{U}$ be a topological vector space. Consider some set-valued mapping

$$\Phi : \mathcal{X} \times \mathcal{U} \rightrightarrows \overline{\mathcal{Y}}, (x, u) \mapsto \Phi(x, u).$$

If Φ is such that, $\Phi(x, 0) = F(x)$, $\forall x \in \mathcal{X}$, then it is called a *perturbation mapping* for F over $\mathcal{X}$ with perturbation space $\mathcal{U}$. The exact choice of perturbation mapping gives rise to different types of dual problems, *e.g.* *Legendre-Fenchel* and *Lagrange* dual problems. See [4, Chapt. 7] for exact choices and reformulations for the specific cases of nondominated and weakly nondominated points, as previously discussed in Sect. 3.4. We may embed

our primal problem in the following family of perturbed primal problems: for any $u \in \mathcal{U}$, we define

$$(P^u) \quad \inf_{x \in \mathcal{X}} \Phi(x, u).$$

Naturally, for $u = 0$, we have $(P^u) = (P)$. The same can be applied to the subproblem $(P_=)$. Based on the perturbation mapping Φ , we can easily construct a perturbation mapping E_Φ for the epigraphical set mapping E_F with

$$E_\Phi : \mathcal{X} \times \mathcal{U} \rightrightarrows \overline{\mathcal{Y}}, (x, u) \mapsto E_\Phi(x, u) := B_R(\Phi(x, u)).$$

We therefor also denote the family of perturbed epi-problems, of which (P_{epi}) is a special case for $u = 0$, by

$$(P_{epi}^u) \quad \inf_{x \in \mathcal{X}} E_\Phi(x, u).$$

Since $\mathcal{X}$ and $\mathcal{U}$ are assumed to be topological vector spaces, so is $\mathcal{X} \times \mathcal{U}$ under pointwise addition and scalar multiplication, and the product topology [1, Thm. 5.2]. Consider an arbitrary linear mapping $L \in \mathcal{L}(\mathcal{X} \times \mathcal{U}, \mathcal{Y})$. Then, by linearity,

$$L(x + 0, 0 + u) = L(x, u) = L(x, 0) + L(0, u), \quad \forall (x, u) \in \mathcal{X} \times \mathcal{U},$$

and L can be decomposed into $Tx := L(x, 0)$ and $\Lambda u := L(0, u)$, with $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$. In the following, we may thus consider linear mappings $(T, \Lambda) \in \mathcal{L}(\mathcal{X} \times \mathcal{U}, \mathcal{Y}) \cong \mathcal{L}(\mathcal{X}, \mathcal{Y}) \times \mathcal{L}(\mathcal{U}, \mathcal{Y})$ with

$$(T, \Lambda) : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{Y}, (x, u) \mapsto Tx + \Lambda u.$$

4.2.3 Weak Duality

Recall how *weak duality* in the scalar case is associated with the dual function value evaluated at any dual feasible point being smaller or equal to the primal function value evaluated at any primal feasible point, making the range of the dual objective function a lower bound of the range of the primal objective function. Additionally, the dual objective function and the dual supremal and maximal values are all lower bounds to the primal objective function and the primal infimal and minimal values. In other words, for a primal objective function $f : \mathcal{X} \rightarrow \overline{\mathbb{R}}$ and dual objective function $\psi : \mathcal{Z} \rightarrow \overline{\mathbb{R}}$,

weak duality holds when $\psi(z) \leq f(x)$ for any $x \in \mathcal{X}$ and $z \in \mathcal{Z}$, and implies:

$$\psi(z) \leq \left\{ \begin{array}{c} \max_{z \in \mathcal{Z}} \psi(z) \\ \sup_{z \in \mathcal{Z}} \psi(z) \end{array} \right\} \leq \left\{ \begin{array}{c} \min_{x \in \mathcal{X}} f(x) \\ \inf_{x \in \mathcal{X}} f(x) \end{array} \right\} \leq f(x). \quad (53)$$

We aim to generalize this idea and construct weakly dual problems to (P) . We shall say that, for a dual problem (D) with *dual objective mapping* $\Psi : \mathcal{Z} \rightrightarrows \overline{\mathcal{Y}}$, $z \mapsto \Psi(z)$, the pair of problems (P) and (D) satisfies weak duality if

$$\Psi(z) \ll_R F(x)$$

for any $x \in \mathcal{X}$ and any $z \in \mathcal{Z}$ (we shall specify $\mathcal{Z}$ and Ψ shortly). In this case, (D) is said to be *weakly dual* to (P) .

To construct a weakly dual problem to (P) , geometrically, we can make use of affine minorants. By Prop. 4.13, the affine minorants of F with slope $0 \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ map to sets of points that are necessarily lower bounds to the range of F . However, this is too restrictive. Instead of applying this idea directly to F , we can consider the affine minorants of the perturbation mapping Φ . More specifically, any affine minorant G of Φ that has a slope of zero in $\mathcal{X}$, maps to lower bounds of Φ , and does so independently of $\mathcal{X}$. Thus, the affine minorants of Φ with slope $(0, \Lambda) \in \mathcal{L}(\mathcal{X} \times \mathcal{U}, \mathcal{Y})$ map, for any $(x, 0) \in \mathcal{X} \times \mathcal{U}$, to lower bounds of $\Phi(x, 0)$, and thus of $F(x)$. This is the cornerstone of conjugate duality.

We proceed with the detailed construction of two dual problems based on the above principle. This section can be thought of as combining the approaches from Boţ et al. [4, Sect. 7.1.2, Sect. 7.4.2]. Consider the general and attained conjugates of Φ :

$$\begin{aligned} \Phi^* : \mathcal{L}(\mathcal{X} \times \mathcal{U}, \mathcal{Y}) &\rightrightarrows \overline{\mathcal{Y}}, \Phi^*(T, \Lambda) := \sup_{(x, u) \in \mathcal{X} \times \mathcal{U}} Tx + \Lambda u - \Phi(x, u), \\ \Phi^\# : \mathcal{L}(\mathcal{X} \times \mathcal{U}, \mathcal{Y}) &\rightrightarrows \mathcal{Y}, \Phi^\#(T, \Lambda) := \max_{(x, u) \in \mathcal{X} \times \mathcal{U}} Tx + \Lambda u - \Phi(x, u). \end{aligned}$$

We specifically have

$$\begin{aligned} \Phi^*(0, \Lambda) &= \sup_{(x, u) \in \mathcal{X} \times \mathcal{U}} \Lambda u - \Phi(x, u), \\ \Phi^\#(0, \Lambda) &= \max_{(x, u) \in \mathcal{X} \times \mathcal{U}} \Lambda u - \Phi(x, u). \end{aligned}$$

Moreover, note that, by Prop. 4.9, Φ^* is also the conjugate mapping of E_Φ . We formulate two dual problems:

$$\begin{aligned} (D^*) \quad & \sup_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} -\Phi^*(0, \Lambda), \\ (D_{\#}^{\#}) \quad & \max_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} -\Phi^{\#}(0, \Lambda), \end{aligned}$$

with $\Lambda \mapsto -\Phi^*(0, \Lambda)$ and $\Lambda \mapsto -\Phi^{\#}(0, \Lambda)$ the respective dual objective mappings of (D^*) and $(D_{\#}^{\#})$. The two dual problems respectively represent the *supremum-based* and *maximum-based* conjugate duality approaches. The sets of supremal and maximal points to (D^*) are respectively given by

$$\sup_R(D^*) := \sup_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} -\Phi^*(0, \Lambda), \quad \max_R(D^*) := \max_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} -\Phi^*(0, \Lambda),$$

and the set of maximal points of $(D_{\#}^{\#})$ by

$$\max_R(D_{\#}^{\#}) := \max_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} -\Phi^{\#}(0, \Lambda).$$

Applying the generalized Young-Fenchel Inequality from Prop. 4.10, the following result relates the primal to the dual problems, showing weak duality holding for any of the primal problems (P) and (P_{epi}) , with any of the dual problems (D^*) and $(D_{\#}^{\#})$.

4.19 Proposition: *Weak Duality*

For any set-valued mapping $F : \mathcal{X} \rightrightarrows \bar{\mathcal{Y}}$, perturbation space $\mathcal{U}$ and perturbation mapping $\Phi : \mathcal{X} \times \mathcal{U} \rightrightarrows \bar{\mathcal{Y}}$ of F , it holds:

$$\left\{ \begin{array}{l} \bigcup_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} -\Phi^*(0, \Lambda) \\ \bigcup_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} -\Phi^{\#}(0, \Lambda) \end{array} \right\} \ll_R \left\{ \begin{array}{l} \bigcup_{x \in \mathcal{X}} F(x), \\ \bigcup_{x \in \mathcal{X}} E_F(x), \end{array} \right.$$

and both (D^*) and $(D_{\#}^{\#})$ are weakly dual to (P) and (P_{epi}) . Moreover, if $\text{dom } F \neq \emptyset$, then $-\infty_R \notin \text{rge } \Phi^* \cup \text{rge } \Phi^{\#}$.

Proof. By Prop. 4.10, we know

$$Tx + \Lambda u - \left\{ \begin{array}{l} \Phi^*(T, \Lambda) \\ \Phi^{\#}(T, \Lambda) \end{array} \right\} \ll_R \left\{ \begin{array}{l} \Phi(x, u), \\ E_\Phi(x, u), \end{array} \right.$$

for any $(x, u) \in \mathcal{X} \times \mathcal{U}$ and $(T, \Lambda) \in \mathcal{L}(\mathcal{X} \times \mathcal{U}, \mathcal{Y})$. Then, for $u = 0$ and $T = 0$ we specifically obtain

$$\left. \begin{array}{l} -\Phi^*(0, \Lambda) \\ -\Phi^\#(0, \Lambda) \end{array} \right\} \ll_R \left\{ \begin{array}{l} \Phi(x, 0) = F(x), \\ E_\Phi(x, 0) = E_F(x), \end{array} \right.$$

for any $x \in \mathcal{X}$ and $\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$. Since the left-hand side of the relation depends only on Λ , and the right-hand side only on x , it follows that:

$$\left. \begin{array}{l} \bigcup_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} -\Phi^*(0, \Lambda) \\ \bigcup_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} -\Phi^\#(0, \Lambda) \end{array} \right\} \ll_R \left\{ \begin{array}{l} \bigcup_{x \in \mathcal{X}} F(x), \\ \bigcup_{x \in \mathcal{X}} E_F(x). \end{array} \right.$$

Moreover, if the domain of F is nonempty, neither is the domain of Φ and $+\infty_R$ is not included in the left-hand side of any of the above, by Prop. 4.10. Thus, $-\infty_R \notin \text{rge } \Phi^* \cup \text{rge } \Phi^\#$, completing the proof. $\square$

Prop. 4.19 displays how the primal objective mappings of (P) and (P_{epi}) are both upper bounds to the dual objective mappings of (D^*) and $(D_\#^*)$. By Lem. 4.4, this relationship is conserved under supremization and maximization over the dual objective mapping, and under infimization and minimization over the primal objective mapping. For the specific pair of problems (P) and (D^*) , this means

$$\left. \begin{array}{l} \text{Sup}_R(D^*) \\ \text{Max}_R(D^*) \\ \bigcup_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} -\Phi^*(0, \Lambda) \end{array} \right\} \ll_R \left\{ \begin{array}{l} \text{Inf}_R(P), \\ \text{Min}_R(P), \\ \bigcup_{x \in \mathcal{X}} F(x), \end{array} \right.$$

and analogous results hold for the pairs $(P)-(D_\#^*)$, $(P_{epi})-(D^*)$ and $(P_{epi})-(D_\#^*)$. This draws a parallel to the effects of weak duality from (53) in the scalar case.

Recall the two solution concepts $\bar{\mathcal{S}}$ and $\tilde{\mathcal{S}}$ being characterized by the minimal pairs of (P) and (P_{epi}) , respectively. The following theorem links optimality conditions established for the supremization and maximization approaches.

4.20 Theorem: Optimality Conditions for Minimal Pairs

Let $F : \mathcal{X} \rightrightarrows \bar{\mathcal{Y}}$ be proper with perturbation mapping $\Phi : \mathcal{X} \times \mathcal{U} \rightrightarrows \bar{\mathcal{Y}}$ and perturbation space $\mathcal{U}$. For any $\bar{x} \in \mathcal{X}$, $\bar{y} \in \bar{\mathcal{Y}}$ and $\bar{\Lambda} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, the following are equivalent:

- (i) $(\bar{x}, \bar{y})$ is a minimal pair of (P) and $(\bar{\Lambda}, \bar{y})$ is a maximal pair of (D^*) ,
- (ii) $(\bar{x}, \bar{y})$ is a minimal pair of (P) and $(\bar{\Lambda}, \bar{y})$ is a maximal pair of $(D^\#)$,
- (iii) $\bar{y} \in F(\bar{x}) \cap -\Phi^*(0, \bar{\Lambda})$,
- (iv) $\bar{y} \in F(\bar{x}) \cap -\Phi^\#(0, \bar{\Lambda})$,
- (v) $(0, \bar{\Lambda}) \in \partial\Phi(\bar{x}, 0 \mid \bar{y})$.

Proof. Recall that $F(\bar{x}) = \Phi(\bar{x}, 0)$ and that $\text{dom } \Phi \neq \emptyset$ by F being proper.

- (iv) $\Leftrightarrow$ (v): In both cases $\bar{y} \neq -\infty_R$. By Prop. 4.19, $+\infty_R \notin -\Phi^\#(0, \bar{\Lambda})$. Therefore, $\bar{y} \in F(\bar{x}) \cap \mathcal{Y}$ and by Prop. 4.11,

$$0\bar{x} + \bar{\Lambda}0 - \bar{y} = -\bar{y} \in \Phi^\#(0, \bar{\Lambda}) \iff (0, \bar{\Lambda}) \in \partial\Phi(\bar{x}, 0 \mid \bar{y}).$$

- (iv) $\Leftrightarrow$ (iii): In both cases $\bar{y} \neq -\infty_R$ and $(\bar{x}, 0, \bar{y}) \in \text{gph } \Phi$. By Lem. 4.12 and $\bar{y} = \bar{y} - 0\bar{x} + \bar{\Lambda}0$ for $(0, \bar{\Lambda}) \in \mathcal{L}(\mathcal{X} \times \mathcal{U}, \mathcal{Y})$,

$$\bar{y} \in -\Phi^\#(0, \bar{\Lambda}) \iff \bar{y} \in -\Phi^*(0, \bar{\Lambda}).$$

- (i) $\Leftrightarrow$ (iii): Obviously if $(\bar{x}, \bar{y})$ is a minimal pair of (P) , then $\bar{y} \in F(\bar{x})$, and if $(\bar{\Lambda}, \bar{y})$ is a maximal pair of (D^*) , then $\bar{y} \in -\Phi^*(0, \bar{\Lambda})$. Assume now $\bar{y} \in F(\bar{x}) \cap -\Phi^*(0, \bar{\Lambda})$. Then

$$\bar{y} \in \underbrace{\bigcup_{x \in \mathcal{X}} F(x)}_{=: S^1} \cap \underbrace{\bigcup_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} -\Phi^*(0, \Lambda)}_{=: S^2}.$$

Following Lem. 4.4 and Prop. 4.19, $S^1 \ll_R S^2$ and

$$\bar{y} \in (\text{Sup}_R S^1 \cap S^1) \cap (\text{Inf}_R S^2 \cap S^2) \\ \stackrel{\text{Prop. 2.11}}{=} \text{Max}_R S^1 \cap \text{Min}_R S^2,$$

showing that $\bar{y}$ is a minimal point of (P) and a maximal point of (D^*) , implying corresponding minimal and maximal pairs.

- (ii) $\Leftrightarrow$ (iv): The equivalence is demonstrated analogously to the equivalence of (i) and (iii).

□

Note that the above result still holds when $\text{dom } F \neq \emptyset$ but $-\infty_R \in F(x)$ at some $x \in \mathcal{X}$, provided the search is restricted to $\bar{y} \in \mathcal{Y}$ instead of $\bar{\mathcal{Y}}$. The above theorem shows that both approaches to constructing dual problems find the same maximal pairs to the minimal pairs of (P) . Condition (v) already hints at the role that subdifferentiability plays in guaranteeing that all minimal points of (P) are also maximal in (D^*) and $(D_{\#}^*)$.

In view of the solution concept $\tilde{\mathcal{S}}$, we also formulate an optimality condition for epi-infimal pairs of (P) . By Prop. 4.19, we know that both dual objective mappings provide lower bounds to the set of infimal points and the set of epi-infimal points of (P) .

4.21 Corollary: Optimality Conditions for Epi-Infimal Pairs

Let $F : \mathcal{X} \rightrightarrows \bar{\mathcal{Y}}$ be proper with perturbation mapping $\Phi : \mathcal{X} \times \mathcal{U} \rightrightarrows \bar{\mathcal{Y}}$ and perturbation space $\mathcal{U}$. For $(\bar{x}, \bar{y}, \bar{\Lambda}) \in \mathcal{X} \times \bar{\mathcal{Y}} \times \mathcal{L}(\mathcal{U}, \mathcal{Y})$, $(\bar{x}, \bar{y})$ is an epi-infimal pair of (P) and $(\bar{\Lambda}, \bar{y})$ is a maximal pair of (D^*) if and only if

$$\bar{y} \in E_F(\bar{x}) \cap -\Phi^*(0, \bar{\Lambda}) \quad \Longleftrightarrow \quad (0, \bar{\Lambda}) \in \partial E_{\Phi}(\bar{x}, 0 \mid \bar{y}).$$

Proof. By definition, $(\bar{x}, \bar{y})$ is an epi-infimal pair of (P) exactly when it is a minimal pair of (P_{epi}) . Consider the dual problem to (P_{epi}) ,

$$(D_{\text{epi}}) \quad \sup_{\Lambda \in \mathcal{L}(\mathcal{X}, \mathcal{Y})} -E_{\Phi}^*(0, \Lambda),$$

based on the perturbation set mapping E_{Φ} of the epigraphical set mapping E_F . Due to F being proper and $E_F(x) := B_R(F(x))$, we know $\emptyset \neq E_F(x) \subseteq \mathcal{Y}, \forall x \in \mathcal{X}$, making E_F proper and Thm. 4.20 applicable. Applying cases (i), (iii) and (v) of Thm. 4.20 to (P_{epi}) and (D_{epi}) yields that $(\bar{x}, \bar{y})$ is an epi-infimal pair of (P) and $(\bar{\Lambda}, \bar{y})$ is a maximal pair of (D_{epi}) if and only if

$$\bar{y} \in E_F(\bar{x}) \cap -E_{\Phi}^*(0, \bar{\Lambda}) \quad \Longleftrightarrow \quad (0, \bar{\Lambda}) \in \partial E_{\Phi}(\bar{x}, 0 \mid \bar{y}).$$

Since, by Prop. 4.9, $\Phi^* = E_{\Phi}^*$ holds, the dual problems (D_{epi}) and (D^*) coincide, and the condition becomes $\bar{y} \in E_F(\bar{x}) \cap -\Phi^*(0, \bar{\Lambda})$. □

Naturally, that there may still exist infimal points of (P) that are also maximal in (D^*) , but are not epi-infimal. Hence,

$$\bar{y} \in E_F(\bar{x}) \cap -\Phi^*(0, \bar{\Lambda})$$

is only a sufficient condition for $\bar{y}$ to be an infimal point of (P) and a maximal point of (D^*) . Since an analogous result may not be directly formulated for $(D^\#)$, the supremization approach to duality is to be preferred to the maximization approach, whenever one uses the solution concept $\tilde{\mathcal{S}}$.

4.2.4 Strong Duality

We have seen how both dual problems (D^*) and $(D^\#)$ are weakly dual to (P) and (P_{epi}) . For functions, we call a dual problem *strongly dual* if the optimal values of the primal and dual problems coincide. Otherwise, we say that there exists a *duality gap* between the primal and dual problem. For vector- and set-valued problems, as there may not exist a unique optimal value, we compare the optimal sets of the primal and dual problems. As in [11, 16, 25], we call a dual problem *strongly dual* whenever the primal optimal set is a subset of the dual optimal set. For instance, given the specific pair of problems (P) and (D^*) , we say (D^*) is strongly dual to (P) whenever

$$\text{Inf}_R(P) \subseteq \text{Sup}_R(D^*).$$

A motivation for this definition of strong duality can be found in the optimality conditions for weakly dual problem and the two proposed solution concepts $\bar{\mathcal{S}}$ and $\tilde{\mathcal{S}}$:

- If $(D^\#)$ is strongly dual to (P_-) , by Thm. 4.20,

$$\text{Min}_R(P) \subseteq \text{Max}_R(D^\#),$$

and the optimality conditions from Thm. 4.20 become both sufficient and necessary for the minimal pairs of (P) .

- If (D^*) is strongly dual to (P) ,

$$\text{Min}_R(P_{epi}) \subseteq \text{Inf}_R(P) \subseteq \text{Max}_R(D^*),$$

and the optimality conditions from Cor. 4.21 are both necessary and sufficient for the epi-infimal pairs of (P) .

Note that, as both the optimality conditions from Thm. 4.20 and Cor. 4.21 are formulated with respect to the maximal pairs of (D^*) and not to its supremal points, strong duality is required for (P) and the subproblem $(D_{=}^*)$ rather than (D^*) directly. Still, by

$$\text{Inf}_R(P) \subseteq \text{Max}_R(D_{=}^*) = \text{Max}_R(D^*) \subseteq \text{Sup}_R(D^*),$$

if $(D_{=}^*)$ is strongly dual to (P) , so is (D^*) .

Whilst the above strong duality results imply necessity and sufficiency of optimality conditions for the minimal and epi-infimal pairs of (P) , the same cannot be said for the dual maximal pairs of (D^*) and $(D_{=}^{\#})$. This will be addressed in Sect. 4.2.5 under *reflexive duality*.

To derive conditions, under which certain dual problems are strongly dual to (P) , we introduce the *infimal* and *minimal set mappings* of the perturbed problem class (P^u) :

$$\begin{aligned} H : \mathcal{U} \rightrightarrows \overline{\mathcal{Y}}, H(u) &:= \text{Inf}_R \Phi(x, u), \\ H_{=} : \mathcal{U} \rightrightarrows \mathcal{Y}, H_{=}(u) &:= \text{Min}_R \Phi(x, u) \subseteq H(u). \end{aligned}$$

So $H(u)$ and $H_{=}(u)$ are respectively the sets of infimal and minimal points of (P^u) . It is also clear that $H(0) = \text{Inf}_R(P)$ and $H_{=}(0) = \text{Min}_R(P)$. Determining general and attained conjugates,

$$\begin{aligned} H^* : \mathcal{L}(\mathcal{U}, \mathcal{Y}) \rightrightarrows \overline{\mathcal{Y}}, H^*(\Lambda) &:= \text{Sup}_R \left(\Lambda u - \text{Inf}_R \Phi(x, u) \right), \\ H_{=}^{\#} : \mathcal{L}(\mathcal{U}, \mathcal{Y}) \rightrightarrows \overline{\mathcal{Y}}, H_{=}^{\#}(\Lambda) &:= \text{Max}_R \left(\Lambda u - \text{Min}_R \Phi(x, u) \right). \end{aligned}$$

We introduce two assumptions on the infimal and minimal set mappings:

$$H(u) \in \left[\bigcup_{x \in \mathcal{X}} \Phi(x, u) \right], \forall u \in \mathcal{U}, \quad (\text{A1})$$

$$H_{=}(u) \in \left[\bigcup_{x \in \mathcal{X}} \Phi(x, u) \right], \forall u \in \mathcal{U}. \quad (\text{A2})$$

The condition (A1) (resp. (A2)) ensures that the image of the infimal (resp. minimal) set mapping is representative of the range of the perturbation mapping Φ over $\mathcal{X}$, for every perturbation $u \in \mathcal{U}$. It thus especially implies that the set of infimal (resp. minimal) points of (P) is representative, *i.e.* that it characterizes $F(\mathcal{X})$ sufficiently well, without losing information on the set, that would be relevant in infimization (resp. minimization). These conditions are difficult to ensure because, even for $u = 0$, in general, it is not

sufficient for $\text{Inf}_R F(x)$ and $\text{Min}_R F(x)$ to be representative of $F(x)$ at every $x \in \mathcal{X}$.

4.22 Lemma:

Let $\Phi : \mathcal{X} \times \mathcal{U} \rightrightarrows \bar{\mathcal{Y}}$ be a perturbation mapping, $H : \mathcal{U} \rightrightarrows \bar{\mathcal{Y}}$ an infimal set mapping and $H_- : \mathcal{U} \rightrightarrows \mathcal{Y}$ a minimal set mapping for (P). For any $\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$:

- (i) If (A1) holds, then $\Phi^*(0, \Lambda) = H^*(\Lambda)$,
- (ii) If (A2) holds, then (A1) holds and $\Phi^\#(0, \Lambda) = H_\#(\Lambda)$,

Proof. (i) Let (A1) hold, i.e.

$$H(u) = \text{Inf}_R \bigcup_{x \in \mathcal{X}} \Phi(x, u) \in \left[\bigcup_{x \in \mathcal{X}} \Phi(x, u) \right],$$

for all $u \in \mathcal{U}$. Then, using Cor. 2.47, we obtain:

$$\begin{aligned} -\Phi^*(0, \Lambda) &= -\text{Sup}_R \bigcup_{(x, u) \in \mathcal{X} \times \mathcal{U}} (\Lambda u - \Phi(x, u)) \\ &= \text{Inf}_R \bigcup_{u \in \mathcal{U}} \left(-\Lambda u + \bigcup_{x \in \mathcal{X}} \Phi(x, u) \right) \\ &\stackrel{\text{Cor. 2.47}}{=} \text{Inf}_R \bigcup_{u \in \mathcal{U}} \left(-\Lambda u + \text{Inf}_R \bigcup_{x \in \mathcal{X}} \Phi(x, u) \right) \\ &= -\text{Sup}_R \bigcup_{u \in \mathcal{U}} \left(\Lambda u - \text{Inf}_R \bigcup_{x \in \mathcal{X}} \Phi(x, u) \right) = -H^*(\Lambda). \end{aligned}$$

(ii) Let (A2) hold, i.e.

$$H_-(u) = \text{Min}_R \bigcup_{x \in \mathcal{X}} \Phi(x, u) \in \left[\bigcup_{x \in \mathcal{X}} \Phi(x, u) \right],$$

for all $u \in \mathcal{U}$. By Prop. 2.31, representativity of minima implies representativity of infima and (A2) implies (A1). Then, using Cor. 2.46,

we obtain:

$$\begin{aligned}
-\Phi^\#(0, \Lambda) &= -\text{Max}_R \bigcup_{(x,u) \in \mathcal{X} \times \mathcal{U}} (\Lambda u - \Phi(x, u)) \\
&= \text{Min}_R \bigcup_{u \in \mathcal{U}} \left(-\Lambda u + \bigcup_{x \in \mathcal{X}} \Phi(x, u) \right) \\
&\stackrel{\text{Cor. 2.46}}{=} \text{Min}_R \bigcup_{u \in \mathcal{U}} \left(-\Lambda u + \text{Min}_R \bigcup_{x \in \mathcal{X}} \Phi(x, u) \right) \\
&= -\text{Max}_R \left(\Lambda u - \text{Min}_R \bigcup_{x \in \mathcal{X}} \Phi(x, u) \right) = -H_\#^\#(\Lambda).
\end{aligned}$$

□

4.23 Proposition: *Strong Dual Problems*

Let $\Phi : \mathcal{X} \times \mathcal{U} \rightrightarrows \overline{\mathcal{Y}}$ be a perturbation mapping, $H : \mathcal{U} \rightrightarrows \overline{\mathcal{Y}}$ an infimal set mapping and $H_- : \mathcal{U} \rightrightarrows \mathcal{Y}$ a minimal set mapping for (P). If (A1) holds,

$$(D^*) \quad \text{Sup}_R \bigcup_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} -H^*(\Lambda),$$

and the set of supremal and maximal points of (D^*) respectively coincide with $H^{**}(0)$ and $H^{*\#}(0)$. If (A2) holds, then so does all of the above,

$$(D_\#^\#) \quad \text{Max}_R \bigcup_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} -H_\#^\#(\Lambda),$$

and the set of maximal points of $(D_\#^\#)$ coincides with $H_\#^{*\#}(0)$.

Proof. The respective general and attained biconjugates of H and H_- are given by

$$\begin{aligned}
H^{**} : \mathcal{U} \rightrightarrows \overline{\mathcal{Y}}, \quad H^{**}(u) &:= \text{Sup}_R \bigcup_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} \Lambda u - H^*(\Lambda), \\
H_\#^{*\#} : \mathcal{U} \rightrightarrows \mathcal{Y}, \quad H_\#^{*\#}(u) &:= \text{Max}_R \bigcup_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} \Lambda u - H_\#^\#(\Lambda).
\end{aligned}$$

Furthermore, we specify the following mixed biconjugate of H :

$$H^{*\#} : \mathcal{U} \rightrightarrows \mathcal{Y}, \quad H^{*\#}(u) := \text{Max}_R \bigcup_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} \Lambda u - H^*(\Lambda).$$

For $u = 0$, we get

$$\begin{aligned} H^{**}(0) &= \sup_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} -H^*(\Lambda), \\ H^{*\#}(0) &= \max_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} -H^*(\Lambda), \\ H_{\#}^{\#}(0) &= \max_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} -H_{\#}^{\#}(\Lambda). \end{aligned}$$

By Lem. 4.22, $\Phi^*(0, \Lambda) = H^*(\Lambda)$ provided (A1) holds, in which case

$$\begin{aligned} \sup_R(D^*) &= \sup_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} -\Phi^*(0, \Lambda) = H^{**}(0), \\ \max_R(D^*) &= \max_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} -\Phi^*(0, \Lambda) = H^{*\#}(0), \end{aligned} \tag{54}$$

and the dual objective mapping of (D^*) can be replaced by $-H^*$. Additionally, under (A2), $\Phi^{\#}(0, \Lambda) = H_{\#}^{\#}(\Lambda)$ and (A1) hold, implying (54),

$$\max_R(D_{\#}^{\#}) = \max_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} -\Phi^{\#}(0, \Lambda) = H_{\#}^{\#}(0),$$

and the dual objective mapping of $(D_{\#}^{\#})$ can be replaced by $-H_{\#}^{\#}$. This concludes the proof. $\square$

Using Prop. 4.23, we may use Prop. 4.14 to determine optimality conditions for the pair of problems $(P)-(D_{\#}^{\#})$ and $(P_{\#})-(D^{\#})$.

4.24 Proposition: *Optimality Conditions*

Let $\Phi : \mathcal{X} \times \mathcal{U} \rightrightarrows \bar{\mathcal{Y}}$ be a perturbation mapping, $H : \mathcal{U} \rightrightarrows \bar{\mathcal{Y}}$ an infimal set mapping and $H_{\#} : \mathcal{U} \rightrightarrows \mathcal{Y}$ a minimal set mapping for (P) . Let $\bar{x} \in \mathcal{X}$, $\bar{y} \in \bar{\mathcal{Y}}$ and $\bar{\Lambda} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$.

- (i) If (A1) holds, $\bar{y}$ is an infimal point of (P) and $(\bar{\Lambda}, \bar{y})$ is a maximal pair of (D^*) if and only if $\bar{\Lambda} \in \partial H(0 \mid \bar{y})$.
- (ii) If (A2) holds, $(\bar{x}, \bar{y})$ is a minimal pair of (P) and $(\bar{\Lambda}, \bar{y})$ is a maximal pair of (D^*) and $(D_{\#}^{\#})$ if and only if $\bar{y} \in F(\bar{x})$ and $\bar{\Lambda} \in \partial H_{\#}(0 \mid \bar{y})$.

Proof. (i) Under (A1), by Prop. 4.23, the set of maximal points of (D^*) coincides with $H^{*\#}(0)$. Since the set of infimal points of (P) coincides with $H(0)$, some $\bar{y}$ is infimal for (P) and maximal for (D^*) , with

associated maximal pair $(\bar{\Lambda}, \bar{y})$, if and only if $\bar{y} \in H(0) \cap H^{*\#}(0)$ and $\bar{y} \in -H^*(\bar{\Lambda})$. Assume this to be true. Being a maximal point, $\bar{y}$ can only lie in $\mathcal{Y}$. By $\bar{y} \in H(0) \cap \mathcal{Y}$, Prop. 4.11 and Lem. 4.12,

$$\begin{aligned} \bar{\Lambda}0 - \bar{y} \in H^*(\bar{\Lambda}) &\xLeftrightarrow{\text{Lem. 4.12}} \bar{\Lambda}0 - \bar{y} \in H^\#(\bar{\Lambda}) \\ &\xLeftrightarrow{\text{Prop. 4.11}} \bar{\Lambda} \in \partial H(0 \mid \bar{y}). \end{aligned} \quad (55)$$

Assume now the existence of some $\bar{\Lambda} \in \partial H(0 \mid \bar{y})$. Then, by Prop. 4.14, $\bar{y} \in H(0) \cap H^{*\#}(0)$, showing that $\bar{y}$ is both an infimal point of (P) and a maximal point of (D^*) . Additionally, by definition of subgradients, $\bar{y} \in \mathcal{Y}$, making Prop. 4.11 and Lem. 4.12 applicable. Then, by (55), $(\bar{\Lambda}, \bar{y}) \in \text{gph}(-H^*)$, such that $(\bar{\Lambda}, \bar{y})$ a maximal pair of (D^*) .

- (ii) Under (A2), by Prop. 4.23, the set of maximal points of $(D_\#^\#)$ coincides with $H_\#^{*\#}(0)$. Since the set of minimal points of (P) coincides with $H_=(0)$, some $\bar{y}$ is minimal for (P) and maximal for $(D_\#^\#)$, with associated maximal pair $(\bar{\Lambda}, \bar{y})$, if and only if $\bar{y} \in H_=(0) \cap H_\#^{*\#}(0)$ and $\bar{y} \in -H_\#^\#(\bar{\Lambda})$. Assume this to be true. Being a maximal point, $\bar{y}$ can only lie in $\mathcal{Y}$. By $\bar{y} \in H_=(0) \cap \mathcal{Y}$ and Prop. 4.11,

$$\bar{\Lambda}0 - \bar{y} \in H_\#^\#(\bar{\Lambda}) \xLeftrightarrow{\text{Prop. 4.11}} \bar{\Lambda} \in \partial H_=(0 \mid \bar{y}). \quad (56)$$

Assume now the existence of some $\bar{\Lambda} \in \partial H_=(0 \mid \bar{y})$. Then, by Prop. 4.14, $\bar{y} \in H_=(0) \cap H_\#^{*\#}(0)$, showing that $\bar{y}$ is both a minimal point of (P) and a maximal point of $(D_\#^\#)$. Additionally, by definition of subgradients, $\bar{y} \in \mathcal{Y}$, making Prop. 4.11 applicable. Then, by (56), $(\bar{\Lambda}, \bar{y}) \in \text{gph}(-H_\#^\#)$, such that $(\bar{\Lambda}, \bar{y})$ a maximal pair of $(D_\#^\#)$.

□

The above result displays how close subdifferentiability of the infimal and minimal set mappings is related to the existence of primal-dual optimal points for the pair of problems $(P)-(D_\#^*)$ and $(P_-)-(D_\#^\#)$. For any perturbation mapping Φ , we call the infimal set mapping H (resp. minimal set mapping H_-) *stable* if it is subdifferentiable at $u = 0$. We may show that one assumption is stronger than the other:

4.25 Theorem: *Stability of Infimal and Minimal Set Mappings*

Let $\Phi : \mathcal{X} \times \mathcal{U} \rightrightarrows \bar{\mathcal{Y}}$ be a perturbation mapping, $H : \mathcal{U} \rightrightarrows \bar{\mathcal{Y}}$ an infimal set mapping and $H_- : \mathcal{U} \rightrightarrows \mathcal{Y}$ a minimal set mapping for (P) .

If H is stable, then so is H_- and $F = \Phi(\cdot, 0)$ is proper.

Proof. If $H_=(0) = \emptyset$, then $H_=-$ is trivially subdifferentiable at zero. Assume there exists some $\bar{y} \in H_=(0) \subseteq H(0)$, implying $\bar{y} \in \text{dom } H_- \cap \text{dom } H$. Let H be stable. Then, there exists some $\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ with $\Lambda \in \partial H(0 \mid \bar{y})$. By definition, Λ is the slope of a supporting affine minorant ray $u \mapsto \Lambda u + c$ of H at $(0, \bar{y})$ with $\bar{y} \in H(0) \cap \mathcal{Y}$. Then, by Prop. 4.6 (ii), the shift of the affine minorant ray is $c = -\bar{y}$ with

$$-\bar{y} \in A_R \left(\bigcup_{u \in \mathcal{U}} H(u) - \Lambda u \right),$$

and by Lem. 2.24 (i) and $H_=(u) \subseteq H(u)$ for every $u \in \mathcal{U}$,

$$-\bar{y} \in A_R \left(\bigcup_{u \in \mathcal{U}} H(u) - \Lambda u \right) \subseteq A_R \left(\bigcup_{u \in \mathcal{U}} H_=(u) - \Lambda u \right).$$

Again, by Prop. 4.6 (ii), $u \mapsto \Lambda u - \bar{y}$ is also a supporting affine minorant ray of H_- at $(0, \bar{y})$, and by definition of subgradients, $\Lambda \in \partial H_=(0 \mid \bar{y})$. As $\bar{y} \in H_=(0)$ was chosen arbitrarily, the above holds for every $y \in H_=(0)$, making H_- subdifferentiable at zero, and therefor stable.

Lastly, by the definition of infima, subdifferentiability of H implies

$$H(0) = \text{Inf}_R \bigcup_{x \in \mathcal{X}} F(x) \neq \{\pm\infty_R\},$$

as there can exist no subgradients at graph points (u, y) of H with $u \in \mathcal{U}$ and $y = \pm\infty_R$. Whenever F is improper, either $-\infty_R \in \text{rge } F$ and $H(0) = \{-\infty_R\}$, or $\text{dom } F = \emptyset$ and $H(0) = \{+\infty_R\}$, both preventing H to be stable. Therefor, F must be proper. $\square$

By Thm. 4.25, stability of H is a stronger assumption than F being proper. Note that the above result relies neither on (A1) nor on (A2). Whether or not H and H_- are stable depends entirely on the choice of perturbation mapping Φ . In the literature, it is not the infimal or minimal set mapping which is called stable, but the primal problem itself. However, as the primal problem has multiple possible conjugate dual problems, we transfer the name and property to the infimal and minimal set mapping.

4.26 Theorem: Strong Duality

Let $\Phi : \mathcal{X} \times \mathcal{U} \rightrightarrows \overline{\mathcal{Y}}$ be a perturbation mapping, $H : \mathcal{U} \rightrightarrows \overline{\mathcal{Y}}$ an infimal set mapping and $H_- : \mathcal{U} \rightrightarrows \mathcal{Y}$ a minimal set mapping for (P) .

If (A1) holds, the following are equivalent:

- (i) H is stable,
- (ii) (D^*) is strongly dual to (P) ,
- (iii) $\text{Inf}_R(P) \subseteq \text{Max}_R(D^*) \subseteq \text{Sup}_R(D^*)$,
- (iv) for any infimal point $\bar{y}$ of (P) , there exists some $\bar{\Lambda} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, such that $(\bar{\Lambda}, \bar{y})$ is a maximal pair of (D^*) .

If (A2) holds, the following are equivalent:

- (i) H_- is stable,
- (ii) $(D^\#)$ is strongly dual to (P_-) ,
- (iii) $\text{Min}_R(P) \subseteq \text{Max}_R(D^\#)$,
- (iv) for any minimal point $\bar{y}$ of (P) , there exists some $\bar{\Lambda} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, such that $(\bar{\Lambda}, \bar{y})$ is a maximal pair of $(D^\#)$.

Proof. By Prop. 4.23, under (A1), the set of infimal points of (P) coincides with $H(0)$ and the set of maximal points of (D^*) coincides with $H^{*\#}(0)$, and, under (A2), the set of minimal points of (P) coincides with $H_-(0)$ and the set of maximal points of $(D^\#)$ coincides with $H_-^{*\#}(0)$. Then, by Cor. 4.16, H and H_- being subdifferentiable at $u = 0$ is respectively equivalent to

$$\begin{aligned} H(0) &= \text{Inf}_R(P) \subseteq \text{Max}_R(D^*) = H^{*\#}(0), \\ H_-(0) &= \text{Min}_R(P) \subseteq \text{Max}_R(D^\#) = H_-^{*\#}(0). \end{aligned}$$

Lastly, under stability, for every $\bar{y} \in H(0)$ (resp. $H_-(0)$), there must exist some $\bar{\Lambda} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ with $\bar{\Lambda} \in \partial H(0 \mid \bar{y})$ (resp. $\partial H_-(0 \mid \bar{y})$). By Prop. 4.24, $(\bar{\Lambda}, \bar{y})$ forms a maximal pair of (D^*) (resp. D_-). The remaining assertions follow from the relevant definitions. $\square$

Thm. 4.26 also implies that, under (A1) and H stable, by Cor. 4.21, every epi-infimal pair $(\bar{x}, \bar{y})$ of (P) has an associated maximal pair $(\bar{\Lambda}, \bar{y})$ of (D^*) satisfying

$$\bar{y} \in E_F(\bar{x}) \cap -\Phi^*(0, \bar{\Lambda}).$$

4.27 Corollary:

Let $\Phi : \mathcal{X} \times \mathcal{U} \rightrightarrows \overline{\mathcal{Y}}$ be a perturbation mapping and $H : \mathcal{U} \rightrightarrows \overline{\mathcal{Y}}$ an infimal set mapping for (P) . Assuming (A1) holds and H is stable:

- (i) for any minimal point $\bar{y}$ of (P) , there exists some $\bar{\Lambda} \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ such that $(\bar{\Lambda}, \bar{y})$ is a maximal pair of (D^*) and $(D_{\#}^{\#})$,
- (ii) for any epi-infimal point $\bar{y}$ of (P) , there exists some $\bar{\Lambda} \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ such that $(\bar{\Lambda}, \bar{y})$ is a maximal pair of (D^*) .

Proof. Recall that every minimal point of (P) is also infimal. From Thm. 4.26, under (A1) and stability of H , for any minimal point $\bar{y}$ of (P) , with associated minimal pair $(\bar{x}, \bar{y}) \in \text{gph } F$, there exists some $\bar{\Lambda} \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ such that $(\bar{\Lambda}, \bar{y})$ is a maximal pair of (D^*) . Since H is stable, F must be proper. Then, by Thm. 4.20, by $(\bar{x}, \bar{y})$ a minimal pair of (P) and $(\bar{\Lambda}, \bar{y})$ a maximal pair of (D^*) , $(\bar{\Lambda}, \bar{y})$ must also be maximal pair of $(D_{\#}^{\#})$.

The second assertion follows immediately from Thm. 4.26 and epi-infimal points being especially infimal points of (P) , *i.e.* $\text{Min}_R(P_{\text{epi}}) \subseteq \text{Inf}_R(P)$ (see Thm. 4.2). $\square$

With regards to Cor. 4.27, the conditions (A1) and a stable infimal set mapping H are necessary and sufficient for $(D_{\#}^{\#})$ (and thus (D^*)) to be strongly dual to (P) . Beware that, without (A2), one cannot assume the maximal points of $(D_{\#}^{\#})$ to be given by $H_{\#}^{\#}$. Nevertheless, solving for the optimality conditions in Thm. 4.20 allows one to identify all minimal points of (P) .

It is left to discuss how one can guarantee stability of H and $H_{\#}$. Essentially, it is through the choice of an appropriate perturbation mapping Φ that one can assure subdifferentiability of H and $H_{\#}$ and thus stability. However, it appears difficult to derive generalized conditions on Φ , that hold for every possible relation R . One approach could be to attempt a generalization of the *convexlike-convex* property described in [4, Def. 7.1.8]. For the specific relations $R = \not\prec_D$ and $\not\prec_D^{\#}$, we refer to the work of Boţ et al. [4].

4.2.5 Reflexive Duality

To conclude our overview of conjugate duality for set-valued problems, we address the symmetry of primal-dual problem pairs. We may call a duality scheme *reflexive*, whenever the roles of the primal and dual problems can be exchanged, *i.e.* when the primal problem is also dual to its dual problem.

We only construct a conjugate dual problem to (D^*) , as a dual problem to $(D^\#)$ may be obtained analogously. Since (D^*) is a supremization problem, we cannot directly use subgradients and conjugates, as these constructs are catered to lower bounds of mappings. Instead, we focus on the negative set of supremal points of (D^*) :

$$-\text{Sup}_R(D^*) = - \sup_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} -\Phi^*(T, \Lambda) = \inf_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} \Phi^*(T, \Lambda).$$

By Prop. 4.10 applied to the mapping $\Phi^* : \mathcal{L}(\mathcal{X} \times \mathcal{U}, \mathcal{Y}) \rightrightarrows \overline{\mathcal{Y}}$, for any $(T, \Lambda) \in \mathcal{L}(\mathcal{X} \times \mathcal{U}, \mathcal{Y})$ and any $(x, u) \in \mathcal{X} \times \mathcal{U}$:

$$Tx + \Lambda u - \Phi^{**}(x, u) \ll_R \Phi^*(T, \Lambda).$$

As such, as in Prop. 4.19, we obtain

$$\bigcup_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} -\Phi^*(0, \Lambda) \ll_R \bigcup_{x \in \mathcal{X}} \Phi^{**}(x, 0), .$$

Then, the pair of problems (P') and (D^*) satisfy weak duality, with (P') defined as:

$$(P') \quad \inf_{x \in \mathcal{X}} \Phi^{**}(x, 0).$$

Optimality conditions similar to those in Thm. 4.20 can be derived. For strong duality, let $V : \mathcal{L}(\mathcal{X}, \mathcal{Y}) \rightrightarrows \overline{\mathcal{Y}}$ be the *negative supremal set mapping* of (D^*) with

$$V(T) := \inf_R \bigcup_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} \Phi^*(T, \Lambda).$$

For $T = 0$, it is clear that $-V(0) = \text{Sup}_R(D^*)$, *i.e.* the set of supremal points of (D^*) . Similarly to (A1), we further assume

$$V(T) \in \left[\bigcup_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} \Phi^*(T, \Lambda) \right], \quad (\text{D1})$$

Then, analogously to Lem. 4.22, using (D1) and Cor. 2.47,

$$\begin{aligned} -V^*(x) &= \inf_R \bigcup_{T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})} \left(-Tx + \inf_R \bigcup_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} \Phi^*(T, \Lambda) \right) \\ &= \inf_R \bigcup_{T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})} \left(-Tx + \bigcup_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} \Phi^*(T, \Lambda) \right) = -\Phi^{**}(x, 0). \end{aligned}$$

The biconjugate of V then becomes

$$V^{**}(T) = \sup_R \bigcup_{x \in \mathcal{X}} Tx - \Phi^{**}(x, 0),$$

and for $T = 0$, we obtain

$$-V^{**}(0) = \inf_R \bigcup_{x \in \mathcal{X}} \Phi^{**}(x, 0) = \inf_R(P'),$$

as well as $-V^{*\#}(0) = \min_R(P')$. By Prop. 4.16, V is subdifferentiable at zero if and only if $V(0) \subseteq V^{*\#}(0) \subseteq V^{**}(0)$ holds, in which case we call V stable. Therefor, as in Thm. 4.26, whenever (D1) holds, V is stable if and only if

$$\sup_R(D^*) \subseteq \min_R(P') \subseteq \inf_R(P').$$

4.28 Theorem: *Reflexive Strong Duality*

Let $\Phi : \mathcal{X} \times \mathcal{U} \rightrightarrows \overline{\mathcal{Y}}$ be a perturbation mapping and $H : \mathcal{U} \rightrightarrows \overline{\mathcal{Y}}$ the infimal set mapping for (P) . Let $V : \mathcal{L}(\mathcal{X}, \mathcal{U}) \rightrightarrows \overline{\mathcal{Y}}$ be the negative supremal set mapping of (D^*) . Assume (A1) and (D1) hold, and $-V^{**}(0) = H(0)$. Then, (P) is also a conjugate dual problem of (D^*) , and if both H and V are stable, then

$$\inf_R(P) = \max_R(D^*) = \sup_R(D^*).$$

Proof. By Prop. 4.23, under (A1),

$$H(0) = \inf_R(P), \quad H^{**}(0) \stackrel{(A1)}{=} \sup_R(D^*) = -V(0),$$

and under (D1), $-V^{**}(0) = \text{Inf}_R(P')$ holds. Further assuming $-V^{**}(0) = H(0)$, which translates to

$$-V^{**}(0) = \text{Inf}_R \bigcup_{x \in \mathcal{X}} \Phi^{**}(x, 0) = \text{Inf}_R \bigcup_{x \in \mathcal{X}} \Phi(x, 0) = H(0),$$

we immediately obtain that (P) is equivalent to (P') in the sense of infimization, *i.e.*

$$\text{Inf}_R(P) = \text{Inf}_R(P'),$$

but not necessarily $\text{Min}_R(P) = \text{Min}_R(P')$. Then, respectively under (A1) and (D1), by Thm. 4.26 and the above discussion,

$$\begin{aligned} H \text{ stable} &\iff H(0) \subseteq H^{\#}(0) \quad (\subseteq H^{**}(0)), \\ V \text{ stable} &\iff V(0) \subseteq V^{\#}(0) \quad (\subseteq V^{**}(0)), \end{aligned}$$

which translates to both H and V stable if

$$\overbrace{\text{Inf}_R(P) \subseteq \text{Max}_R(D^*) \subseteq \text{Sup}_R(D^*) \subseteq \text{Inf}_R(P)}^{\text{under (A1), iff } H \text{ stable}},$$

under (D1), if V stable

and implies $\text{Inf}_R(P) = \text{Max}_R(D^*) = \text{Sup}_R(D^*)$. □

Regarding the assumption that $-V^{**}(0) = H(0)$, *i.e.*

$$\text{Inf}_R \bigcup_{x \in \mathcal{X}} \Phi^{**}(x, 0) = \text{Inf}_R \bigcup_{x \in \mathcal{X}} \Phi(x, 0),$$

it is sufficient to verify that, for every $x \in \mathcal{X}$, by Thm. 2.39,

$$\Phi^{**}(x, 0) = \text{Inf}_R \Phi(x, 0) \in [\Phi(x, 0)].$$

For a set-valued mapping F , it is unlikely that one can choose a perturbation mapping such that $\Phi = \Phi^{**}$ or $\Phi(x, 0) = \Phi^{**}(x, 0)$. For instance, if F maps to non-infimal points, the condition cannot hold (see Rem. 4.17). Nevertheless, whenever F is only single-valued, this may not be an issue. Strengthening the assumption $-V^{**}(0) = H(0)$ to $\Phi(x, 0) = \Phi^{**}(x, 0)$ for every $x \in \mathcal{X}$, then also $-V^{\#}(0) = \text{Min}_R(P)$ holds, and the result of Thm. 4.28 can be strengthened to

$$\text{Inf}_R(P) = \text{Min}_R(P) = \text{Max}_R(D^*) = \text{Sup}_R(D^*).$$

Something similar can be established for the set of primal minimal points $\text{Min}_R(P)$ and the dual problem $(D_{\#})$. For a primal problem

$$(P'_{\#}) \quad \text{Min}_R \Phi^{\#\#}(x, 0),$$

the *negative maximal set mapping* $V_{\#} : \mathcal{L}(\mathcal{X}, \mathcal{Y}) \rightrightarrows \mathcal{Y}$ with

$$V_{\#}(T) := \text{Min}_R \bigcup_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} \Phi^{\#}(T, \Lambda),$$

and under condition

$$V_{\#}(T) \in \left[\bigcup_{\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})} \Phi^{\#}(T, \Lambda) \right], \quad (\text{D2})$$

we can show that, $V_{\#}$ is stable if and only if $\text{Max}_R(D_{\#}) \subseteq \text{Min}_R(P'_{\#})$. Then, as in Thm. 4.28, provided $-V_{\#}(0) = H_{\#}(0)$, (A2) and (D2) hold, $H_{\#}$ and $V_{\#}$ are stable if and only if $\text{Max}_R(D_{\#}) = \text{Min}_R(P)$.

4.3 Duality for Weak Infima

We have seen how, for a common primal problem (P) , the dual problems (D^*) and $(D_{\#})$ are closely related. Regarding the set of minimal points of (P) , Thm. 4.20 has shown how the set of points that are primal minimal and dual maximal coincide, *i.e.*

$$\text{Min}_R(P) \cap \text{Max}_R(D^*) = \text{Min}_R(P) \cap \text{Max}_R(D_{\#}).$$

Regarding strong duality, we have seen that, under (A1), $\text{Inf}_R(P) \subseteq \text{Max}_R(D^*)$ holds if and only if H is stable, and that, under (A2), $\text{Min}_R(P) \subseteq \text{Max}_R(=D)$ holds if and only if $H_{\#}$ is stable (see Thm. 4.26).

Consider a nontrivial pointed and convex ordering cone $D \subseteq \mathcal{Y}$, containing zero. In the literature on conjugate duality with respect to efficiency, the considered problems are

$$(P_{\#}^{\text{eff}}) \quad \text{Min}_D F(x) \quad \text{and} \quad (D_{\#}^{\text{eff}}) \quad \text{Max}_D -\Phi^{\#}(0, \Lambda),$$

with the aim of finding the efficient solutions $\bar{x} \in \mathcal{X}$ of $(P_{\#}^{\text{eff}})$, *i.e.* points for which there exist some $\bar{y} \in \text{Min}_D(P_{\#}^{\text{eff}})$ with $(\bar{x}, \bar{y})$ a minimal pairs of $(P_{\#}^{\text{eff}})$. Then, (A2) is equivalent to $\text{Min}_D \bigcup_{x \in \mathcal{X}} \Phi(x, u)$ being either empty or

externally stable for every $u \in \mathcal{U}$ (see the discussion preceding Thm. 3.23). It is this property, which may prove difficult to show. Still, the optimality conditions established in Thm. 4.20 are always sufficient for finding minimal pairs of $(P_{=}^{\text{eff}})$.

The approach is different for weak efficiency. The primal and dual problems are formulated as

$$(P^w) \quad \text{w-Inf}_D F(x) \quad \text{and} \quad (D^w) \quad \text{w-Sup}_D -\Phi^*(0, \Lambda),$$

$x \in \mathcal{X}$ $\Lambda \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$

but the focus is usually on determining the set of weakly efficient solutions $\bar{x} \in \mathcal{X}$ of (P^w) , *i.e.* points for which there exist some $\bar{y} \in \text{w-Min}_D(P^w)$ with $(\bar{x}, \bar{y})$ a minimal pair of (P^w) . The set $\text{w-Inf}_D(P^w)$ of weakly infimal points of (P^w) is only used as a tool to obtain $\text{w-Min}_D(P^w)$. This follows the solution concept $\bar{\mathcal{S}}$.

To understand why supremal-based dual problems are preferred for weak minimization, recall that in Sect. 3.4.2, we discussed some of the specific properties of weak infima, that are not generally true for other types of underlying relations. The most important property is the representativity of every weak infimum from Thm. 3.32. This implies that the assumption (A1) is immediately satisfied for the weak infimal set mapping H with

$$H(u) := \text{w-Inf}_D \bigcup_{x \in \mathcal{X}} \Phi(x, u).$$

Thus, by Prop. 4.23, we can rewrite the set of primal infimal and dual supremal points as

$$\text{w-Inf}_D(P^w) = H(0) \quad \text{and} \quad \text{w-Sup}_D(D^w) = H^{**}(0),$$

and by Thm. 4.26, stability of H is always equivalent to

$$\text{w-Inf}_D(P^w) \subseteq \text{w-Max}_D(D^w) \subseteq \text{w-Sup}_D(D^w).$$

An exemplary condition on Φ based on notions of convexity to guarantee that H is stable can be found with similar notation in [4, Prop. 7.4.14]. The optimality conditions established in Thm. 4.20 are then both necessary and sufficient for identifying every primal minimal pair.

A second specificity of weak infima can be summarized to “infima are separating the space $\mathcal{Y}$ ”. This property is addressed in Lem. 3.31 when one recalls that, for any $Y \subseteq \bar{\mathcal{Y}}$, $A_{\nearrow_D}(Y) \cup B_{\nearrow_D}(Y) = \mathcal{Y}$. This is one of the core observations supporting the analysis of solution concepts and duality

for weakly nondominated points in *Vector Optimization with Infimum and Supremum* by Löhne [16], which we have referenced multiple times. This property leads to a class of set-valued mappings which coincide with their general biconjugate, and a more powerful strong duality result concerning specifically the infimal points of (P^w) .

4.29 Proposition: [27, Thm. 4.1]

Let $F : \mathcal{X} \rightrightarrows \overline{\mathcal{Y}}$. Then, $\text{w-Inf}_D F$ is subdifferentiable at $\hat{x}$ if and only if

$$\text{w-Inf}_D F(\hat{x}) = F^{*\#}(\hat{x}) = F^{**}(\hat{x}).$$

Proof. Keep in mind that, for weak infima, $R = \not\prec_D$ and $-R = \not\prec_D$. We show that the conjugates of $\text{w-Inf}_D F$ coincide with the conjugates of F . By Cor. 2.47 and $\text{w-Inf}_D F(x) \in [F(x)]$, for any $x \in \mathcal{X}$ by Thm. 3.32,

$$\begin{aligned} F^*(T) &= -\text{w-Inf}_D \bigcup_{x \in \mathcal{X}} -Tx + F(x) \\ &\stackrel{\text{Cor. 2.47}}{=} -\text{w-Inf}_D \bigcup_{x \in \mathcal{X}} -Tx + \text{w-Inf}_D F(x) = (\text{w-Inf}_D F)^*(T). \end{aligned}$$

The same naturally follows for the biconjugates F^{**} and $F^{*\#}$. By the above and Cor. 4.16, $\text{w-Inf}_D F$ is subdifferentiable at $\hat{x}$ if and only if

$$\text{w-Inf}_D F(\hat{x}) \subseteq F^{*\#}(\hat{x}) \subseteq F^{**}(\hat{x}). \quad (57)$$

It remains to show that the above inclusion is only ever satisfied with equality. Assume (57) holds and that there exists some $y^* \in F^{**}(\hat{x}) \setminus \text{w-Inf}_D F(\hat{x})$. By the above, $\text{w-Inf}_D F \neq \{\pm\infty_D\}$, as otherwise it would not be a subset of $F^{*\#}(\hat{x})$, and even $\emptyset \neq \text{w-Inf}_D F(x) \subseteq \mathcal{Y}$, as by Cor. 3.33, weak suprema and infima are never empty. This additionally shows, that $\emptyset \neq F^{**}(\hat{x}) \subseteq \mathcal{Y}$, as the inclusion could otherwise not hold. From the discussion following Prop. 4.10, $F^{**}(x) \ll_R F(x)$ for any $x \in \mathcal{X}$. By $+\infty_D \notin F^{**}(\hat{x})$ and Lem. 4.3 (ii),

$$F^{**}(\hat{x}) \subseteq A_{\not\prec_D}(F(\hat{x})),$$

implying that $y^* \in F^{**}(\hat{x}) \setminus B_{\not\prec_D}(F(\hat{x}))$, by the definition of infimum and $y^* \in A_{\not\prec_D}(F(\hat{x}))$. Choose some $u^0 \in B_{\not\prec_D}(F(\hat{x}))$, nonempty as $\text{w-Inf}_D F(\hat{x})$ also nonempty. By $-\mathcal{D}^c = \text{int } D$ being a nonempty convex cone, it is generating by Lem. 1.13, and by Lem. 1.12,

$$\exists u \in (y^* + \text{int } D) \cap (u^0 + \text{int } D),$$

i.e. $u >_D y^*$ and $u >_D u^0$, such that u is dominated by both y^* and u^0 . From $u >_D y^*$, we obtain $\lambda(u - y^*) \in \text{int } D$ for any $\lambda > 0$. By Lem. 3.31, with slight reformulation, there exists some $\bar{\lambda} \in [0, 1]$ such that

$$\hat{y} := y^* + \bar{\lambda}(u - y^*) \in \text{w-Inf}_D F(\hat{x}),$$

and since $y^* \notin \text{w-Inf}_D F(\hat{x})$, $\bar{\lambda} \neq 0$ and $\hat{y} \in y^* + \text{int } D$, i.e. $\hat{y} >_D y^*$. However, then $\hat{y}$ is not a lower bound to F^{**} , i.e.

$$\hat{y} \notin A_{\not\geq_D}(F^{**}(\hat{x})).$$

Since $F^{**}(\hat{x}) \subseteq \mathcal{Y}$ is self-infimal by Prop. 3.35, it follows that

$$\hat{y} \notin \text{w-Inf}_D F^{**}(\hat{x}) = F^{**}(\hat{x}),$$

contradicting $\text{w-Inf}_D F(\hat{x}) \subseteq F^{**}(\hat{x})$, and subdifferentiability of $\text{w-Inf}_D F$ at $\hat{x}$. This shows $\text{w-Inf}_D F(\hat{x}) \supseteq F^{**}(\hat{x})$ and, together with (57), concludes the proof. $\square$

The previous result can be shown in several different ways, but the property is essentially a consequence of representativity and nonemptiness of any infimum, combined with infima separating the space in two. Alternative proofs can be found, e.g. in [27, Thm. 4.1] and [4, Prop. 7.4.12]. Prop. 4.29 allows to strengthen the strong duality result from Thm. 4.26.

4.30 Theorem: *Strong Duality for weak Infimization*

Let $\Phi : \mathcal{X} \times \mathcal{U} \rightrightarrows \bar{\mathcal{Y}}$ be a perturbation mapping and $H : \mathcal{U} \rightrightarrows \bar{\mathcal{Y}}$ an infimal set mapping of (P^w) . H is stable if and only if

$$\text{w-Inf}_D(P^w) = \text{w-Max}_D(D^w) = \text{w-Sup}_D(D^w),$$

i.e. every weakly infimal point $\bar{y}$ of (P^w) is also a supremal and maximal point of (D^w) .

Proof. By Thm. 3.32, (A1) is immediately satisfied and the infimal set mapping $H(u) := \text{w-Inf}_D \bigcup_{x \in \mathcal{X}} \Phi(x, u)$ is self-infimal. Then, by Prop. 4.29, whenever H is stable, i.e. subdifferentiable at zero,

$$H(0) = \text{w-Inf}_D H(0) = H^{*\#}(0) = H^{**}(0).$$

By definition, $\text{w-Inf}_D(P^w) = H(0)$. As (A1) is satisfied, by Prop. 4.23,

$$\text{w-Max}_D(D^w) = H^{*\#}(0), \quad \text{w-Sup}_D(D^w) = H^{**}(0),$$

showing $\text{w-Inf}_D(P^w) = \text{w-Max}_D(D^w) = \text{w-Sup}_D(D^w)$. $\square$

The above result shows how every dual maximal point of (D^w) is also primal infimal, and the pair (P^w) and (D^w) satisfies a more powerful version of strong duality: solving the primal problem also solves the dual problem. However, whenever primal minimal pairs $(\bar{x}, \bar{y})$ of (P^w) are concerned, *i.e.* pairs of a weakly efficient solution $\bar{x}$ and a nondominated point $\bar{y}$ of (P^w) , we still only have

$$\text{w-Inf}_D(P^w) \subseteq \text{w-Inf}_D(P^w) = \text{w-Max}_D(D^w).$$

The same is true for epi-infimal pairs. Even if Φ can be chosen to be self infimal, $\text{w-Inf}_D(P^w) = \text{w-Inf}_D(P^w)$ may still not hold. We conclude this section with conditions on Φ for (P^w) – (D^w) and (P_{epi}^w) – (D^w) to satisfy reflexive strong duality.

4.31 Theorem: Reflexive Strong Duality for Weak Infimization

Let $\Phi : \mathcal{X} \times \mathcal{U} \rightrightarrows \bar{\mathcal{Y}}$ be a perturbation mapping and $H : \mathcal{U} \rightrightarrows \bar{\mathcal{Y}}$ an infimal set mapping of (P^w) . Let $V : \mathcal{L}(\mathcal{X}, \mathcal{U}) \rightrightarrows \bar{\mathcal{Y}}$ be the negative supremal set mapping of (D^w) . Assume $\text{w-Inf}_D \Phi$ is subdifferentiable at every $(x, 0) \in \mathcal{X} \times \{0\}$. Then, (P^w) is also a conjugate dual problem of (D^w) and if either H or V is stable, then

$$\text{w-Inf}_D(P^w) = \text{w-Max}_D(D^w) = \text{w-Sup}_D(D^w).$$

Additionally, if both H and V are stable and $\text{w-Inf}_D(P^w) \neq \{\pm\infty_D\}$:

$$\text{w-Min}_D(P_{\text{epi}}^w) = \text{w-Inf}_D(P^w) = \text{w-Max}_D(D^w) = \text{w-Sup}_D(D^w),$$

i.e. all infimal points of (P^w) are epi-infimal.

Proof. It is clear from Thm. 3.32, that both (A1) and (D1) always hold. By $\text{w-Inf}_D \Phi$ subdifferentiable at $(x, 0)$ for every $x \in \mathcal{X}$ and Prop. 4.29,

$$\text{w-Inf}_D \Phi(x, 0) = \Phi^{*\#}(x, 0) = \Phi^{**}(x, 0), \quad \forall x \in \mathcal{X}, \quad (58)$$

such that, by $\text{w-Inf}_D \Phi(x, 0) \in [\Phi(x, 0)]$ and Thm. 2.39,

$$\begin{aligned} -V^{**}(0) &= \text{w-Inf}_D \bigcup_{x \in \mathcal{X}} \Phi^{**}(x, 0) \stackrel{(58)}{=} \text{w-Inf}_D \bigcup_{x \in \mathcal{X}} \text{w-Inf}_D \Phi(x, 0) \\ &\stackrel{\text{Thm. 2.39}}{=} \text{w-Inf}_D \bigcup_{x \in \mathcal{X}} \Phi(x, 0) = H(0) = \text{w-Inf}_D(P^w). \end{aligned} \quad (59)$$

By Thm. 4.28, (P^w) is therefor a conjugate dual problem of (D^w) . By representativity of infima from Thm. 3.32, since the negative supremal set

mapping is characterized by a weak infimum, it is self-infimal, *i.e.* $V(T) = \text{w-Inf}_D V(T)$ (see *e.g.* Lem. 2.30). Let V be stable, that is, subdifferentiable at zero. Then, by Prop. 4.29,

$$V(0) = V^{**}(0) \stackrel{(D1)}{\Longleftrightarrow} \text{w-Sup}_D(D^w) = \text{w-Inf}_D(P^w),$$

proving the first assertion. If H is stable, the same result follows from Thm. 4.30.

Consider now the infimal sets $\text{w-Inf}_D \Phi(x, 0)$ from (59). Assume $\text{w-Inf}_D(P^w) \neq \{\pm\infty_D\}$. If $\text{w-Inf}_D \Phi(x, 0) = \{-\infty_D\}$ for some $x \in \mathcal{X}$, then $\text{w-Inf}_D(P^w) = \{-\infty_D\}$, contradicting the assumption. If now $\text{w-Inf}_D \Phi(x, 0) = \{+\infty_D\}$ for all $x \in \mathcal{X}$, then $\text{w-Inf}_D(P^w) = \{+\infty_D\}$, also yielding a contradiction. Let $\mathcal{X}'$ be the set of $x \in \mathcal{X}$ for which $\text{w-Inf}_D \Phi(x, 0) \subseteq \mathcal{Y}$. For every $x \in \mathcal{X} \setminus \mathcal{X}'$, we have $\text{w-Inf}_D \Phi(x, 0) = \{+\infty_D\}$ and $\text{Cl}_+ \Phi(x, 0) = E_F(x) = \emptyset$ (see Thm. 3.30). Then, by Prop. 2.22,

$$\begin{aligned} -V^{*\#}(0) &= \text{w-Min}_D \bigcup_{x \in \mathcal{X}} \Phi^{**}(x, 0) = \text{w-Min}_D \bigcup_{x \in \mathcal{X}} \text{w-Inf}_D \Phi(x, 0) \cup \{+\infty_D\} \\ &= \text{w-Min}_D \left(\bigcup_{x \in \mathcal{X}'} \text{w-Min}_D E_F(x) \cup \{+\infty_D\} \right), \\ &= \text{w-Min}_D \bigcup_{x \in \mathcal{X}'} \text{w-Min}_D E_F(x). \end{aligned}$$

Furthermore, by

$$\text{w-Min}_D E_F(x, 0) = \text{w-Inf}_D \Phi(x, 0) \in [\Phi(x, 0)] = [E_F(x)],$$

for all $x \in \mathcal{X}'$ (see Lem. 2.30), $E_F(x) = \emptyset$ for all $x \in \mathcal{X} \setminus \mathcal{X}'$ and Prop. 2.40:

$$-V^{*\#}(0) = \text{w-Min}_D \bigcup_{x \in \mathcal{X}'} E_F(x) = \text{w-Min}_D \bigcup_{x \in \mathcal{X}} E_F(x) = \text{w-Min}_D(P_{epi}^w).$$

Thus, by Prop. 4.14, V is stable if and only if

$$V(0) \subseteq V^{*\#}(0) \Longleftrightarrow \text{w-Sup}_D(D^w) \subseteq \text{w-Min}_D(P_{epi}^w),$$

and H is stable, by Thm. 4.30, if and only if

$$\text{w-Min}_D(P_{epi}^w) \subseteq \text{w-Inf}_D(P^w) = \text{w-Max}_D(D^w) = \text{w-Sup}_D(D^w).$$

Altogether we obtain

$$\text{w-Min}_D(P_{epi}^w) = \text{w-Inf}_D(P^w) = \text{w-Max}_D(D^w) = \text{w-Sup}_D(D^w).$$

□

4.4 Notes on the Literature

Terminology for set-valued mappings is mainly due to Rockafellar and Wets [24]. For extended set-valued mappings $F : \mathcal{X} \rightrightarrows \overline{\mathcal{Y}}$, we adopted similar terminology to [4], with slight changes regarding the definition of domain. Since most of the theory in which a domain is required for an extended space like $\overline{\mathcal{Y}}$ focuses on minimization, it is not uncommon to admit $-\infty_R$ in the definition. This is in a direct parallel to the *effective domain* of functions $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ with $\text{dom } f := \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$. Here, we chose to use the same definition as Tanino [27], and define the domain in a way that is compatible with both the relation R and its inverse $-R$.

Our proposed definition of epigraph also differs from that of several authors when considering nondominated and weakly nondominated points, *e.g.* [4, 25, 27]. For D a pointed nonempty convex cone in $\mathcal{Y}$ containing zero, the concept being used appears under the term *cone-epigraph*, in this case *D-epigraph*, for which the epigraphical set mapping is defined as $F(x) + D$ (provided $-\infty_D \notin F(x)$). The epigraph is mostly used as a tool to derive subdifferentiability conditions using forms of convexity, whereas we use the epigraphical set mapping to find infimal points through minimization (compare to $\tilde{\mathcal{S}}$ in Sect. 4.2.1). For comparison, in the context of weak nondomination with $R = \not\leq_D$, by our definition, assuming $F(x) \subseteq \mathcal{Y}$:

$$E_F = B_{\not\leq_D}(F(x)) = \text{cl}(F(x) + D).$$

The notation for the extended set relation $\ll_R$ is directly inspired by Kawasaki [12], originally in the context of weak nondomination. Additional relations like $\ll_R$ were defined, *e.g.* for expressing that two sets satisfy the domination property for one-another. Being only somewhat applicable to general relations, we chose to abstain from generalizing them. $\ll_R$ is really an extension to the binary relation R , in the sense that it integrates fully with sets from $\overline{\mathcal{Y}}$. The motivation behind R being only somewhat compatible with $\pm\infty_R$, was to allow tracing parallels to relations characterized by a domination structure, as described in Sect. 1.2. The definition of $\ll_R$ and the properties observed in Lem. 4.3 and Lem. 4.4 expand the generalized infimum framework from Sect. 2.

The connections between supporting affine minorants, conjugates and subgradients of functions are well established in convex analysis. Indeed, a closed convex function f corresponds to the pointwise supremum over all its affine minorants [22, Thm. 12.1]. The construction of subgradients and of

conjugates based on collections of affine minorant rays aims to give a better geometric intuition of how the two concepts relate to each other, instead of starting from a generalization of the respective definitions. Whilst the notion of subgradients of functions is a local concept, for the purpose of conjugate duality, only global subgradients are considered (see Remark 4.7). Some of the first proposed definitions of (global) subgradients and conjugates of set-valued mappings, with the intent of developing a version of conjugate duality, are due to Tanino and Sawaragi [28] and later Kawasaki [12], for nondominated and weakly nondominated points, respectively.

Approaches restricting the space $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ from which slopes of affine minorants are considered, appear in [25] and [4]. Boş et al. [4] characterize *k-conjugacy* through replacing $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ by families of subsets

$$\{T \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \mid T = \langle x^*, \cdot \rangle_k\},$$

characterized by the dual space X^* and parametrized by some $k \in \mathcal{Y}$. For example, given $\mathcal{X} = \mathbb{R}^n$, $\mathcal{X}^* = \mathbb{R}^n$ characterizes the space of linear mappings $x \mapsto \langle x^*, x \rangle$, with $x^* \in \mathcal{X}^*$, whilst $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^p) = \mathbb{R}^{p \times n}$. In turn, this rejoins the concept of *scalarizability* from [12, 13, 28], used to characterize the family $\Gamma^p(\mathcal{X})$ (for $\mathcal{Y} = \mathbb{R}^p$) of set-valued mappings, for which $F = F^{**}$ in the sense of nondomination and weak nondomination. For the latter, F is called *scalarizable* if it is self(-weakly)-supremal, and thus also self(-weakly)-infimal, which is equivalent to F being representable by a collection of *scalarized functions* f_a , that are all contained in $\Gamma^1(\mathcal{X})$ [12, Thm. 4.1]. Omitting the details, conjugates of F can then be obtained via the conjugates of the scalarized functions [12, Prop. 4.1], for which $\mathcal{L}(\mathcal{X}, \mathbb{R})$ is fully characterized by $\mathcal{X}^*$.

Sawaragi et al. [25] calls $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the *paired space* to $\mathcal{X}$ with respect to $\mathcal{Y}$. Whilst current considerations tend to try and reduce the size of this paired space, Rockafellar and Wets [24, Chapt. 11-L] offers a perspective on using *generalized conjugacy* to construct conjugates. For example, instead of linear mappings, one may attempt to use concave mappings to construct an extension to generalized relations and set-valued mappings.

The framework for minimality and infimality from Sect. 2 allowed us to simultaneously consider attained conjugates, based on maximization, and general conjugates, based on supremization, respectively found in the contexts of nondomination [28] and weak nondomination [12]. Using the proposed mixed biconjugate $F^{*\#}$, specific properties of general conjugates could be further strengthened, *e.g.* Cor. 4.16 coincides with Prop. 7.1.4(a) and strengthens the first part of Prop. 7.4.12 in [4] from $F(x) \subseteq F^{**}(x)$ if

F is subdifferentiable at x to: F subdifferentiable at x if and only if $F(x) \subseteq F^{*\#}(x) (\subseteq F^{**}(x))$.

Regarding Sect. 4.2, conjugate duality originated from Rockafellar [23] as an overarching duality concept containing both Lagrangian- and Fenchel-Duality. Expansions for vector-valued functions and set-valued mappings were proposed by [28] and [12, 13], respectively for nondominated and weakly nondominated points in real finite vector spaces. Sect. 4.2 essentially applies the same constructs as the generalization to topological vector spaces by Boş et al. [4, Chapt. 7], whilst allowing now for general relations R .

Apart from Löhne [16], few authors consider infimality in the solutions of vector-valued (let alone set-valued) problems. In [16], the considered vector-valued problems (V) are, with adjusted notation, of the form:

$$\text{minimize } F : \mathcal{X} \rightarrow \overline{\mathcal{Y}} \text{ with respect to } \not\prec_D \text{ over } S \subseteq \mathcal{X}.$$

Following [16, Thm. 2.21], a set $\overline{X} \subset \mathcal{X}$ would be called a solution to (V) , if $\overline{X} \subseteq S$, $F(\overline{X}) = \text{w-Inf}_D F(S)$ and, notably different to other authors,

$$\text{w-Inf}_D F(\overline{X}) = \text{w-Inf}_D F(S).$$

Whilst the first condition is specific to constrained problems, the second condition supports our solution concept $\overline{\mathcal{S}}$ of finding minimal pairs. The third condition is a consequence of wanting to ensure representativity of the solution. To this end, Löhne [16] uses a notion of *attainment of infimum*, which is coherent with our proposed concept of representativity (see Sect. 2.2). Indeed, under weak nondomination, by weak infima being representative of their respective sets, both $\text{w-Inf}_D F(\overline{X})$ and $\text{w-Inf}_D F(S)$ have respectively the same lower bounds as $F(\overline{X})$ and $F(S)$, such that

$$\text{w-Inf}_D F(\overline{X}) = \text{w-Inf}_D F(S) \iff F(\overline{X}) \in [F(S)].$$

Note that “attainment of infimum” as understood in [16] is not the same as the attained infimal points used for the solution concept $\overline{\mathcal{S}}$. Nevertheless, considerations for representativity may be added to $\overline{\mathcal{S}}$ and $\tilde{\mathcal{S}}$ by respectively demanding that

$$\left. \begin{array}{l} \text{Min}_R(P) \\ \text{Min}_R(P_{epi}) \end{array} \right\} \in [F(\mathcal{X})].$$

Whilst the solution concept $\overline{\mathcal{S}}$ may be widely accepted as a solution of minimization and infimization problems, the proposed solution concept $\tilde{\mathcal{S}}$,

being based on epi-infimal pairs, may be seen as solving a form of closure of (P) . More specifically, under weak nondomination, since $E_F(x) = \text{Cl}_+ F(x) = \text{cl}(F(x) + D)$, solving for $\text{w-Inf}_D(P_{\text{epi}})$ would entail the weak minimization of the pointwise upper closure of F over $\mathcal{X}$.

Comparing the two conjugate duality approaches from Tanino and Sawaragi [28] and Kawasaki [13] simultaneously in a common framework allowed to highlight the equivalence of the optimality conditions from Thm. 4.20, in the sense of minimal pairs of (P) .

Other definitions for strong duality than the one provided exist for vector optimization problems. Göpfert et al. [9] assumes $(D_{\underline{\underline{\#}}})$ is strongly dual to (P) whenever, for any open neighborhood U of zero

$$\text{Inf}_R(P) + U \cap \text{Max}_R(D_{\underline{\underline{\#}}}) + U \neq \emptyset,$$

i.e. the two solution sets (almost) intersect. If they do intersect, then there exists a primal dual optimal point.

On the other hand, Löhne [16] does not attempt to extend the definition of weak and strong duality for partially ordered spaces directly. Instead, the definitions are formalized for complete-lattice-valued optimization problems (of which scalar minimization is a special case). From there, every set-valued problem is assigned a complete-lattice-valued optimization problem $(P_{\mathcal{I}})$ and a complete-lattice-valued dual problem $(D_{\mathcal{I}})$ can be constructed, with strong duality being verified on the pair $(P_{\mathcal{I}})$ – $(D_{\mathcal{I}})$. The decisive observation is that a complete lattice $(\mathcal{I}, \preceq)$ can be constructed on the space of self-weakly-infimal sets. For our problem (P) formulated with $R = \not\prec_D$, we obtain by representativity of weak infima (see Thm. 3.32):

$$(P^{\text{w}}) \quad \text{w-Inf}_{\underset{x \in \mathcal{X}}{D}} F(x) = \text{w-Inf}_{\underset{x \in \mathcal{X}}{D}} \text{w-Inf}_D F(x).$$

The solution set of (P^{w}) can thus be viewed as taking a uniquely existing infimum with respect to $\preceq$ over the set of self-infimal sets $\{\text{w-Inf}_D F(x) \in \mathcal{I} \mid x \in \mathcal{X}\}$. Löhne [16] then proposes dual problems and conditions for Fenchel- and Lagrange-type dual problems. A drawback of this approach to duality is that not every dual problem $(D_{\mathcal{I}})$ can be re-interpreted as a set-valued optimization problem.

Regarding stability, Boţ et al. [4] gives an overview of conditions on the perturbation mapping Φ to ensure subdifferentiability of the minimal and infimal set mappings at zero. The first approach taken by Tanino and Sawaragi [28], was to use complete scalarizability, as has been discussed above,

combined with Φ assumed to be convex. Later, Boţ et al. [4] used the notion of *convexlike-convexity* to account for nonconvex primal objective mappings. Conceptually, for single-valued mappings, Φ is to be chosen such that $\Phi(x, \cdot)$ be convex at every $x \in \mathcal{X}$ and in such a way, that its projection $\bigcup_{x \in \mathcal{X}} \Phi(x, \cdot)$ is also convex. Specific results may be found in [4, Chapt. 7]. More research is required to formulate a form of generalized “convexlike-convexity” criterion for Φ , with respect to an arbitrary relation R .

Reflexive duality has been addressed by Kawasaki [13] in the context of strong duality for weak nondomination. Having chosen Φ from the the family of set-valued mappings Γ^p , *i.e.* with $\Phi = \Phi^{**}$, as part of conditions for stability to hold, the conditions of Thm. 4.31 were also satisfied, making (P^w) and (D^w) conjugate dual problems to eachother. With Sect. 4.2.5 and Thm. 4.28, we attempted to isolate the conditions for strong duality to hold.

The supremal based conjugate duality approach under weak nondomination has been first proposed by Kawasaki [13], and subsequently refined by [27]. Sect. 4.3 shows how specific assumptions from Sect. 4.2 on arbitrary relations, being automatically satisfied for $R = \not\prec_R$, become hidden. This also highlights the advantages of supremum-based conjugate duality, compared to the maximum-based duality approach, and of weak infima in general. Given more properties of the nondominated infimum Min_D , it may be possible to establish an analogous duality scheme with respect to $R = \not\prec_D$.

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