

Decay of the Maxwell System Caused by Conductivity

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INTRODUCTION

While magnetic and electrostatic phenomena have been observed for millennia, it was not until the early 19th century that the connection between these domains was found. This discovery is usually attributed to Hans Christian Ørsted, who in 1820 observed the deflection of a compass needle by an electric current running across it. However, there is some debate about whether he was the first to make this observation, see [61]. Regardless, it was Ørsted's publication [47] that led to further development on the theory of electromagnetism. Over the following years the experiments and mathematical theories of André-Marie Ampère, Michael Faraday, and others culminated 1865 in James Clerk Maxwell's famed paper "A Dynamical Theory of the Electromagnetic Field" [39], unifying the theory of electromagnetism and proposing light as an electromagnetic phenomenon. In the modern notation, credited to Oliver Heaviside, the *Maxwell equations* connect the electric fields E and D to the magnetic fields H and B , and the electric current J via

$$\partial_t D = \text{curl } H - J \quad \text{and} \quad \partial_t B = -\text{curl } E,$$

also known as the *Ampère-Maxwell law* and *Faraday's law*, as well as *Gauß' laws*

$$\text{div } D = \rho \quad \text{and} \quad \text{div } B = 0,$$

relating the electric field D to the charge density ρ , alongside the absence of magnetic charges. For further historical information, see for example [54] or [57].

The fields E and B can be observed by their effect on moving charges. They exert the *Lorentz force*

$$F = q(E + v \times B)$$

on a point particle of charge $q \in \mathbb{R}$ moving along the velocity vector $v \in \mathbb{R}^3$.

CONSTITUTIVE RELATIONS

In order to uniquely determine solutions to the system above one has to impose additional so-called *constitutive relations* between the fields. Here, we focus on the most common class, which relate the fields E and D , as well as H and B . When an electric field E is applied to a material it causes a displacement of the charges present, inducing the polarization P . If this effect is aligned to the applied field lines, we call the material *isotropic*. However, for example in non-trivial crystal structures due to restrictions along lattices, see for example [8] or [9], *anisotropic* effects also occur in nature. Moreover, nonlinear effects can be observed when subjecting dielectrics to strong electric fields. Historically, this was first observed by John Kerr when studying the effect of electric fields on glass, resin, or quartz and observing polarized light passing through those materials, see [32]. Such effects can also be caused by the incident fields themselves, see [40]. The electric displacement field D is then given as

$$D = \varepsilon_0 E + P,$$

where ε_0 is the permittivity of free space. Note that by convention P is the negative of the induced field, explaining the sign in the equation. Similarly, the interaction of a material with the magnetic field causes the magnetization M . Then, the magnetic fields H and B are related by

$$H = \frac{1}{\mu_0} B - M,$$

where μ_0 is the permeability of free space. By custom, the polarization P is usually expressed as a function of E , whereas the magnetization M is given as a function of H . For further information see, for example, Chapters 4 and 6 in [27]. In general, P and M may include retardation effects and therefore depend on the history of a solution (E, H) . For example, P may be of the form

$$P(E)(t, x) = \sum_{n, \alpha} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \chi_{\alpha_1, \dots, \alpha_n}^{(n)}(t, x; \tau_1, \dots, \tau_n) E_{\alpha_1}(t, x) \cdots E_{\alpha_n}(t, x) d\tau_1 \dots d\tau_n,$$

see Chapter 1 in [9]. However, in some situations it is reasonable to approximate such materials by instantaneous laws, see Equation (2.3.6) and the subsequent paragraph in [2].

In this work, we restrict ourselves to materials with an instantaneous response. Assuming that $P = \varepsilon_0 \chi_E(E)E$ and $M = \chi_H(H)H$, we set the permittivity $\varepsilon := \varepsilon_0(1 + \chi_E)$ and permeability $\mu := \mu_0(1 + \chi_H)$. These material parameters then lead to the constitutive equations

$$D = \varepsilon(E)E \quad \text{and} \quad B = \mu(H)H.$$

While matrix-valued material laws naturally occur, they can also appear in the analysis of the isotropic, quasilinear Maxwell system, as in [35], [46], or [51], which we will discuss in detail in Chapters 4 and 7. We stress that the anisotropic Maxwell system is significantly stronger coupled than in the case of isotropic material laws with scalar coefficients. For instance, if the coefficients are constant, the system can be reduced to decoupled scalar wave equations for the components E (if $\operatorname{div}(\varepsilon E) = 0$) or H (if $\sigma = 0$). Moreover, since anisotropic material laws change the direction of the fields, the arising boundary and especially interface conditions are much harder to handle in this setting.

DECAY MECHANISM

Throughout, we study these equations on a bounded domain $\Omega \subset \mathbb{R}^3$ to establish global existence and exponential decay of solutions. We consider two cases:

In the first one, we assume perfectly conducting boundary conditions, which are derived under the assumption that the relevant traces of the fields E and H outside Ω vanish at $\partial\Omega$, leading to the boundary conditions

$$E \times \nu = 0 \quad \text{and} \quad B \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

where ν denotes the outer unit normal. In the absence of a current J , this system will preserve the weighted L^2 -norm of our fields. In order to observe decay, we then assume our material is conductive, which by *Ohm's law* means that

$$J = \sigma E,$$

where σ is the given conductivity. The *Maxwell–Ampère law* then yields

$$\rho(t) := \operatorname{div}(\varepsilon E(t)) = \operatorname{div}(\varepsilon E_0) - \int_0^t \operatorname{div}(\sigma E(\tau)) d\tau, \quad t \geq 0, \quad (1.1)$$

under very mild conditions on the solution. The time integral in (1.1) is a serious obstacle for the study of the long-time behavior. For isotropic, linear coefficients one can reduce (1.1) to the scalar ODE $\partial_t \rho = -\frac{\sigma}{\varepsilon} \rho - \nabla \frac{\sigma}{\varepsilon} \cdot \varepsilon E$ for fixed $x \in \Omega$. In the anisotropic case this approach fails.

We will see that one obtains exponential decay if $\sigma \geq \sigma_0 > 0$ on the support of σ , provided it contains a neighborhood of the boundary $\partial\Omega$. In this setting an interface Γ separates $(\operatorname{supp} \sigma)^\circ$ and $\Omega \setminus \operatorname{supp} \sigma$.

Across the interface one can derive the conditions

$$\begin{aligned} \llbracket E \times \nu^\Gamma \rrbracket &= 0, & \llbracket H \times \nu^\Gamma \rrbracket &= J_\Gamma, \\ \llbracket \varepsilon E \cdot \nu^\Gamma \rrbracket &= \llbracket \varepsilon E(t_0) \cdot \nu^\Gamma \rrbracket - \int_{t_0}^t \left(\llbracket J \cdot \nu^\Gamma \rrbracket - \operatorname{div}_\Gamma J_\Gamma \right) d\tau, & \llbracket \mu H \cdot \nu^\Gamma \rrbracket &= 0, \end{aligned} \quad (1.2)$$

for a given surface current J_Γ , see Chapter I.5 in [30] and Equations (1.2) in [56]. Here, $\llbracket \cdot \rrbracket$ denotes the jump across the interface and ν^Γ its normal, see Section 1.3 below. We will assume $J_\Gamma = 0$ in the following. These conditions will play an important role in the analysis of the quasilinear system in Chapter 4.

In the second case we study, the decay is caused by absorbing boundary conditions. In this case we set $\sigma = 0$, so we have no internal damping. Assuming that the exterior field has zero trace, the interface condition across $\partial\Omega$ yields

$$(H - 0) \times \nu = J_{\partial\Omega}.$$

By the surface version of Ohm's law, cf. (7.155) in [24], the surface current $J_{\partial\Omega}$ is given by to the tangential electric field E via

$$J_{\partial\Omega} = (\lambda(\nu \times E)) \times \nu,$$

where $\lambda = \lambda(E \times \nu)$ denotes the surface conductivity on $\partial\Omega$. Combining these equations leads to the absorbing boundary conditions

$$H \times \nu + (\lambda(E \times \nu)) \times \nu = 0 \quad \text{on } \partial\Omega.$$

In our analysis of both cases, we first focus on linear, autonomous coefficients, which allow for a simpler presentation of the techniques involved. In particular, the well-posedness theory easily follows from semigroup theory, which yields global solutions and provides a convenient framework for our analysis. See [5], [12], or [24] for a systematic treatment of the linear Maxwell equations from a mathematical point of view.

Based on insights from the linear case, we proceed by considering also quasilinear materials, in which the permittivity and permeability may depend on the fields themselves. Mathematically, the dependence of our coefficients on the solutions itself already complicates the local well-posedness theory considerably, see [55], [56], [58], [59], and [60].

In order to show exponential decay we rely on energy estimates for our solutions, which we combine with suitable observability estimates. These are shown using a multiplier technique based upon a Helmholtz decomposition and Morawetz multipliers. Obtaining these estimates is the main challenge in the linear case. In the quasilinear case additional error terms appear in these estimates, which pose additional challenges when closing the decay result.

The thesis is thus split into two parts. In the first part we focus on the case where we have damping in the interior and in the second part we discuss damping at the boundary. We proceed by providing a more detailed overview concerning the approach and the relevant literature.

1.1. PART I – DECAY DUE TO INTERIOR CONDUCTIVITY

In the first main result of Part I, Theorem 2.3.1, we consider anisotropic, instantaneous, time-independent, linear material laws $\varepsilon, \mu \in C^1(\overline{\Omega}, \mathbb{R}_{\text{sym}}^{3 \times 3})$ and the solutions to the system

$$\begin{aligned} \partial_t(\varepsilon(x)E(t, x)) &= \text{curl } H(t, x) - \sigma(x)E(t, x), & t \geq 0, x \in \Omega, \\ \partial_t(\mu(x)H(t, x)) &= -\text{curl } E(t, x), & \\ \text{div}(\mu(x)H(t, x)) &= 0, & t \geq 0, x \in \Omega, \\ E(t, x) \times \nu(x) &= 0, \quad \nu(x) \cdot \mu(x)H(t, x) = 0, & t \geq 0, x \in \partial\Omega, \\ E(0, x) &= E_0(x), \quad H(0, x) = H_0(x), & x \in \Omega, \end{aligned} \tag{1.3}$$

with a non-negative conductivity $\sigma \in L^\infty(\Omega, \mathbb{R}_{\text{sym}}^{3 \times 3})$ which is strictly positive near the boundary of the bounded domain $\Omega \subset \mathbb{R}^3$. It is assumed that the permittivity ε and permeability μ satisfy the non-trapping condition (2.6), which limits the decay of coefficients in radial direction. Heuristically, this property reduces back reflections, so that waves can reach the boundary in a sufficient way. We allow for a multiply connected Ω and disconnected boundary $\partial\Omega$. Under these assumptions, we obtain exponential decay to 0 if there are no magnetic charges on Ω (which agrees with observations in physics), no electric charges on $\Omega \setminus \text{supp } \sigma$, and we exclude stationary solutions. In Corollary 2.3.8, we remove the condition on initial charges as well as the magnetic boundary condition and obtain exponential convergence to equilibria instead.

For the charge-free system with $\sigma = 0$, we also prove exact observability and controllability with respect to a collar at $\partial\Omega$, see Theorem 2.2.8. In particular, one can steer the system from every initial to any final state by applying a suitable external current at a collar near $\partial\Omega$.

In the case of anisotropic coefficients, these seem to be the first results in this context if the damping or observability region is not the full domain. Moreover, for exponential stability, so far only constant ε and μ have been treated in this context.

OVERVIEW OF THE LITERATURE

Compared to the damped wave equation, see for example [7], [36], [53], and the references therein, the theory of decay properties of the Maxwell system is far less developed. Indeed, the huge kernel of the curl operator, the strong coupling in the Maxwell system, and the complicated boundary conditions pose significant challenges compared to the scalar wave equation. Still, the results and methods for the wave equation are the starting point for the treatment of the Maxwell system.

For constant coefficients $\varepsilon, \mu \in \mathbb{R}_+$, Theorem 5.5 in [49] establishes that solutions decay exponentially to 0 provided that the conductivity σ is strictly positive on its support and satisfies the Geometric Control Condition (GCC) derived from the wave equation framework. The GCC requires the existence of a critical time $T_c > 0$ such that every ray of geometric optics intersects $\text{supp } \sigma$ within the time interval $(0, T_c)$, see [7] for interior control and [6] for boundary control. The proof in [49] relies on microlocal analysis to exploit the elliptic properties of the Laplacian.

Alternative techniques using Morawetz multipliers and Helmholtz decompositions have also been successfully deployed to study these systems. For instance, for interior damping and scalar coefficients, [44] utilizes a multiplier argument to establish exact observability on a collar around $\partial\Omega$, which is an important step to proving exponential decay.

For matrix-valued coefficients, the literature on exponential decay is limited. Existing results typically assume the strict positivity of the conductivity tensor throughout the domain, as seen in [19] for linear Maxwell systems and [35] for quasilinear variants.

Weaker notations of decay, such as polynomial stability, have also been studied. For smoothly vanishing conductivity, [49] establishes polynomial decay given that $\text{supp } \sigma$ still satisfies the GCC. In [42] and [50] polynomial decay is shown for an internal conductivity restricted to a strip of a cube, violating the GCC. In contrast to [50], the rate is explicitly given in [42].

With the exception of Theorem 5.1 in [49], which assumes positive, scalar coefficients, the aforementioned works impose various topological constraints, such as the simple connectedness of Ω or the connectedness of the boundary $\partial\Omega$. These topological assumptions are typically required to improve the mapping properties of the curl operator, see Lemma A.1.3 in the appendix. However, we can circumvent these restrictions by excluding certain stationary solutions that belong to the kernel of the curl. Such stationary solutions can arise when the domain Ω encloses electrical charges or when an external current runs through loops of Ω , as illustrated in Examples 2.1.3 and 2.1.4.

To our knowledge the paper [35], which we generalize, contains the only known result considering exponential decay of a quasilinear Maxwell system with internal damping on a bounded domain. We note that on the full space $\mathbb{R}^3$ polynomial decay results for the quasilinear system and small data are known. There the decay is caused by dispersion rather than damping, see for example Chapter 11.6 in [52] for coefficients with a constant linear part.

PROOF TECHNIQUES

In order to establish the results outlined above, we rely on an observability estimate for the charge-free, homogeneous system without conductivity, see system (2.23). So, in order to apply it to solutions of system (1.3) we split (E, H) , which may contain electric charges on $\omega = \text{supp } \sigma^\circ$, into the solution (V_h, W_h) of this system and further correction terms. Specifically, to account for the charges present on ω , we introduce a gradient field ∇p , where the potential $p \in H^1(\Omega)$ solves the elliptic PDE (2.13). While ∇p itself cannot be bounded in L^2 in a useful way, its time derivative $\partial_t \nabla p$ can be controlled by the dissipation term $\|\sigma^{1/2} E\|_2$, which is subsequently bounded via the energy equality. This decomposition originates from [49] for the constant-coefficient case with $p \in H_0^1(\Omega)$, and connected $\partial\Omega$ and $\text{supp } \sigma$. Our formulation of system (2.13) resolves several of the topological obstructions to the invertibility of the curl operator. To address the remaining topological constraints, we restrict the electric and magnetic fields of the Maxwell system to an invariant subsystem of finite codimension, defined in (2.7). The remaining components of the solution, denoted by (V_i, W_i) , then satisfy the inhomogeneous system (2.27) incorporating the inhomogeneity $-\sigma E - \varepsilon \nabla \partial_t p$. A related splitting, omitting the potential p , was utilized in [44]. Finally, to account for the influence of the initial potential $p(0)$ within these systems, the initial data will be chosen according to (2.21). Due to technical considerations some of the initial fields have to be approximated by more regular ones.

In Proposition 2.2.2 we show the crucial observability-type estimate for V_h . Dissipation terms only occur for the electric part, so that it is useful to work with a second-order formulation involving only V_h , see (2.32). Though, a first-order approach is also feasible and used in showing an observability-type estimate for the quasilinear system which we discuss below, see (4.57). Moreover, the behavior of ∇p and ρ in (1.1) suggests to work with time derivatives. Therefore, we first estimate $\partial_t V_h$ and only later pass to V_h in Corollary 2.2.6 by integration. (This result is then reformulated as Theorem 2.2.8 on observability and controllability.) In a lengthy calculation also involving vector analysis, Proposition 2.2.2 is then shown employing a Morawetz multiplier and by a tailor-made multiplier using Lax–Milgram (see Lemma 2.2.3), as well as the energy equality for V_h . For this reasoning it is important that $\text{supp } \sigma$ contains a collar at $\partial\Omega$ and that the coefficients satisfy the non-trapping condition. For scalar coefficients

similar arguments are found in [44], but it is quite sophisticated to extend them to the matrix-valued case.

To pass from the observability estimate of the homogeneous system to the decay result for solutions, including charges on ω , we estimate the other correction terms of (E, H) in several steps. First, the energy of $\partial_t(E, H)$ is bounded by that of (E, H) for which it is crucial that σ is strictly positive on its support. This was done for connected $\partial\Omega$ and ω in [49], using the auxiliary function ∇p . In our more general setting we have to proceed differently, based on a lower estimate for the curl operator in Lemma 2.3.2. One can control $\partial_t H$ in space-time by $\partial_t E$ and dissipation terms using the splitting of E and energy-type estimates. The inhomogeneous part (V_i, W_i) is handled just by means of Duhamel's formula, exploiting properties of the initial data of (2.27). Several of these inequalities on the time interval $[0, T]$ depend on $T > 0$. Nevertheless, they can be combined into a proof of exponential stability employing a strategy that goes back to [29] in the wave case. The convergence result for data with electrical charges on $\Omega \setminus \text{supp } \sigma$ (and possibly magnetic charges) then follows by a projection argument. These results are presented in Chapter 2 and in the joint preprint [45] with Roland Schnaubelt.

Next, we extend the methods from the linear case to the quasilinear case. In this setting, the coefficients $\varepsilon(\cdot, E)$ and $\mu(\cdot, H)$ in (1.3) can depend directly on the solution itself. We consider the system

$$\begin{aligned} \partial_t(\varepsilon(x, E(t, x)) E(t, x)) &= \text{curl } H(t, x) - \sigma(x)E(t, x), & t \geq 0, x \in \Omega, \\ \partial_t(\mu(x, H(t, x)) H(t, x)) &= -\text{curl } E(t, x), & \\ \text{div}(\mu(x, H(t, x)) H(t, x)) &= 0, & t \geq 0, x \in \Omega, \\ E(t, x) \times \nu(x) &= 0, \quad \nu(x) \cdot \mu(x)H(t, x) = 0, & t \geq 0, x \in \partial\Omega, \\ E(0, x) &= E_0(x), \quad H(0, x) = H_0(x), & x \in \Omega. \end{aligned} \tag{1.4}$$

In particular, since the conductivity σ is either uniformly positive or vanishing, this introduces a jump in our solutions and therefore also in our coefficients. Thus, we have to treat the quasilinear system as an interface problem. In our analysis, we will therefore rely on the interface conditions (1.2).

As alluded to above, even the local well-posedness in this case is quite challenging, see [56], [58], [59], and [60]. (It is known that in quasilinear Maxwell systems for non-small data blow-up may occur, see [14] and the references therein.) It forces us to work in the solution space $\cap_{j=0}^3 C^j((T_{\min}, T_{\max}), H_{\Gamma}^{3-j}(\Omega_{\pm}))$, since the local well-posedness requires additional control on $\partial_t(E, H)$ through Sobolev embeddings. Here, H_{Γ}^k denotes Sobolev regularity on either side of the interface, see Section 1.3 for the definition. Thus, we also have to consider time derived versions of system (1.4), which lead to additional error terms in the analysis, and to require additional compatibility conditions of the initial values at the boundary and interface. We stress that the coefficients of the time derived system are matrix-valued, even if $\varepsilon(\cdot, E)$ and $\mu(\cdot, H)$ are scalar.

In order to show exponential decay for small initial data, we adapt a strategy similar to the linear case. Such a smallness condition is common in nonlinear theory. However, in order to justify a Morawetz estimate for the highest order term $\partial_t^3(E, H)$, we require additional regularity. We obtain this by an approximation argument in Chapter 3. There, we first approximate coefficients and initial data smoothly and afterwards correct the approximation in such a way, that it satisfies the compatibility conditions. It is then shown that the corrected approximation converges back to the original solution. So far, such regularization procedures for (1.4) were only known for very basic cases, see the proofs of Lemma 4.2 in [35] or Lemma 3.2.3.

This finally allows us to obtain decay results in Chapter 4. For this, we consider $\sigma \in C^3(\Omega_{\pm}, \mathbb{R}_{\text{sym}}^{3 \times 3})$ to be uniformly positive on its support containing a collar of the boundary, as well as uniformly positive definite coefficients $\varepsilon, \mu \in C^3(\overline{\Omega}_{\pm} \times \mathbb{R}^3, \mathbb{R}_{\text{sym}}^{3 \times 3})$ which, in addition to the non-trapping condition from the linear case in the interior, also have to satisfy a non-trapping at the interface in order to prevent internal reflection, see (4.21). In the linear case, for piecewise constant coefficients a similar condition was already used in [31].

Considering each time derivative $\partial_t^k(E, H)$ individually, we first split them into a divergence- and a curl-free part, using gradient fields $\nabla(p^k, q^k)$ where $p^k, q^k \in H^1(\Omega)$ solve the elliptic PDEs (4.40) and (4.47) which we adapt to the interface problem at hand. As in the linear case, we show suitable estimates for (p^k, q^k) and establish an observability-type estimate for the time derivatives of the divergence-free part of the solution (V^k, W^k) . In this estimate we rely on the non-trapping conditions in the interior and at the interface.

In order to utilize the observability estimate, we require additional regularity estimates for various error terms. These error terms contain higher Sobolev norms $\|\partial_t^k(E, H)\|_{H_{\Gamma}^{3-k}}$ which have to be bounded by the L^2 -norms $\|\partial_t^k(E, H)\|_{L^2}$, to which we can apply the observability estimate, and superlinear terms. It is crucial that the constants in this regularity estimate do not depend on time. In the quite sophisticated and lengthy proof, we employ ideas from [35] and [51], but the interface requires new ideas. Finally, we can combine these estimates to show the exponential decay of small data in Theorem 4.6.1.

1.2. PART II – DECAY DUE TO ABSORBING BOUNDARY CONDITIONS

In Part II, we focus on the effect of a strictly positive surface conductivity λ , which may also depend on the electric field E , investigating the system

$$\begin{aligned} \partial_t(\varepsilon(x, E(t, x))E(t, x)) &= \text{curl } H(t, x), & t \geq 0, x \in \Omega, \\ \partial_t(\mu(x, H(t, x))H(t, x)) &= -\text{curl } E(t, x), & t \geq 0, x \in \Omega, \\ \text{div}(\varepsilon(x, E(t, x))E(t, x)) &= 0, & t \geq 0, x \in \Omega, \\ \text{div}(\mu(x, H(t, x))H(t, x)) &= 0, & t \geq 0, x \in \Omega, \\ H(t, x) \times \nu + \left(\lambda(x, E(t, x) \times \nu)(E(t, x) \times \nu) \right) \times \nu &= 0, & t \geq 0, x \in \partial\Omega, \\ E(0, x) &= E_0(x), \quad H(0, x) = H_0(x), & x \in \Omega, \end{aligned} \tag{1.5}$$

on a bounded, smooth domain $\Omega \subset \mathbb{R}^3$ with nonlinear, inhomogeneous, and anisotropic material coefficients, and given initial fields (E_0, H_0) satisfying the compatibility conditions.

OVERVIEW OF THE LITERATURE

In the case of linear material laws, such nonlinear λ have been studied in, for instance, [20], [21], [43], and for delayed problems in [4]. For state-independent λ and small initial fields it is shown in [51] that the solutions of the quasilinear Maxwell system (1.5) exist globally in time and decay exponentially as $t \rightarrow \infty$. However, in [51] it is assumed that the spatial domain Ω is strictly starshaped.

In related problems it is known that this assumption can be removed provided one can show extra regularity of the normal trace of the solutions, see [37] for the wave equation and [18] for the linear, autonomous Maxwell system with perfectly conducting boundary conditions (which we studied in Part I). Regarding trace regularities, a variant of our normal trace estimate already has been shown in [11] using completely different methods.

Prior observability results have been shown in [51] for strictly starshaped, simply connected domains and in [46] (jointly with Roland Schnaubelt) for domains with connected complement. Based on our estimates, we can extend the analysis of [51] to the case of a nonlinear boundary conductivity $\lambda(x, E)$ and show in our main Theorem 7.2.1 that solutions for small data converge exponentially to 0 without assuming starshapedness or connectedness. However, we have to exclude stationary solutions from our initial values that become possible by removing the topological assumptions on Ω .

PROOF TECHNIQUES

In Chapter 5 we prove a normal trace estimate for the solutions of linear, non-autonomous, anisotropic Maxwell systems with absorbing boundary conditions and inhomogeneities, see Corollary 5.1.9 and Proposition 5.1.11. We use the so-called collar operator studied in [18] in the context of different boundary conditions, cf. [37] for earlier work on the wave equation. This pseudodifferential operator allows us to trade space for time regularity in the course of a sophisticated regularity argument. We further employ the div-curl estimate from Theorem A.2.9 and exploit heavily the structure of the (time-differentiated) Maxwell system and the absorbing boundary condition. In the trace estimates it is crucial that constants do not depend on time, see Corollary 5.1.9. We also derive a more concise variant of the estimate in Proposition 5.1.11, which has time dependent constants, though, reproving a result of [11].

Next, we restrict ourselves in Chapter 6 to the linear, autonomous, anisotropic case (6.1). This allows us to further improve the trace estimates from Chapter 5 in Theorem 6.1.2, following the approach of [34] for the wave equation, which involves a compactness argument to get rid of lower-order terms, see also [18] for the Maxwell case. We obtain a bound on the normal traces of E and H in $L^2((0, T) \times \partial\Omega)$ in terms of the tangential trace of E , with constants being independent of the end time T . In order to perform this argument, we furthermore need the div-curl estimate from Theorem A.2.9 in $H^{-\theta}(\Omega)$ for $\theta \in (0, \frac{1}{2})$, which requires several properties of the Helmholtz decomposition in $H^{-\theta}$ also established in the appendix.

As a by-product, this allows us to derive exponential stability of the system (6.1), which seems to be a new result for matrix-valued coefficients (and is not a special case of our nonlinear decay result).

In Chapter 7 we again consider the quasilinear system (7.1)–(7.5) for small solutions in $\cap_{j=0}^3 C^j((T_{\min}, T_{\max}), H^{3-j}(\Omega))$. Using the trace regularity established in Chapter 5 we derive an observability-type estimate for this system in Proposition 7.2.3. To bound the solutions, we again consider the time-derived Maxwell systems (7.20)–(7.21). Moreover, the time-dependent coefficients also introduce error terms at the boundary, which will be treated as inhomogeneities.

As in Part I, [35], and [51], the decay result follows from three propositions dealing with the time derivatives $\partial_t^k(E, H)$ for $k \in \{0, 1, 2, 3\}$ —an energy estimate, an observability-type estimate, and a regularity analysis of error terms. The fields $\partial_t^k(E, H)$ satisfy the boundary condition (7.3) up to lower-order terms. This fact is crucial for the analysis. An energy estimate and an observability-type estimate will allow us to control the squared L^2 -norm of $\partial_t^k(E, H)$ by a dissipation term plus an error term which is small for small data, but contains space and time derivatives of higher order. The core observability-type estimate in Proposition 7.2.3 is based on a Morawetz multiplier argument as in [51], which uses ideas from [18] or [44] treating different boundary conditions, also utilizing the normal trace estimates of Chapter 5. Tangential traces of solutions are bounded by the energy estimate, see also [11] or [55]. For this result we again have to assume the non-trapping condition (7.24) on ε

and μ . Finally, the regularity result in Proposition 7.2.6 for the nonlinear problem allows us to absorb the error terms. The decay result of Theorem 7.2.1 then follows by a standard bootstrapping procedure, given at the end.

CONTENT OF THIS THESIS

The linear analysis of the first part is based on [45]. In Chapter 3, we show an approximation result in order to justify several formal calculations. This allows us to proceed with the quasilinear case in Chapter 4. These two chapters contain unpublished results.

The second part of thesis is largely based upon [46], but generalizes it regarding the topology of our domain. There the connectedness assumption on $\mathbb{R}^3 \setminus \Omega$ can be removed by additional assumptions on our initial values.

Finally, the appendix contains results pertaining to the curl operator and related estimates. These results can be found mainly in [5], [10], [12], [17], and [46].

ACKNOWLEDGMENTS

AI tools, mainly *Gemini 3.1 Pro*, were used to generate the initial TikZ code for some of the figures below and were further used to improve them. Additionally, the tools were employed in the manipulation and the typesetting in L^AT_EX of some of the equations present in the text. They were consulted at various points to check results and to improve the flow of text, when writing up this thesis. However, the final arguments and explanations in this thesis are human-generated and the author assumes full responsibility for the integrity of the work.

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1.3. NOTATION

Let us start by setting up general notation used throughout this entire thesis. In the following $\Omega \subset \mathbb{R}^3$ will always be a bounded domain¹ with at least Lipschitz boundary. Moreover, the boundary $\partial\Omega$ has either C^2 - or $C^{2,1}$ -regularity when studying the linear Maxwell system and C^5 -regularity when considering the quasilinear Maxwell system.

A bounded domain $\Omega \subset \mathbb{R}^3$ is *multiply connected* if there are disjoint C^k -surfaces $\Sigma_1, \dots, \Sigma_L$ such that $\Omega \setminus \bigcup_{l=1}^L \Sigma_l$ is simply connected, see [5], [12], or [16]. For the connected components of the boundary $\partial\Omega$ we write $\partial\Omega_0, \dots, \partial\Omega_K$.

Let

$$\mathcal{N}_a := \{x \in \overline{\Omega} \mid \text{dist}(x, \partial\Omega) < a\}$$

be the collar of width $a > 0$ around the boundary $\partial\Omega$ of Ω .

In the first part our domain will also contain an interface $\Gamma \subset \mathbb{R}^3$ which has the same regularity as the boundary, as well as positive distance to it, i.e., $\text{dist}(\Gamma, \partial\Omega) > 0$. Thus, the interface splits Ω into two subdomains. The one bordering $\partial\Omega$ will be denoted by $\omega \subseteq \Omega$ and the other one we call $v \subseteq \Omega$. For the union of these we write $\Omega_{\pm} := \omega \cup v$. Hence, we get

$$\Omega = \Omega_{\pm} \cup \Gamma = v \cup \Gamma \cup \omega.$$

¹*domain* meaning a non-empty, open, connected set

A collar around Γ of width $a > 0$ is denoted by

$$\mathcal{N}_a^\Gamma := \{x \in \overline{\Omega} \mid \text{dist}(x, \Gamma) < a\}.$$

We write ν for the outer unit normal to $\partial\Omega$ and ν^Γ for the unit normal at the interface, pointing from v to ω .

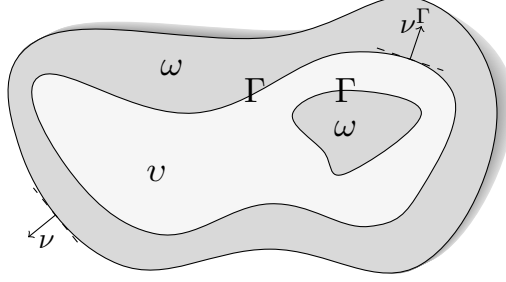
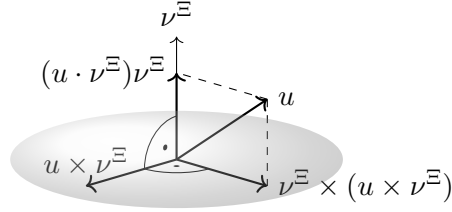


Figure 1.1.: 2d-cut through $\Omega \subset \mathbb{R}^3$.

Note that $\nu^\Gamma = \nu^{\partial v} = -\nu^{\partial\omega}$ on Γ . The surface measure on $\partial\Omega$ or $\partial\omega$ is denoted by ς . To emphasize that we are integrating a function f over a surface, we write ‘ $\oint f d\varsigma$ ’.

In order to study functions at the boundary and interface we define, first for $u \in C(\overline{\Omega})$, the following vector-valued traces

$$\begin{aligned} \text{tr}_n u &= \text{tr}_{n, \Xi} u := (u \cdot \nu^\Xi) \nu^\Xi, \\ \text{tr}_t u &= \text{tr}_{t, \Xi} u := u \times \nu^\Xi, \\ \text{tr}_\tau u &= \text{tr}_{\tau, \Xi} u := \nu^\Xi \times (u \times \nu^\Xi), \end{aligned}$$



where Ξ is usually ∂v , $\partial\omega$ or $\partial\Omega$ and $\nu = \nu^\Xi$ denotes the corresponding outer normal. We set $\llbracket f \rrbracket := \text{tr}_{\partial v} f - \text{tr}_{\partial\omega} f$ to be the jump across the interface. Here and below we often omit the arguments of functions.

FUNCTION SPACES

We use the real, L^2 -based Sobolev spaces H^s on the domains and their boundaries. If the domain of H^s is omitted, we implicitly mean $H^s(\Omega) = H^s$. Let

$$\begin{aligned} H^\theta(\text{div}_\alpha) &:= \left\{ f \in (H^\theta(\Xi))^3 \mid \text{div}(\alpha f) =: \text{div}_\alpha f \in H^\theta(\Xi) \right\}, \\ H^\theta(\text{curl}_\beta) &:= \left\{ f \in (H^\theta(\Xi))^3 \mid \text{curl}(\beta f) =: \text{curl}_\beta f \in (H^\theta(\Xi))^3 \right\}, \end{aligned}$$

for $\theta \in \mathbb{R}$, $\Xi \in \{v, \omega, \Omega\}$, and suitable matrix-valued $\alpha, \beta: \Xi \rightarrow \mathbb{R}^{3 \times 3}$. For $\theta = 0$ we simply write $H(\text{div}_\alpha) = H^0(\text{div}_\alpha)$ and $H(\text{curl}_\beta) = H^0(\text{curl}_\beta)$ and if $\alpha = \beta = I_3$ is the identity matrix, we write $H^\theta(\text{div}) = H^\theta(\text{div}_{I_3})$ and $H^\theta(\text{curl}) = H^\theta(\text{curl}_{I_3})$, etc. Then $H(\text{div})$ and $H(\text{curl})$ are the maximal domains of the distributional operators div and curl in $L^2(\Xi)$ equipped with the graph norm, respectively. (We often omit the range spaces for vector fields etc.)

The kernels of these operators are written as $H(\text{div } 0)$ and $H(\text{curl } 0)$ and are endowed with the L^2 -norm.

On the boundary $\partial\Xi$, we introduce the negative fractional Sobolev space $H^{-1/2}(\partial\Xi)$ as the continuous dual of $H^{1/2}(\partial\Xi)$ with respect to the pivot space $L^2(\partial\Xi)$. The normal trace

tr_n and the tangential traces $\text{tr}_t, \text{tr}_\tau$ can be extended to maps from $H(\text{div})$ and $H(\text{curl})$ to $H^{-1/2}(\partial\Xi)^3$. We denote the kernels of these trace mappings by $H_{n0}(\text{div})$ and $H_{t0}(\text{curl})$, respectively, which coincide with the closures of the space of test functions $C_c^\infty(\Xi)^3$ under the respective graph norms. The intersection with the kernels we then write as

$$H_{t0}(\text{curl}0) := H_{t0}(\text{curl}) \cap H(\text{curl}0) \quad \text{and} \quad H_{n0}(\text{div}0) := H_{n0}(\text{div}) \cap H(\text{div}0).$$

We define the refinements

$$\begin{aligned} H^{\partial\Xi}(\text{div}_\alpha 0) &:= \{u \in L^2(\Xi)^3 \mid \text{div}_\alpha u = 0, \langle \text{tr}_n(\alpha u), \mathbb{1} \rangle_{H^{-\frac{1}{2}}(\partial\Xi_k)} = 0, k \in \{0, \dots, K_\Xi\}\}, \\ H_{n0}^\Sigma(\text{div}_\alpha 0) &:= \{u \in L^2(\Omega)^3 \mid \text{div}_\alpha u = 0, \text{tr}_n(\alpha u) = 0, \langle \text{tr}_n(\alpha u), \mathbb{1} \rangle_{H^{-\frac{1}{2}}(\Sigma_l)} = 0, \\ &\quad l \in \{1, \dots, L\}\}. \end{aligned}$$

where the boundary $\partial\Xi$ has the connected components $\partial\Xi_0, \dots, \partial\Xi_{K_\Xi}$. (We will need the space $H^\Sigma(\text{div}_\alpha 0)_{n0}$ only for Ω . For $\Xi = v$ we write $H^{\partial\Xi}(\text{div}_\alpha 0) = H^\Gamma(\text{div}_\alpha 0)$.) For $\alpha = I$ we simply write $H_{n0}^\Sigma(\text{div}0)$ and $H^{\partial\Xi}(\text{div}0)$. The divergence theorem yields $H^{\partial\Xi}(\text{div}_\alpha 0) = H(\text{div}_\alpha 0)$ if $\partial\Xi$ is connected, and for simply connected Ω we have $H_{n0}^\Sigma(\text{div}_\alpha 0) = H_{n0}(\text{div}_\alpha 0)$. See [5], [12], or [16] for these and related facts.

To describe regularity on either side of an interface we introduce the space

$$H_\Gamma^k = H_\Gamma^k(\Omega) := \left\{ f \in L^2(\Omega) \mid f|_\omega \in H^k(\omega) \text{ and } f|_v \in H^k(v) \right\}.$$

When considering non-autonomous coefficients with an interface as in Chapter 3, we will consider solutions in

$$\begin{aligned} \mathcal{G}_\Gamma^k &= \mathcal{G}_\Gamma^k((T_{\min}, T_{\max}) \times \Omega) := \bigcap_{j=0}^k C^j((T_{\min}, T_{\max}), H_\Gamma^{k-j}(\Omega)), \\ \text{with norm} \quad \|u\|_{\mathcal{G}_\Gamma^k} &:= \sup_{t \in (T_{\min}, T_{\max})} \sum_{j=0}^k \|\partial_t^j u(t)\|_{H_\Gamma^{m-j}}. \end{aligned}$$

In the absence of an interface, as in Chapter 7, we drop the Γ in the subscripts above, denoting these spaces by $\mathcal{G}^k$.

Finally, we introduce the notation “ $\lesssim$ ”, whenever an inequality holds up to a constant not depending on time or the explicit norm of the solution.

PART I

DECAY CAUSED BY INTERIOR CONDUCTIVITY

CHAPTER 2

EXPONENTIAL DECAY OF THE LINEAR MAXWELL SYSTEM DUE TO CONDUCTIVITY NEAR THE BOUNDARY

We start our discussion by considering a linear Maxwell system with anisotropic coefficients ε , μ , and σ on a bounded domain $\Omega \subset \mathbb{R}^3$. On the boundary we assume perfectly conducting boundary conditions. The precise assumptions are given at the beginning of Section 2.1 below. Namely, the Maxwell system is given by

$$\begin{aligned} \partial_t(\varepsilon(x)E(t, x)) &= \operatorname{curl} H(t, x) - \sigma(x)E(t, x), & t \geq 0, x \in \Omega, \end{aligned} \quad (2.1)$$

$$\begin{aligned} \partial_t(\mu(x)H(t, x)) &= -\operatorname{curl} E(t, x), \\ \operatorname{div}(\mu(x)H(t, x)) &= 0, \end{aligned} \quad t \geq 0, x \in \Omega, \quad (2.2)$$

$$E(t, x) \times \nu(x) = 0, \quad \nu(x) \cdot \mu(x)H(t, x) = 0, \quad t \geq 0, x \in \partial\Omega, \quad (2.3)$$

$$E(0, x) = E_0(x), \quad H(0, x) = H_0(x), \quad x \in \Omega. \quad (2.4)$$

By standard semigroup theory we first show that a solution exists and then proceed by studying its long-term behavior. In order to achieve convergence to some steady state, we require damping in the system. This is realized through the conductivity σ , which we will assume to be supported near the boundary $\partial\Omega$. Excluding magnetic charges, electric charges on $(\operatorname{supp} \sigma)^c$, and external stationary fields, we can show that by Theorem 2.3.1 this suffices to show exponential decay to 0.

The proof is based on a Helmholtz decomposition and an observability estimate, i.e., an estimate which allows us to bound the initial state through an integral over a set $\mathcal{O} \subseteq [0, T] \times \operatorname{supp} \sigma \subseteq [0, T] \times \Omega$

$$\int_{\Omega} (|E_0|^2 + |H_0|^2) dx \lesssim \int_{\mathcal{O}} |E|^2 d(\tau, x),$$

cf. Proposition 2.2.6. Notice that if the right-hand side vanishes on $\mathcal{O}$, then our initial value was already 0. This estimate is shown using a Morawetz multiplier argument. By control theory this is equivalent to showing exact controllability, see Theorem 2.2.8 and [64] for a systematic treatment.

Finally, in the presence of charges we will show that we converge to an orthogonal projection of the initial values.

2.1. ASSUMPTIONS AND AUXILIARY RESULTS

In this section we first collect the main hypotheses in this chapter. Afterwards, we show the existence of a solution by semigroup theory and derive an energy equality. Finally, we introduce the required in the proof of our observability estimate.

ASSUMPTIONS

Throughout Chapter 2, we assume the following conditions.

- (H_ℓ) Let $\Omega \subset \mathbb{R}^3$ be open, bounded and multiply connected with cuts $\Sigma_1, \dots, \Sigma_L$, and $\partial\Omega \in C^2$ have connected components $\partial\Omega_0, \dots, \partial\Omega_K$. Let $\varepsilon, \mu \in C^1(\overline{\Omega}, \mathbb{R}_{\text{sym}}^{3 \times 3})$ be uniformly positive definite, $\sigma \in L^\infty(\Omega, \mathbb{R}_{\text{sym}}^{3 \times 3})$, $\omega := \{x \in \Omega \mid \sigma(x) > 0\}$ have a C^1 -boundary and let $\operatorname{dist}(\partial\omega, \partial\Omega) > 0$.

Our core assumption for exponential stability is that there are constants $a, \sigma_0 > 0$ such that

$$\mathcal{N}_a \subseteq \omega \quad \text{and} \quad \sigma|_\omega \geq \sigma_0 > 0. \quad (2.5)$$

In particular, the boundaries $\partial\omega = \partial v \cup \partial\Omega$ are disjoint, see Section 1.3.

Let $m(x) := x - x_0$ for some $x_0 \in \mathbb{R}^3$. We also require the ‘non-trapping’ condition

$$\tilde{\varepsilon} := \varepsilon + (m \cdot \nabla)\varepsilon \geq \tilde{\eta}\varepsilon, \quad \tilde{\mu} := \mu + (m \cdot \nabla)\mu \geq \tilde{\eta}\mu \quad (2.6)$$

for some $\tilde{\eta} > 0$; i.e., the coefficients cannot decay too strongly in radial direction.

GENERATION RESULTS

Let $L_\alpha^2(\Omega)$ be the space of measurable f fulfilling $\alpha^{1/2}f \in L^2(\Omega)$, endowed with the canonical norm. We define

$$X_0 = \{(e, h) \in L_\varepsilon^2(\Omega) \times L_\mu^2(\Omega) \mid \varepsilon e|_v \in H^\Gamma(\operatorname{div} 0, v), \mu h \in H_{n0}^\Sigma(\operatorname{div} 0)\}. \quad (2.7)$$

The operator

$$D(A_0) = (H_{t0}(\operatorname{curl}) \times H(\operatorname{curl})) \cap X_0, \quad A_0 = \begin{pmatrix} -\varepsilon^{-1}\sigma & \varepsilon^{-1}\operatorname{curl} \\ -\mu^{-1}\operatorname{curl} & 0 \end{pmatrix}, \quad (2.8)$$

is well-defined as an operator on X_0 Theorem 6.1.4 in [5] and Proposition IX.1.3 in [16], the latter applied on v , see also Lemma A.1.3. Note that the magnetic component of $(e, h) \in D(A_0)$ belongs to $H^1(\Omega)$ by Proposition 6.1 of [35], and thus to $H^1(v)$.

We first show that A_0 is in fact a generator proceeding as in, for example, Lemma 2.1 of [42] or [44].

Lemma 2.1.1. *The operator A_0 defined in (2.8) is densely defined and maximally dissipative. Hence, A_0 is closed and generates a contractive C_0 -semigroup $S_0(\cdot)$ on X_0 . Moreover, if $\sigma = 0$ the operator A_0 generates a unitary group.*

Proof. 1) First, we show that A_0 is maximally dissipative. Let $(E, H) \in D(A_0)$. Integration by parts yields

$$\operatorname{Re}(A_0(E, H)|(E, H))_{X_0} = - \int_\Omega \sigma E \cdot E \, dx,$$

using that $\operatorname{tr}_t E = 0$. The right-hand side is non-positive since $\sigma \geq 0$ by assumption. Hence, A_0 is dissipative.

Next, we establish the maximality of A_0 . Let $\lambda > 0$. To show the surjectivity of $\lambda - A_0$, we take $(F, G) \in X_0$ and show that there exists some $(E, H) \in D(A_0)$ such that

$$\lambda(E, H) - A_0(E, H) = (F, G),$$

or equivalently,

$$\lambda\varepsilon E - \operatorname{curl} H + \sigma E = \varepsilon F, \quad \lambda\mu H + \operatorname{curl} E = \mu G. \quad (2.9)$$

We solve the second equation for H

$$H = \lambda^{-1}G - \lambda^{-1}\mu^{-1}\operatorname{curl} E$$

and insert it into the first, which leads to second order formulation for E

$$\lambda\varepsilon E + \lambda^{-1}\operatorname{curl}(\mu^{-1}\operatorname{curl} E) + \sigma E = \varepsilon F + \lambda^{-1}\operatorname{curl} G \quad (2.10)$$

or in its weak formulation

$$a(E, \Phi) = \int_{\Omega} (\varepsilon F \cdot \Phi + \lambda^{-1} G \cdot \operatorname{curl} \Phi) \, dx, \quad \forall \Phi \in V,$$

for the bounded bilinear form

$$a(u, v) := \int_{\Omega} ((\lambda \varepsilon + \sigma) u \cdot v + \lambda^{-1} \mu^{-1} \operatorname{curl} u \cdot \operatorname{curl} v) \, dx$$

on the space $V := \{v \in H_{t0}(\operatorname{curl}) \mid \varepsilon v|_v \in H^\Gamma(\operatorname{div} 0, v)\}$. Moreover, the form a is coercive in V since

$$a(u, u) \geq \int_{\Omega} (\lambda \eta |u|^2 + (\lambda \|\mu\|_\infty)^{-1} |\operatorname{curl} u|^2) \, dx \cong \|u\|_V^2.$$

The Lax-Milgram lemma then provides a unique $E \in V$ solving (2.9). We set $H := \lambda^{-1} G - \lambda^{-1} \mu^{-1} \operatorname{curl} E$ to solve the second part of (2.9). To see that $H \in H(\operatorname{curl})$, we test the weak formulation with $\Phi \in C_c^\infty(\Omega) \subset V$, which yields

$$\int_{\Omega} \lambda^{-1} (\mu^{-1} \operatorname{curl} E - G) \cdot \operatorname{curl} \Phi \, dx = \int_{\Omega} (\varepsilon F - (\lambda \varepsilon + \sigma) E) \cdot \Phi \, dx.$$

The left term in the parentheses is $-H$, and we get

$$\int_{\Omega} H \cdot \operatorname{curl} \Phi \, dx = - \int_{\Omega} (\varepsilon F - (\lambda \varepsilon + \sigma) E) \cdot \Phi \, dx,$$

i.e., $\operatorname{curl} H = -\varepsilon F + (\lambda \varepsilon + \sigma) E$ in the sense of distributions as required. Since the right-hand side belongs to $L^2(\Omega)$, it follows that $\operatorname{curl} H \in L^2(\Omega)$ and thus $H \in H(\operatorname{curl})$. Since $\mu G \in H_{n0}^\Sigma(\operatorname{div} 0)$ by assumption and $\operatorname{curl} E \in H_{n0}^\Sigma(\operatorname{div} 0)$ by Lemma A.1.3 a), the pair (E, H) indeed lies in $D(A_0)$. Therefore, A_0 is maximally dissipative.

2) By Theorem 4.6 in [48] the domain $D(A_0)$ is dense in X_0 . Furthermore, the Lumer-Phillips theorem, see for example Theorem 1.4.3 in [48], shows that the operator A_0 generates a C_0 -semigroup of contractions.

3) For $\sigma = 0$ the dissipativity calculation leads to

$$(A_0(E, H) | (E, H))_{X_0} = 0 \quad \text{for all } (E, H) \in D(A_0),$$

i.e., the operator A_0 is skew-symmetric and in particular also $-A_0$ is dissipative. Repeating the above argument for $-A_0$, we obtain the maximal dissipativity of $-A_0$, which therefore also generates a contractive semigroup. By the proof of Stone's theorem, see Theorem II.3.24 in [22], it follows that S_0 is a unitary group for $\sigma = 0$. $\square$

Given $(E_0, H_0) \in D(A_0)$, we thus obtain a unique solution of (2.1)–(2.4) satisfying $\operatorname{div}(\varepsilon E(t)) = 0$ on $\Omega \setminus \bar{\omega}$ for $t \geq 0$ and

$$(E, H) \in C(\mathbb{R}_{\geq 0}, D(A_0)) \cap C^1(\mathbb{R}_{\geq 0}, X_0). \quad (2.11)$$

We assume that (E_0, H_0) belongs to $D(A_0)$ except for the proofs of Theorems 2.2.8 and 2.3.1, where we pass to general $(E_0, H_0) \in X_0$. Also in this case $(E(t), H(t)) = S_0(t)(E_0, H_0)$ is called a solution of the system.

Remark 2.1.2. The proof above can be easily modified to show that on $X := L_\varepsilon^2(\Omega) \times L_\mu^2(\Omega)$ the extension A of A_0 with domain $D(A) := H_{t0}(\operatorname{curl}) \times H(\operatorname{curl})$ generates a contraction semigroup $S(\cdot)$ which leaves X_0 invariant and coincides there with $S_0(\cdot)$ as it solves the same Cauchy problem. Therefore, the divergence condition on v for εE_0 and the divergence and boundary conditions for μH_0 are also invariant under the evolution of the system within X whenever $(E_0, H_0) \in D(A_0) \subset D(A)$. Moreover, as we see in Lemma 2.3.7, the kernel of A is orthogonal to X_0 in X so that our restrictions on charges and boundary conditions in (2.7) exclude the stationary fields which obstruct exponential stability.

Let us discuss the choice of X_0 in (2.7) in more detail. For this we consider

$$\tilde{X} = H(\operatorname{div}_\varepsilon 0, v) \times H_{n0}(\operatorname{div}_\mu 0)$$

and the operator

$$D(\tilde{A}) = (H_{t0}(\operatorname{curl}) \times H(\operatorname{curl})) \cap \tilde{X}, \quad \tilde{A} = \begin{pmatrix} -\varepsilon^{-1}\sigma & \varepsilon^{-1}\operatorname{curl} \\ -\mu^{-1}\operatorname{curl} & 0 \end{pmatrix},$$

on $\tilde{X}$. Conceptually, by assuming that $E \in H(\operatorname{div}_\varepsilon 0, v)$ we exclude charges in the interior of Ω away from the support of the conductivity σ . Furthermore, by using $H^\Gamma(\operatorname{div}_\varepsilon 0, v)$ in the definition of X_0 instead we assume additionally that there are no charges on the connected components of the support of the conductivity. This for examples occurs in a capacitor. These charges generate stationary electric fields that do not decay in time. The following example illustrates this.

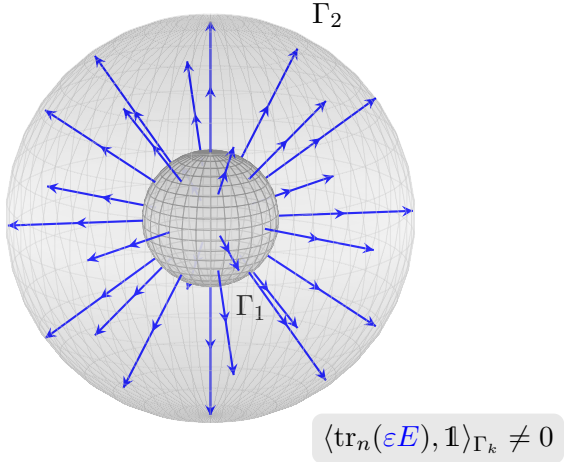


Figure 2.1.: A spherical shell serves as example for v as in Example 2.1.3.

Thus, we have constructed a non-zero stationary field, which would contradict an exponential decay result for disconnected ω .

The restrictions of $H_{n0}^\Sigma(\operatorname{div} 0)$ on the other hand exclude certain magnetic fields. By Ampère's law in (2.1) an electric current induces a magnetic field. When such currents pass through the 'holes' of Ω , one can obtain stationary, non-zero magnetic fields, satisfying the boundary condition on $\partial\Omega$. By Faraday's law, these fields cause currents in conductive loops along the boundaries $\partial\Sigma_i$ of the cuts of Ω . In the following example, we illustrate this by considering a torus with an external current running through the middle.

Example 2.1.4. Assume that μ is constant and consider a torus Ω lying in the x_1x_2 -plane centered at the origin. Take a current I running upwards along the x_3 -axis. By the *Biot-Savart law* this creates a stationary magnetic field, which is proportional to

$$H_0(x) = \frac{1}{x_1^2 + x_2^2} \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix}.$$

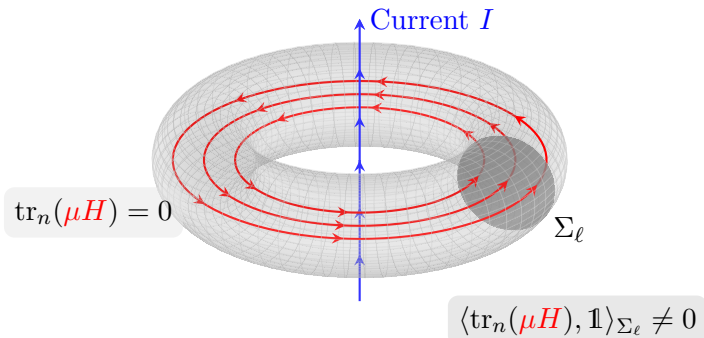


Figure 2.2.: Torus as in Example 2.1.4.

Example 2.1.3. Assume that ω consists of two connected components ω_1 and ω_2 . Let us call the interfaces to v $\Gamma_1 = \partial\omega_1 \cap \Omega$ and $\Gamma_2 = \partial\omega_2 \cap \Omega$. Let ϕ solve

$$\begin{aligned} \operatorname{div}(\varepsilon \nabla \phi) &= 0 \text{ on } v, \\ \phi|_{\Gamma_1} &= 1, \quad \phi|_{\Gamma_2} = 0. \end{aligned}$$

Note that ϕ cannot be constant and hence $\nabla \phi$ does not vanish. However, $\nabla \phi$ belongs to $\mathbb{H}_2^\varepsilon = H_{t0}(\operatorname{curl}, v) \cap H(\operatorname{div}_\varepsilon 0)$ by Lemma A.1.1. We can extend $\nabla \phi$ by 0 to $H_{t0}(\operatorname{curl}, \Omega)$. Let us call this extension E_0 .

Then $0 \neq (E_0, 0) \in \ker(\tilde{A})$. Therefore, $(E, H) = (E_0, 0)$ is a constant solution to the Cauchy problem associated with $\tilde{A}$.

It satisfies $\operatorname{curl} H_0 = 0$ and $\operatorname{div}(\mu H_0) = 0$ on Ω as well as the boundary condition $\operatorname{tr}_n(\mu H_0) = 0$ on $\partial\Omega$, i.e., $H_0 \in \mathbb{H}_1^\mu$.

Thus, $0 \neq (0, H_0) \in \ker(\tilde{A})$ and $(E, H) = (0, H_0)$ is a constant, non-zero solution to the associated Cauchy problem. We have constructed a non-zero stationary magnetic field, highlighting that the space $H_{n0}^\Sigma(\operatorname{div} 0)$ in the definition of X_0 is necessary to achieve exponential stability.

ENERGIES

Our arguments heavily use estimates of the ‘energies’

$$\begin{aligned} e_0(t) &:= \int_{\Omega} (\varepsilon E(t) \cdot E(t) + \mu H(t) \cdot H(t)) \, dx = \|(E(t), H(t))\|_{X_0}^2, \\ e_1(t) &:= \int_{\Omega} (\varepsilon \partial_t E(t) \cdot \partial_t E(t) + \mu \partial_t H(t) \cdot \partial_t H(t)) \, dx = \|(\partial_t E(t), \partial_t H(t))\|_{X_0}^2, \end{aligned} \quad (2.12)$$

and in particular the following energy equality with dissipation related to σ .

Lemma 2.1.5. *Let (H_ℓ) hold and (E, H) as in (2.11) solve (2.1). We then obtain*

$$\begin{aligned} e_0(s) - e_0(t) &= 2 \int_s^t \int_{\Omega} |\sigma^{1/2} E|^2 \, dx \, d\tau, \\ e_1(s) - e_1(t) &= 2 \int_s^t \int_{\Omega} |\sigma^{1/2} \partial_t E|^2 \, dx \, d\tau, \quad t \geq s \geq 0. \end{aligned}$$

Proof. We only consider e_1 as the estimate for e_0 is shown in a similar, but simpler way. After regularizing (E_0, H_0) in $D(A_0^2)$, we pass to the time differentiated version of the Maxwell system (2.1). Here the energy inequality is shown in a standard way.

So, take fields $(E_0^n, H_0^n) \in D(A_0^2)$ converging to (E_0, H_0) in $D(A_0)$ with respect to the graph norm. These initial values yield solutions

$$(E^n, H^n) \in C^2(\mathbb{R}_{\geq 0}, X_0) \cap C^1(\mathbb{R}_{\geq 0}, D(A_0)) \cap C(\mathbb{R}_{\geq 0}, D(A_0^2))$$

of the equations

$$\partial_t^2 E^n = \varepsilon^{-1} \operatorname{curl} \partial_t H^n - \varepsilon^{-1} \sigma \partial_t E^n, \quad \partial_t^2 H^n = -\mu^{-1} \operatorname{curl} \partial_t E^n.$$

Then $(E^n, H^n) = S_0(\cdot)(E_0^n, H_0^n)$ tends to (E, H) in the space $C_b(\mathbb{R}_{\geq 0}, D(A_0))$ of bounded functions, and $\partial_t(E^n, H^n) = S_0(\cdot)A_0(E_0^n, H_0^n)$ to $\partial_t(E, H)$ in $C_b(\mathbb{R}_{\geq 0}, X_0)$.

Let $e_1^n = \|(\partial_t E^n, \partial_t H^n)\|_{X_0}^2$. The above system and integration by parts lead to

$$\begin{aligned} \partial_t e_1^n &= 2 \int_{\Omega} (\partial_t E^n \cdot \varepsilon \partial_t^2 E^n + \partial_t H^n \cdot \mu \partial_t^2 H^n) \, dx \\ &= 2 \int_{\Omega} (\partial_t E^n \cdot (\operatorname{curl} \partial_t H^n - \sigma \partial_t E^n) - \partial_t H^n \cdot \operatorname{curl} \partial_t E^n) \, dx \\ &= -2 \int_{\Omega} |\sigma^{1/2} \partial_t E^n|^2 \, dx, \end{aligned}$$

since $\operatorname{tr}_t \partial_t E^n = 0$. Integrating in time, we derive

$$e_1^n(s) - e_1^n(t) = 2 \int_s^t \int_{\Omega} |\sigma^{1/2} \partial_t E^n|^2 \, dx \, ds$$

for $t \geq s \geq 0$. The result follows in the limit $n \rightarrow \infty$. □

HELMHOLTZ DECOMPOSITION

Our arguments rely on a splitting of the electric field $E \in C^1(\mathbb{R}_{\geq 0}, L^2(\Omega)) \cap C(\mathbb{R}_{\geq 0}, H_{t0}(\text{curl}))$ into a div_ε -free and a curl-free part. The following construction is inspired by Chapter 5 in [49], where constant ε and connected $\partial\Omega$ and ω were treated using $p \in H_0^1(\Omega)$. We consider the elliptic problem

$$\begin{aligned} \text{div}(\varepsilon \nabla p) &= \text{div}(\varepsilon E) \quad \text{on } \Omega, \quad p = 0 \text{ on } \partial\Omega_0, \\ p &\text{ constant on } \partial\Omega_k \quad \text{and} \quad \langle \text{tr}_n \varepsilon(\nabla p - E), \mathbb{1} \rangle_{H^{-\frac{1}{2}}(\partial\Omega_k)} = 0, \quad \forall k \in \{1, \dots, K\}. \end{aligned} \quad (2.13)$$

For the weak formulation of the problem, we define the Hilbert space

$$\mathcal{H}_c := \{\varphi \in H^1(\Omega) \mid \varphi = 0 \text{ on } \partial\Omega_0, \varphi \text{ constant on } \partial\Omega_k, \forall k \in \{1, \dots, K\}\}$$

endowed with the norm of $H^1(\Omega)$. We introduce the form

$$B[u, v] := \int_{\Omega} \nabla u \cdot \varepsilon \nabla v \, dx \quad \text{on } \mathcal{H}_c,$$

and set

$$\ell(v) = \ell_E(v) := \int_{\Omega} \varepsilon E \cdot \nabla v \, dx.$$

Then, the weak formulation of (2.13) reads as

$$B[p, \varphi] = \ell(\varphi) \quad \text{for all } \varphi \in \mathcal{H}_c. \quad (2.14)$$

Clearly, B and ℓ are bounded on $\mathcal{H}_c$, and B is coercive by Theorem 13.6.9 in [64]. The Lax–Milgram lemma then yields a unique solution $p = p(t) \in \mathcal{H}_c$ of (2.14) with

$$\|p\|_{H^1(\Omega)} \lesssim \|\ell\|_{\mathcal{H}_c^*} = \sup_{\|\varphi\|_{\mathcal{H}_c}=1} |\ell(\varphi)| \lesssim \|\varepsilon E\|_{L^2(\Omega)}. \quad (2.15)$$

We thus have bounded linear maps $L^2(\Omega) \rightarrow \mathcal{H}_c^*$; $E \mapsto \ell_E$, and $L^2(\Omega) \rightarrow \mathcal{H}_c$; $E \mapsto p$. Because of (2.11), we can continuously differentiate $t \mapsto \ell_{E(t)}$ in $\mathcal{H}_c^*$ and thus also p in $\mathcal{H}_c$. Therefore, p belongs to $C^1(\mathbb{R}_{\geq 0}, \mathcal{H}_c)$.

Note that $\varepsilon E - \varepsilon \nabla p$ belongs to $H(\text{div})$ with $\text{div}(\varepsilon E - \varepsilon \nabla p) = 0$, since (2.14) holds in particular for $H_0^1(\Omega) \subseteq \mathcal{H}_c$. Hence, the last boundary condition in (2.13) is well-defined, and it follows by inserting $\varphi_k \in \mathcal{H}_c$ with $\varphi_k|_{\partial\Omega_j} = \delta_{kj}$ for $k \neq 0$ into (2.14) via

$$0 = \int_{\Omega} (\varepsilon E - \varepsilon \nabla p) \cdot \nabla \varphi_k \, dx = \langle \text{tr}_n(\varepsilon E - \varepsilon \nabla p), \mathbb{1} \rangle_{H^{-\frac{1}{2}}(\partial\Omega_k)}.$$

Finally, the PDE in (2.13) is understood in $\mathcal{H}_c^*$.

To pass to charge-free fields, we define

$$V := E - \nabla p \in C^1(\mathbb{R}_{\geq 0}, L^2(\Omega)) \cap C(\mathbb{R}_{\geq 0}, H_{t0}(\text{curl})) \quad (2.16)$$

using Lemma A.1.1 and (2.11). By the properties of p , this function satisfies

$$\text{curl } V = \text{curl } E, \quad \varepsilon V \in H^{\partial\Omega}(\text{div } 0). \quad (2.17)$$

For the last assertion, the condition $\langle \text{tr}_n \varepsilon V, \mathbb{1} \rangle_{H^{-\frac{1}{2}}(\partial\Omega_k)} = 0$ is clear by (2.13) for $k > 0$. For $k = 0$ it then follows from $\text{div}(\varepsilon V) = 0$ which yields $\langle \text{tr}_n \varepsilon V, \mathbb{1} \rangle_{H^{-\frac{1}{2}}(\partial\Omega)} = 0$.

In other main arguments we use the time derivatives of p since they can be estimated solely by dissipation terms, in contrast to p in (2.15). This fact corresponds to the behavior of the charges $\text{div}(\varepsilon E)$ in (1.1), where the time derivative is better suited for the study of the long-term behavior, too.

Lemma 2.1.6. *Let (E, H) satisfy (2.11) and solve (2.1). Then the solution p of (2.13) belongs to $C^2(\mathbb{R}_{\geq 0}, \mathcal{H}_c)$ and satisfies*

$$\|\partial_t p\|_{H^1(\Omega)} \lesssim \|\partial_t \nabla p\|_{L^2(\Omega)} \lesssim \|\sigma E\|_{L^2(\omega)} \quad \text{and} \quad \|\partial_t^2 p\|_{H^1} \lesssim \|\partial_t^2 \nabla p\|_{L^2(\Omega)} \lesssim \|\sigma \partial_t E\|_{L^2(\omega)}.$$

Moreover, $\varepsilon \partial_t V$ is contained in $H^{\partial\Omega}(\text{div } 0)$.

Proof. Equation (2.1) and Lemma A.1.3 b) yield

$$\partial_t \ell_{E(t)}(\varphi) = \int_{\Omega} \varepsilon \partial_t E \cdot \nabla \varphi \, dx = - \int_{\Omega} \sigma E \cdot \nabla \varphi \, dx \quad (2.18)$$

for $\varphi \in \mathcal{H}_c$. Estimate (2.15) then implies the first assertion, using also the Poincaré inequality in Theorem 13.6.9 in [64]. The last claim now follows from (2.17). Due to (2.18), we can differentiate in time once more and derive the remaining part. $\square$

2.2. OBSERVABILITY ESTIMATE

Above we have split E into the charge-free part V and the curl-free part ∇p . To obtain observability results, we decompose (V, H) further into fields (V_h, W_h) and (V_i, W_i) solving a homogeneous and an inhomogeneous system, respectively. Here we combine and extend the approaches from [44] and [49]. In Section 3 of [44] an observability estimate is shown for scalar coefficients and a homogeneous system with charge-free initial data. In its proof also a Morawetz multiplier is used. Note that for $\sigma \neq 0$ the solutions of the Maxwell system (2.1) will in general not stay charge-free. In particular the time integral in the Gauß law (1.1)

$$\text{div}(\varepsilon E(t)) = \text{div}(\varepsilon E_0) - \int_0^t \text{div}(\sigma E(\tau)) \, d\tau, \quad t \geq 0 \quad (2.19)$$

is a major problem for showing exponential decay.

By reducing the exponential decay result to the time derived solutions as in Chapter 5.2 of [49], see Lemma 2.3.3, it suffices to bound $\partial_t(\nabla p, \nabla q)$, $\partial_t(V_h, W_h)$ and $\partial_t(V_i, W_i)$ by suitable dissipation terms, see Lemma 2.1.6 and Proposition 2.2.2 as well as Lemma 2.3.5 below. The energy equality in Lemma 2.1.5 then allows us to control dissipation terms. We come back to (V_h, W_h) in Corollary 2.2.6.

We proceed by deriving the systems solved by (V_h, W_h) and (V_i, W_i) .

In the stability analysis it will be crucial that $\partial_t V_i(0)$ and $\partial_t W_i(0)$ vanish, see Lemma 2.3.5. To this aim, we choose suitable initial values for the inhomogeneous problem. By Lemma 2.1.6 the derivative $\varepsilon \partial_t V$ belongs to $H^{\partial\Omega}(\text{div } 0)$, which implies

$$\sigma E + \varepsilon \partial_t \nabla p = -\varepsilon \partial_t E + \text{curl } H + \varepsilon \partial_t \nabla p = -\varepsilon \partial_t V + \text{curl } H \in H^{\partial\Omega}(\text{div } 0) \quad (2.20)$$

since $\text{curl } H \in H^{\partial\Omega}(\text{div } 0)$ due to Lemma A.1.3 b) as well as (2.2) and (2.3). Lemma A.1.3 b) then provides a field $W_{i0} \in H(\text{curl}) \cap H_{n0}(\text{div}_\mu 0)$ which is orthogonal to $\ker(\text{curl}_H) = \mathbb{H}_1^\mu$ in L_μ^2 and satisfies

$$\text{curl } W_{i0} = \sigma E_0 + \varepsilon \partial_t \nabla p(0). \quad (2.21)$$

To derive the observability estimate for $\partial_t V_h$, we consider a second-order equation for our homogeneous problem using the generator

$$D(A_h) = (H_{i0}(\text{curl}) \times H(\text{curl})) \cap X_h, \quad A_h = \begin{pmatrix} 0 & \varepsilon^{-1} \text{curl} \\ -\mu^{-1} \text{curl} & 0 \end{pmatrix} \quad (2.22)$$

without conductivity acting in the charge-free subspace

$$X_h := \{(v, w) \in L_\varepsilon^2(\Omega) \times L_\mu^2(\Omega) \mid \varepsilon v \in H^{\partial\Omega}(\operatorname{div} 0), \mu w \in H_{n0}^\Sigma(\operatorname{div} 0)\}$$

of X_0 . The operator A_h maps into X_h by Lemma A.1.3. Its maximal dissipativity is a special case of that of A_0 in (2.8) for $\sigma = 0$ and $\omega = \emptyset$. By (2.7), (2.16), (2.17) and $W_{i0} \in H_{n0}(\operatorname{div}_\mu 0)$, the fields $(E_0 - \nabla p(0), H_0 - W_{i0})$ belong to $D(A_h)$.

We can approximate $(E_0 - \nabla p(0), H_0 - W_{i0})$ in $D(A_h)$ via $(V_{h0}^{(n)}, W_{h0}^{(n)}) \in D(A_h^2)$ by means of Yosida approximations. For these initial fields we can derive a second-order problem. First note that the homogeneous Maxwell system

$$\begin{aligned} \partial_t(\varepsilon V_h^{(n)}) &= \operatorname{curl} W_h^{(n)}, & \partial_t(\mu W_h^{(n)}) &= -\operatorname{curl} V_h^{(n)}, \\ \operatorname{div}(\varepsilon V_h^{(n)}) &= 0, & \operatorname{div}(\mu W_h^{(n)}) &= 0, \\ \operatorname{tr}_t V_h^{(n)} &= 0, & \operatorname{tr}_n(\mu W_h^{(n)}) &= 0, \\ V_h^{(n)}(0) &= V_{h0}^{(n)}, & W_h^{(n)}(0) &= W_{h0}^{(n)}, \end{aligned} \quad (2.23)$$

has the solution

$$(V_h^{(n)}, W_h^{(n)}) \in C^2(\mathbb{R}_{\geq 0}, X_h) \cap C^1(\mathbb{R}_{\geq 0}, D(A_h)) \cap C(\mathbb{R}_{\geq 0}, D(A_h^2)). \quad (2.24)$$

It converges to the solution (V_h, W_h) of (2.23) with initial data $(E_0 - \nabla p(0), H_0 - W_{i0})$ in the space $C_b^1(\mathbb{R}_{\geq 0}, X_h) \cap C_b(\mathbb{R}_{\geq 0}, D(A_h))$. We stress that $\varepsilon V_h^{(n)}$ is divergence-free. The system (2.23) leads to the second-order equation

$$\begin{aligned} \partial_t(\varepsilon \partial_t V_h^{(n)}) &= -\operatorname{curl}(\mu^{-1} \operatorname{curl} V_h^{(n)}), \\ \operatorname{tr}_t V_h^{(n)} &= 0, \quad \operatorname{div}(\varepsilon V_h^{(n)}) = 0, \\ V_h^{(n)}(0) &= V_{h0}^{(n)}, \quad \partial_t V_h^{(n)}(0) = \varepsilon^{-1} \operatorname{curl} W_{h0}^{(n)}. \end{aligned} \quad (2.25)$$

Corresponding to (2.24), in (2.25) we assume that

$$\begin{aligned} V_h^{(n)} &\in C^2(\mathbb{R}_{\geq 0}, H^{\partial\Omega}(\operatorname{div}_\varepsilon 0)) \cap C^1(\mathbb{R}_{\geq 0}, H_{t0}(\operatorname{curl})), \\ \mu^{-1} \operatorname{curl} V_h^{(n)} &\in C(\mathbb{R}_{\geq 0}, H(\operatorname{curl})). \end{aligned} \quad (2.26)$$

To account for the missing terms in (2.23), we consider the inhomogeneous system

$$\begin{aligned} \partial_t(\varepsilon V_i) &= \operatorname{curl} W_i - \sigma E - \varepsilon \partial_t \nabla p, \\ \partial_t(\mu W_i) &= -\operatorname{curl} V_i, \\ \operatorname{div}(\mu W_i) &= 0, \\ \operatorname{tr}_t V_i &= 0, \quad \operatorname{tr}_n(\mu W_i) = 0, \\ V_i(0) &= 0, \quad W_i(0) = W_{i0}. \end{aligned} \quad (2.27)$$

Observe that $(-\varepsilon^{-1} \sigma E - \partial_t \nabla p, 0)$ belongs to $C^1(\mathbb{R}_{\geq 0}, X_h)$ due to (2.11), (2.20) and Lemma 2.1.6 and that $(0, W_{i0})$ is contained in $D(A_h)$. Hence, system (2.27) has a unique solution

$$(V_i, W_i) \in C^1(\mathbb{R}_{\geq 0}, X_h) \cap C(\mathbb{R}_{\geq 0}, D(A_h)). \quad (2.28)$$

For later use we record that the definition of W_{i0} in (2.21) yields

$$\begin{aligned} \partial_t(\varepsilon V_i)(0) &= \operatorname{curl} W_{i0} - \sigma E_0 - \varepsilon \partial_t \nabla p(0) = 0, \\ \partial_t(\mu W_i)(0) &= -\operatorname{curl} V_i(0) = 0. \end{aligned} \quad (2.29)$$

Note that the fields $(V_h + V_i + \nabla p, W_h + W_i)$ solve the original system (2.1)–(2.4) with initial values

$$(E_0 - \nabla p(0) + \nabla p(0), H_0 - W_{i0} + W_{i0}) = (E_0, H_0),$$

and hence coincide with (E, H) by uniqueness. The convergence noted after (2.24) then leads to the limit

$$(V_h^{(n)} + V_i + \nabla p, W_h^{(n)} + W_i) \longrightarrow (V_h + V_i + \nabla p, W_h + W_i) = (E, H)$$

in $C_b^1(\mathbb{R}_{\geq 0}, X_0) \cap C_b(\mathbb{R}_{\geq 0}, D(A_0))$.

To simplify notation, we drop the superscripts in this section and write $(V_h^{(n)}, W_h^{(n)}) = (V_h, W_h)$, assuming that (V_h, W_h) satisfies (2.24). The final result will then follow by the approximation argument above, see Remark 2.2.7. We next state the ‘energy’ identity for V_h .

Lemma 2.2.1. *Let V_h satisfy (2.26) and solve (2.25). We then obtain*

$$\begin{aligned} & \int_{\Omega} (\varepsilon \partial_t V_h(t) \cdot \partial_t V_h(t) + \mu^{-1} \operatorname{curl} V_h(t) \cdot \operatorname{curl} V_h(t)) \, dx \\ &= \int_{\Omega} (\varepsilon \partial_t V_h(0) \cdot \partial_t V_h(0) + \mu^{-1} \operatorname{curl} V_h(0) \cdot \operatorname{curl} V_h(0)) \, dx, \quad t \geq 0. \end{aligned}$$

Proof. System (2.25) and integration by parts imply

$$\begin{aligned} 0 &= \int_0^t \int_{\Omega} (\partial_t^2 (\varepsilon V_h) + \operatorname{curl} \mu^{-1} \operatorname{curl} V_h) \cdot \partial_t V_h \, dx \, d\tau \\ &= \frac{1}{2} \int_0^t \partial_t \int_{\Omega} (\partial_t (\varepsilon V_h) \cdot \partial_t V_h + \mu^{-1} \operatorname{curl} V_h \cdot \operatorname{curl} V_h) \, dx \, d\tau \\ &= \frac{1}{2} \left[\int_{\Omega} \varepsilon \partial_t V_h(t) \cdot \partial_t V_h(t) + \mu^{-1} \operatorname{curl} V_h(t) \cdot \operatorname{curl} V_h(t) \, dx \right. \\ &\quad \left. - \int_{\Omega} \varepsilon \partial_t V_h(0) \cdot \partial_t V_h(0) + \mu^{-1} \operatorname{curl} V_h(0) \cdot \operatorname{curl} V_h(0) \, dx \right]. \quad \square \end{aligned}$$

The following observability estimate for $\partial_t V_h$ is the core step in our arguments.

Proposition 2.2.2. *Assume that (H_ℓ) and the non-trapping condition (2.6) hold. Let Then for any $a > 0$ there exists a time $T_O > 0$ such that for $T \geq T_O$ we have*

$$\int_{\Omega} (|\partial_t V_h(0)|^2 + |\operatorname{curl} V_h(0)|^2) \, dx \lesssim \int_0^T \int_{\mathcal{N}_a} |\partial_t V_h|^2 \, dx \, d\tau$$

for any V_h satisfying (2.26) and solving (2.25).

For the proof of Proposition 2.2.2 we need several auxiliary results. In the first one we use the Lax–Milgram lemma to construct a multiplier, cf. [44]. We endow the space $\mathcal{H} := H_{t0}(\operatorname{curl}) \cap H^{\partial\Omega}(\operatorname{div}_\varepsilon 0)$ with the norm $\|\cdot\|_{H(\operatorname{curl})}$. By Lemma A.1.3 the operator $\operatorname{curl}_E: \mathcal{H} \rightarrow H_{n0}^\Sigma(\operatorname{div} 0)$ is boundedly invertible.

Lemma 2.2.3. *Take a map $\tilde{\vartheta} \in L^\infty(\Omega)$ with $\tilde{\vartheta} = 1$ on $\mathcal{N}_{a/2}$ and $\operatorname{supp} \tilde{\vartheta} \subseteq \mathcal{N}_a$ for some $a > 0$ and $f \in L^2(\Omega)$. Then there is a unique solution $w \in \mathcal{H} \subseteq H^1(\Omega)$ of*

$$\forall \psi \in \mathcal{H}: \quad \int_{\Omega} \mu^{-1} \operatorname{curl} w \cdot \operatorname{curl} \psi \, dx = \int_{\Omega} \tilde{\vartheta} f \psi \, dx. \quad (2.30)$$

It satisfies

$$\|w\|_{H^1(\Omega)} \lesssim \|w\|_{\mathcal{H}} \lesssim \|\tilde{\vartheta} f\|_{L^2(\Omega)}. \quad (2.31)$$

Proof. Theorem A.2.9 (with $v = 0$, $u \in H_{t0}(\text{curl})$ and $\theta = 0$) yields

$$\|u\|_{H^1(\Omega)} \lesssim \|u\|_{L^2(\Omega)} + \|\text{curl } u\|_{L^2(\Omega)} + \|\text{div}(\varepsilon u)\|_{L^2(\Omega)}.$$

Since the divergence of εu vanishes for $u \in \mathcal{H}$ and the curl is invertible on $\mathcal{H}$ by Lemma A.1.3, we see that

$$\|u\|_{H^1(\Omega)} \lesssim \|\text{curl } u\|_{L^2(\Omega)} \quad \text{for } u \in \mathcal{H}.$$

Hence, the bilinear form on the left-hand side is coercive and bounded. The assertion then follows from the Lax–Milgram lemma, using that $L^2(\Omega) \hookrightarrow \mathcal{H}^*$. $\square$

For a time dependent $f \in C^1(\mathbb{R}_{\geq 0}, L^2(\Omega))$ we obtain a solution $w(t)$ of (2.30) for each $t \geq 0$. The regularity of f transfers to w as shown next.

Remark 2.2.4. Let $f \in C^1(\mathbb{R}_{\geq 0}, L^2(\Omega))$ in Lemma 2.2.3. We then obtain

- a) $w \in C^1(\mathbb{R}_{\geq 0}, \mathcal{H})$ and
- b) $\|w(t)\|_{H^1}^2 \lesssim \int \tilde{\vartheta} |f(t)|^2 dx$, $\|\partial_t w(t)\|_{H^1}^2 \lesssim \int \tilde{\vartheta} |\partial_t f(t)|^2 dx$.

Proof. We first check that w is C^1 . Its Lipschitz continuity follows from (2.31) and linearity via

$$\left\| \frac{1}{h} (w(t+h) - w(t)) \right\|_{H^1(\Omega)} \lesssim \left\| \frac{1}{h} (f(t+h) - f(t)) \right\|_{L^2(\Omega)}.$$

Hence, w is differentiable in $H^1(\Omega)$ for a.e. $t \geq 0$. We can now differentiate (2.30) in t a.e., obtaining

$$\int_{\Omega} \mu^{-1} \text{curl } \partial_t w \cdot \text{curl } \psi \, dx = \int_{\Omega} \tilde{\vartheta} \partial_t f \psi \, dx.$$

As above, we then infer the continuity of $\partial_t w$ from (2.31), which also yields the second estimate. The first one follows from (2.31) for w . $\square$

We now turn to the proof of the core result of this section.

Proof of Proposition 2.2.2. 1) Let $T > 0$, $\chi \in C^1(\overline{\Omega})$ and $m(x) = x - x_0$ for some $x_0 \in \mathbb{R}^3$. We multiply (2.25) with the Morawetz multiplier $\chi m \times \text{curl } V_h$ obtaining

$$\begin{aligned} 0 &= \int_0^T \int_{\Omega} (\partial_t(\varepsilon \partial_t V_h) + \text{curl } \mu^{-1} \text{curl } V_h) \cdot (\chi m \times \text{curl } V_h) \, dx \, d\tau \\ &= \int_0^T \int_{\Omega} \partial_t((\varepsilon \partial_t V_h) \cdot (\chi m \times \text{curl } V_h)) \, dx \, d\tau - \int_0^T \int_{\Omega} (\varepsilon \partial_t V_h) \cdot (\chi m \times \text{curl } \partial_t V_h) \, dx \, d\tau \\ &\quad + \int_0^T \int_{\Omega} \mu^{-1} \text{curl } V_h \cdot \text{curl}(\chi m \times \text{curl } V_h) \, dx \, d\tau \\ &\quad - \int_0^T \oint_{\partial\Omega} \mu^{-1} \text{curl } V_h \cdot (\nu \times (\chi m \times \text{curl } V_h)) \, d\zeta \, d\tau \\ &=: I_i + I_t + I_c + I_c^{\partial\Omega}. \end{aligned} \tag{2.32}$$

In the above four integrands we have to eliminate the second derivatives of V_h , which is easy for I_i . In the following we freely use standard formulas from vector analysis. Also integrating by parts and exploiting $\text{tr}_t \partial_t V_h = 0$, we compute

$$I_t = \int_0^T \int_{\Omega} \text{curl } \partial_t V_h \cdot (\chi m \times \varepsilon \partial_t V_h) \, dx \, d\tau = \int_0^T \int_{\Omega} \partial_t V_h \cdot \text{curl}(\chi m \times \varepsilon \partial_t V_h) \, dx \, d\tau.$$

Since $\text{div}(\varepsilon V_h) = 0$, the integrand can be rewritten as

$$\text{curl}(\chi m \times \varepsilon \partial_t V_h)$$

$$\begin{aligned}
&= (\varepsilon \partial_t V_h \cdot \nabla)(\chi m) - \chi(m \cdot \nabla)(\varepsilon \partial_t V_h) + \chi m \partial_t \operatorname{div}(\varepsilon V_h) - \varepsilon \partial_t V_h \operatorname{div}(\chi m) \\
&= (\varepsilon \partial_t V_h \cdot \nabla \chi)m + \chi \varepsilon \partial_t V_h - \chi(m \cdot \nabla)(\varepsilon \partial_t V_h) - (\varepsilon \partial_t V_h) \nabla \chi \cdot m - 3\chi \varepsilon \partial_t V_h.
\end{aligned}$$

Only the middle term involves second derivatives of V_h . In this summand of the integrand we can take out the gradient via

$$\chi(m \cdot \nabla)(\varepsilon \partial_t V_h) \cdot \partial_t V_h = \frac{1}{2} \chi(m \cdot \nabla)(\varepsilon \partial_t V_h \cdot \partial_t V_h) + \frac{1}{2} \chi((m \cdot \nabla) \varepsilon) \partial_t V_h \cdot \partial_t V_h$$

by means of the product rule. The remaining problematic part of I_t is now integrated by parts resulting in

$$\begin{aligned}
&- \int_0^T \int_{\Omega} \frac{1}{2} \chi(m \cdot \nabla)(\varepsilon \partial_t V_h \cdot \partial_t V_h) \, dx \, d\tau \\
&= \int_0^T \int_{\Omega} \left(\frac{1}{2} (\nabla \chi \cdot m)(\varepsilon \partial_t V_h \cdot \partial_t V_h) + \frac{3}{2} \chi(\varepsilon \partial_t V_h \cdot \partial_t V_h) \right) \, dx \, d\tau \\
&\quad - \int_0^T \oint_{\partial\Omega} \frac{1}{2} \chi \nu \cdot m (\varepsilon \partial_t V_h \cdot \partial_t V_h) \, d\varsigma \, d\tau.
\end{aligned}$$

Recalling $\tilde{\varepsilon} = \varepsilon + (m \cdot \nabla) \varepsilon$ from (2.6), we conclude

$$\begin{aligned}
I_t &= \int_0^T \int_{\Omega} \left[-\frac{1}{2} \chi \tilde{\varepsilon} \partial_t V_h \cdot \partial_t V_h - \frac{1}{2} (\nabla \chi \cdot m)(\varepsilon \partial_t V_h \cdot \partial_t V_h) + \partial_t V_h \cdot (\varepsilon \partial_t V_h \cdot \nabla \chi) m \right] \, dx \, d\tau \\
&\quad - \int_0^T \oint_{\partial\Omega} \frac{1}{2} \chi \nu \cdot m (\varepsilon \partial_t V_h \cdot \partial_t V_h) \, d\varsigma \, d\tau. \tag{2.33}
\end{aligned}$$

We next treat I_c in a similar way, reformulating it as

$$\begin{aligned}
I_c &= \int_0^T \int_{\Omega} \mu^{-1} \operatorname{curl} V_h \cdot [\operatorname{div} \operatorname{curl}(V_h) \chi m - \operatorname{div}(\chi m) \operatorname{curl} V_h] \\
&\quad + (\operatorname{curl} V_h \cdot \nabla)(\chi m) - (\chi m \cdot \nabla) \operatorname{curl} V_h \, dx \, d\tau.
\end{aligned}$$

The last term is the only problematic one. As above, it is equal to

$$\begin{aligned}
&- \int_0^T \int_{\Omega} (\mu^{-1} \operatorname{curl} V_h) \cdot ((\chi m \cdot \nabla) \operatorname{curl} V_h) \, dx \, d\tau \\
&= \frac{1}{2} \int_0^T \int_{\Omega} \left[(\chi m \cdot \nabla \mu^{-1}) \operatorname{curl} V_h \cdot \operatorname{curl} V_h - \chi(m \cdot \nabla)(\mu^{-1} \operatorname{curl} V_h \cdot \operatorname{curl} V_h) \right] \, dx \, d\tau.
\end{aligned}$$

We integrate by parts the last term and obtain

$$\begin{aligned}
I_c &= \int_0^T \int_{\Omega} \mu^{-1} \operatorname{curl} V_h \cdot \left((\operatorname{curl} V_h \cdot \nabla \chi) m + \chi \operatorname{curl} V_h - \frac{1}{2} \operatorname{div}(\chi m) \operatorname{curl} V_h \right) \, dx \, d\tau \\
&\quad + \frac{1}{2} \int_0^T \int_{\Omega} ((\chi m \cdot \nabla) \mu^{-1}) \operatorname{curl} V_h \cdot \operatorname{curl} V_h \, dx \, d\tau \\
&\quad - \frac{1}{2} \int_0^T \oint_{\partial\Omega} \chi(m \cdot \nu)(\mu^{-1} \operatorname{curl} V_h \cdot \operatorname{curl} V_h) \, d\varsigma \, d\tau \\
&= \int_0^T \int_{\Omega} \left[-\frac{1}{2} \chi \tilde{\mu}_{-1} \operatorname{curl} V_h \cdot \operatorname{curl} V_h + \mu^{-1} \operatorname{curl} V_h \cdot ((\operatorname{curl} V_h \cdot \nabla \chi) m) \right. \\
&\quad \left. - \frac{1}{2} (m \cdot \nabla \chi)(\operatorname{curl} V_h \cdot \mu^{-1} \operatorname{curl} V_h) \right] \, dx \, d\tau \\
&\quad - \frac{1}{2} \int_0^T \oint_{\partial\Omega} \chi(m \cdot \nu)(\mu^{-1} \operatorname{curl} V_h \cdot \operatorname{curl} V_h) \, d\varsigma \, d\tau. \tag{2.34}
\end{aligned}$$

Note that (2.6) implies $\tilde{\mu}_{-1} := \mu^{-1} - m \cdot \nabla \mu^{-1} \geq \tilde{\eta} \mu^{-1}$.

Using Lemma A.1.6a), the last integral in (2.32) can be rewritten as

$$\begin{aligned}
I_c^{\partial\Omega} &= - \int_0^T \oint_{\partial\Omega} \mu^{-1} \operatorname{curl} V_h \cdot (\nu \times (\chi m \times \operatorname{curl} V_h)) \, d\varsigma \, d\tau \\
&= \int_0^T \oint_{\partial\Omega} \nu \cdot (\mu^{-1} \operatorname{curl} V_h \times (\chi m \times \operatorname{curl} V_h)) \, d\varsigma \, d\tau \\
&= \int_0^T \oint_{\partial\Omega} \nu \cdot (\chi m (\mu^{-1} \operatorname{curl} V_h \cdot \operatorname{curl} V_h) - \operatorname{curl} V_h (\mu^{-1} \operatorname{curl} V_h \cdot \chi m)) \, d\varsigma \, d\tau \\
&= \int_0^T \oint_{\partial\Omega} \chi (\nu \cdot m) (\mu^{-1} \operatorname{curl} V_h \cdot \operatorname{curl} V_h) \, d\varsigma \, d\tau.
\end{aligned} \tag{2.35}$$

Equations (2.32), (2.33), (2.34), and (2.35) then yield the core identity

$$\begin{aligned}
&\int_0^T \int_{\Omega} \frac{1}{2} \chi (\tilde{\varepsilon} \partial_t V_h \cdot \partial_t V_h + \tilde{\mu}^{-1} \operatorname{curl} V_h \cdot \operatorname{curl} V_h) \, dx \, d\tau \\
&= \left[\int_{\Omega} \varepsilon \partial_t V_h \cdot (\chi m \times \operatorname{curl} V_h) \, dx \right]_0^T \\
&\quad - \int_0^T \int_{\Omega} \frac{1}{2} (\nabla \chi \cdot m) (\varepsilon \partial_t V_h \cdot \partial_t V_h + \mu^{-1} \operatorname{curl} V_h \cdot \operatorname{curl} V_h) \, dx \, d\tau \\
&\quad + \int_0^T \int_{\Omega} ((m \cdot \partial_t V_h) (\partial_t \varepsilon V_h \cdot \nabla \chi) + (m \cdot \mu^{-1} \operatorname{curl} V_h) (\operatorname{curl} V_h \cdot \nabla \chi)) \, dx \, d\tau \\
&\quad + \int_0^T \oint_{\partial\Omega} \frac{1}{2} \chi (m \cdot \nu) (\mu^{-1} \operatorname{curl} V_h \cdot \operatorname{curl} V_h - \varepsilon \partial_t V_h \cdot \partial_t V_h) \, d\varsigma \, d\tau.
\end{aligned} \tag{2.36}$$

2) We first take $\chi = 1$. Then (2.36) and the non-trapping condition (2.6) imply

$$\begin{aligned}
&\tilde{\eta} \int_0^T \int_{\Omega} (|\partial_t V_h|^2 + |\operatorname{curl} V_h|^2) \, dx \, d\tau \\
&\lesssim \int_{\Omega} |\partial_t V_h(T)| |\operatorname{curl} V_h(T)| \, dx + \int_{\Omega} |\partial_t V_h(0)| |\operatorname{curl} V_h(0)| \, dx \\
&\quad + \int_0^T \oint_{\partial\Omega} (\nu \cdot m) (\mu^{-1} \operatorname{curl} V_h \cdot \operatorname{curl} V_h - \varepsilon \partial_t V_h \cdot \partial_t V_h) \, d\varsigma \, d\tau.
\end{aligned}$$

Next, for $\chi \in C^1(\bar{\Omega})$ with $\chi = 1$ on $\partial\Omega$ and support in $\mathcal{N}_{a/4}$, Equation (2.36) yields

$$\begin{aligned}
&\left| \int_0^T \oint_{\partial\Omega} (\nu \cdot m) (\mu^{-1} \operatorname{curl} V_h \cdot \operatorname{curl} V_h - \varepsilon \partial_t V_h \cdot \partial_t V_h) \, d\varsigma \, d\tau \right| \\
&\lesssim \int_{\Omega} |\partial_t V_h(T)| |\operatorname{curl} V_h(T)| \, dx + \int_{\Omega} |\partial_t V_h(0)| |\operatorname{curl} V_h(0)| \, dx \\
&\quad + \int_0^T \int_{\mathcal{N}_{a/4}} (|\partial_t V_h|^2 + |\operatorname{curl} V_h|^2) \, dx \, d\tau.
\end{aligned}$$

The last two estimates lead to

$$\begin{aligned}
&\int_0^T \int_{\Omega} (|\partial_t V_h|^2 + |\operatorname{curl} V_h|^2) \, dx \, d\tau \lesssim \int_0^T \int_{\mathcal{N}_{a/4}} (|\partial_t V_h|^2 + |\operatorname{curl} V_h|^2) \, dx \, d\tau \\
&\quad + \int_{\Omega} |\partial_t V_h(T)| |\operatorname{curl} V_h(T)| \, dx + \int_{\Omega} |\partial_t V_h(0)| |\operatorname{curl} V_h(0)| \, dx.
\end{aligned} \tag{2.37}$$

3) In light of Lemma 2.2.1 we only have to estimate the curl term on $\mathcal{N}_{a/4}$ to obtain the desired result, see (2.41). To this aim, take $\vartheta \in C^1(\bar{\Omega})$ with $\text{supp } \vartheta \subseteq \mathcal{N}_{a/2}$ and $\vartheta = 1$ on $\mathcal{N}_{a/4}$. Equation (2.25) and integration by parts yield

$$\begin{aligned} 0 &= \int_0^T \int_{\Omega} (\partial_t^2(\varepsilon V_h) + \text{curl}(\mu^{-1} \text{curl } V_h)) \cdot (\vartheta V_h) \, dx \, d\tau \\ &= - \int_0^T \int_{\Omega} \varepsilon \partial_t V_h \cdot \partial_t(\vartheta V_h) \, dx \, d\tau + \left[\int_{\Omega} (\varepsilon \partial_t V_h) \cdot \vartheta V_h \, dx \right]_0^T \\ &\quad + \int_0^T \int_{\Omega} \mu^{-1} \text{curl } V_h \cdot (\vartheta \text{curl } V_h + \nabla \vartheta \times V_h) \, dx \, d\tau \\ &\quad + \int_0^T \oint_{\partial\Omega} (\nu \times \mu^{-1} \text{curl } V_h) \cdot (\vartheta V_h) \, d\varsigma \, d\tau. \end{aligned} \quad (2.38)$$

The boundary integral vanishes by (2.25) since

$$(\nu \times \mu^{-1} \text{curl } V_h) \cdot (\vartheta V_h) = (\vartheta V_h \times \nu) \cdot (\mu^{-1} \text{curl } V_h) = 0$$

on $\partial\Omega$. As $\mu^{-1} \geq 1/\|\mu\|_{\infty}$, for any $\delta > 0$ we derive

$$\begin{aligned} \int_0^T \int_{\mathcal{N}_{a/4}} |\text{curl } V_h|^2 \, dx \, d\tau &\leq \int_0^T \int_{\mathcal{N}_{a/2}} \vartheta |\text{curl } V_h|^2 \, dx \, d\tau \\ &\lesssim \delta \int_0^T \int_{\mathcal{N}_{a/2}} |\text{curl } V_h|^2 \, dx \, d\tau + c_{\delta} \int_0^T \int_{\mathcal{N}_{a/2}} |V_h|^2 \, dx \, d\tau + \int_0^T \int_{\mathcal{N}_{a/2}} |\partial_t V_h|^2 \, dx \, d\tau \\ &\quad + \int_{\Omega} (|\partial_t V_h(T)|^2 + |V_h(T)|^2) \, dx + \int_{\Omega} (|\partial_t V_h(0)|^2 + |V_h(0)|^2) \, dx. \end{aligned}$$

We insert this inequality into (2.37), bound $\|V_h(\tau)\|_{L^2} \lesssim \|\text{curl } V_h(\tau)\|_{L^2}$ by Lemma A.1.3 and (2.26), and absorb the time integral with the curl term fixing a small $\delta > 0$. Using also Lemma 2.2.1, it follows

$$\begin{aligned} \int_0^T \int_{\Omega} (|\partial_t V_h|^2 + |\text{curl } V_h|^2) \, dx \, d\tau \\ \lesssim \int_0^T \int_{\mathcal{N}_{a/2}} (|V_h|^2 + |\partial_t V_h|^2) \, dx \, d\tau + \int_{\Omega} (|\partial_t V_h(0)|^2 + |\text{curl } V_h(0)|^2) \, dx. \end{aligned} \quad (2.39)$$

4) We have to get rid of the new term involving V_h . Taking $w \in H_{t0}(\text{curl})$ from Lemma 2.2.3 with $f = V_h$ as a multiplier, like in step 3) we derive

$$\begin{aligned} 0 &= \int_0^T \int_{\Omega} (\partial_t^2(\varepsilon V_h) + \text{curl}(\mu^{-1} \text{curl } V_h)) \cdot w \, dx \, d\tau \\ &= \int_0^T \int_{\Omega} ((\mu^{-1} \text{curl } V_h) \cdot \text{curl } w - \varepsilon \partial_t V_h \cdot \partial_t w) \, dx \, d\tau + \left[\int_{\Omega} \varepsilon \partial_t V_h \cdot w \, dx \right]_0^T. \end{aligned}$$

As V_h belongs to $\mathcal{H}$ by (2.26), Equation (2.30) with $V_h = \psi$ then yields

$$0 = \int_0^T \int_{\Omega} (\tilde{\vartheta} |V_h|^2 - \varepsilon \partial_t V_h \cdot \partial_t w) \, dx \, d\tau + \left[\int_{\Omega} \varepsilon \partial_t V_h \cdot w \, dx \right]_0^T.$$

The properties of $\tilde{\vartheta}$, Remark 2.2.4, Lemma 2.2.1 and Lemma A.1.3 a) then imply

$$\int_0^T \int_{\mathcal{N}_{a/2}} |V_h|^2 \, dx \, d\tau \lesssim \delta \int_0^T \int_{\Omega} |\partial_t V_h|^2 \, dx \, d\tau + c_{\delta} \int_0^T \int_{\Omega} \tilde{\vartheta} |\partial_t V_h|^2 \, dx \, d\tau$$

$$\begin{aligned}
& + \int_{\Omega} (|\partial_t V_h(T)|^2 + |V_h(T)|^2) dx + \int_{\Omega} (|\partial_t V_h(0)|^2 + |V_h(0)|^2) dx \\
& \lesssim \delta \int_0^T \int_{\Omega} |\partial_t V_h|^2 dx d\tau + c_{\delta} \int_0^T \int_{\Omega} \tilde{\vartheta} |\partial_t V_h|^2 dx d\tau \\
& + \int_{\Omega} (|\partial_t V_h(0)|^2 + |\operatorname{curl} V_h(0)|^2) dx.
\end{aligned}$$

We insert this inequality in (2.39). For a fixed small $\delta > 0$, the space-time term without localization can be absorbed by the left-hand side resulting in

$$\begin{aligned}
& \int_0^T \int_{\Omega} (|\partial_t V_h|^2 + |\operatorname{curl} V_h|^2) dx d\tau \\
& \lesssim \int_0^T \int_{\mathcal{N}_a} |\partial_t V_h|^2 dx d\tau + \int_{\Omega} (|\partial_t V_h(0)|^2 + |\operatorname{curl} V_h(0)|^2) dx.
\end{aligned} \tag{2.40}$$

5) In a final step we simplify the time integral on the left-hand side of (2.40) by means of Lemma 2.2.1, and obtain

$$\begin{aligned}
T \int_{\Omega} (|\partial_t V_h(0)|^2 + |\operatorname{curl} V_h(0)|^2) dx & \leq c_0 \int_{\Omega} (|\partial_t V_h(0)|^2 + |\operatorname{curl} V_h(0)|^2) dx \\
& + c_1 \int_0^T \int_{\mathcal{N}_a} |\partial_t V_h|^2 d\tau dx
\end{aligned} \tag{2.41}$$

for some constants $c_j > 0$. Taking $T > c_0$, we conclude

$$\int_{\Omega} (|\partial_t V_h(0)|^2 + |\operatorname{curl} V_h(0)|^2) dx \leq \frac{c_1}{T - c_0} \int_0^T \int_{\mathcal{N}_a} |\partial_t V_h|^2 d\tau dx. \quad \square$$

Remark 2.2.5. Instead of considering a second order formulation in (2.32) one can also derive an observability estimate for (V_h, W_h) instead using the Morawetz multiplier

$$0 = \int_0^T \int_{\Omega} (\partial_t(\varepsilon V_h) - \operatorname{curl} W_h) \cdot ((\mu W_h) \times \chi m) dx d\tau. \tag{2.42}$$

We use this approach when deriving an observability estimate for the quasilinear system, see (4.57). Unfortunately, this leads to a term $\int_0^T \int_{\mathcal{N}_a} |W_h|^2 dx d\tau$ on the right-hand side of our observability estimate, cf. Proposition 2.2.2, which we cannot control by the dissipation acting only on the electric field. For this reason, we pass to the second order formulation (2.25). One could also replace the fields (V_h, W_h) in (2.42) by $(\partial_t V_h, \partial_t W_h)$, which also solve (2.23). However, this way one also arrives at (2.32). We remove the extra derivatives below, see also Lemma 2.3.3.

It is however still possible to show an observability estimate for V_h instead of $\partial_t V_h$; that is, for the solutions to a homogeneous Maxwell system ($\sigma = 0$) with divergence free initial values. To do so, as in [44], we pass to an antiderivative in the proof.

Corollary 2.2.6. *Assume that (H_{ℓ}) and the non-trapping condition (2.6) hold. Let (V_h, W_h) satisfy (2.26) and solve (2.23). For each $a > 0$ there exists a time $T_O > 0$ such that for $T \geq T_O$ we have*

$$\int_{\Omega} (|V_h(0)|^2 + |W_h(0)|^2) dx \lesssim \int_0^T \int_{\mathcal{N}_a} |V_h|^2 dx d\tau.$$

Proof. Since $\mu W_h(0) \in H_{n0}^{\Sigma}(\operatorname{div} 0)$, Lemma A.1.3 a) yields a field u_0 in $H_{t0}(\operatorname{curl}) \cap H^{\partial\Omega}(\operatorname{div}_{\varepsilon} 0)$ with $\operatorname{curl} u_0 = -\mu W_h(0)$. We set

$$u(t) := u_0 + \int_0^t V_h(s) ds, \quad t \geq 0,$$

obtaining $u(0) = u_0$ and $\partial_t u(0) = V_h(0)$. Note that u also satisfies (2.26). We check that u solves (2.25) with different initial values. Indeed, the definition of u_0 and (2.23) yield

$$\begin{aligned}\partial_t^2(\varepsilon u(t)) &= \partial_t(\varepsilon V_h(t)) = \operatorname{curl} W_h(t) = \operatorname{curl} \left(\mu^{-1} \left(\mu W_h(0) + \int_0^t \partial_t(\mu W_h(s)) \, ds \right) \right) \\ &= -\operatorname{curl} \left(\mu^{-1} \operatorname{curl} \left(u_0 + \int_0^t V_h(s) \, ds \right) \right) = -\operatorname{curl} (\mu^{-1} \operatorname{curl} u(t)), \\ \operatorname{tr}_t u(t) &= \operatorname{tr}_t u_0 + \int_0^t \operatorname{tr}_t V_h(s) \, ds = 0, \\ \operatorname{div}(\varepsilon u(t)) &= \operatorname{div}(\varepsilon u_0) + \int_0^t \operatorname{div}(\varepsilon V_h(s)) \, ds = 0.\end{aligned}$$

The corollary now follows from Proposition 2.2.2 as $\partial_t u = V_h$ and $\operatorname{curl} u = -\mu W_h$. $\square$

Remark 2.2.7. In Lemma 2.2.1, Proposition 2.2.2 and Corollary 2.2.6 we have assumed that $(V_h(0), W_h(0)) = (V_{h0}^{(n)}, W_{h0}^{(n)})$ belongs to $D(A_h^2)$ and so the solutions $(V_h, W_h) = (V_h^{(n)}, W_h^{(n)})$ satisfy (2.24). By the approximation argument discussed after (2.27), the lemma and the proposition can be extended to $(V_h(0), W_h(0))$ in $D(A_h)$. Similarly, one derives the corollary for $(V_h(0), W_h(0)) \in X_h$.

We reformulate the above corollary as (exact) observability and controllability of (2.1)–(2.4) with $\operatorname{div}(\varepsilon E) = 0$ and $\sigma = 0$. For $E \in H^{\partial\Omega}(\operatorname{div}_\varepsilon 0)$ we have $p = 0$ and $(V_h, W_h) = (E, H)$, see (2.13), (2.21), (2.23), and (2.27). The equivalence of observability and controllability is shown in Theorem 11.2.1 in [64], noting that A_h is skew-adjoint in $X_h = H^{\partial\Omega}(\operatorname{div}_\varepsilon 0) \times H_{n0}^\Sigma(\operatorname{div}_\mu 0)$, see (2.22).

Theorem 2.2.8. *Assume that (H_ℓ) with $\sigma = 0$ and the non-trapping condition (2.6) hold. Let $a > 0$. Then there is a time $T_O > 0$ and a constant $c > 0$ such that for $(E_0, H_0) \in X_h$ we have*

$$\int_\Omega (|E_0|^2 + |H_0|^2) \, dx \leq c \int_0^{T_O} \int_{\mathcal{N}_a} |E|^2 \, dx \, d\tau,$$

where $(E, H) \in C(\mathbb{R}_{\geq 0}, X_h)$ solves (2.1)–(2.4) with $\sigma = 0$.

Moreover, for each $T \geq T_O$ and $(E_1, H_1) \in X_h$ we find a current $J \in L^2((0, T) \times \mathcal{N}_a)$ with values in $H^{\partial\Omega}(\operatorname{div}_\varepsilon 0)$ such that $(\hat{E}(T), \hat{H}(T)) = (E_1, H_1)$ for the solution $(\hat{E}, \hat{H}) \in C(\mathbb{R}_{\geq 0}, X_h)$ of (2.1)–(2.4) with σE replaced by J .

2.3. EXPONENTIAL DECAY

We now present the second main result of the chapter.

Theorem 2.3.1. *Let (H_ℓ) , (2.5) and (2.6) hold. Then there exist constants $M \geq 1$ and $\omega > 0$ such that*

$$\|(E(t), H(t))\|_{L^2(\Omega)} \leq M e^{-\omega t} \|(E_0, H_0)\|_{L^2(\Omega)}, \quad t \geq 0,$$

for the solution (E, H) of (2.1)–(2.4) with initial value $(E_0, H_0) \in X_0$.

Our proof relies on Lemma 2.3.3 which estimates the usual energy through time derivatives. Theorem 5.4 of [49] provides such an inequality for constant ε and μ and for connected $\partial\Omega$ and ω . The argument there is based on the splitting $E = V + \nabla p$. We could not extend this approach to our setting and proceed in a more direct way using the following estimate of E on v .

Lemma 2.3.2. *For $E \in H^\Gamma(\operatorname{div}_\varepsilon 0, v) \cap H(\operatorname{curl}, v)$, we have*

$$\|E\|_{L^2(v)} \lesssim \|\operatorname{curl} E\|_{L^2(v)} + \|\operatorname{tr}_t E\|_{H^{-1/2}(\partial v)}.$$

Proof. By Proposition IX.1.3 of [16] the field E belongs to $\varepsilon^{-1} \operatorname{curl} H^1(v) = H^\Gamma(\operatorname{div}_\varepsilon 0, v)$, see also Lemma A.1.3. In view the decomposition (A.2), we can thus compute its norm in $L_\varepsilon^2(v)$ by testing with $\varphi = \varepsilon^{-1} \operatorname{curl} \Phi$ for $\Phi \in H^1(v)$. Theorem 3.4.1 in [5] allows us to choose the vector potential Φ such that $\|\Phi\|_{H^1(v)} \lesssim \|\varepsilon \varphi\|_{L^2(v)}$. It follows

$$\|E\|_{L_\varepsilon^2(v)} = \sup_{\substack{\varphi \in H^\Gamma(\operatorname{div}_\varepsilon 0, v), \\ \|\varphi\|_{L_\varepsilon^2(v)}=1}} \int_v E \cdot \varepsilon \varphi \, dx \lesssim \sup_{\substack{\Phi \in H^1(v), \\ \|\Phi\|_{H^1(v)}=1}} \int_v E \cdot \operatorname{curl} \Phi \, dx. \quad (2.43)$$

Integration by parts then yields

$$\begin{aligned} \int_v E \cdot \operatorname{curl} \Phi \, dx &= \int_v \operatorname{curl} E \cdot \Phi \, dx + \langle \operatorname{tr}_t E, \Phi \rangle_{H^{-1/2}(\partial v)} \\ &\leq \|\operatorname{curl} E\|_{L^2(v)} \|\Phi\|_{L^2(v)} + \|\operatorname{tr}_t E\|_{H^{-1/2}(\partial v)} \|\Phi\|_{H^1(v)} \end{aligned}$$

We infer the assertion by inserting this inequality into (2.43). $\square$

The next proof relies on the assumption that σ either vanishes or is uniformly positive.

Lemma 2.3.3. *Let (H_ℓ) and (2.5) hold. Take a solution (E, H) of (2.1)–(2.4) as in (2.11). The energies e_0 and e_1 from (2.12) then satisfy*

$$e_0(T) \lesssim e_1(T), \quad T \geq 0.$$

Proof. We adopt some ideas from the proof of Theorem 5.4 in [49]. Condition (H_ℓ) , the Maxwell equations (2.1), (2.3) and integration by parts lead to

$$\begin{aligned} \sigma_0 \int_\omega |E(t)|^2 \, dx &\leq \int_\Omega \sigma E(t) \cdot E(t) = \int_\Omega (-\varepsilon \partial_t E(t) + \operatorname{curl} H(t)) \cdot E(t) \, dx \\ &= - \int_\Omega \varepsilon \partial_t E(t) \cdot E(t) - \int_\Omega H(t) \cdot \mu \partial_t H(t) \, dx \\ &\lesssim \sqrt{e_0(t)} \sqrt{e_1(t)}. \end{aligned} \quad (2.44)$$

Using also (2.11), Lemma A.1.3 b) and (2.1), we then estimate the magnetic field by

$$\begin{aligned} \|H(t)\|_{L^2(\Omega)}^2 &\lesssim \|\operatorname{curl} H(t)\|_{L^2(\Omega)}^2 \lesssim \|\varepsilon \partial_t E(t)\|_{L^2(\Omega)}^2 + \|\sigma E(t)\|_{L^2(\Omega)}^2 \\ &\lesssim e_1(t) + \sqrt{e_0(t) e_1(t)}. \end{aligned} \quad (2.45)$$

To control E on v , we recall

$$\|E(t)\|_{L^2(v)} \lesssim \|\operatorname{curl} E(t)\|_{L^2(v)} + \|\operatorname{tr}_t E(t)\|_{H^{-1/2}(\partial v)}$$

from Lemma 2.3.2. Since $E \in H(\operatorname{curl})$, Proposition 2.2.32 in [5], see Lemma A.1.2 shows that $\operatorname{tr}_{t, \partial v} E = \operatorname{tr}_{t, \partial \omega} E$ on ∂v . From the usual trace estimate and Equation (2.1), we thus deduce

$$\begin{aligned} \|E(t)\|_{L^2(v)} &\lesssim \|\operatorname{curl} E(t)\|_{L^2(v)} + \|\operatorname{tr}_t E(t)\|_{H^{-\frac{1}{2}}(\partial \omega)} \lesssim \|\mu \partial_t H(t)\|_{L^2(\Omega)} + \|E(t)\|_{H(\operatorname{curl}, \omega)} \\ &\lesssim \|\mu \partial_t H(t)\|_{L^2(\Omega)} + \|E(t)\|_{L^2(\omega)}. \end{aligned}$$

Combined with (2.44) and (2.45), we arrive at

$$e_0(t) \leq c_1 \sqrt{e_0(t)} \sqrt{e_1(t)} + c_2 e_1(t) \leq \frac{1}{2} e_0(t) + C e_1(t),$$

which yields the assertion. $\square$

The time derivative of magnetic field can be estimated by $\partial_t E$ up to a dissipation term, using our Helmholtz decomposition. We proceed similarly as in the proof of Lemma 5.2 of [49], see also Proposition 4.4 in [35]. Here and below the estimates depend on the end time, which fortunately does not cause problems in the main argument.

Lemma 2.3.4. *Let (H_ℓ) hold. Take a solution (E, H) of (2.1)–(2.4) as in (2.11). For $T > 0$ we then obtain*

$$\int_0^T \int_\Omega |\partial_t H|^2 dx d\tau \lesssim (1+T) \int_0^T \int_\Omega |\partial_t E|^2 dx d\tau + \int_0^T \int_\Omega \sigma E \cdot E dx d\tau.$$

Proof. Fix $\Phi \in C_c^\infty((0, T))$ such that $0 \leq \Phi \leq 1$ and $\Phi = 1$ on $[\frac{1}{3}T, \frac{2}{3}T]$. In (2.16) and (2.17), we have decomposed $E = V + \nabla p$ with $V \in H_{t0}(\text{curl}) \cap H^{\partial\Omega}(\text{div}_\varepsilon 0)$. Furthermore, (after regularization) E solves the second-order problem

$$\varepsilon \partial_t^2 E = -\text{curl}(\mu^{-1} \text{curl} E) - \sigma \partial_t E.$$

Starting from (2.1) and integrating by parts, we then compute

$$\begin{aligned} \int_0^T \int_\Omega \Phi^2 \mu \partial_t H \cdot \partial_t H dx d\tau &= \int_0^T \int_\Omega \Phi^2 \text{curl} E \cdot \mu^{-1} \text{curl} E dx d\tau \\ &= \int_0^T \int_\Omega \Phi^2 \text{curl} V \cdot \mu^{-1} \text{curl} E dx d\tau \\ &= \int_0^T \int_\Omega \Phi^2 V \cdot \text{curl}(\mu^{-1} \text{curl} E) dx d\tau \\ &= - \int_0^T \int_\Omega \Phi^2 V \cdot (\varepsilon \partial_t^2 E + \sigma \partial_t E) dx d\tau \\ &= \int_0^T \int_\Omega ((2\Phi \partial_t \Phi) V + \Phi^2 \partial_t V) \cdot \varepsilon \partial_t E dx d\tau \\ &\quad - \int_0^T \int_\Omega \Phi^2 V \cdot \sigma \partial_t E dx d\tau. \end{aligned}$$

Lemma A.1.3 a) yields $\|V\|_{L^2} \lesssim \|\text{curl} V\|_{L^2}$. We also insert $\partial_t V = \partial_t E - \partial_t \nabla p$. Hölder's inequality thus implies

$$\begin{aligned} \int_0^T \int_\Omega \Phi^2 \mu \partial_t H \cdot \partial_t H dx d\tau &\lesssim \int_0^T \left[\|\Phi'\|_\infty (\delta \|\Phi \mu^{-1/2} \text{curl} V\|_{L^2}^2 + \frac{1}{\delta} \|\partial_t E\|_{L^2}^2) \right. \\ &\quad \left. + (\|\partial_t \nabla p\|_{L^2}^2 + \|\partial_t E\|_{L^2}^2) + (\delta \|\Phi \mu^{-1/2} \text{curl} V\|_{L^2}^2 + \frac{1}{\delta} \|\sigma \partial_t E\|_{L^2}^2) \right] d\tau. \end{aligned}$$

As $\text{curl} V = -\mu \partial_t H$ by (2.1), we can absorb the curl terms by the left-hand side. Lemma 2.1.6 now leads to

$$\int_0^T \int_\Omega \Phi^2 \mu \partial_t H \cdot \partial_t H dx d\tau \lesssim \int_0^T \int_\Omega |\partial_t E|^2 dx d\tau + \int_0^T \int_\Omega |\sigma E|^2 dx d\tau.$$

By means of the energy equality from Lemma 2.1.5, we conclude

$$\begin{aligned} \int_0^T \int_\Omega \mu \partial_t H \cdot \partial_t H dx d\tau &\leq T e_1(0) = T \left(e_1(T) + 2 \int_0^T \sigma \partial_t E \cdot \partial_t E \right) dx d\tau \\ &\leq 3 \frac{T}{3} e_1\left(\frac{2}{3}T\right) + 2T \int_0^T \sigma \partial_t E \cdot \partial_t E dx d\tau \\ &\lesssim \int_{T/3}^{2T/3} \left| \mu^{\frac{1}{2}} \partial_t H \right|^2 dx d\tau + (1+T) \int_0^T |\partial_t E|^2 dx d\tau \\ &\lesssim (1+T) \int_0^T \int_\Omega |\partial_t E|^2 dx d\tau + \int_0^T \int_\Omega \sigma E \cdot E dx d\tau. \quad \square \end{aligned}$$

We next treat the inhomogeneous part of the fields. Duhamel's formula and our choice of the initial values lead to the following estimate.

Lemma 2.3.5. *Let (H_ℓ) hold and (V_i, W_i) as in (2.28) solve (2.27). We then obtain*

$$\int_0^T \int_\Omega (|\partial_t V_i|^2 + |\partial_t W_i|^2) dx d\tau \leq CT^2 \int_0^T \int_\Omega |\sigma \partial_t E|^2 dx d\tau.$$

Proof. We use the generator A_h of $T_h(\cdot)$ from (2.22). Equation (2.27) and the subsequent comments imply that

$$\begin{pmatrix} V_i(t) \\ W_i(t) \end{pmatrix} = T_h(t) \begin{pmatrix} 0 \\ W_{i0} \end{pmatrix} - \int_0^t T_h(s) \begin{pmatrix} \varepsilon^{-1} \sigma E(t-s) + \partial_t \nabla p(t-s) \\ 0 \end{pmatrix} ds.$$

Because of (2.11) and Lemma 2.1.6 we can differentiate this formula in L^2 with respect to t , where the resulting initial values vanish due to (2.29); i.e.,

$$\partial_t \begin{pmatrix} V_i(t) \\ W_i(t) \end{pmatrix} = - \int_0^t T_h(t-s) \begin{pmatrix} \varepsilon^{-1} \sigma \partial_t E(s) + \partial_t^2 \nabla p(s) \\ 0 \end{pmatrix} ds.$$

Lemma 2.1.6 and Hölder's inequality now yield

$$\begin{aligned} \int_0^T \left\| \begin{pmatrix} \partial_t V_i \\ \partial_t W_i \end{pmatrix} \right\|_{L^2}^2 d\tau &\lesssim \int_0^T \left(\int_0^t \|\sigma \partial_t E\|_{L^2} ds \right)^2 d\tau \lesssim \int_0^T t \int_0^t \|\sigma \partial_t E\|_{L^2}^2 ds d\tau \\ &\leq CT^2 \int_0^T \int_\Omega |\sigma \partial_t E|^2 dx d\tau. \end{aligned} \quad \square$$

The above estimates lead to the core inequality.

Proposition 2.3.6. *Assume that (H_ℓ) , (2.5) and (2.6) hold. Let (E, H) as in (2.11) solve (2.1)–(2.4), and $T_O > 0$ be given by Proposition 2.2.2. Then there exists a constant $\gamma \in [0, 1)$ such that*

$$e_0(T) + e_1(T) \leq \gamma(e_0(0) + e_1(0)), \quad T \geq \max\{T_O, 1\}.$$

Proof. Lemmas 2.3.3 and 2.1.5 imply

$$T(e_0(T) + e_1(T)) \lesssim T e_1(T) \lesssim \int_0^T \int_\Omega (|\partial_t E|^2 + |\partial_t H|^2) dx d\tau.$$

We can first eliminate H by means of Lemma 2.3.4 via

$$T(e_0(T) + e_1(T)) \lesssim (1+T) \int_0^T \int_\Omega |\partial_t E|^2 dx d\tau + \int_0^T \int_\Omega \sigma E \cdot E dx d\tau.$$

Here we will insert the decomposition $\partial_t E = \partial_t V_h + \partial_t V_i + \partial_t \nabla p$ established before Lemma 2.2.1. Next, the homogeneous system (2.23) for (V_h, W_h) , Lemma 2.2.1, and Remark 2.2.7 show that the integrals

$$\int_\Omega (\varepsilon \partial_t V_h \cdot \partial_t V_h + \mu \partial_t W_h \cdot \partial_t W_h) dx = \int_\Omega (\varepsilon \partial_t V_h \cdot \partial_t V_h + \mu^{-1} \operatorname{curl} V_h \cdot \operatorname{curl} V_h) dx$$

are constant in time. As $T \geq 1$, we thus obtain

$$T(e_0(T) + e_1(T)) \lesssim T \int_0^T \int_\Omega (|\partial_t V_h|^2 + |\partial_t V_i|^2 + |\partial_t \nabla p|^2) dx d\tau + \int_0^T \int_\Omega \sigma E \cdot E dx d\tau$$

$$\begin{aligned} &\lesssim T^2 \int_{\Omega} (|\partial_t V_h(0)|^2 + |\operatorname{curl} V_h(0)|^2) dx \\ &\quad + T \int_0^T \int_{\Omega} (|\partial_t \nabla p|^2 + |\partial_t V_i|^2) dx d\tau + \int_0^T \int_{\Omega} \sigma E \cdot E dx d\tau. \end{aligned}$$

The observability estimate of Proposition 2.2.2 and Remark 2.2.7 then yield

$$\begin{aligned} T(e_0(T) + e_1(T)) &\lesssim T^2 \int_0^T \int_{\mathcal{N}_a} \partial_t V_h \cdot \partial_t V_h dx d\tau + T \int_0^T \int_{\Omega} (|\partial_t \nabla p|^2 + |\partial_t V_i|^2) dx d\tau \\ &\quad + \int_0^T \int_{\Omega} \sigma E \cdot E dx d\tau \end{aligned}$$

for $T \geq T_O$. After replacing again $V_h = E - V_i - \nabla p$, condition (2.5) leads to

$$\begin{aligned} T(e_0(T) + e_1(T)) &\lesssim T^2 \int_0^T \int_{\Omega} \sigma \partial_t E(t) \cdot \partial_t E(t) dx d\tau + T^2 \int_0^T \int_{\Omega} (|\partial_t \nabla p|^2 + |\partial_t V_i|^2) dx d\tau \\ &\quad + \int_0^T \int_{\Omega} \sigma E \cdot E dx d\tau. \end{aligned}$$

Using Lemmas 2.3.5, 2.1.6 and 2.1.5, we finally deduce

$$\begin{aligned} T(e_0(T) + e_1(T)) &\lesssim T^4 \int_0^T \int_{\Omega} (\sigma E \cdot E + \sigma \partial_t E \cdot \partial_t E) dx d\tau \\ &\leq CT^4 (e_0(0) - e_0(T) + e_1(0) - e_1(T)). \end{aligned}$$

Setting $\gamma = \frac{CT^3}{1+CT^3} < 1$, we conclude

$$e_0(T) + e_1(T) \leq \gamma(e_0(0) + e_1(0)). \quad \square$$

Theorem 2.3.1 now follows by a simple argument.

Proof of Theorem 2.3.1. First let $(E_0, H_0) \in D(A_0)$. Iterating the estimate from Proposition 2.3.6, we obtain constants $\tilde{M} \geq 1$ and $\omega > 0$ such that

$$e_0(t) + e_1(t) \leq \tilde{M} e^{-\omega t/2} (e_0(0) + e_1(0))$$

for all $t \geq 0$. Since $e_0(t) = \|S_0(t)(E_0, H_0)\|_{X_0}^2$ and $e_1(t) = \|A_0 S_0(t)(E_0, H_0)\|_{X_0}^2$ by (2.12), we have shown that $S_0(\cdot)$ exponentially decays in $D(A_0)$ with the graph norm. This space is isomorphic to X_0 by $(I - A_0)^{-1}$, so that the assertion follows. $\square$

Finally, we remove the divergence constraints in Theorem 2.3.1 by projecting in $X = L_{\varepsilon}^2(\Omega) \times L_{\mu}^2(\Omega)$ onto $\ker(A)^{\perp} = \overline{\operatorname{ran}(A)}$ for the extension A of A_0 , see Remark 2.1.2. Here we proceed similar to [42]. Then the theorem will imply the exponential decay of the extended semigroup $S(\cdot)$ to the kernel $\ker(A)$.

Lemma 2.3.7. *Let (H_{ℓ}) and (2.5) hold. We then have*

$$\ker(A) = \{(E, H) \in H_{t0}(\operatorname{curl} 0) \times H(\operatorname{curl} 0) \mid E = 0 \text{ on } \omega\} = \operatorname{ran}(A)^{\perp}.$$

Moreover, the orthogonal projection P onto $\ker(A)$ commutes with $S(\cdot)$, $S(t)P = I$ for $t \geq 1$, and $\ker(A)^{\perp} = \operatorname{ran}(A)$ is contained in X_0 .

Proof. 1) Take $w = (E, H) \in \ker(A)$. This means that $\operatorname{curl} E = 0$ and $\operatorname{curl} H = \sigma E$. As $\operatorname{tr}_t E = 0$, integration by parts yields

$$0 = (Aw|w)_X = \int_{\Omega} (\operatorname{curl} H \cdot E - \sigma E \cdot E - \operatorname{curl} E \cdot H) dx = - \int_{\Omega} |\sigma^{\frac{1}{2}} E|^2 dx,$$

so that $\sigma E = 0$ and thus $E = 0$ on ω by (2.5). This shows ‘ $\subseteq$ ’ in the first asserted identity. The converse inclusion is clear because of $\operatorname{supp} \sigma = \bar{\omega}$.

2) Step 1) and integration by parts imply that the kernel $\ker(A)$ is orthogonal to the range $\text{ran}(A)$. To show $\ker(A) = \text{ran}(A)^\perp$, take $h = (f, g) \in X$ with

$$0 = (h|Aw)_X = \int_{\Omega} (f \cdot \text{curl } H - f \cdot \sigma E - g \cdot \text{curl } E) \, dx$$

for all $w = (E, H) \in D(A)$. Choosing $(0, H)$ and $(E, 0)$ with $E, H \in H_0^1(\Omega)$, we see that $\text{curl } f = 0$ and $\sigma f + \text{curl } g = 0$. If we insert $(0, H)$ with $H \in H^1(\Omega)$, it follows $\langle \text{tr}_t f, H \rangle_{H^{-1/2}(\partial\Omega)} = 0$ so that $h \in D(A)$. The formula in display with $h = w$ then implies $\int_{\omega} \sigma f \cdot f \, dx = 0$ which yields $f = 0$ on ω and $\text{curl } g = 0$; i.e., $h \in \ker(A)$ by step 1) as needed.

3) Hence, $\overline{\text{ran}(A)} = \ker(A)^\perp$ is the kernel of P , implying $PA = 0 = AP$ on $D(A)$, and thus $PS(t) = S(t)P = P$ for $t \geq 0$. Let $(f, g) = A(E, H)$ for some $(E, H) \in D(A)$. Then $g = -\mu^{-1} \text{curl } E$ belongs to $H_{n0}^\Sigma(\text{div}_\mu 0)$ by Lemma A.1.3, and $\varepsilon f|_v = (\text{curl } H)|_v$ to $H^\Gamma(\text{div } 0, v)$ because of (2.5) and Lemma A.1.3. As a result, $\overline{\text{ran}(A)}$ is contained in X_0 . $\square$

Corollary 2.3.8. *Let (H_ℓ) , (2.5), and (2.6) hold. Then there exist constants $M' \geq 1$ and $\omega > 0$ such that for $(E_0, H_0) \in X = L_\varepsilon^2(\Omega) \times L_\mu^2(\Omega)$ we have*

$$\|(S(t) - P)(E_0, H_0)\|_{L^2(\Omega)} \leq M' e^{-\omega t} \|(E_0, H_0)\|_{L^2(\Omega)}, \quad t \geq 0.$$

Proof. Lemma 2.3.7 yields $S(t) - P = S(t)(I - P) = S_0(t)(I - P)$ so that the result follows from Theorem 2.3.1. $\square$

APPROXIMATION OF THE MAXWELL INTERFACE PROBLEM WITH PERFECTLY CONDUCTING ELECTRICAL BOUNDARY CONDITIONS

In the previous chapter, we required additional regularity to justify several calculations when deriving the observability estimate, see (2.24). In the autonomous, linear case one can use here the density of the domains $D(A^n)$ in $D(A)$ of the generator A of the system. In this chapter we consider the Maxwell interface problem with non-autonomous coefficients,

$$\begin{aligned}
 \partial_t (\varepsilon(t, x) E(t, x)) &= \operatorname{curl} H(t, x) - \sigma(t, x) E(t, x) - f(t, x), & t \geq 0, x \in \Omega_{\pm}, \\
 \partial_t (\mu(t, x) H(t, x)) &= -\operatorname{curl} E(t, x) - g(t, x), \\
 E(t, x) \times \nu(x) &= h(t, x), & t \geq 0, x \in \partial\Omega, \\
 \llbracket E(t, x) \times \nu^\Gamma(x) \rrbracket &= \kappa(t, x), \quad \llbracket H(t, x) \times \nu^\Gamma(x) \rrbracket = \varkappa(t, x), & t \geq 0, x \in \Gamma, \\
 E(0, x) &= E_0(x), \quad H(0, x) = H_0(x), & x \in \Omega_{\pm},
 \end{aligned} \tag{3.1}$$

for which this approach is no longer viable. This occurs in particular for quasilinear problems considered in Chapter 4, where the coefficients $\varepsilon(E)$ and $\mu(H)$ depend on the fields. Furthermore, establishing well-posedness in this context, as done in [56], is significantly more complex.

In order to obtain solutions of higher regularity we will approximate the data and coefficients by more regular ones. A major obstacle with this approach is that the initial values of a Maxwell system must satisfy specific compatibility conditions depending on the regularity level, see Definition 3.1.2, which the approximations must also preserve. We therefore adjust the initial data of an approximating sequence to satisfy the compatibility conditions in Lemma ???. Then, the well-posedness theory from Theorem 3.1 in [56] yields solutions with additional regularity, but the a-priori estimate given there does not guarantee convergence in the right norm, as outlined at the beginning of Section 3.3. Therefore, to establish the convergence, we first use a mollification argument to show the convergence of the highest-order time derivatives. We then extend this result to the full space-time-norm by employing a div-curl estimate from [10] and [17].

3.1. PROBLEM SETTING AND COMPATIBILITY CONDITIONS

Let $m, k \in \mathbb{N}_0$ with $m \leq k$. In the following, we allow for an interface Γ such that $\partial\Omega \cap \Gamma = \emptyset$, splitting Ω into two disjoint sets v and ω with $\partial\Omega \subseteq \partial\omega$. Let ω have a C^{k+1} -boundary.

In view of (3.1) we consider the non-autonomous, anisotropic Maxwell system

$$\begin{aligned}
 \partial_t (\alpha(t, x) u(t, x)) &= \operatorname{curl} v(t, x) - \tilde{\sigma}(t, x) u(t, x) - f(t, x), & t \geq 0, x \in \Omega_{\pm}, \\
 \partial_t (\beta(t, x) v(t, x)) &= -\operatorname{curl} u(t, x) - \tilde{\rho}(t, x) v(t, x) - g(t, x),
 \end{aligned} \tag{3.2}$$

$$u(t, x) \times \nu(x) = h(t, x), \quad t \geq 0, x \in \partial\Omega, \tag{3.3}$$

$$\llbracket u(t, x) \times \nu^\Gamma(x) \rrbracket = \kappa(t, x), \quad \llbracket v(t, x) \times \nu^\Gamma(x) \rrbracket = \varkappa(t, x), \quad t \geq 0, x \in \Gamma, \tag{3.4}$$

$$u(0, x) = u_0(x), \quad v(0, x) = v_0(x), \quad x \in \Omega_{\pm}, \tag{3.5}$$

where $\llbracket f \rrbracket := \operatorname{tr}_{\partial v} f - \operatorname{tr}_{\partial \omega} f$ denotes the jump across the interface. We may also include a (purely mathematical) magnetic conductivity $\tilde{\rho}$. For energy estimates it is favorable to

commute α and β with ∂_t , which yields a symmetric, hyperbolic system. In this setting the well-posedness theory has been established in [56]. We thus rewrite the above system as

$$\alpha(t, x) \partial_t u(t, x) = \operatorname{curl} v(t, x) - \sigma(t, x) u(t, x) - f(t, x), \quad t \geq 0, x \in \Omega_{\pm}, \quad (3.6)$$

$$\beta(t, x) \partial_t v(t, x) = -\operatorname{curl} u(t, x) - \rho(t, x) v(t, x) - g(t, x), \quad t \geq 0, x \in \partial\Omega, \quad (3.7)$$

$$u(t, x) \times \nu^\Gamma(x) = h(t, x), \quad t \geq 0, x \in \Gamma, \quad (3.8)$$

$$\llbracket u(t, x) \times \nu^\Gamma(x) \rrbracket = \kappa(t, x), \quad \llbracket v(t, x) \times \nu^\Gamma(x) \rrbracket = \varkappa(t, x), \quad t \geq 0, x \in \Gamma, \quad (3.9)$$

where

$$\sigma = \tilde{\sigma} + \partial_t \alpha \quad \text{and} \quad \rho = \tilde{\rho} + \partial_t \beta. \quad (3.10)$$

We use the spaces

$$\begin{aligned} \mathcal{C}^\ell &:= \bigcap_{j=0}^\ell C^{\ell-j}([0, T], H_\Gamma^j(\Omega, \mathbb{R}_{\text{sym}}^{3 \times 3})), \\ \mathcal{C}_\eta^\ell &:= \left\{ A \in \mathcal{C}^\ell \mid A(t) \geq \eta \text{ for all } t \in [0, T] \right\}, \quad \text{for } \ell \in \mathbb{N}_0, \\ \|A\|_{\mathcal{C}_\eta^\ell} &= \|A\|_{\mathcal{C}^\ell} := \sup_{\substack{j=0, \dots, \ell \\ t \in [0, T]}} \|\partial_t^{\ell-j} A(t)\|_{H_\Gamma^j}, \end{aligned}$$

The coefficients are required to satisfy

$$\alpha, \beta \in \mathcal{C}_\eta^{\tilde{m}} \text{ for some } \eta > 0 \quad \text{and} \quad \sigma, \rho \in \mathcal{C}^{\tilde{m}}, \quad (3.11)$$

with $\tilde{m} := \max\{m, 3\}$, cf. Chapter 2 in [56]. Now in view of (3.10) we additionally assume that

$$\partial_t \alpha, \partial_t \beta \in \mathcal{C}^{\hat{m}} \quad (3.12)$$

with $\hat{m} := \max\{m, 2\}$. The latter we use to control error terms when estimating the divergence of αu and βv in the proof of the convergence of the approximation by means of a div-curl lemma, see (3.41). Note that for $m \leq 2$, i.e. $\tilde{m} = 3$ and $\hat{m} = 2$, condition (3.11) already implies (3.12). In Remark 3.3.6 we discuss the application of our results to a time-derived, quasilinear system in the context of condition (3.12). We further assume

$$f, g \in H^m([0, T] \times \Omega_{\pm}), \quad h \in T^m(\partial\Omega), \quad \kappa, \varkappa \in J^m(\Gamma), \quad \text{and} \quad u_0, v_0 \in H_\Gamma^m(\Omega), \quad (3.13)$$

where the space of spatial traces

$$T^m(\partial\Omega) := \operatorname{tr}_{t, \partial\Omega} \mathcal{G}_\Gamma^{m+1}, \quad J^m(\Gamma) := \llbracket \mathcal{G}_\Gamma^{m+1} \times \nu^\Gamma \rrbracket,$$

are subsets of $E^m(\Xi) := \bigcap_{j=0}^m H^j([0, T], H^{m+1/2-j}(\Xi))$, for $\Xi \in \{\partial\Omega, \Gamma\}$, see [56] and [59], allowing us to extend $h, \kappa, \varkappa$ to the interior as functions $h^\circ, \kappa^\circ, \varkappa^\circ$ in $\mathcal{G}_\Gamma^{m+1}$. These spaces naturally occur, when considering tangential derivatives of solutions to the Maxwell system with homogeneous boundary and interface conditions as in Section 4.5 below.

Remark 3.1.1. a) Note that the coefficients $\alpha, \beta \in \mathcal{C}_\eta^{\tilde{m}}$ satisfy

$$\begin{aligned} \alpha|_\Xi, \beta|_\Xi, \sigma|_\Xi, \rho|_\Xi &\in \{A \in W^{1, \infty}([0, T] \times \Xi)^{3 \times 3} : \\ &\partial^\alpha A \in L^\infty([0, T], L^2(\Xi)) \text{ for all } \alpha \in \mathbb{N}_0^4 \text{ with } 1 \leq |\alpha| \leq \tilde{m}\} \end{aligned}$$

for $\Xi \in \{v, \omega\}$ and that α, β are uniformly positive definite. Hence, we may apply the local well-posedness result from Theorem 3.1 in [56] to our system (3.6) above.

b) By extending the coefficients, using the usual construction of Theorem 5.19 in [1] in time, and a mollification argument, one can show that the space $C^\infty([0, T] \times \overline{\Omega_{\pm}}, \mathbb{R}_{\text{sym}}^{3 \times 3})$ is dense in $C^{\tilde{m}-j}([0, T], H_\Gamma^j(\Omega, \mathbb{R}_{\text{sym}}^{3 \times 3}))$. Thus, we can approximate our coefficients smoothly, see (App).

Assume for a moment that we have a solution $(u, v) \in \mathcal{G}_\Gamma^m([0, T])$ of (3.6) satisfying the boundary and interface conditions

$$\text{tr}_{t, \partial\Omega} u(t) = h, \quad \llbracket u(t) \times \nu^\Gamma \rrbracket = \kappa, \quad \llbracket v(t) \times \nu^\Gamma \rrbracket = \varkappa,$$

for all $t \geq 0$. Since the trace $\text{tr}_{t, \partial\Omega}$ and the interface jump $\llbracket \cdot \times \nu^\Gamma \rrbracket$ act only on spatial variables, they commute with ∂_t^i for $i < m$ resulting in

$$\text{tr}_{t, \partial\Omega} \partial_t^i u(t) = \partial_t^i h, \quad \llbracket \partial_t^i u(t) \times \nu^\Gamma \rrbracket = \partial_t^i \kappa, \quad \llbracket \partial_t^i v(t) \times \nu^\Gamma \rrbracket = \partial_t^i \varkappa.$$

Now, the system (3.6) relates time derivatives to spatial derivatives yielding the conditions

$$\begin{aligned} \text{tr}_{t, \partial\Omega} \left(\partial_t^{i-1} \left(\alpha^{-1} (\text{curl } v - \sigma u - f) \right) \right) &= \partial_t^i h, \\ \llbracket \partial_t^{i-1} \left(\alpha^{-1} (\text{curl } v - \sigma u - f) \right) \times \nu^\Gamma \rrbracket &= \partial_t^i \kappa, \\ \llbracket \partial_t^{i-1} \left(\beta^{-1} (-\text{curl } u - \rho v - g) \right) \times \nu^\Gamma \rrbracket &= \partial_t^i \varkappa, \end{aligned}$$

for $1 \leq i < m$. By applying (3.6) again we can recursively translate these conditions on the time derivatives into conditions on the spatial derivatives of (u, v) . In particular, they have to hold at $t = 0$, i.e., the initial values $u_0, v_0 \in H_\Gamma^m(\Omega)$ of a solution $(u, v) \in \mathcal{G}_\Gamma^m$ to (3.6)–(3.9) must satisfy additional *compatibility conditions*.

The compatibility conditions of order m are then defined as follows, see Equation (2.6) in [56]. As usual the sum from 1 to 0 is empty.

Definition 3.1.2 (Compatibility Condition of order m). *Let $(u_{(0)}, v_{(0)}) := (u_0, v_0) \in H_\Gamma^m(\Omega)$. For $1 \leq i \leq m$, we define $u_{(i)}, v_{(i)}$ by recursively differentiating the system at $t = 0$, obtaining*

$$\begin{aligned} u_{(i)} &:= \alpha(0)^{-1} \left(\text{curl } v_{(i-1)} - \sigma(0) u_{(i-1)} - \partial_t^{i-1} f(0) \right. \\ &\quad \left. - \sum_{\ell=1}^{i-1} \binom{i-1}{\ell} \left((\partial_t^\ell \alpha)(0) u_{(i-\ell)} + \partial_t^\ell \sigma(0) u_{(i-1-\ell)} \right) \right), \\ v_{(i)} &:= \beta(0)^{-1} \left(-\text{curl } u_{(i-1)} - \rho(0) v_{(i-1)} - \partial_t^{i-1} g(0) \right. \\ &\quad \left. - \sum_{\ell=1}^{i-1} \binom{i-1}{\ell} \left((\partial_t^\ell \beta)(0) v_{(i-\ell)} + \partial_t^\ell \rho(0) v_{(i-1-\ell)} \right) \right). \end{aligned}$$

Then, the compatibility conditions of order m are given by

$$\text{tr}_{t, \partial\Omega} u_{(i)} = \partial_t^i h, \quad \llbracket u_{(i)} \times \nu^\Gamma \rrbracket = \partial_t^i \kappa \quad \text{and} \quad \llbracket v_{(i)} \times \nu^\Gamma \rrbracket = \partial_t^i \varkappa \quad \text{for } i < m, \quad (3.14)$$

which we denote by $(CC_{\alpha, \beta, \sigma, \rho, f, g, h, \kappa, \varkappa, m}) = (CC_m)$. The compatibility conditions (CC_0) of order zero do not require any conditions.

Note that the compatibility conditions (CC_m) are conditions on the time derivatives up to order $m - 1$. Hence, the traces in (3.14) are well-defined in $H^{1/2}$. Theorem 3.1 and the comment after (1.7) in [56] then provide the following well-posedness result and a-priori estimate.

Proposition 3.1.3. *Let $\eta > 0$, $m, k \in \mathbb{N}_0$ with $\tilde{m} + 2 \leq k$ and Ω as above. Fix $T, r > 0$. Take coefficients $(\alpha, \beta, \sigma, \rho)$ satisfying (3.11) with $\|(\alpha, \beta, \sigma, \rho)\|_{C^{\tilde{m}}} < r$, and data (u_0, v_0) , $(h, \kappa, \varkappa)$ and (f, g) satisfying (3.13). Let (u_0, v_0) fulfill the compatibility conditions (CC_m) . Then the system (3.6)–(3.9) has a unique solution $(u, v) \in \mathcal{G}_\Gamma^m([0, T])$, such that*

$$\|(u, v)\|_{\mathcal{G}_\Gamma^m}^2 \leq C_{r,T} \left(\|(u_0, v_0)\|_{H_\Gamma^m}^2 + \|(f, g)\|_{H^m([0,T] \times \Omega_\pm)}^2 + \|h\|_{E^m(\partial\Omega)}^2 + \|(\kappa, \varkappa)\|_{E^m(\Gamma)}^2 \right). \quad (3.15)$$

for a constant $C_{r,T} > 0$ depending on r and T , but not on u_0, v_0, f, g or $h, \kappa, \varkappa$.

3.2. APPROXIMATION OF DATA

We first show that we can always approximate first-order compatibility conditions. These are much simpler than higher order compatibility conditions, since they do not involve derivatives of other fields.

Lemma 3.2.1. *Let $m, a \in \mathbb{N}_0$ with $m + a \leq k$, $h \in T^{m+a}(\partial\Omega)$, and $\kappa, \varkappa \in J^{m+a}(\Gamma)$. Then the set*

$$H_{\Gamma, CC_1}^{m+a}(\Omega) := \left\{ (u, v) \in H_\Gamma^{m+a}(\Omega) \mid \text{tr}_{t, \partial\Omega} u = h, \left\| (u \times \nu^\Gamma, v \times \nu^\Gamma) \right\| = (\kappa, \varkappa) \right\}$$

is dense in $L^2(\Omega)$ as well as in

$$H_{\Gamma, CC_1}^m(\Omega) = \left\{ (u, v) \in H_\Gamma^m(\Omega) \mid \text{tr}_{t, \partial\Omega} u = h, \left\| (u \times \nu^\Gamma, v \times \nu^\Gamma) \right\| = (\kappa, \varkappa) \right\}.$$

This means that we can approximate adding or preserving the compatibility condition of first-order.

Proof. We first extend (h, κ) and $(0, \varkappa)$ to functions $W^u, W^v \in H^{m+a}(\omega)$ satisfying

$$\text{tr}_{t, \Gamma}(W^u, W^v) = (\kappa, \varkappa) \quad \text{and} \quad \text{tr}_{t, \partial\Omega}(W^u, W^v) = (h, 0)$$

simultaneously by using cutoffs $\chi_\Xi \in C_c^\infty(\bar{\omega})$, for $\Xi \in \{\Gamma, \partial\Omega\}$, such that $\chi_\Xi|_\Xi = 1$ and $\text{supp } \chi_\Gamma \cap \text{supp } \chi_{\partial\Omega} = \emptyset$ and setting

$$W^u = \chi_{\partial\Omega} h^\circ + \chi_\Gamma \kappa^\circ \quad \text{and} \quad W^v = \chi_\Gamma \varkappa^\circ. \quad (3.16)$$

We then extend W^u and W^v by 0 to v and denote the extensions also by $W^u, W^v \in H_\Gamma^{m+a}(\Omega_\pm)$. The density in L^2 is then clear since $(W^u, W^v) + C_c^\infty(\Omega_\pm)$ is dense in $L^2(\Omega_\pm) = L^2(\Omega)$.

Next, we want to approximate $(u, v) \in H_{\Gamma, CC_1}^m(\Omega)$ by $(u_n, v_n) \in H_\Gamma^{m+a}(\Omega)$ satisfying the compatibility condition of order 1, namely

$$\text{tr}_{t, \partial\Omega}(u_n) = h, \quad \left\| (u_n \times \nu^\Gamma, v_n \times \nu^\Gamma) \right\| = (\kappa, \varkappa). \quad (3.17)$$

For the construction of these maps, we choose $(u_n^v, v_n^v) \in H^{m+a}(v)$ such that $(u_n^v, v_n^v) \rightarrow (u, v)$ in $H^m(v)$ and set

$$(u_n, v_n)|_v := (u_n^v, v_n^v).$$

We next approximate (u, v) in $H^m(\omega)$ by $(u_n^\omega, v_n^\omega) \in H^{m+a}(\omega)$. Now we have to adjust the trace terms in order to satisfy the compatibility conditions. The map

$$c_n := \begin{cases} (\text{tr}_{t, \partial\Omega}(u_n^\omega) - h, 0), & \text{on } \partial\Omega, \\ \text{tr}_{t, \Gamma}((u_n^\omega, v_n^\omega)) - \text{tr}_{t, \Gamma}((u_n^v, v_n^v)) + (\kappa, \varkappa), & \text{on } \partial v, \end{cases}$$

then belongs to $(H^{m+a-1/2}(\partial\omega))^6$. Applying the extension operator R componentwise, we then define

$$(u_n, v_n)|_\omega := (u_n^\omega, v_n^\omega) - Rc_n.$$

Clearly, $(u_n, v_n) \in H_\Gamma^{m+a}$ satisfies the compatibility condition (3.17). Since (u, v) satisfies the compatibility conditions by definition, we have $\text{tr}_{t, \partial\Omega} u = h$ and $\text{tr}_{t, \partial v}(u, v) - \text{tr}_{t, \partial\omega}(u, v) = (\kappa, \varkappa)$. Substituting these into our bound for c_n allows us to split the jump difference back into the separate domain errors. So, we can estimate

$$\begin{aligned} & \| (u, v) - (u_n, v_n) \|_{H_\Gamma^m} \\ & \leq \| (u, v) - (u_n^v, v_n^v) \|_{H^m(v)} + \| (u, v) - (u_n^\omega, v_n^\omega) \|_{H^m(\omega)} \\ & \quad + \| Rc_n \|_{H^m(\omega)} \\ & \lesssim \| (u, v) - (u_n^v, v_n^v) \|_{H^m(v)} + \| (u, v) - (u_n^\omega, v_n^\omega) \|_{H^m(\omega)} \\ & \quad + \| \text{tr}_{t, \partial\Omega} (u_n^\omega) - h \|_{H^{m-1/2}(\partial\Omega)} \\ & \quad + \| \text{tr}_{t, \partial\omega} (u_n^\omega, v_n^\omega) - \text{tr}_{t, \partial v} (u_n^v, v_n^v) + (\kappa, \varkappa) \|_{H^{m-1/2}(\partial v)} \\ & = \| (u, v) - (u_n^v, v_n^v) \|_{H^m(v)} + \| (u, v) - (u_n^\omega, v_n^\omega) \|_{H^m(\omega)} \\ & \quad + \| \text{tr}_{t, \partial\Omega} (u_n^\omega - u) \|_{H^{m-1/2}(\partial\Omega)} \\ & \quad + \| \text{tr}_{t, \partial\omega} (u_n^\omega, v_n^\omega) - \text{tr}_{t, \partial\omega}(u, v) + \text{tr}_{t, \partial v}(u, v) - \text{tr}_{t, \partial v}(u_n^v, v_n^v) \|_{H^{m-1/2}(\partial v)} \\ & \lesssim \| (u, v) - (u_n^v, v_n^v) \|_{H^m(v)} + \| (u, v) - (u_n^\omega, v_n^\omega) \|_{H^m(\omega)} \\ & \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

□

Remark 3.2.2. For a given solution (u, v) of (3.6)–(3.9) in $\mathcal{G}_\Gamma^0([0, T])$, sequences $(f_n, g_n) \in H^1([0, T] \times \Omega_\pm)$ such that $(f_n, g_n) \rightarrow (f, g)$ in $L^2([0, T] \times \Omega)$ and $h_n \in T^1(\partial\Omega)$, $(\kappa_n, \varkappa_n) \in J^1(\Gamma)$ such that $h_n \rightarrow h$ in $T^0(\partial\Omega)$ and $(\kappa_n, \varkappa_n) \rightarrow (\kappa, \varkappa)$ in $J^0(\Gamma)$, Lemma 3.2.1 allows us to approximate its initial values $(u_0, v_0) \in L^2(\Omega)$ by (u_{0, n_k}, v_{0, n_k}) in $H_{\Gamma, CC_1}^1(\Omega)$ with respect to $(h_n, \kappa_n, \varkappa_n)$.

By the a-priori estimate (3.15), the corresponding diagonal sequence of solutions $(u_{n_n}, v_{n_n}) \in \mathcal{G}_\Gamma^1([0, T])$ of (3.6)–(3.9) converge to (u, v) in $\mathcal{G}_\Gamma^0([0, T])$.

This result can be used to justify energy equalities for $\mathcal{G}_\Gamma^0$ solutions to the Maxwell system with homogeneous boundary and interface data.

Lemma 3.2.3 (Energy Equality). *Let $(u, v) \in \mathcal{G}_\Gamma^0([0, T])$ be a solution to (3.6)–(3.9), where here we only assume $\alpha, \beta \in \mathcal{C}_\eta^{\tilde{m}}$ for some $\eta > 0$, $\sigma, \rho \in \mathcal{C}^{\tilde{m}}$ and (3.13) with $h = \kappa = \varkappa = 0$. The solution then satisfies the energy equality*

$$\begin{aligned} & \frac{1}{2} \int_\Omega \left(\alpha(t)u(t) \cdot u(t) + \beta(t)v(t) \cdot v(t) \right) dx + \int_s^t \int_\Omega \sigma u \cdot u + \rho v \cdot v dx d\tau \\ & = \frac{1}{2} \int_\Omega \left(\alpha(s)u(s) \cdot u(s) + \beta(s)v(s) \cdot v(s) \right) dx \\ & \quad + \int_s^t \int_\Omega \left(\frac{1}{2} \partial_t \alpha u \cdot u + \frac{1}{2} \partial_t \beta v \cdot v - f \cdot u - g \cdot v \right) dx d\tau \end{aligned}$$

for $0 \leq s \leq t \leq T$.

Proof. As in Remark 3.2.2 we choose $f_n, g_n \in H^1([0, T] \times \Omega_\pm)$, and $u_{0, n}, v_{0, n} \in H_\Gamma^1(\Omega)$, satisfying the compatibility conditions of order 1, such that the corresponding solution $(u_n, v_n) \in \mathcal{G}_\Gamma^1([0, T])$ of (3.6)–(3.9) converges to (u, v) in $L^2([0, T] \times \Omega)$ as $n \rightarrow \infty$.

This allows us to differentiate in time and apply integration by parts, leading to

$$\begin{aligned}
& \frac{d}{dt} \frac{1}{2} \int_{\Omega} \left(\alpha(t) u_n(t) \cdot u_n(t) + \beta(t) v_n(t) \cdot v_n(t) \right) dx \\
&= \int_{\Omega} \left(\frac{1}{2} \partial_t \alpha u_n \cdot u_n + \frac{1}{2} \partial_t \beta v_n \cdot v_n + \alpha \partial_t u_n \cdot u_n + \beta \partial_t v_n \cdot v_n \right) dx \\
&= \int_{\Omega} \left(\frac{1}{2} \partial_t \alpha u_n \cdot u_n + \frac{1}{2} \partial_t \beta v_n \cdot v_n + (\operatorname{curl} v_n - \sigma u_n - f_n) \cdot u_n + (-\operatorname{curl} u_n - \rho v_n - g_n) \cdot v_n \right) dx \\
&= \int_{\Omega} \left(\frac{1}{2} \partial_t \alpha u_n \cdot u_n + \frac{1}{2} \partial_t \beta v_n \cdot v_n - f_n \cdot u_n - g_n \cdot v_n - \sigma u_n \cdot u_n - \rho v_n \cdot v_n \right) dx.
\end{aligned}$$

Note that the boundary terms cancel because of (3.7) and (3.8) with $h = \kappa = \varkappa = 0$. It follows that

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega} \left(\alpha(t) u_n(t) \cdot u_n(t) + \beta(t) v_n(t) \cdot v_n(t) \right) dx + \int_s^t \int_{\Omega} (\sigma u_n \cdot u_n + \rho v_n \cdot v_n) dx d\tau \\
&= \frac{1}{2} \int_{\Omega} \left(\alpha(s) u_n(s) \cdot u_n(s) + \beta(s) v_n(s) \cdot v_n(s) \right) dx \\
&\quad + \int_s^t \int_{\Omega} \left(\frac{1}{2} \partial_t \alpha u_n \cdot u_n + \frac{1}{2} \partial_t \beta v_n \cdot v_n - f_n \cdot u_n - g_n \cdot v_n \right) dx d\tau.
\end{aligned}$$

Now letting $n \rightarrow \infty$ we obtain the result by Remark 3.2.2. $\square$

In view of Remark 3.2.2, we assume that $m \geq 1$ in the following. In order to obtain solutions of higher regularity than $\tilde{m}$, we need additional regularity the coefficients and inhomogeneities, so that the coefficients also have to be approximated. These functions exists by Remark 3.1.1. They also enter the compatibility conditions of the approximation.

(App) We choose $(\alpha_n, \beta_n, \sigma_n, \rho_n) \in C^\infty([0, T] \times \overline{\Omega_\pm}; \mathbb{R}_{\text{sym}}^{3 \times 3})^4 \cap (\mathcal{C}_\eta^{\tilde{m}} \times \mathcal{C}_\eta^{\tilde{m}} \times \mathcal{C}^{\tilde{m}} \times \mathcal{C}^{\tilde{m}})$ with $\alpha_n \rightarrow \alpha$, $\beta_n \rightarrow \beta$ in $\mathcal{C}^{\tilde{m}}$, $\partial_t \alpha_n \rightarrow \partial_t \alpha$, $\partial_t \beta_n \rightarrow \partial_t \beta$ in $\mathcal{C}^{\tilde{m}}$, $\sigma_n \rightarrow \sigma$, and $\rho_n \rightarrow \rho$ in $\mathcal{C}^{\tilde{m}}$, $(f_n, g_n) \in C^\infty([0, T] \times \overline{\Omega_\pm})$ with $f_n \rightarrow f$, $g_n \rightarrow g$ in $H^m([0, T] \times \Omega_\pm)$. Furthermore, we approximate the $\mathcal{G}_\Gamma^{m+1}$ -extensions h°, κ° and $\varkappa^\circ$ by smooth functions $h_n^\circ, \kappa_n^\circ, \varkappa_n^\circ \in C^\infty([0, T] \times \overline{\Omega_\pm})$ in $\mathcal{G}_\Gamma^{m+1}$. We set $h_n = \operatorname{tr}_{\partial\Omega} h_n^\circ \in C^\infty([0, T] \times \partial\Omega)$. Then $h_n \rightarrow h$ in $E^m(\partial\Omega)$. Similarly, we set $\kappa_n := \llbracket \kappa_n^\circ \rrbracket, \varkappa_n := \llbracket \varkappa_n^\circ \rrbracket \in C^\infty([0, T] \times \Gamma)$ such that $(\kappa_n, \varkappa_n) \rightarrow (\kappa, \varkappa)$ in $E^m(\Gamma)$.

In a core step, we start from fields (u_0, v_0) which satisfy the m -th order compatibility condition up to an error of size r . The next lemma then provides fields fulfilling the $(m+a)$ -th order compatibility condition whose distance to (u_0, v_0) in H_Γ^m is less than Cr .

Proposition 3.2.4. *Let $a \in \mathbb{N}_0$, $m \geq 1$ with $m + a \leq k$ and $r > 0$. Let $(u_{(0)}^{app}, v_{(0)}^{app}) := (u_0, v_0) \in H_{\Gamma, CC_1}^{m+a}(\Omega)$, and set recursively*

$$\begin{aligned}
u_{(i)}^{app} &= \alpha_n(0)^{-1} \left(\operatorname{curl} v_{(i-1)}^{app} - \sigma_n(0) u_{(i-1)}^{app} - \partial_t^{i-1} f_n(0) \right. \\
&\quad \left. - \sum_{\ell=1}^{i-1} \binom{i-1}{\ell} \left((\partial_t^\ell \alpha_n)(0) u_{(i-\ell)}^{app} + (\partial_t^\ell \sigma_n)(0) u_{(i-1-\ell)}^{app} \right) \right), \\
v_{(i)}^{app} &= \beta_n(0)^{-1} \left(-\operatorname{curl} u_{(i-1)}^{app} - \rho_n(0) v_{(i-1)}^{app} - \partial_t^{i-1} g_n(0) \right. \\
&\quad \left. - \sum_{\ell=1}^{i-1} \binom{i-1}{\ell} \left((\partial_t^\ell \beta_n)(0) v_{(i-\ell)}^{app} + (\partial_t^\ell \rho_n)(0) v_{(i-1-\ell)}^{app} \right) \right),
\end{aligned}$$

for $1 \leq i \leq m$. Assume that (u_0, v_0) satisfy the compatibility conditions of order m approximately, i.e.,

$$\begin{aligned} & \left\| \text{tr}_{t, \partial\Omega} u_{(i)}^{app} - \partial_t^i h_n(0) \right\|_{H^{m-i-1/2}(\partial\Omega)} < r, \\ & \left\| \llbracket u_{(i)}^{app} \times \nu^\Gamma \rrbracket - \partial_t^i \kappa_n(0) \right\|_{H^{m-i-1/2}(\Gamma)} < r \quad \text{and} \quad \left\| \llbracket v_{(i)}^{app} \times \nu^\Gamma \rrbracket - \partial_t^i \varkappa_n(0) \right\|_{H^{m-i-1/2}(\Gamma)} < r, \end{aligned} \quad (3.18)$$

for $0 \leq i < m$. Then, there exists a constant $C > 0$, independent of r , and functions $(u_0^{m+a}, v_0^{m+a}) \in H_\Gamma^{m+a}(\Omega)$ satisfying $(CC_{\alpha_n, \beta_n, \sigma_n, \rho_n, f_n, g_n, h_n, \kappa_n, \varkappa_n, m+a})$ such that

$$\left\| (u_0, v_0) - (u_0^{m+a}, v_0^{m+a}) \right\|_{H_\Gamma^m} \leq Cr. \quad (3.19)$$

For a given $r > 0$ the condition (3.18) can be satisfied when approximating coefficients and initial data satisfying the compatibility condition of order m as in (App) for large n and then applying Lemma 3.2.1.

Proof of Proposition 3.2.4. We proceed iteratively over the order j of the compatibility condition from 0 to m . The cases $m + \hat{j}$ (for $0 < \hat{j} \leq a$) require a different argument at the end.

1) Assume that for some $0 < j < m$, the pair $(u_0^j, v_0^j) \in H_\Gamma^{m+a}(\Omega)$ satisfies the exact compatibility condition (CC_j) and the estimate $\|(u_0, v_0) - (u_0^j, v_0^j)\|_{H^m(\Omega)} \leq Cr$. We construct (u_0^{j+1}, v_0^{j+1}) that fulfill the condition of order $j+1$, while preserving the conditions of order k for $k < j$ by adding a correction term $(c_0^j, d_0^j) \in H^{m+a}(\omega) \cap H_0^j(\omega)$ (and 0 on v) to (u_0^j, v_0^j) . Set $(u_0^{j+1}, v_0^{j+1}) := (u_0^j, v_0^j) + (c_0^j, d_0^j)$.

We first want to study the effect of adding $(c_0^j, d_0^j) \in H^{m+a}(\omega) \cap H_0^j(\omega)$. We will choose it explicitly later on. For $i < j$ the compatibility conditions

$$\text{tr}_{t, \partial\Omega} u_{(i)}^{j+1} = \partial_t^i h_n(0), \quad \llbracket u_{(i)}^{j+1} \times \nu^\Gamma \rrbracket = \partial_t^i \kappa_n(0) \quad \text{and} \quad \llbracket v_{(i)}^{j+1} \times \nu^\Gamma \rrbracket = \partial_t^i \varkappa_n(0)$$

only depend on the traces of derivatives of order $\leq i$ of (u_0^{j+1}, v_0^{j+1}) . The fields $(c_0^j, d_0^j) \in H_0^j(\omega)$ do not influence these traces for $i < j$. So, we obtain $\text{tr}_{t, \Xi}(u_{(i)}^{j+1}, v_{(i)}^{j+1}) = \text{tr}_{t, \Xi}(u_{(i)}^j, v_{(i)}^j)$ for $\Xi \in \{\partial\Omega, \partial\omega \cap \Omega, \partial v\}$ and $i < j$.

The j -th compatibility condition of (CC_{j+1}) for $u_{(j)}^{j+1}$ imposes on $\text{curl } v_{(j-1)}^{j+1}$ the identity

$$\begin{aligned} \llbracket u_{(j)}^{j+1} \times \nu^\Gamma \rrbracket &\stackrel{!}{=} \partial_t^j \kappa_n(0) \iff \llbracket (\alpha_n(0)^{-1} \text{curl } v_{(j-1)}^{j+1}) \times \nu^\Gamma \rrbracket \stackrel{!}{=} I_j^v \\ \text{with } I_j^v &= I_j^v \begin{pmatrix} u_0^{j+1} \\ v_0^{j+1} \end{pmatrix} := \partial_t^j \kappa_n(0) + \left[\alpha_n(0)^{-1} \left(\sigma_n(0) u_{(j-1)}^{j+1} + \partial_t^{j-1} f_n(0) \right. \right. \\ &\quad \left. \left. + \sum_{\ell=1}^{j-1} \binom{j-1}{\ell} \left((\partial_t^\ell \alpha_n)(0) u_{(j-\ell)}^{j+1} + (\partial_t^\ell \sigma_n)(0) u_{(j-1-\ell)}^{j+1} \right) \right) \times \nu^\Gamma \right]. \end{aligned}$$

Likewise, the compatibility condition for $v_{(j)}^{j+1}$ requires the equation

$$\begin{aligned} \llbracket v_{(j)}^{j+1} \times \nu^\Gamma \rrbracket &\stackrel{!}{=} \partial_t^j \varkappa_n(0) \iff \llbracket (\beta_n(0)^{-1} \text{curl } u_{(j-1)}^{j+1}) \times \nu^\Gamma \rrbracket \stackrel{!}{=} I_j^u \\ \text{with } I_j^u &= I_j^u \begin{pmatrix} u_0^{j+1} \\ v_0^{j+1} \end{pmatrix} := \partial_t^j \varkappa_n(0) - \left[\beta_n(0)^{-1} \left(\rho_n(0) v_{(j-1)}^{j+1} + \partial_t^{j-1} g_n(0) \right. \right. \\ &\quad \left. \left. + \sum_{\ell=1}^{j-1} \binom{j-1}{\ell} \left((\partial_t^\ell \beta_n)(0) v_{(j-\ell)}^{j+1} + (\partial_t^\ell \rho_n)(0) v_{(j-1-\ell)}^{j+1} \right) \right) \times \nu^\Gamma \right]. \end{aligned}$$

At the outer boundary we obtain

$$\begin{aligned} \operatorname{tr}_{t,\partial\Omega} u_{(j)}^{j+1} &\stackrel{!}{=} \partial_t^j h_n(0) \iff \operatorname{tr}_{t,\partial\Omega} \left(\alpha_n(0)^{-1} \operatorname{curl} v_{(j-1)}^{j+1} \right) \stackrel{!}{=} B_j^v \\ \text{with } B_j^v &= B_j^v \begin{pmatrix} u_0^{j+1} \\ v_0^{j+1} \end{pmatrix} := \partial_t^j h_n(0) + \operatorname{tr}_{t,\partial\Omega} \left(\alpha_n(0)^{-1} \left(\sigma_n(0) u_{(j-1)}^{j+1} + \partial_t^{j-1} f_n(0) \right. \right. \\ &\quad \left. \left. + \sum_{\ell=1}^{j-1} \binom{j-1}{\ell} \left((\partial_t^\ell \alpha_n)(0) u_{(j-\ell)}^{j+1} + (\partial_t^\ell \sigma_n)(0) u_{(j-1-\ell)}^{j+1} \right) \right) \right). \end{aligned}$$

Note that the right-hand sides B_j^v , I_j^u , and I_j^v do not depend on (c_0^j, d_0^j) , since the traces of $(u_{(i)}^{j+1}, v_{(i)}^{j+1})$ for $i < j$ appearing therein remain unaffected by the correction. So, to summarize we have to choose $(c_0^j, d_0^j) \in H_0^j(\omega)$ such that

$$\begin{aligned} \operatorname{tr}_{t,\partial\Omega} \left(\alpha_n(0)^{-1} \operatorname{curl} v_{(j-1)}^{j+1} \right) &\stackrel{!}{=} B_j^v(u_0^j, v_0^j), \\ \llbracket (\alpha_n(0)^{-1} \operatorname{curl} v_{(j-1)}^{j+1}) \times \nu^\Gamma \rrbracket &\stackrel{!}{=} I_j^v(u_0^j, v_0^j), \\ \llbracket (\beta_n(0)^{-1} \operatorname{curl} u_{(j-1)}^{j+1}) \times \nu^\Gamma \rrbracket &\stackrel{!}{=} I_j^u(u_0^j, v_0^j). \end{aligned} \tag{3.20}$$

Let us take a closer look at the dependence of the left-hand side on (c_0^j, d_0^j) . Recalling the iterative Definition 3.1.2 of $(u_{(j-1)}^{j+1}, v_{(j-1)}^{j+1})$, we realize that

$$\begin{pmatrix} \alpha_n(0)^{-1} \operatorname{curl} v_{(j-1)}^{j+1} \\ \beta_n(0)^{-1} \operatorname{curl} u_{(j-1)}^{j+1} \end{pmatrix} = P^j \begin{pmatrix} u_0^{j+1} \\ v_0^{j+1} \end{pmatrix} + R^j \begin{pmatrix} u_0^{j+1} \\ v_0^{j+1} \end{pmatrix} \tag{3.21}$$

where the principal part is given for even j by

$$P^j(u, v) = (-1)^{j/2} \begin{pmatrix} (\prod_{i=1}^{j/2} \alpha_n^{-1}(0) \operatorname{curl} \beta_n^{-1}(0) \operatorname{curl}) u \\ -(\prod_{i=1}^{j/2} \beta_n^{-1}(0) \operatorname{curl} \alpha_n^{-1}(0) \operatorname{curl}) v \end{pmatrix}$$

and for odd j

$$P^j(u, v) = (-1)^{(j-1)/2} \begin{pmatrix} (\prod_{i=1}^{(j-1)/2} \alpha_n^{-1}(0) \operatorname{curl} \beta_n^{-1}(0) \operatorname{curl}) \alpha_n^{-1}(0) \operatorname{curl} v \\ (\prod_{i=1}^{(j-1)/2} \beta_n^{-1}(0) \operatorname{curl} \alpha_n^{-1}(0) \operatorname{curl}) \beta_n^{-1}(0) \operatorname{curl} u \end{pmatrix}.$$

The point being that the principal part P^j is just a composition of curls and matrices, whereas the remainder $R^j(u_0^{j+1}, v_0^{j+1}) = R^j(u_0^{j+1}, v_0^{j+1}, \dots, D^{j-1} u_0^{j+1}, D^{j-1} v_0^{j+1})$ only depends in lower order on $\begin{pmatrix} u_0^{j+1} \\ v_0^{j+1} \end{pmatrix}$. (Note that the remainder also depends on (f_n, g_n) .) Since $(c_0^j, d_0^j) \in H_0^j(\Omega)$, the trace thus satisfies

$$\operatorname{tr}_{t,\Xi} R^j(u_0^{j+1}, v_0^{j+1}) = \operatorname{tr}_{t,\Xi} R^j(u_0^j + c_0^j, v_0^j + d_0^j) = \operatorname{tr}_{t,\Xi} R^j(u_0^j, v_0^j)$$

for $\Xi \in \{\partial\Omega, \partial\omega, \partial v\}$. Therefore, the linearity of P^j and (3.21) for j imply

$$\begin{aligned} \operatorname{tr}_{t,\Xi} \begin{pmatrix} \alpha_n(0)^{-1} \operatorname{curl} v_{(j-1)}^{j+1} \\ \beta_n(0)^{-1} \operatorname{curl} u_{(j-1)}^{j+1} \end{pmatrix} &= \operatorname{tr}_{t,\Xi} \left(P^j(u_0^{j+1}, v_0^{j+1}) + R^j(u_0^j, v_0^j) \right) \\ &= \operatorname{tr}_{t,\Xi} \left(P^j(c_0^j, d_0^j) \right) + \operatorname{tr}_{t,\Xi} \begin{pmatrix} \alpha_n(0)^{-1} \operatorname{curl} v_{(j-1)}^j \\ \beta_n(0)^{-1} \operatorname{curl} u_{(j-1)}^j \end{pmatrix} \end{aligned}$$

Thus, the compatibility condition (3.20) becomes

$$\mathrm{tr}_{t,\partial\Omega} \left(P^j(c_0^j, d_0^j) \right)_u \stackrel{!}{=} B_j^v(u_0^j, v_0^j) - \mathrm{tr}_{t,\partial\Omega} \left(\alpha_n(0)^{-1} \mathrm{curl} v_{(j-1)}^j \right) = \partial_t^j h_n(0) - \mathrm{tr}_{t,\partial\Omega} u_{(j)}^j, \quad (3.22)$$

$$\llbracket P^j(c_0^j, d_0^j) \times \nu^\Gamma \rrbracket \stackrel{!}{=} \begin{pmatrix} I_j^v(u_0^j, v_0^j) - \llbracket (\alpha_n(0)^{-1} \mathrm{curl} v_{(j-1)}^j) \times \nu^\Gamma \rrbracket \\ I_j^u(u_0^j, v_0^j) - \llbracket (\beta_n(0)^{-1} \mathrm{curl} v_{(j-1)}^j) \times \nu^\Gamma \rrbracket \end{pmatrix} = \begin{pmatrix} \partial_t^j \kappa_n(0) - \llbracket u_{(j)}^j \times \nu^\Gamma \rrbracket \\ \partial_t^j \varkappa_n(0) - \llbracket v_{(j)}^j \times \nu^\Gamma \rrbracket \end{pmatrix},$$

where $(\cdot)_u$ denotes the first three components corresponding to the compatibility condition on $\mathrm{tr}_{t,\partial\Omega} u_{(j)}^{j+1}$.

Moreover, by Lemma 3.2.5 below, the (full) trace of the principal part is given by

$$\mathrm{tr}_\Xi \left(P^j \begin{pmatrix} c_0^j \\ d_0^j \end{pmatrix} \right) = (-1)^{\lfloor j/2 \rfloor} \begin{pmatrix} \Pi_{i=1}^j T_{\zeta_i, x} & 0 \\ 0 & \Pi_{i=1}^j T_{\xi_i, x} \end{pmatrix} \begin{pmatrix} 1-\gamma & \gamma \\ -\gamma & 1-\gamma \end{pmatrix} \begin{pmatrix} \mathrm{tr}(\partial_\nu^j c_0^j) \\ \mathrm{tr}(\partial_\nu^j d_0^j) \end{pmatrix},$$

where $\gamma = j \bmod 2$, $T_{\zeta_i, x}, T_{\xi_i, x}$ are given by (3.26), and $\zeta_i, \xi_i \in \{\alpha_n^{-1}(0), \beta_n^{-1}(0)\}$ are smooth, uniformly positive definite matrices. We want to solve for $\begin{pmatrix} \mathrm{tr}(\partial_\nu^j c_0^j) \\ \mathrm{tr}(\partial_\nu^j d_0^j) \end{pmatrix}$ after taking the tangential trace in the equality above. For this, note that the linear map

$$A_\tau(x) \begin{pmatrix} u \\ v \end{pmatrix} := (-1)^{\lfloor j/2 \rfloor} \begin{pmatrix} \mathrm{tr}_{t,\Xi} & 0 \\ 0 & \mathrm{tr}_{t,\Xi} \end{pmatrix} \begin{pmatrix} \Pi_{i=1}^j T_{\zeta_i, x} & 0 \\ 0 & \Pi_{i=1}^j T_{\xi_i, x} \end{pmatrix} \begin{pmatrix} 1-\gamma & \gamma \\ -\gamma & 1-\gamma \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

is invertible on the tangential plane for each $x \in \Xi$ for $\Xi \in \{\partial\Omega, \partial\omega \cap \Omega\}$ since the matrices ζ_i and ξ_i are positive definite. Let us denote the inverse on the tangential plane as

$$A_\tau^{-1}(x) : \left(\langle \{\nu(x)\} \rangle^\perp \right)^2 \rightarrow \left(\langle \{\nu(x)\} \rangle^\perp \right)^2 \subseteq \mathbb{R}^6.$$

Furthermore, the matrices ζ_i and ξ_i are smooth and the normal lies in $C^k(\partial\omega)$ because of $\partial\omega \in C^{k+1}$. Thus, A_τ^{-1} boundedly maps $\{w \in H^\ell(\partial\omega) \mid w \cdot \nu = w \cdot \nu^\Gamma = 0\}$ into itself for $\ell \in [1, k]$.

Hence, when setting $(c_0^j, d_0^j)|_v = 0$ the compatibility condition in (3.22) is satisfied if and only if

$$\begin{aligned} \mathrm{tr}_{\partial\Omega} \begin{pmatrix} \partial_\nu^j c_0^j \\ \partial_\nu^j d_0^j \end{pmatrix} &\stackrel{!}{=} A_\tau^{-1} \begin{pmatrix} \partial_t^j h_n(0) - \mathrm{tr}_{t,\partial\Omega} u_{(j)}^j \\ 0 \end{pmatrix}, \\ \mathrm{tr}_{\partial\omega \cap \Omega} \begin{pmatrix} \partial_\nu^j c_0^j \\ \partial_\nu^j d_0^j \end{pmatrix} &\stackrel{!}{=} -A_\tau^{-1} \begin{pmatrix} \partial_t^j \kappa_n(0) - \llbracket u_{(j)}^j \times \nu^\Gamma \rrbracket \\ \partial_t^j \varkappa_n(0) - \llbracket v_{(j)}^j \times \nu^\Gamma \rrbracket \end{pmatrix}, \end{aligned} \quad (3.23)$$

where the minus sign in the second equality stems from the definition of $\llbracket \cdot \rrbracket = \mathrm{tr}_{\partial v} - \mathrm{tr}_{\partial\omega}$.

As $\partial\omega \in C^{k+1}$, by Theorem 1.5.1.2 in [28] the map

$$\begin{aligned} T : H^{m+a}(\omega) &\rightarrow \Pi_{s=0}^j H^{m+a-s-1/2}(\partial\omega); \\ u &\mapsto (\mathrm{tr} u, \mathrm{tr}(\partial_\nu u), \dots, \mathrm{tr}(\partial_\nu^j u)) \end{aligned} \quad (3.24)$$

is surjective with continuous right inverse T_{-1} . Such an inverse can be explicitly constructed, see for example the proof of Lemma 3.36 and Theorem 3.37 in [41]. One can then show that

$$\left\| T_{-1}(\mathrm{tr} u, \mathrm{tr}(\partial_\nu u), \dots, \mathrm{tr}(\partial_\nu^j u)) \right\|_{H^{m+a-\ell}(\omega)} \lesssim \sum_{s=0}^j \left\| \mathrm{tr}(\partial_\nu^s u) \right\|_{H^{m+a-s-1/2-\ell}(\partial\omega)},$$

for $\ell \in \{0, \dots, m+a-j-1\}$. Hence, we find $c_0^j, d_0^j \in H^{m+a}(\omega) \cap H_0^j(\omega)$ satisfying (3.23). Then, $(u_0^{j+1}, v_0^{j+1}) = (u_0^j + c_0^j, v_0^j + d_0^j)$ in turn fulfill (3.20). Moreover, the estimates above, (3.23), and the usual Sobolev trace theorem yield

$$\begin{aligned}
& \|(u_0, v_0) - (u_0^{j+1}, v_0^{j+1})\|_{H_\Gamma^m(\Omega)} \leq \|(c_0^j, d_0^j)\|_{H^m(\omega)} + \|(u_0, v_0) - (u_0^j, v_0^j)\|_{H_\Gamma^m(\Omega)} \\
& \lesssim \|\text{tr } \partial_\nu^j(c_0^j, d_0^j)\|_{H^{m-j-1/2}(\partial\omega)} + \|(u_0, v_0) - (u_0^j, v_0^j)\|_{H_\Gamma^m(\Omega)} \\
& \lesssim \left\| \left[(u_{(j)}^j \times \nu^\Gamma, v_{(j)}^j \times \nu^\Gamma) \right] - (\partial_t^j \kappa_n(0), \partial_t^j \varkappa_n(0)) \right\|_{H^{m-j-1/2}(\Gamma)} \\
& \quad + \|\text{tr}_t u_{(j)}^j - \partial_t^j h_n(0)\|_{H^{m-j-1/2}(\partial\Omega)} + \|(u_0, v_0) - (u_0^j, v_0^j)\|_{H_\Gamma^m(\Omega)} \\
& \lesssim \left\| \left[(u_{(j)}^{\text{app}} \times \nu^\Gamma, v_{(j)}^{\text{app}} \times \nu^\Gamma) \right] - (\partial_t^j \kappa_n(0), \partial_t^j \varkappa_n(0)) \right\|_{H^{m-j-1/2}(\Gamma)} \\
& \quad + \|\text{tr}_t u_{(j)}^{\text{app}} - \partial_t^j h_n(0)\|_{H^{m-j-1/2}(\partial\Omega)} + \|(u_0, v_0) - (u_0^j, v_0^j)\|_{H_\Gamma^m(\Omega)} \\
& \leq \hat{C}r,
\end{aligned} \tag{3.25}$$

where we also used the assumption and induction hypothesis.

Iteratively, this allows us to construct

$$(u_0^m, v_0^m) = (u_0, v_0) + \sum_{j=1}^{m-1} (c_0^j, d_0^j) \in H_\Gamma^{m+a}(\Omega),$$

satisfying the compatibility condition (CC_m) of order m and (3.19).

2) In order to pass to (CC_{m+a}) , for $a > 0$, we can construct $(\hat{c}_0^{m+a-1}, \hat{d}_0^{m+a-1})$ correcting the compatibility condition iteratively as above. However, we cannot control the norm through the (approximate) compatibility condition of (u_0, v_0) as in (3.25) before. Instead, we rely on a scaling argument in this case. In the following we let the $a = 1$ in the compatibility condition, i.e., we show how to obtain (CC_{m+1}) . We will only focus on $\hat{c}_0^m$ as the argument for $\hat{d}_0^m$ follows similarly.

We take a covering $\{U_i\}_{i=0}^M$ of ω and coordinate charts $\{(U_i, \Psi_i)\}_{i=1}^M$ at the boundary of ω such that $(\mathcal{N}_b^{\partial\Omega} \cup \mathcal{N}_b^\Gamma) \cap \omega \subseteq \cup_{i=1}^M U_i$, $\Psi_i(U_i \cap \omega) = V_i \cap \mathbb{R}_+^3$ and $\Psi_i(U_i \cap \partial\omega) = V_i \cap \{y \in \mathbb{R}^3 \mid y_3 = 0\}$ for $i \geq 1$. Here

$$\mathcal{N}_b^\Xi := \{x \in \bar{\Omega} \mid \text{dist}(x, \Xi) < b\}$$

denotes collars of width $b > 0$ around $\Xi \in \{\partial\Omega, \Gamma\}$. Let ϕ_i be a partition of unity subordinate to $\{U_i \mid i = 0, \dots, M\}$.

Let $\delta \in (0, 1)$. For $i = 1, \dots, M$, let

$$p_i: \mathbb{R}_+^3 \rightarrow \mathbb{R}^3; \quad p_i(y) := \begin{cases} (\phi_i \hat{c}_0^m)(\Psi_i^{-1}(y)), & \text{for } y \in V_i \\ 0, & \text{for } y \notin V_i, \end{cases}$$

denote the local pushforward of $\phi_i \hat{c}_0^m$ extended by zero. We use the scaling matrix $S_\delta :=$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\delta} \end{pmatrix} \text{ and set}$$

$$c_0^{m,\delta}(x) := \delta^m \sum_{i=1}^M p_i(S_\delta \Psi_i(x)), \quad x \in \omega,$$

where $p_i(S_\delta \Psi_i(x))$ is understood to be 0 for $x \notin U_i$. Thus, by substitution

$$\|c_0^{m,\delta}\|_{L^2(\Omega)}^2 \lesssim \sum_{i=1}^M \int_{\mathbb{R}_+^3} \left| \delta^m p_i\left(y', \frac{y_3}{\delta}\right) \right|^2 dy$$

$$\begin{aligned}
&= \sum_{i=1}^M \delta^{2m} \int_{\mathbb{R}_+^3} |p_i(u)|^2 \delta \, du \\
&\lesssim \delta^{2m+1} \|\hat{c}_0^m\|_{L^2(\Omega)}^2.
\end{aligned}$$

Likewise, for the Sobolev norm of order m , we consider derivatives D^α with $|\alpha| \leq m$. The dominant terms in δ arise from the normal derivatives $\partial_{x_3}^\ell$ where $\ell = \alpha_3 \leq m$. Namely, for $\alpha = (\alpha', \ell)$ with $\ell \in \{1, \dots, m\}$, we get

$$\begin{aligned}
\|D^\alpha c_0^{m,\delta}\|_{L^2(\Omega)}^2 &\lesssim \sum_{i=1}^M \int_{\mathbb{R}_+^3} \left| \delta^m \delta^{-\ell} (\partial_{y'}^{\alpha'} \partial_{y_3}^\ell p_i) \left(y', \frac{y_3}{\delta} \right) \right|^2 dy \\
&= \sum_{i=1}^M \delta^{2(m-\ell)} \int_{\mathbb{R}_+^3} |(D^\alpha p_i)(u)|^2 \delta \, du \\
&\leq C \delta^{2(m-\ell)+1} \|\hat{c}_0^m\|_{H_\Gamma^{|\alpha|}(\Omega)}^2
\end{aligned}$$

for some constant $C > 0$. Since $\ell \leq m$, the exponent $2(m-\ell)+1$ is larger or equal to 1. Thus, for $\delta \rightarrow 0$ we have $\|c_0^{m,\delta}\|_{H^m(\omega)} \rightarrow 0$. Without loss of generality we may therefore assume that $\|c_0^m\|_{H^m(\omega)} \leq r$ by choosing $c_0^m := c_0^{m,\delta}$ with $\delta := \frac{r^2}{C\|\hat{c}_0^m\|_{H^m(\Omega)}^2}$.

Note that the m -th order normal derivative at $\partial\omega = \partial\Omega \cup \Gamma$ is unaffected by the scaling. To see this, let us consider the partial derivatives of order m . The derivatives of $(\phi_i \hat{c}_0^m) \left(\Psi_i^{-1}(S_\delta \Psi_i(x)) \right)$ involving derivatives of $\hat{c}_0^m$ of order strictly less than m vanish on $\partial\omega$, since $\hat{c}_0^m \in H_0^m(\omega)$. Hence, for $|\alpha| = m$

$$D^\alpha c_0^{m,\delta}(x) = \delta^m \sum_{i=1}^M \phi_i(x) D^\alpha \left(\hat{c}_0^m \left(\Psi_i^{-1}(S_\delta \Psi_i(x)) \right) \right)$$

on $\partial\omega$. In the following we are thus only interested in terms where ‘all derivatives fall upon $\hat{c}_0^m$ ’.

Let $p_{\delta,i}$ be the scaled function in local coordinates, i.e.,

$$p_{\delta,i}: \mathbb{R}^3 \rightarrow \mathbb{R}^3; \quad p_{\delta,i}(y) = \delta^m p_i(S_\delta y).$$

Note that by the chain rule applied to the m -th derivative, only the normal derivatives (in local coordinates) scale with δ^{-1} by the definition of S_δ . Specifically, on the boundary $\{y_3 = 0\}$

$$\partial_{y_3}^m p_{\delta,i}(y) = \delta^m \left(\frac{1}{\delta} \right)^m (\partial_{y_3}^m p_i)(y) = (\partial_{y_3}^m p_i)(y).$$

By the chain rule we get

$$D c_0^{m,\delta}(x) = \sum_{i=1}^M D|_{(\Psi_i(x))} p_{\delta,i} \cdot D \Psi_i(x) = \sum_{i=1}^M (\partial_{y_3} p_{\delta,i})|_{y=\Psi_i(x)} (\nabla(\Psi_i)_3)^\top \quad \text{on } \partial\omega,$$

since $p_{\delta,i}|_{\{y_3=0\}} = 0$ and hence $D p_{\delta,i} = (0 \mid 0 \mid \partial_{y_3} p_{\delta,i})$ on $\{y_3 = 0\}$. Here $(\cdot)_3$ denotes the third component. Thus, for $x \in \partial\omega$

$$\partial_\nu c_0^{m,\delta}(x) = (\nu \cdot \nabla) c_0^{m,\delta}(x) = \sum_{i=1}^M (\nu \cdot \nabla(\Psi_i)_3) (\partial_{y_3} p_{\delta,i})|_{y=\Psi_i(x)},$$

i.e., the normal derivative corresponds to ∂_{y_3} in local coordinates times the factor $(\nu \cdot \nabla(\Psi_i)_3)$ depending on the chart. (Since $\partial\omega \cap U_i = \{x \in U_i \mid (\Psi_i(x))_3 = 0\}$ the gradient $\nabla(\Psi_i)_3$ is normal to $\partial\omega$.) Similarly, for the m -th derivative we obtain

$$\begin{aligned} \partial_\nu^m c_0^{m,\delta}(x) &= \sum_{i=1}^M (\nu \cdot \nabla(\Psi_i)_3)^m (\partial_{y_3}^m p_{\delta,i})|_{y=\Psi_i(x)} \\ &= \sum_{i=1}^M \phi_i(x) (\nu \cdot \nabla(\Psi_i)_3)^m \partial_{y_3}^m \left(\hat{c}_0^m \left(\Psi_i^{-1}(y) \right) \right)|_{y=\Psi_i(x)} \\ &= \partial_\nu^m \hat{c}_0^m(x). \end{aligned}$$

Note that here we can again neglect terms where derivatives fall upon $\phi_i(x)(\nu \cdot \nabla(\Psi_i)_3)$ and also that $D^\beta p_{\delta,i}|_{\{y_3=0\}} = 0$ for $|\beta| \leq m-1$ because $\hat{c}_0^m \in H_0^m(\omega)$.

For d_0^m we repeat the same argument as for c_0^m . Thus, $(u_0^{m+1}, v_0^{m+1}) := (u_0^m, v_0^m) + (c_0^m, d_0^m)$ satisfies $(CC_{\alpha_n, \beta_n, \sigma_n, \rho_n, f_n, g_n, h_n, \kappa_n, \varkappa_n, m+1})$. Iterating this procedure for $\hat{i} \leq a$ yields (u_0^{m+a}, v_0^{m+a}) satisfying $(CC_{\alpha_n, \beta_n, \sigma_n, \rho_n, f_n, g_n, h_n, \kappa_n, \varkappa_n, m+a})$. $\square$

We used the following lemma in the proof above.

Lemma 3.2.5 (Simplification of $(\alpha \operatorname{curl})^m$ for vanishing traces). *Let $m \in \mathbb{N}$ and let $\alpha_i \in C^\infty(\bar{\omega}; \mathbb{R}_{\text{sym}}^{3 \times 3})$ be positive definite matrices. Let $u \in (H^{m+1}(\omega) \cap H_0^m(\omega))^3$. Define the operator $L_i u := \alpha_i(\operatorname{curl} u)$. Then*

$$\operatorname{tr}(\Pi_{i=1}^m L_i u) = (\Pi_{i=1}^m T_{\alpha_i, x}) \operatorname{tr}(\partial_\nu^m u) \quad \text{on } \partial\omega,$$

where $T_{\alpha_i, x}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the linear map defined for $x \in \partial\omega$ by

$$T_{\alpha_i, x}(\xi) := \alpha_i(x)(\nu(x) \times \xi). \quad (3.26)$$

Proof. 1) We first consider the case $m = 1$. Let $u \in (H^2(\omega) \cap H_0^1(\omega))^3$ and let $u_n \in C^\infty(\bar{\omega})^3$ with $\operatorname{tr} u_n = 0$ approximating u in $H^2(\omega)$. Since $\operatorname{tr} u_n = 0$, the gradient runs orthogonal to $\partial\omega$, i.e.,

$$\partial_j(u_n)_k = (\nabla(u_n)_k)_j = (\nu \cdot \nabla(u_n)_k) \nu_j \quad \text{for } j, k \in \{1, 2, 3\}.$$

By approximation this equality also holds for u . The curl now becomes

$$(\operatorname{curl} u)_i = \epsilon_{ijk} \partial_j u_k = \epsilon_{ijk} \nu_j (\nu \cdot \nabla u_k) = (\nu \times \partial_\nu u)_i,$$

where ϵ_{ijk} for $i, j, k \in \{1, 2, 3\}$ denotes the Levi-Civita symbol. Thus,

$$\operatorname{tr}(\alpha_1 \operatorname{curl} u) = T_{\alpha_1, x} \partial_\nu u.$$

2) Now let us consider the general case $u \in H^{m+1} \cap H_0^m$. We aim to compute the trace of $\Pi_{i=1}^m L_i u = \Pi_{i=1}^m (\alpha_i \operatorname{curl} u)$. For $u \in H_0^m(\omega)$ the normal derivatives up to order $m-1$ are zero

$$\partial_\nu^\ell u|_{\partial\omega} = 0 \quad \text{for } \ell = 0, 1, \dots, m-1.$$

By the product rule

$$\Pi_{i=1}^m L_i u = P^m u + R(u, Du, \dots, D^{m-1} u),$$

where P^m is the principal part of $\Pi_{i=1}^m L_i$ and R is a linear lower order differential operator. Since $u \in H_0^m(\omega)$ the remainder $R(u, Du, \dots, D^{m-1} u)$ vanishes on $\partial\omega$.

So, an iteration of Step 1) yields

$$\operatorname{tr}(\Pi_{i=1}^m L_i u) = T_{\alpha_m, x}(\partial_\nu(T_{\alpha_{m-1}, x}(\partial_\nu(\dots T_{\alpha_1, x}(\partial_\nu u) \dots)))).$$

Moreover, the commutator $[\partial_\nu, T_{\alpha_i, x}]$ results in lower order terms, which again vanish on the boundary, i.e.,

$$\partial_\nu(T_{\alpha_i, x}\psi) = (\partial_\nu T_{\alpha_i, x})\psi + T_{\alpha_i, x}(\partial_\nu\psi) = T_{\alpha_i, x}(\partial_\nu\psi).$$

Thus, we commute all normal derivatives to the right obtaining

$$\text{tr}(\Pi_{i=1}^m L_i u) = (T_{\alpha_m, x} \circ \cdots \circ T_{\alpha_1, x}) \partial_\nu^m u \quad \text{on } \partial\omega. \quad \square$$

3.3. CONVERGENCE RESULT

Based on the previous section, we will work in the following setting:

(H_a) Let $a \in \mathbb{N}_0$, $m \in \mathbb{N}$ with $\tilde{m} + 2 + a \leq k$. Let $(u, v) \in \mathcal{G}_\Gamma^m$ be a solution to (3.6)–(3.9) with initial values (u_0, v_0) and coefficients satisfying (3.11) – (3.13). We approximate our coefficients as in (App). By Lemma 3.2.1 and Proposition 3.2.4 we may choose initial values $(u_{0,n}^{m+a}, v_{0,n}^{m+a}) \in H_\Gamma^{m+a}(\Omega)$ satisfying $(CC_{\alpha_n, \beta_n, \sigma_n, \rho_n, f_n, g_n, h_n, \kappa_n, \varkappa_n, m+a})$ converging to the original initial data $(u_0, v_0) \in H_\Gamma^m(\Omega)$. We denote the corresponding solutions by $(u_n, v_n) \in \mathcal{G}_\Gamma^{m+a}$.

We want to show the convergence to our original solution (u, v) in $\mathcal{G}_\Gamma^m$. Unfortunately, considering the difference between the approximate solution and the exact solution

$$U := u_n - u \quad \text{and} \quad V := v_n - v$$

leads to the system

$$\begin{aligned} \alpha \partial_t U &= \text{curl } V - \sigma U - \mathfrak{f}_n, & t \geq 0, x \in \Omega_\pm, \\ \beta \partial_t V &= -\text{curl } U - \rho V - \mathfrak{g}_n, \\ U \times \nu &= h_n - h, & t \geq 0, x \in \partial\Omega, \\ \llbracket U \times \nu^\Gamma \rrbracket &= \kappa_n - \kappa, \quad \llbracket V \times \nu^\Gamma \rrbracket = \varkappa_n - \varkappa, & t \geq 0, x \in \Gamma, \\ U(0) &= u_{0,n} - u_0, \quad V(0) = v_{0,n} - v_0, & x \in \Omega_\pm. \end{aligned}$$

The inhomogeneities $\mathfrak{f}_n$ and $\mathfrak{g}_n$ lie in $L^2(0, T; L^2(\Omega)^3)$ and are given by

$$\begin{aligned} \mathfrak{f}_n &:= (f_n - f) - (\alpha - \alpha_n) \partial_t u_n + (\sigma - \sigma_n) u_n, \\ \mathfrak{g}_n &:= (g_n - g) - (\beta - \beta_n) \partial_t v_n + (\rho - \rho_n) v_n. \end{aligned}$$

In this direct approach the terms $(\alpha - \alpha_n) \partial_t u_n$ and $(\beta - \beta_n) \partial_t v_n$ cause trouble. While the difference of the coefficients vanishes in the right norm, one lacks control over the time derivative. However, the a-priori estimate (3.15) at least yields convergence in $\mathcal{G}_\Gamma^{m-1}$.

To obtain $\mathcal{G}_\Gamma^m$ -convergence, we first mollify the original solution in time direction. This allows us to show that the time derivatives converge. Afterwards, we inductively reduce the convergence of spatial norms to the temporal ones using a div-curl lemma, see Theorem A.2.2, and the Maxwell system.

Definition 3.3.1 (Mollifier). In the following, let $\eta_\varepsilon(t) = \frac{1}{\varepsilon} \eta(\frac{t}{\varepsilon})$ for the usual mollifier

$$\eta(t) := \begin{cases} C \exp\left(\frac{1}{|t|^2 - 1}\right), & |t| \leq 1, \\ 0, & |t| > 1, \end{cases}$$

where $C > 0$ is chosen such that $\int_{\mathbb{R}} \eta(t) dt = 1$. Let $(u)_\varepsilon := u * \eta_\varepsilon$ denote the mollification in time.

We will need the following elementary mollification result later on.

Lemma 3.3.2. *Let $T > 0$, $\varepsilon_0 > 0$, $u \in C([- \varepsilon_0, T + \varepsilon_0]; L^2(\Omega)^3)$ and $\alpha \in C^1([- \varepsilon_0, T + \varepsilon_0]; H_\Gamma^2(\Omega, \mathbb{R}_{\text{sym}}^{3 \times 3}))$. Then,*

$$\partial_t(\alpha u)_\varepsilon - \alpha \partial_t u_\varepsilon \rightarrow (\partial_t \alpha)u \quad \text{in } C([0, T]; L^2(\Omega)^3),$$

for $\varepsilon < \varepsilon_0$ and $\varepsilon \rightarrow 0$.

Proof. In the following, let $\varepsilon < \varepsilon_0$. Let $\mathcal{M}_\varepsilon(t) = \partial_t(\alpha u)_\varepsilon(t) - \alpha(t) \partial_t u_\varepsilon(t)$. Expanding the convolutions and applying the change of variables $\tau = \frac{t-s}{\varepsilon}$, the commutator evaluated at time $t \in [0, T]$ is given by

$$\mathcal{M}_\varepsilon(t) = \int_{-1}^1 \frac{\alpha(t - \varepsilon\tau) - \alpha(t)}{\varepsilon} u(t - \varepsilon\tau) \eta'(\tau) d\tau.$$

We rewrite the limit as

$$\partial_t \alpha(t) u(t) = -\partial_t \alpha(t) u(t) \int_{-1}^1 \tau \eta'(\tau) d\tau$$

and consider the difference

$$\mathcal{M}_\varepsilon(t) - (\partial_t \alpha(t))u(t) = I_1(t, \varepsilon) + I_2(t, \varepsilon),$$

where

$$\begin{aligned} I_1(t, \varepsilon) &= \int_{-1}^1 \left[\frac{\alpha(t - \varepsilon\tau) - \alpha(t)}{\varepsilon} + \tau \partial_t \alpha(t) \right] u(t - \varepsilon\tau) \eta'(\tau) d\tau, \\ I_2(t, \varepsilon) &= - \int_{-1}^1 \tau \partial_t \alpha(t) (u(t - \varepsilon\tau) - u(t)) \eta'(\tau) d\tau. \end{aligned}$$

Since $\alpha \in C^1([- \varepsilon_0, T + \varepsilon_0]; L^\infty(\Omega)^{3 \times 3})$ by the Sobolev embeddings, we see that

$$\begin{aligned} \sup_{t \in [0, T], \tau \in [-1, 1]} \left\| \frac{\alpha(t - \varepsilon\tau) - \alpha(t)}{\varepsilon} + \tau \partial_t \alpha(t) \right\|_{L^\infty(\Omega)} &\rightarrow 0, \\ \sup_{t \in [0, T], \tau \in [-1, 1]} \|u(t - \varepsilon\tau) - u(t)\|_{L^2(\Omega)} &\rightarrow 0, \end{aligned}$$

for $\varepsilon \rightarrow 0$. Hence,

$$\sup_{t \in [0, T]} I_1(t, \varepsilon), \sup_{t \in [0, T]} I_2(t, \varepsilon) \rightarrow 0 \quad \text{for } \varepsilon \rightarrow 0$$

and the claim follows. $\square$

Recall that the $\mathcal{G}_\Gamma^m$ norm is given by

$$\|u\|_{\mathcal{G}_\Gamma^m([0, T] \times \Omega)} := \sup_{t \in [0, T]} \sum_{j=0}^m \|\partial_t^j u(t)\|_{H_\Gamma^{m-j}}.$$

We first focus on the highest time derivative, i.e., $\sup_{t \in [0, T]} \|\partial_t^m u(t)\|_{L^2(\Omega)}$.

Lemma 3.3.3 (Convergence of Time Derivatives). *Let (H_a) hold. Then,*

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left(\|\partial_t^m u_n(t) - \partial_t^m u(t)\|_{L^2(\Omega)} + \|\partial_t^m v_n(t) - \partial_t^m v(t)\|_{L^2(\Omega)} \right) = 0.$$

Proof. 1) In order to apply a mollifier on $[0, T]$, we continuously extend the coefficients $\alpha, \alpha_n, \beta, \beta_n \in \mathcal{C}_\eta^m$, $\sigma, \sigma_n, \rho, \rho_n \in \mathcal{C}^m$, $f, f_n, g, g_n \in H^m([0, T] \times \Omega_\pm)$, and the extensions of the boundary and interface data $h^\circ, h_n^\circ, \kappa^\circ, \kappa_n^\circ, \varkappa^\circ, \varkappa_n^\circ \in \mathcal{G}_\Gamma^{m+1}$ to a slightly larger time interval $[-\varepsilon_0, T + \varepsilon_0]$ by the usual extension operator as in Theorem 5.19 in [1]. Theorem 3.1 in [56] then provides extensions of the solutions $(u, v), (u_n, v_n)$ in $\mathcal{G}_\Gamma^m([0, T])$ respectively $\mathcal{G}_\Gamma^{m+a}([0, T])$ to $\mathcal{G}_\Gamma^m([-\varepsilon_0, T + \varepsilon_0])$ respectively $\mathcal{G}_\Gamma^{m+a}([-\varepsilon_0, T + \varepsilon_0])$. From now on, we denote the extensions by the same letters.

Let $U_n := \partial_t^m u_n$, $V_n := \partial_t^m v_n$ and $U := \partial_t^m u$, $V := \partial_t^m v$ denote the highest-order time derivatives. We mollify U_n on $[0, T]$, yielding $U_{n,\delta} := (U_n)_\delta$ for $0 < \delta < \varepsilon_0$. Likewise, we mollify U on $[0, T]$ and obtain $U_\varepsilon := (U)_\varepsilon$ for $0 < \varepsilon < \varepsilon_0$. (For the approximating sequence this has to be done to include the case of $a = 0$. Otherwise, the proof simplifies and only requires the mollification of (U, V) . Regardless, the additional arguments are quite similar to that case.)

We define the differences $D_{n,\delta,\varepsilon}^u := U_{n,\delta} - U_\varepsilon$ and $D_{n,\delta,\varepsilon}^v := V_{n,\delta} - V_\varepsilon$, and set

$$e_{n,\delta,\varepsilon}^D(t) := \frac{1}{2} \int_\Omega \left(D_{n,\delta,\varepsilon}^u \cdot \alpha_n D_{n,\delta,\varepsilon}^u + D_{n,\delta,\varepsilon}^v \cdot \beta_n D_{n,\delta,\varepsilon}^v \right) dx.$$

Moreover, as in (3.16) there exist extensions (W_n^u, W_n^v) and (W^u, W^v) in $\mathcal{G}_\Gamma^{m+1}$ matching the data simultaneously, i.e.,

$$\begin{aligned} W^u \times \nu &= h \quad \text{on } \partial\Omega, & \llbracket W^u \times \nu^\Gamma \rrbracket &= \kappa, & \llbracket W^v \times \nu^\Gamma \rrbracket &= \varkappa, \\ W_n^u \times \nu &= h_n \quad \text{on } \partial\Omega, & \llbracket W_n^u \times \nu^\Gamma \rrbracket &= \kappa_n, & \text{and } \llbracket W_n^v \times \nu^\Gamma \rrbracket &= \varkappa_n. \end{aligned}$$

Because the extensions lie in $\mathcal{G}_\Gamma^{m+1}$, their m -th time derivatives inherently belong to $H_\Gamma^1(\Omega) \subset H(\text{curl}, \Omega_\pm)$ with the $(m+1)$ -th time derivatives in $L^2(\Omega)$. Furthermore,

$$\begin{aligned} \sup_{t \in [-\varepsilon_0, T + \varepsilon_0]} \left(\|\partial_t^m W_n^u - \partial_t^m W^u\|_{H_\Gamma^1(\Omega)} + \|\partial_t^m W_n^v - \partial_t^m W^v\|_{H_\Gamma^1(\Omega)} \right) \\ \lesssim \|(h_n^\circ - h^\circ, \kappa_n^\circ - \kappa, \varkappa_n^\circ - \varkappa)\|_{\mathcal{G}_\Gamma^{m+1}} \rightarrow 0 \end{aligned}$$

for $n \rightarrow \infty$.

Let $E_{n,\delta,\varepsilon}^u := (\partial_t^m W_n^u)_\delta - (\partial_t^m W^u)_\varepsilon$ and $E_{n,\delta,\varepsilon}^v := (\partial_t^m W_n^v)_\delta - (\partial_t^m W^v)_\varepsilon$. We define the shifted differences

$$\tilde{D}_{n,\delta,\varepsilon}^u := D_{n,\delta,\varepsilon}^u - E_{n,\delta,\varepsilon}^u \quad \text{and} \quad \tilde{D}_{n,\delta,\varepsilon}^v := D_{n,\delta,\varepsilon}^v - E_{n,\delta,\varepsilon}^v,$$

which by construction satisfy homogeneous boundary and jump conditions, allowing us to integrate by parts.

We set the shifted energy

$$\tilde{e}_{n,\delta,\varepsilon}^D(t) := \frac{1}{2} \int_\Omega \left(\tilde{D}_{n,\delta,\varepsilon}^u \cdot \alpha_n \tilde{D}_{n,\delta,\varepsilon}^u + \tilde{D}_{n,\delta,\varepsilon}^v \cdot \beta_n \tilde{D}_{n,\delta,\varepsilon}^v \right) dx.$$

Applying the time derivative in the classical sense yields

$$\begin{aligned} \frac{d}{dt} \tilde{e}_{n,\delta,\varepsilon}^D(t) &= \int_\Omega \left(\tilde{D}_{n,\delta,\varepsilon}^u \cdot \alpha_n \partial_t \tilde{D}_{n,\delta,\varepsilon}^u + \tilde{D}_{n,\delta,\varepsilon}^v \cdot \beta_n \partial_t \tilde{D}_{n,\delta,\varepsilon}^v \right) dx \\ &\quad + \frac{1}{2} \int_\Omega \left(\tilde{D}_{n,\delta,\varepsilon}^u \cdot (\partial_t \alpha_n) \tilde{D}_{n,\delta,\varepsilon}^u + \tilde{D}_{n,\delta,\varepsilon}^v \cdot (\partial_t \beta_n) \tilde{D}_{n,\delta,\varepsilon}^v \right) dx. \end{aligned} \tag{3.27}$$

In the following, we will only consider u and u_n , since the calculations for v and v_n are done similarly. We intend to apply a Gronwall argument to (3.27). For this, we first expand $\alpha_n \partial_t \tilde{D}_{n,\delta,\varepsilon}^u = \alpha_n \partial_t D_{n,\delta,\varepsilon}^u - \alpha_n \partial_t E_{n,\delta,\varepsilon}^u$ to substitute expressions for the time derivatives.

The $(m-1)$ -th level system for u_n is

$$\alpha_n U_n = \operatorname{curl}(\partial_t^{m-1} v_n) - \sigma_n \partial_t^{m-1} u_n - \partial_t^{m-1} f_n - \mathcal{D}_n^u, \quad (3.28)$$

where $\mathcal{D}_n^u := [\partial_t^{m-1}, \alpha_n \partial_t] u_n + [\partial_t^{m-1}, \sigma_n] u_n$ is the commutator with the derivative. We further set $\mathcal{M}_\delta^n := \partial_t(\alpha_n U_n)_\delta - \alpha_n \partial_t U_{n,\delta}$, mollify (3.28) and take the time derivative. It follows that

$$\alpha_n \partial_t U_{n,\delta} = \operatorname{curl} V_{n,\delta} - \partial_t(\sigma_n \partial_t^{m-1} u_n)_\delta - (\partial_t^m f_n)_\delta - \partial_t(\mathcal{D}_n^u)_\delta - \mathcal{M}_\delta^n. \quad (3.29)$$

Similarly, for U we mollify the $(m-1)$ -th level equation

$$\alpha U = \operatorname{curl}(\partial_t^{m-1} v) - \sigma \partial_t^{m-1} u - \partial_t^{m-1} f - \mathcal{D}^u$$

with $\mathcal{D}^u := [\partial_t^{m-1}, \alpha \partial_t] u + [\partial_t^{m-1}, \sigma] u$. Also applying the time derivative and introducing $\mathcal{M}_\varepsilon := \partial_t(\alpha U)_\varepsilon - \alpha \partial_t U_\varepsilon$ yields

$$\alpha \partial_t U_\varepsilon = \operatorname{curl} V_\varepsilon - \partial_t(\sigma \partial_t^{m-1} u)_\varepsilon - (\partial_t^m f)_\varepsilon - \partial_t(\mathcal{D}^u)_\varepsilon - \mathcal{M}_\varepsilon. \quad (3.30)$$

Writing $\alpha_n \partial_t U_\varepsilon = \alpha \partial_t U_\varepsilon + (\alpha_n - \alpha) \partial_t U_\varepsilon$, Formulas (3.29) and (3.30) provide the equation

$$\alpha_n \partial_t D_{n,\delta,\varepsilon}^u = \operatorname{curl} D_{n,\delta,\varepsilon}^v - (\sigma_n + m \partial_t \alpha_n) D_{n,\delta,\varepsilon}^u + R_{n,\delta,\varepsilon}^u$$

for the unshifted difference, where the remainder $R_{n,\delta,\varepsilon}^u$ is given by

$$\begin{aligned} R_{n,\delta,\varepsilon}^u := & -(\alpha_n - \alpha) \partial_t U_\varepsilon + \sigma_n U_{n,\delta} - \sigma_n U_\varepsilon + m(\partial_t \alpha_n) U_{n,\delta} - m(\partial_t \alpha_n) U_\varepsilon \\ & + \partial_t(\sigma \partial_t^{m-1} u)_\varepsilon - \partial_t(\sigma_n \partial_t^{m-1} u_n)_\delta + (\partial_t^m f)_\varepsilon - (\partial_t^m f_n)_\delta \\ & + \partial_t(\mathcal{D}^u)_\varepsilon - \partial_t(\mathcal{D}_n^u)_\delta + \mathcal{M}_\varepsilon - \mathcal{M}_\delta^n. \end{aligned} \quad (3.31)$$

We added the $-(\sigma_n + m \partial_t \alpha_n) D_{n,\delta,\varepsilon}^u$ -term to rebalance the remainder later on.

Substituting the shifted variables, $(\tilde{D}_{n,\delta,\varepsilon}^u, \tilde{D}_{n,\delta,\varepsilon}^v)$ satisfy

$$\alpha_n \partial_t \tilde{D}_{n,\delta,\varepsilon}^u = \operatorname{curl} \tilde{D}_{n,\delta,\varepsilon}^v - (\sigma_n + m \partial_t \alpha_n) \tilde{D}_{n,\delta,\varepsilon}^u + \tilde{R}_{n,\delta,\varepsilon}^u,$$

with the modified remainder

$$\tilde{R}_{n,\delta,\varepsilon}^u := R_{n,\delta,\varepsilon}^u - \alpha_n \partial_t E_{n,\delta,\varepsilon}^u + \operatorname{curl} E_{n,\delta,\varepsilon}^v - (\sigma_n + m \partial_t \alpha_n) E_{n,\delta,\varepsilon}^u. \quad (3.32)$$

Analogously, the difference of the magnetic terms satisfies

$$\beta_n \partial_t \tilde{D}_{n,\delta,\varepsilon}^v = -\operatorname{curl} \tilde{D}_{n,\delta,\varepsilon}^u - (\rho_n + m \partial_t \beta_n) \tilde{D}_{n,\delta,\varepsilon}^v + \tilde{R}_{n,\delta,\varepsilon}^v,$$

where $\tilde{R}_{n,\delta,\varepsilon}^v := R_{n,\delta,\varepsilon}^v - \operatorname{curl} E_{n,\delta,\varepsilon}^u - \beta_n \partial_t E_{n,\delta,\varepsilon}^v - (\rho_n + m \partial_t \beta_n) E_{n,\delta,\varepsilon}^v$.

We substitute these equations into (3.27) and integrate by parts. Since $\tilde{D}_{n,\delta,\varepsilon}^u$ and $\tilde{D}_{n,\delta,\varepsilon}^v$ satisfy homogeneous boundary and jump conditions

$$\tilde{D}_{n,\delta,\varepsilon}^u \times \nu = 0 \text{ on } \partial\Omega \quad \text{and} \quad \llbracket (\tilde{D}_{n,\delta,\varepsilon}^u \times \nu^\Gamma, \tilde{D}_{n,\delta,\varepsilon}^v \times \nu^\Gamma) \rrbracket = 0,$$

all surface integrals vanish. It follows that

$$\begin{aligned} \frac{d}{dt} \tilde{e}_{n,\delta,\varepsilon}^D(t) = & \int_{\Omega} \left(\tilde{D}_{n,\delta,\varepsilon}^u \cdot (\operatorname{curl} \tilde{D}_{n,\delta,\varepsilon}^v - (\sigma_n + m \partial_t \alpha_n) \tilde{D}_{n,\delta,\varepsilon}^u + \tilde{R}_{n,\delta,\varepsilon}^u) \right) dx \\ & + \int_{\Omega} \left(\tilde{D}_{n,\delta,\varepsilon}^v \cdot (-\operatorname{curl} \tilde{D}_{n,\delta,\varepsilon}^u - (\rho_n + m \partial_t \beta_n) \tilde{D}_{n,\delta,\varepsilon}^v + \tilde{R}_{n,\delta,\varepsilon}^v) \right) dx \\ & + \frac{1}{2} \int_{\Omega} \left(\tilde{D}_{n,\delta,\varepsilon}^u \cdot (\partial_t \alpha_n) \tilde{D}_{n,\delta,\varepsilon}^u + \tilde{D}_{n,\delta,\varepsilon}^v \cdot (\partial_t \beta_n) \tilde{D}_{n,\delta,\varepsilon}^v \right) dx \end{aligned}$$

$$\begin{aligned}
&= \int_{\Omega} \left(\tilde{D}_{n,\delta,\varepsilon}^u \cdot (-\sigma_n \tilde{D}_{n,\delta,\varepsilon}^u + \tilde{R}_{n,\delta,\varepsilon}^u) + \tilde{D}_{n,\delta,\varepsilon}^v \cdot (-\rho_n \tilde{D}_{n,\delta,\varepsilon}^v + \tilde{R}_{n,\delta,\varepsilon}^v) \right) dx \\
&\quad + \int_{\Omega} \left(\tilde{D}_{n,\delta,\varepsilon}^u \cdot \left(\frac{1}{2} - m\right) (\partial_t \alpha_n) \tilde{D}_{n,\delta,\varepsilon}^u + \tilde{D}_{n,\delta,\varepsilon}^v \cdot \left(\frac{1}{2} - m\right) (\partial_t \beta_n) \tilde{D}_{n,\delta,\varepsilon}^v \right) dx \\
&\leq C_{\eta} (1 + \|(\partial_t \alpha_n, \partial_t \beta_n, \sigma_n, \rho_n)(t)\|_{L^\infty}) \tilde{e}_{n,\delta,\varepsilon}^D(t) \\
&\quad + \|\tilde{R}_{n,\delta,\varepsilon}^u(t)\|_{L^2(\Omega)}^2 + \|\tilde{R}_{n,\delta,\varepsilon}^v(t)\|_{L^2(\Omega)}^2
\end{aligned}$$

for some constant $C_{\eta} > 0$.

Gronwall's inequality then yields

$$\tilde{e}_{n,\delta,\varepsilon}^D(t) \leq e^{C_G t} \left(\tilde{e}_{n,\delta,\varepsilon}^D(0) + \int_0^t \left(\|\tilde{R}_{n,\delta,\varepsilon}^u(s)\|_{L^2(\Omega)}^2 + \|\tilde{R}_{n,\delta,\varepsilon}^v(s)\|_{L^2(\Omega)}^2 \right) ds \right) \quad (3.33)$$

with the constant $C_G := \sup_{n \in \mathbb{N}, t \in [0, T]} C_{\eta} (1 + \|(\partial_t \alpha_n, \partial_t \beta_n, \sigma_n, \rho_n)(t)\|_{L^\infty}) > 0$.

2) For fixed n and ε , we take the limit $\delta \rightarrow 0$. We define the limits $E_{n,\varepsilon}^u := \partial_t^m W_n^u - (\partial_t^m W^u)_\varepsilon$ and $E_{n,\varepsilon}^v := \partial_t^m W_n^v - (\partial_t^m W^v)_\varepsilon$, as well as the shifted differences $\tilde{D}_{n,\varepsilon}^u := D_{n,\varepsilon}^u - E_{n,\varepsilon}^u$ and $\tilde{D}_{n,\varepsilon}^v := D_{n,\varepsilon}^v - E_{n,\varepsilon}^v$, alongside the corresponding energy $\tilde{e}_{n,\varepsilon}^D(t) := \frac{1}{2} \int_{\Omega} (\tilde{D}_{n,\varepsilon}^u \cdot \alpha_n \tilde{D}_{n,\varepsilon}^u + \tilde{D}_{n,\varepsilon}^v \cdot \beta_n \tilde{D}_{n,\varepsilon}^v) dx$.

Since $(u_n, v_n) \in \mathcal{G}_{\Gamma}^m$, and hence $\partial_t^m u_n, \partial_t^m v_n \in C([- \varepsilon_0, T + \varepsilon_0]; L^2(\Omega))$, standard results on mollifiers yield the uniform convergence $U_{n,\delta} \rightarrow U_n$ and $V_{n,\delta} \rightarrow V_n$ in $C([0, T]; L^2(\Omega))$ for $\delta \rightarrow 0$. Furthermore, $(\partial_t^m f_n)_\delta$ tends to $\partial_t^m f_n$ in $L^2([0, T] \times \Omega)$, $\partial_t(\sigma_n \partial_t^{m-1} u_n)_\delta$ to $\partial_t(\sigma_n \partial_t^{m-1} u_n)$ in $C([0, T], L^2(\Omega))$, $\partial_t(\mathcal{D}_n^u)_\delta$ to $\partial_t \mathcal{D}_n^u$ in $C([0, T], L^2(\Omega))$, and by Lemma 3.3.2, the commutator $\mathcal{M}_\delta^n$ to $(\partial_t \alpha_n) U_n$ in $C([0, T], L^2(\Omega))$ for $\delta \rightarrow 0$.

Additionally, since $\partial_t^m(W_n^u, W_n^v)$ and $\partial_t^m(W^u, W^v)$ belong to $\mathcal{G}_{\Gamma}^1$ the mollified functions $E_{n,\delta,\varepsilon}^u \rightarrow E_{n,\varepsilon}^u$ and $E_{n,\delta,\varepsilon}^v \rightarrow E_{n,\varepsilon}^v$ converge in $\mathcal{G}_{\Gamma}^1$ for $\delta \rightarrow 0$. In particular, this guarantees the pointwise convergence of the energy $\tilde{e}_{n,\delta,\varepsilon}^D(t) \rightarrow \tilde{e}_{n,\varepsilon}^D(t)$ for $t \in [0, T]$.

Let us next focus on the electric remainder $\tilde{R}_{n,\delta,\varepsilon}^u$. We expand the time derivative of the commutator $\partial_t \mathcal{D}_n^u = \partial_t(\partial_t^{m-1}(\alpha_n \partial_t u_n) - \alpha_n \partial_t^m u_n + \partial_t^{m-1}(\sigma_n u_n) - \sigma_n \partial_t^{m-1} u_n)$. Observe that

$$\begin{aligned}
\partial_t(\partial_t^{m-1}(\alpha_n \partial_t u_n) - \alpha_n \partial_t^m u_n) &= \partial_t \left(\sum_{j=1}^{m-1} \binom{m-1}{j} \partial_t^j \alpha_n \partial_t^{m-j} u_n \right) \\
&= \sum_{j=2}^{m-1} \left(\binom{m-1}{j-1} + \binom{m-1}{j} \right) \partial_t^j \alpha_n \partial_t^{m-j+1} u_n \\
&\quad + \binom{m-1}{m-1} \partial_t^m \alpha_n \partial_t u_n + \binom{m-1}{1} \partial_t \alpha_n \partial_t^m u_n \\
&= \sum_{j=1}^m \binom{m}{j} \partial_t^j \alpha_n \partial_t^{m-j+1} u_n - (\partial_t \alpha_n) \partial_t^m u_n.
\end{aligned}$$

A similar calculation for $\partial_t(\partial_t^{m-1}(\sigma_n u_n) - \sigma_n \partial_t^{m-1} u_n)$ allows us to rewrite $\partial_t \mathcal{D}_n^u = \mathcal{L}_n^u + (m-1)(\partial_t \alpha_n) U_n - (\partial_t \sigma_n) \partial_t^{m-1} u_n$ with

$$\mathcal{L}_n^u := \sum_{j=2}^m \binom{m}{j} \partial_t^j \alpha_n \partial_t^{m-j+1} u_n + \sum_{j=1}^m \binom{m}{j} \partial_t^j \sigma_n \partial_t^{m-j} u_n,$$

Thus, all remaining terms containing U_n in (3.31), namely $\sigma_n U_{n,\delta} - \partial_t(\sigma_n \partial_t^{m-1} u)_\delta - \mathcal{M}_\delta^n + m(\partial_t \alpha_n) U_{n,\delta}$ cancel in the limit with $(m-1)(\partial_t \alpha_n) U_n - (\partial_t \sigma_n) \partial_t^{m-1} u_n$, leaving the lower order term $\mathcal{L}_n^u$. Thus, $\tilde{R}_{n,\delta,\varepsilon}^u$ converges to $\tilde{R}_{n,\varepsilon}^u$ in $L^2([0, T] \times \Omega)$ as $\delta \rightarrow 0$ given by

$$\tilde{R}_{n,\varepsilon}^u := -(\alpha_n - \alpha) \partial_t U_\varepsilon - \sigma_n U_\varepsilon - m(\partial_t \alpha_n) U_\varepsilon$$

$$\begin{aligned}
& + \partial_t(\sigma \partial_t^{m-1} u)_\varepsilon + (\partial_t^m f)_\varepsilon - \partial_t^m f_n \\
& + \partial_t(\mathcal{D}^u)_\varepsilon - \mathcal{L}_n^u + \mathcal{M}_\varepsilon \\
& - \alpha_n \partial_t E_{n,\varepsilon}^u + \operatorname{curl} E_{n,\varepsilon}^v - (\sigma_n + m \partial_t \alpha_n) E_{n,\varepsilon}^u,
\end{aligned}$$

cf. Equations (3.31) and (3.32). The magnetic terms are treated analogously. Hence, (3.33) reduces to

$$\tilde{e}_{n,\varepsilon}^D(t) \leq e^{C_G t} \left(\tilde{e}_{n,\varepsilon}^D(0) + \int_0^t \left(\|\tilde{R}_{n,\varepsilon}^u(s)\|_{L^2(\Omega)}^2 + \|\tilde{R}_{n,\varepsilon}^v(s)\|_{L^2(\Omega)}^2 \right) ds \right) \quad (3.34)$$

for $\delta \rightarrow 0$.

3) Next, we show that $\tilde{e}_{n,\varepsilon}^D(0)$ vanishes for $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$. We have

$$\begin{aligned}
\|\tilde{D}_{n,\varepsilon}^u(0)\|_{L^2(\Omega)} & \leq \|D_{n,\varepsilon}^u(0)\|_{L^2(\Omega)} + \|E_{n,\varepsilon}^u(0)\|_{L^2(\Omega)} \\
& \leq \|\partial_t^m u_n(0) - \partial_t^m u(0)\|_{L^2(\Omega)} + \|\partial_t^m u(0) - (\partial_t^m u)_\varepsilon(0)\|_{L^2(\Omega)} \\
& \quad + \|\partial_t^m W_n^u(0) - \partial_t^m W^u(0)\|_{L^2(\Omega)} + \|\partial_t^m W^u(0) - (\partial_t^m W^u)_\varepsilon(0)\|_{L^2(\Omega)}.
\end{aligned}$$

We treat these limits separately.

To control the first term, we evaluate the time derivatives of the solutions at $t = 0$ recursively by applying the Maxwell system. So, we derive

$$\partial_t^m(u_0, v_0) = ((u_0)_{(m)}, (v_0)_{(m)}) \quad \text{and} \quad \partial_t^m(u_{0,n}, v_{0,n}) = ((u_{0,n})_{(m)}, (v_{0,n})_{(m)}),$$

see Definition 3.1.2, which only depend on the initial data and its spatial derivatives at $t = 0$. By (H_a), the initial data converge $(u_{0,n}, v_{0,n}) \rightarrow (u_0, v_0)$ in $H_\Gamma^m(\Omega)$, and similarly for the inhomogeneities, coefficients, boundary, and interface data. We thus obtain the convergence of $\partial_t^m(u_n, v_n)(0) \rightarrow \partial_t^m(u, v)(0)$ in $L^2(\Omega)$.

For the second term, we proceed as above, using the extensions to $[-\varepsilon_0, T + \varepsilon_0]$. By standard properties of mollifiers, the convergence is uniform in time. Thus,

$$\lim_{\varepsilon \rightarrow 0} \|\partial_t^m u(0) - (\partial_t^m u)_\varepsilon(0)\|_{L^2(\Omega)} = 0, \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \|\partial_t^m v(0) - (\partial_t^m v)_\varepsilon(0)\|_{L^2(\Omega)} = 0.$$

Moreover, since the extensions are chosen such that $W_n^u \rightarrow W^u$ in $\mathcal{G}_\Gamma^{m+1}$, we in particular have $\partial_t^m W_n^u(0) \rightarrow \partial_t^m W^u(0)$. Likewise, one can show the convergence of the magnetic difference $D_{n,\varepsilon}^v(0)$. Hence,

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \tilde{e}_{n,\varepsilon}^D(0) = 0.$$

4) For fixed $\varepsilon \in (0, \varepsilon_0)$, we now take the limit $n \rightarrow \infty$. Note that $(\alpha_n - \alpha) \partial_t U_\varepsilon \rightarrow 0$ in $L^2([0, T] \times \Omega)$. The boundedness of (u_n, v_n) in $\mathcal{G}_\Gamma^m$, convergence $(u_n, v_n) \rightarrow (u, v)$ in $\mathcal{G}_\Gamma^{m-1}$ from the a-priori estimate (3.15) and the Sobolev embeddings shows that the commutator

$$\mathcal{L}_n^u \rightarrow \mathcal{L}^u := \sum_{j=2}^m \binom{m}{j} \partial_t^j \alpha \partial_t^{m-j+1} u + \sum_{j=1}^m \binom{m}{j} \partial_t^j \sigma \partial_t^{m-j} u$$

converges in $L^2([0, T] \times \Omega)$ for $n \rightarrow \infty$. Additionally, since $W_n^u \rightarrow W^u$, we have $E_{n,\varepsilon}^u \rightarrow E_\varepsilon^u := \partial_t^m W^u - (\partial_t^m W^u)_\varepsilon$ in $H^1(\Omega)$, and its time derivative in $L^2(\Omega)$.

Thus, $\tilde{R}_{n,\varepsilon}^u$ tends to

$$\tilde{R}_\varepsilon^u := R_\varepsilon^u - \alpha \partial_t E_\varepsilon^u + \operatorname{curl} E_\varepsilon^v - (\sigma + m \partial_t \alpha) E_\varepsilon^u$$

where

$$R_\varepsilon^u := \partial_t(\sigma \partial_t^{m-1} u)_\varepsilon - \sigma U_\varepsilon + (\partial_t^m f)_\varepsilon - \partial_t^m f + \partial_t(\mathcal{D}^u)_\varepsilon - m(\partial_t \alpha) U_\varepsilon - \mathcal{L}^u + \mathcal{M}_\varepsilon. \quad (3.35)$$

in $L^2([0, T] \times \Omega)$ for $n \rightarrow \infty$.

Finally, we use that $(\partial_t^m f)_\varepsilon \rightarrow \partial_t^m f$ in $L^2([0, T] \times \Omega)$ for $\varepsilon \rightarrow 0$, which thus vanishes in the limit in (3.35). Lemma 3.3.2 then yields

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} R_\varepsilon^u &= (\partial_t \sigma) \partial_t^{m-1} u + \partial_t \mathcal{D}^u - m(\partial_t \alpha) U - \mathcal{L}^u + (\partial_t \alpha) U \\ &= \partial_t \mathcal{D}^u - \left(\sum_{j=1}^m \binom{m}{j} \partial_t^j \alpha \partial_t^{m-j+1} u + \sum_{j=1}^m \binom{m}{j} \partial_t^j \sigma \partial_t^{m-j} u \right) \\ &\quad + (\partial_t \alpha) U + (\partial_t \sigma) \partial_t^{m-1} u \end{aligned}$$

in $L^2([0, T] \times \Omega)$.

The summand $\partial_t(\mathcal{D}^u)$ is expanded similar to Step 2). We rewrite the first term as

$$\begin{aligned} \partial_t \mathcal{D}^u &= \partial_t (\partial_t^{m-1} (\alpha \partial_t u) - \alpha \partial_t^m u + \partial_t^{m-1} (\sigma u) - \sigma \partial_t^{m-1} u) \\ &= \left(\sum_{j=1}^m \binom{m}{j} \partial_t^j \alpha \partial_t^{m-j+1} u + \sum_{j=1}^m \binom{m}{j} \partial_t^j \sigma \partial_t^{m-j} u \right) - (\partial_t \alpha) U - (\partial_t \sigma) \partial_t^{m-1} u, \end{aligned}$$

and thus $\lim_{\varepsilon \rightarrow 0} \int_0^t \|R_\varepsilon^u\|_{L^2(\Omega)}^2 d\tau = 0$. Furthermore, since $E_\varepsilon^u \rightarrow 0$ in $H^1(\Omega)$ and $\partial_t E_\varepsilon^u \rightarrow 0$ in $L^2(\Omega)$, the additional extension terms in $\tilde{R}_\varepsilon^u$ vanish as well. Therefore, the limit of the electric remainder terms $\lim_{\varepsilon \rightarrow 0} \int_0^t \|\tilde{R}_\varepsilon^u\|_{L^2(\Omega)}^2 d\tau = 0$. Likewise, one shows the analogous estimate for the magnetic remainder terms $\lim_{\varepsilon \rightarrow 0} \int_0^t \|\tilde{R}_\varepsilon^v\|_{L^2(\Omega)}^2 d\tau = 0$.

5) Therefore, the right-hand side of (3.34) vanishes and the claim follows, since

$$\|\partial_t^m u_n(t) - \partial_t^m u(t)\|_{L^2(\Omega)} \leq \|\tilde{D}_{n,\varepsilon}^u(t)\|_{L^2(\Omega)} + \|E_{n,\varepsilon}^u(t)\|_{L^2(\Omega)} + \|(\partial_t^m u)_\varepsilon(t) - \partial_t^m u(t)\|_{L^2(\Omega)} \rightarrow 0,$$

and analogously for v . $\square$

Next, we have to transform the convergence in time into spatial convergence using the div-curl estimate from Theorem A.2.2.

Lemma 3.3.4 (Convergence of Spatial Derivatives). *Let (H_a) hold. Suppose that $(u_n, v_n) \rightarrow (u, v)$ in $\mathcal{G}_\Gamma^{m-1}$ and for some $j \in \{1, \dots, m\}$, we have*

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left(\|\partial_t^j (u_n - u)(t)\|_{H_\Gamma^{m-j}} + \|\partial_t^j (v_n - v)(t)\|_{H_\Gamma^{m-j}} \right) = 0. \quad (3.36)$$

Then, the $(j-1)$ -th time derivative converges in H_Γ^{m-j+1} , i.e.,

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left(\|\partial_t^{j-1} (u_n - u)(t)\|_{H_\Gamma^{m-j+1}} + \|\partial_t^{j-1} (v_n - v)(t)\|_{H_\Gamma^{m-j+1}} \right) = 0.$$

Proof. Let $r := m - j + 1$. We define the differences $D_n^u := \partial_t^{j-1} u_n - \partial_t^{j-1} u$ and $D_n^v := \partial_t^{j-1} v_n - \partial_t^{j-1} v$. We have to show that

$$e_n^D(t) := \|D_n^u(t)\|_{H_\Gamma^r} + \|D_n^v(t)\|_{H_\Gamma^r} \rightarrow 0$$

uniformly for $t \in [0, T]$ and $n \rightarrow \infty$.

In order to apply the div-curl Theorem A.2.2, the fields must belong to $H_{t0}(\text{curl}, \Omega)$, respectively $H(\text{curl}, \Omega)$. Since D_n^u and D_n^v satisfy inhomogeneous boundary and interface conditions, we again extend these errors to the interior.

As in the proof of the previous lemma there exist extensions (W_n^u, W_n^v) of $(h_n, \kappa_n, \varkappa_n)$ and (W^u, W^v) of $(h, \kappa, \varkappa)$ in $\mathcal{G}_\Gamma^{m+1}$ converging in $\mathcal{G}_\Gamma^{m+1}$ following the approximation in (App). Let $E_n^u := \partial_t^{j-1}(W_n^u - W^u)$ and $E_n^v := \partial_t^{j-1}(W_n^v - W^v)$. Since the extensions belong to $\mathcal{G}_\Gamma^{m+1}$, these $(j-1)$ -th time derivatives belong to $H_\Gamma^{(m+1)-(j-1)}(\Omega) = H_\Gamma^{r+1}(\Omega)$. We have

$$\sup_{t \in [0, T]} \left(\|E_n^u(t)\|_{H_\Gamma^{r+1}} + \|E_n^v(t)\|_{H_\Gamma^{r+1}} \right) \rightarrow 0$$

as $n \rightarrow \infty$.

We define the shifted differences $\tilde{D}_n^u := D_n^u - E_n^u$ and $\tilde{D}_n^v := D_n^v - E_n^v$. By construction, these satisfy exactly homogeneous tangential traces and jumps:

$$\tilde{D}_n^u \times \nu = 0 \text{ on } \partial\Omega, \quad \llbracket \tilde{D}_n^u \times \nu^\Gamma \rrbracket = 0, \quad \llbracket \tilde{D}_n^v \times \nu^\Gamma \rrbracket = 0.$$

We start by deriving the Maxwell equations satisfied by the unshifted differences (D_n^u, D_n^v) . For (u_n, v_n) we have

$$\begin{aligned} \alpha_n \partial_t^j u_n &= \text{curl } \partial_t^{j-1} v_n - \sigma_n \partial_t^{j-1} u_n - \partial_t^{j-1} f_n - \mathcal{L}_n^u, \\ \beta_n \partial_t^j v_n &= -\text{curl } \partial_t^{j-1} u_n - \rho_n \partial_t^{j-1} v_n - \partial_t^{j-1} g_n - \mathcal{L}_n^v, \end{aligned}$$

with $\mathcal{L}_n^u := [\partial_t^{j-1}, \alpha_n \partial_t] u_n + [\partial_t^{j-1}, \sigma_n] u_n$ and $\mathcal{L}_n^v := [\partial_t^{j-1}, \beta_n \partial_t] v_n + [\partial_t^{j-1}, \rho_n] v_n$.

Similarly, the limit (u, v) satisfies

$$\begin{aligned} \alpha \partial_t^j u &= \text{curl } \partial_t^{j-1} v - \sigma \partial_t^{j-1} u - \partial_t^{j-1} f - \mathcal{L}^u, \\ \beta \partial_t^j v &= -\text{curl } \partial_t^{j-1} u - \rho \partial_t^{j-1} v - \partial_t^{j-1} g - \mathcal{L}^v, \end{aligned}$$

with commutators $\mathcal{L}^u := [\partial_t^{j-1}, \alpha \partial_t] u + [\partial_t^{j-1}, \sigma] u$ and $\mathcal{L}^v := [\partial_t^{j-1}, \beta \partial_t] v + [\partial_t^{j-1}, \rho] v$. Thus, their differences satisfy

$$\begin{aligned} \partial_t(\alpha_n D_n^u) &= \text{curl } D_n^v - J_n^u, \\ \partial_t(\beta_n D_n^v) &= -\text{curl } D_n^u - J_n^v, \end{aligned}$$

where we set

$$\begin{aligned} J_n^u &:= (\sigma_n - \partial_t \alpha_n) D_n^u + (\alpha_n - \alpha) \partial_t^j u + (\sigma_n - \sigma) \partial_t^{j-1} u + \partial_t^{j-1} (f_n - f) + (\mathcal{L}_n^u - \mathcal{L}^u), \\ J_n^v &:= (\rho_n - \partial_t \beta_n) D_n^v + (\beta_n - \beta) \partial_t^j v + (\rho_n - \rho) \partial_t^{j-1} v + \partial_t^{j-1} (g_n - g) + (\mathcal{L}_n^v - \mathcal{L}^v). \end{aligned}$$

Substituting the shifted variables, the system becomes

$$\partial_t(\alpha_n \tilde{D}_n^u) = \text{curl } \tilde{D}_n^v - \tilde{J}_n^u, \tag{3.37}$$

$$\partial_t(\beta_n \tilde{D}_n^v) = -\text{curl } \tilde{D}_n^u - \tilde{J}_n^v, \tag{3.38}$$

where the modified remainders are given by

$$\begin{aligned} \tilde{J}_n^u &:= J_n^u + \partial_t(\alpha_n E_n^u) - \text{curl } E_n^v, \\ \tilde{J}_n^v &:= J_n^v + \partial_t(\beta_n E_n^v) + \text{curl } E_n^u. \end{aligned}$$

Because $\tilde{D}_n^u$ and $\tilde{D}_n^v$ possess strictly zero tangential jumps, their curls have no surface divergence. Due to Lemma 8.1 in [56], the normal jumps are thus given by

$$\llbracket \alpha_n \tilde{D}_n^u \cdot \nu^\Gamma \rrbracket(t) = \llbracket \alpha_n \tilde{D}_n^u \cdot \nu^\Gamma \rrbracket(0) - \int_0^t \llbracket \tilde{J}_n^u(\tau) \cdot \nu^\Gamma \rrbracket d\tau,$$

$$\llbracket \beta_n \tilde{D}_n^v \cdot \nu^\Gamma \rrbracket(t) = \llbracket \beta_n \tilde{D}_n^v \cdot \nu^\Gamma \rrbracket(0) - \int_0^t \llbracket \tilde{J}_n^v(\tau) \cdot \nu^\Gamma \rrbracket d\tau,$$

and

$$\mathrm{tr}_n(\beta_n \tilde{D}_n^v)(t) = \mathrm{tr}_n(\beta_n \tilde{D}_n^v)(0) - \int_0^t \mathrm{tr}_n \tilde{J}_n^v(\tau) d\tau.$$

Thus, in view of Theorem A.2.2 (corresponding to v there), we define the correction terms

$$\tilde{C}_n^u := -\alpha_n(0)\tilde{D}_n^u(0) + \int_0^t \tilde{J}_n^u(\tau) d\tau, \quad \tilde{C}_n^v := -\beta_n(0)\tilde{D}_n^v(0) + \int_0^t \tilde{J}_n^v(\tau) d\tau. \quad (3.39)$$

Proposition 2.2.30 in [5], see Lemma A.1.2, then shows that $\alpha_n \tilde{D}_n^u + \tilde{C}_n^u \in H(\mathrm{div}, \Omega)$ and $\beta_n \tilde{D}_n^v + \tilde{C}_n^v \in H_{n0}(\mathrm{div}, \Omega)$. We further obtain $\tilde{D}_n^u \in H_{t0}(\mathrm{curl}, \Omega)$ and $\tilde{D}_n^v \in H(\mathrm{curl}, \Omega)$.

Theorem A.2.2 then yields for the shifted energy $\tilde{e}_n^D(t) := \|\tilde{D}_n^u(t)\|_{H_\Gamma^r} + \|\tilde{D}_n^v(t)\|_{H_\Gamma^r}$

$$\begin{aligned} \tilde{e}_n^D(t) &\lesssim \|\tilde{D}_n^u(t)\|_{L^2(\Omega)} + \|\mathrm{curl} \tilde{D}_n^u(t)\|_{H_\Gamma^{r-1}} + \|\mathrm{div}(\alpha_n \tilde{D}_n^u(t))\|_{H_\Gamma^{r-1}} + \|\tilde{C}_n^u(t)\|_{H_\Gamma^r} \\ &\quad + \|\tilde{D}_n^v(t)\|_{L^2(\Omega)} + \|\mathrm{curl} \tilde{D}_n^v(t)\|_{H_\Gamma^{r-1}} + \|\mathrm{div}(\beta_n \tilde{D}_n^v(t))\|_{H_\Gamma^{r-1}} + \|\tilde{C}_n^v(t)\|_{H_\Gamma^r}. \end{aligned} \quad (3.40)$$

To show convergence, we now consider the terms on the right-hand side individually. Again, we only consider u as the estimates for v follow in the same way.

1) First, the L^2 norms converge. We have $\|\tilde{D}_n^u(t)\|_{L^2(\Omega)} \leq \|D_n^u(t)\|_{L^2(\Omega)} + \|E_n^u(t)\|_{L^2(\Omega)}$. By assumption $(u_n, v_n) \rightarrow (u, v)$ in $\mathcal{G}_\Gamma^{m-1}$. Thus, in particular $D_n^u(t) \rightarrow 0$ in $L^2(\Omega)$. Since $E_n^u(t) \rightarrow 0$ in H_Γ^{r+1} , we obtain $\|\tilde{D}_n^u(t)\|_{L^2(\Omega)} \rightarrow 0$ uniformly for $t \in [0, T]$.

2) Next, we consider the correction terms $(\tilde{C}_n^u, \tilde{C}_n^v)$. We decompose $\tilde{J}_n^u$ as follows. To bound the commutator difference $\mathcal{L}_n^u - \mathcal{L}^u$, we extract its highest-order term $(j-1)(\partial_t \alpha_n) \partial_t^{j-1} u_n - (j-1)(\partial_t \alpha) \partial_t^{j-1} u$. Using $D_n^u = \tilde{D}_n^u + E_n^u$, we move the $(j-1)(\partial_t \alpha_n) \tilde{D}_n^u$ component into the Gronwall term.¹ Thus, we write

$$\tilde{J}_n^u = (\sigma_n + (j-2)\partial_t \alpha_n) \tilde{D}_n^u + \tilde{F}_n^u,$$

with

$$\begin{aligned} \tilde{F}_n^u &:= (\alpha_n - \alpha) \partial_t^j u + (\sigma_n - \sigma) \partial_t^{j-1} u + \partial_t^{j-1} (f_n - f) \\ &\quad + (\mathcal{L}_n^u - \mathcal{L}^u - (j-1)(\partial_t \alpha_n) \tilde{D}_n^u) \\ &\quad + \sigma_n E_n^u + \alpha_n \partial_t E_n^u - \mathrm{curl} E_n^v, \end{aligned}$$

where we have subtracted $(j-1)(\partial_t \alpha_n) \tilde{D}_n^u$ such that the commutator difference

$$\begin{aligned} \mathcal{L}_n^u - \mathcal{L}^u - (j-1)(\partial_t \alpha_n) \tilde{D}_n^u &= (j-1)(\partial_t \alpha_n) E_n^u + (j-1)(\partial_t \alpha_n - \partial_t \alpha) \partial_t^{j-1} u \\ &\quad + \sum_{\ell=2}^{j-1} \binom{j-1}{\ell} [(\partial_t^\ell \alpha_n) \partial_t^{j-\ell} u_n - (\partial_t^\ell \alpha) \partial_t^{j-\ell} u] \\ &\quad + \sum_{\ell=1}^{j-1} \binom{j-1}{\ell} [(\partial_t^\ell \sigma_n) \partial_t^{j-1-\ell} u_n - (\partial_t^\ell \sigma) \partial_t^{j-1-\ell} u]. \end{aligned}$$

no longer depends on the difference $D_n^u = \partial_t^{j-1}(u_n - u)$ of order $j-1$ in time. The first term in $\tilde{J}_n^u$ will be treated by a Gronwall argument. To bound the norm of the integral of $\tilde{F}_n^u$ in

¹This highest-order term was overlooked in a draft of this section. The AI tool *Claude Fable 5.1* caught that mistake.

H_Γ^r , we note that $\partial_t^j u \in H_\Gamma^{r-1}$, which prevents a direct bound. However, integrating by parts in time yields

$$\int_0^t (\alpha_n - \alpha) \partial_t (\partial_t^{j-1} u) \, d\tau = [(\alpha_n - \alpha) \partial_t^{j-1} u]_0^t - \int_0^t \partial_t (\alpha_n - \alpha) \partial_t^{j-1} u \, d\tau. \quad (3.41)$$

Since $\partial_t^{j-1} u \in C([0, T]; H_\Gamma^r)$ and $\partial_t \alpha_n \rightarrow \partial_t \alpha$ in $\mathcal{C}^{\hat{m}}$, by Lemma 4.2.2 below, the H_Γ^r -norm of the right-hand side converges to zero uniformly. The remaining terms in $\tilde{F}_n^u$, except the inhomogeneity and the extension terms, similarly vanish after integration in H_Γ^r due to the convergence of the coefficients and commutators, since $u_n \rightarrow u$ in $\mathcal{G}_\Gamma^{m-1}$ and $u \in \mathcal{G}_\Gamma^m$. For the inhomogeneity, since $f \in H^m([0, T] \times \Omega)$, we have $\partial_t^{j-1}(f_n - f) \in L^2([0, T]; H^r(\Omega))$ and

$$\int_0^t \left\| \partial_t^{j-1}(f_n - f)(\tau) \right\|_{H_\Gamma^r} \, d\tau \leq \sqrt{T} \left\| \partial_t^{j-1}(f_n - f) \right\|_{L^2([0, T]; H_\Gamma^r)} \rightarrow 0$$

as $n \rightarrow \infty$. Furthermore, the terms $\sigma_n E_n^u, \alpha_n \partial_t E_n^u, \operatorname{curl} E_n^v$ vanish uniformly in H_Γ^r because $E_n^u, E_n^v \rightarrow 0$ in H_Γ^{r+1} and $\partial_t E_n^u \rightarrow 0$ in H_Γ^r . Thus, keeping the norm strictly outside the time integral for $\tilde{F}_n^u$, we obtain

$$\begin{aligned} \|\tilde{C}_n^u(t)\|_{H_\Gamma^r} &\lesssim \|\alpha_n(0) \tilde{D}_n^u(0)\|_{H_\Gamma^r} + \left\| \int_0^t \tilde{F}_n^u(\tau) \, d\tau \right\|_{H_\Gamma^r} + \int_0^t \|\tilde{D}_n^u(\tau)\|_{H_\Gamma^r} \, d\tau \\ &\leq \tilde{\delta}_{u,n} + \int_0^t \tilde{e}_n^D(\tau) \, d\tau, \end{aligned}$$

for $\tilde{\delta}_{u,n} := (1 + \|\alpha_n(0)\|_{H_\Gamma^{\hat{m}}}) \|\tilde{D}_n^u(0)\|_{H_\Gamma^r} + \sup_{t \in [0, T]} \left\| \int_0^t \tilde{F}_n^u(\tau) \, d\tau \right\|_{H_\Gamma^r} \rightarrow 0$ as $n \rightarrow \infty$.

3) We consider the divergence terms next. Integrating (3.37) and substituting (3.39) yields $\alpha_n \tilde{D}_n^u = -\tilde{C}_n^u - \operatorname{curl} \int_0^t \tilde{D}_n^u \, d\tau$ on $\Omega_\pm$. Therefore, taking the norm of the divergence yields

$$\|\operatorname{div}(\alpha_n \tilde{D}_n^u(t))\|_{H_\Gamma^{r-1}} \lesssim \|\tilde{C}_n^u(t)\|_{H_\Gamma^r}.$$

This is the term we have already considered in the previous step. Thus,

$$\|\operatorname{div}(\alpha_n \tilde{D}_n^u(t))\|_{H_\Gamma^{r-1}} \lesssim \tilde{\delta}_{u,n} + \int_0^t \tilde{e}_n^D(\tau) \, d\tau.$$

In the same way, we bound $\|\operatorname{div}(\beta_n \tilde{D}_n^v)\|_{H_\Gamma^{r-1}}$.

4) Finally, let us consider the curl terms. By (3.38), and the definition of $\tilde{J}_n^v$, we write as in Step 2)

$$\operatorname{curl} \tilde{D}_n^u = -\partial_t(\beta_n \tilde{D}_n^v) - \tilde{J}_n^v = -\beta_n \partial_t \tilde{D}_n^v - (\rho_n + (j-1)\partial_t \beta_n) \tilde{D}_n^v - \tilde{F}_n^v,$$

with

$$\begin{aligned} \tilde{F}_n^v &:= (\beta_n - \beta) \partial_t^j v + (\rho_n - \rho) \partial_t^{j-1} v + \partial_t^{j-1}(g_n - g) \\ &\quad + (\mathcal{L}_n^v - \mathcal{L}^v - (j-1)(\partial_t \beta_n) \tilde{D}_n^v) \\ &\quad + \rho_n E_n^v + \beta_n \partial_t E_n^v + \operatorname{curl} E_n^u. \end{aligned}$$

This allows us to trade space against time regularity of $\partial_t \tilde{D}_n^v = \partial_t^j(v_n - v) - \partial_t E_n^v$. By assumption (3.36), $\partial_t^j(v_n - v) \rightarrow 0$ uniformly in $H_\Gamma^{m-j} = H_\Gamma^{r-1}$ for $n \rightarrow \infty$. Since $\partial_t E_n^v \rightarrow 0$ in $H_\Gamma^r \subset H_\Gamma^{r-1}$, we deduce $\partial_t \tilde{D}_n^v \rightarrow 0$ in H_Γ^{r-1} . Additionally, $\partial_t^j v \in H_\Gamma^{m-j}$, so the uniform

convergence of β_n, ρ_n , the inhomogeneities, the extensions, and the commutators implies that $\|\tilde{F}_n^v\|_{H_\Gamma^{m-j}} \rightarrow 0$. Thus,

$$\sup_{t \in [0, T]} \|\operatorname{curl} \tilde{D}_n^u(t)\|_{H_\Gamma^{r-1}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Based on (3.37), an analogous argument shows $\sup_{t \in [0, T]} \|\operatorname{curl} \tilde{D}_n^v(t)\|_{H_\Gamma^{r-1}} \rightarrow 0$.

Inserting the above estimates into (3.40) we arrive at the inequality

$$\tilde{e}_n^D(t) \leq \Delta_n + C \int_0^t \tilde{e}_n^D(\tau) d\tau,$$

where $C > 0$ and

$$\Delta_n := \sup_{t \in [0, T]} \left(\left\| (\tilde{D}_n^u, \tilde{D}_n^v)(t) \right\|_{L^2(\Omega)} + \left\| \operatorname{curl}(\tilde{D}_n^u, \tilde{D}_n^v)(t) \right\|_{H_\Gamma^{r-1}} \right) + 2(\tilde{\delta}_{u,n} + \tilde{\delta}_{v,n}).$$

Since $\Delta_n \rightarrow 0$ as $n \rightarrow \infty$, Gronwall's inequality yields

$$\sup_{t \in [0, T]} \tilde{e}_n^D(t) \leq \Delta_n e^{CT} \rightarrow 0 \quad \text{for } n \rightarrow \infty.$$

Finally, for the unshifted variables, we have

$$\sup_{t \in [0, T]} e_n^D(t) \leq \sup_{t \in [0, T]} \left(\tilde{e}_n^D(t) + \|E_n^u(t)\|_{H_\Gamma^r} + \|E_n^v(t)\|_{H_\Gamma^r} \right) \rightarrow 0 \quad \text{for } n \rightarrow \infty,$$

and the claim follows. $\square$

Now, we combine Lemma 3.3.3 and Lemma 3.3.4 to obtain our approximation result.

Theorem 3.3.5. *Let $m \in \mathbb{N}_0, k, a \in \mathbb{N}$ with $\tilde{m} + a + 2 \leq k$, and $\partial\Omega \subseteq \partial\omega \in C^{k+1}$. Furthermore, let $\alpha, \beta \in C_\eta^{\tilde{m}}, \partial_t \alpha, \partial_t \beta \in C^{\tilde{m}}, \sigma, \rho \in C^{\tilde{m}}, f, g \in H^m([0, T] \times \Omega_\pm), h \in T^m(\partial\Omega), \kappa, \varkappa \in J^m(\Gamma)$, and $u_0, v_0 \in H_\Gamma^m(\Omega)$ satisfy the compatibility conditions $(CC_{\alpha, \beta, \sigma, \rho, f, g, h, \kappa, \varkappa, m})$ of order m . By Theorem 3.1 in [56], see Proposition 3.1.3, there exists a solution $(u, v) \in \mathcal{G}_\Gamma^m([0, T])$ to (3.6)–(3.9).*

Choose $\alpha_n, \beta_n, \sigma_n, \rho_n \in C^\infty([0, T] \times \Omega_\pm)$, α_n, β_n uniformly positive definite, $f_n, g_n \in C^\infty([0, T] \times \Omega_\pm)$, $h_n \in C^\infty([0, T] \times \partial\Omega)$, $\kappa_n, \varkappa_n \in C^\infty([0, T] \times \Gamma)$ satisfying (App). Then there exist $u_{0,n}, v_{0,n} \in H_\Gamma^{m+a}(\Omega)$ satisfying the compatibility conditions $(CC_{\alpha_n, \beta_n, \sigma_n, \rho_n, f_n, g_n, h_n, \kappa_n, \varkappa_n, m+a})$ of order $m+a$, such that there exists a corresponding solution $(u_n, v_n) \in \mathcal{G}_\Gamma^{m+a}([0, T])$ of (3.6) with

$$(u_n, v_n) \rightarrow (u, v) \quad \text{in } \mathcal{G}_\Gamma^m([0, T])$$

for $n \rightarrow \infty$.

Proof. By Remark 3.2.2 we may assume $m \geq 1$ in the following. Given $\alpha, \beta, \sigma, \rho, f, g$, boundary and jump data $h, \kappa, \varkappa$, and initial values u_0, v_0 , we choose an approximating sequence as outlined in (H_a) . Recall that

$$\|u\|_{\mathcal{G}_\Gamma^m([0, T] \times \Omega)} := \sup_{t \in [0, T]} \sum_{j=0}^m \|\partial_t^j u(t)\|_{H^{m-j}(\Omega_\pm)}$$

and consider the differences $(u_n - u, v_n - v)$.

Firstly, we consider the full time derivatives, corresponding to the $j = m$ term in the $\mathcal{G}_\Gamma^m$ -norm. Lemma 3.3.3 yields

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left(\|\partial_t^m(u_n - u)(t)\|_{L^2(\Omega)} + \|\partial_t^m(v_n - v)(t)\|_{L^2(\Omega)} \right) = 0.$$

Secondly, we use the time convergence as a starting point to iteratively recover the spatial derivatives using Lemma 3.3.4. Considering $j = m$ down to $j = 0$, we show that for every $j \in \{0, \dots, m\}$, we have

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left(\|\partial_t^j(u_n - u)(t)\|_{H^{m-j}(\Omega)} + \|\partial_t^j(v_n - v)(t)\|_{H^{m-j}(\Omega)} \right) = 0,$$

which then shows

$$\lim_{n \rightarrow \infty} \left(\|u_n - u\|_{\mathcal{G}_\Gamma^m([0, T] \times \Omega)} + \|v_n - v\|_{\mathcal{G}_\Gamma^m([0, T] \times \Omega)} \right) = 0. \quad \square$$

Remark 3.3.6. Theorem 3.3.5 is an approximation result for the non-autonomous, linear system under the assumption (3.12). However, we can apply it after differentiating the quasilinear system

$$\begin{aligned} \partial_t(\alpha(u, x)u) &= \operatorname{curl} v - \tilde{\sigma}(t, x)u - f, & t \geq 0, x \in \Omega_\pm, \\ \partial_t(\beta(v, x)v) &= -\operatorname{curl} u - \tilde{\rho}(t, x)v - g, & t \geq 0, x \in \Omega_\pm, \\ u \times \nu(x) &= h, & t \geq 0, x \in \partial\Omega, \\ \llbracket u \times \nu^\Gamma(x) \rrbracket &= \kappa, \quad \llbracket v \times \nu^\Gamma(x) \rrbracket = \varkappa, & t \geq 0, x \in \Gamma, \\ u(0, x) &= u_0(x), \quad v(0, x) = v_0(x), & x \in \Omega_\pm, \end{aligned}$$

cf. (4.1)–(4.4), with coefficients $\alpha(x, u(t, x)), \beta(x, v(t, x))$ depending on (u, v) , where usually one only assumes

$$\partial_{\xi_i} \alpha, \partial_{\xi_i} \beta \in \mathcal{C}^{\tilde{m}}, \quad (3.42)$$

see for example (2.2) in [35] and (4.7) below.

Given coefficients and initial values satisfying (3.11), (3.13) and (3.42) for some $m \leq k$ and $r \in \{0, \dots, m\}$, we may consider the r -times time differentiated system

$$\begin{aligned} \alpha_d(t, x) \partial_t(\partial_t^r u(t, x)) &= \operatorname{curl} \partial_t^r v(t, x) - \tilde{\sigma}(t, x) \partial_t^r u(t, x) - f_r(t, x), & t \geq 0, x \in \Omega_\pm, \\ \beta_d(t, x) \partial_t(\partial_t^r v(t, x)) &= -\operatorname{curl} \partial_t^r u(t, x) - \tilde{\rho}(t, x) \partial_t^r v(t, x) - g_r(t, x), & t \geq 0, x \in \Omega_\pm, \\ \partial_t^r u(t, x) \times \nu(x) &= \partial_t^r h(t, x), & t \geq 0, x \in \partial\Omega, \\ \llbracket \partial_t^r u \times \nu^\Gamma \rrbracket &= \partial_t^r \kappa(t, x), \quad \llbracket \partial_t^r v \times \nu^\Gamma \rrbracket = \partial_t^r \varkappa(t, x), & t \geq 0, x \in \Gamma, \end{aligned}$$

where

$$\begin{aligned} \alpha_d(t, x) &= \alpha(x, u(t, x)) + \partial_\xi \alpha(x, \xi)|_{\xi=u(t, x)} u(t, x) = \alpha(x, u) + \left(\sum_{\ell=1}^3 \partial_{\xi_j} \alpha_{i\ell}(x, u) u_\ell \right)_{ij}, \\ \beta_d(t, x) &= \beta(x, v(t, x)) + \partial_\xi \beta(x, \xi)|_{\xi=v(t, x)} v(t, x) = \beta(x, v) + \left(\sum_{\ell=1}^3 \partial_{\xi_j} \beta_{i\ell}(x, v) v_\ell \right)_{ij}, \\ f_r(t, x) &= \partial_t^r f(t, x) + \sum_{j=1}^r \binom{r}{j} \partial_t^j \tilde{\sigma}(t, x) \partial_t^{r-j} u(t, x) \\ &\quad + \sum_{j=1}^r \binom{r}{j} \partial_t^j \alpha_d(t, x) \partial_t^{r-j+1} u(t, x), \end{aligned}$$

$$\begin{aligned}
g_r(t, x) &= \partial_t^r g(t, x) + \sum_{j=1}^r \binom{r}{j} \partial_t^j \tilde{\rho}(t, x) \partial_t^{r-j} v(t, x) \\
&\quad + \sum_{j=1}^r \binom{r}{j} \partial_t^j \beta_d(t, x) \partial_t^{r-j+1} v(t, x).
\end{aligned}$$

Hence, the time derived system has initial data and inhomogeneities which possess regularity of level $m - r$, with coefficients

$$\alpha_d, \beta_d \in \mathcal{C}^{\tilde{m}} \quad \text{and thus} \quad \partial_t \alpha_d, \partial_t \beta_d \in \mathcal{C}^{\tilde{m}-1}.$$

In this setting we can apply Theorem 3.3.5, assuming that α_d, β_d are uniformly positive definite, to show convergence of approximations to $\partial_t^r(u, v)$ in $\mathcal{G}_\Gamma^{m-r}$ for $r \geq 1$ and $m \in \mathbb{N}$. Moreover, if $m \leq 2$ then $\mathcal{C}^{\tilde{m}-1} = \mathcal{C}^{\hat{m}}$ and we obtain convergence in $\mathcal{G}_\Gamma^m$, which we use in Section 4.4.

In order to obtain convergence in $\mathcal{G}_\Gamma^m$ for general $m \in \mathbb{N}$, one would have to approximate $\alpha(u), \beta(v)$ preserving the quasilinear structure, which is not possible with our current approach, since we assume α_n, β_n to be independent of the solution.

EXPONENTIAL DECAY OF THE QUASILINEAR MAXWELL SYSTEM DUE TO INTERIOR CONDUCTIVITY ON A COLLAR

In Chapter 2, we discussed the long-term behavior of the linear system. We now extend this framework by allowing the material coefficients to depend on the solutions themselves. This leads to the quasilinear Maxwell system

$$\begin{aligned} \partial_t(\varepsilon(x, E(t, x))E(t, x)) &= \operatorname{curl} H(t, x) - \sigma(x)E(t, x), & t \geq 0, x \in \Omega_{\pm}, \end{aligned} \quad (4.1)$$

$$\begin{aligned} \partial_t(\mu(x, H(t, x))H(t, x)) &= -\operatorname{curl} E(t, x), \\ \operatorname{div}(\mu(x)H(t, x)) &= 0, & t \geq 0, x \in \Omega_{\pm}, \end{aligned} \quad (4.2)$$

$$E(t, x) \times \nu(x) = 0, \quad \nu(x) \cdot \mu(x, H(t, x))H(t, x) = 0, \quad t \geq 0, x \in \partial\Omega, \quad (4.3)$$

$$E(0, x) = E_0(x), \quad H(0, x) = H_0(x), \quad x \in \Omega_{\pm}, \quad (4.4)$$

on the bounded, multiply connected domain $\Omega \subset \mathbb{R}^3$. This nonlinearity introduces significant mathematical challenges. Most notably, we can no longer rely on standard semigroup theory for local well-posedness. Instead, we use the results of [56] on the interface problem above to guarantee the existence of solutions (E, H) in $\mathcal{G}^3$. To obtain control in this norm, we have to consider the time derivatives $\partial_t^k(E, H)$ as well. We will show exponential decay to 0 for sufficiently small data as in Chapter 2. To do so, we adapt our previous strategy from the linear problem by splitting $\partial_t^k(E, H)$ via a Helmholtz decomposition and subsequently proving an observability-type estimate. This approach leads to additional error terms, which require careful treatment at the end of this chapter.

4.1. PROBLEM SETTING

For the conductivity we require that

$$\omega \subseteq \Omega \text{ is open, } \partial\omega \in C^5, \sigma \in C^3(\bar{\omega}, \mathbb{R}_{\text{sym}}^{3 \times 3}) \text{ with } \omega \supseteq \mathcal{N}_a, \text{ and } \sigma|_{\omega} \geq \sigma_0 > 0, \quad (4.5)$$

for some $a > 0$. In particular, we assume that $\partial\Omega \in C^5$ and that it has positive distance to $\partial\omega$, where we recall $v = \Omega \setminus \bar{\omega}$ and $\Omega_{\pm} = v \cup \omega$.

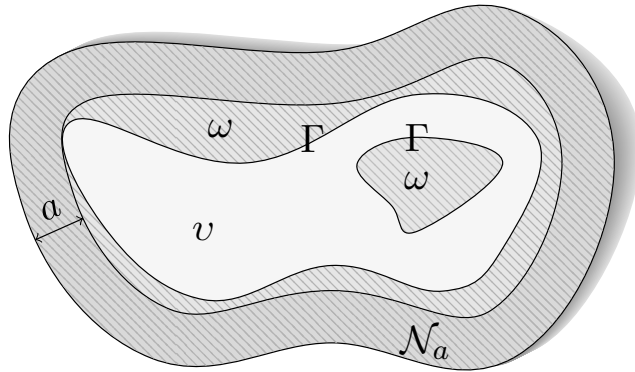


Figure 4.1.: $2d$ -cut through $\Omega \subset \mathbb{R}^3$. Note that $\operatorname{supp} \sigma = \bar{\omega}$ contains the set $\mathcal{N}_a$.

We extend σ discontinuously by 0 to v , leading to an interface problem on $\Gamma = \partial v$ with outer unit normal ν^Γ of v .

The Maxwell equations (4.1)–(4.4) then imply in the absence of surface currents the interface conditions (see Equation (1.2) in [56])

$$\begin{aligned} \llbracket E \times \nu^\Gamma \rrbracket &= 0, & \llbracket H \times \nu^\Gamma \rrbracket &= 0, \\ \llbracket \varepsilon E \cdot \nu^\Gamma \rrbracket &= \llbracket \varepsilon E(t_0) \cdot \nu^\Gamma \rrbracket - \int_{t_0}^t \llbracket \sigma E \cdot \nu^\Gamma \rrbracket d\tau, & \llbracket \mu H \cdot \nu^\Gamma \rrbracket &= 0. \end{aligned} \quad (4.6)$$

ASSUMPTIONS ON THE COEFFICIENT

The material parameters satisfy

$$\varepsilon, \mu, \partial_{\xi_k} \varepsilon, \partial_{\xi_k} \mu \in C^3(\overline{\Omega}_\pm \times \mathbb{R}^3, \mathbb{R}_{\text{sym}}^{3 \times 3}), \quad (4.7)$$

for $k \in \{1, 2, 3\}$, where $C^3(\overline{\Omega}_\pm \times \mathbb{R}^3, \mathbb{R}_{\text{sym}}^{3 \times 3})$ is the space

$$\left\{ \alpha \in C^3(\Omega_\pm \times \mathbb{R}^3, \mathbb{R}_{\text{sym}}^{3 \times 3}) \mid \alpha|_v \in C^3(\overline{v} \times \mathbb{R}^3, \mathbb{R}_{\text{sym}}^{3 \times 3}), \alpha|_\omega \in C^3(\overline{\omega} \times \mathbb{R}^3, \mathbb{R}_{\text{sym}}^{3 \times 3}) \right\},$$

so the coefficients are C^3 on either side of the interface. They are assumed to be uniformly positive definite for vanishing fields in the sense that

$$\varepsilon(x, 0), \mu(x, 0) \geq 2\eta > 0 \quad (4.8)$$

for some $\eta > 0$.

Note that the nonlinearities $\varepsilon(x, \xi)$ and $\mu(x, \xi)$ are not required to be bounded in $\xi \in \mathbb{R}^3$. Later on, however, the arguments E and H will be bounded and so will $\varepsilon(x, E)$ as well as $\mu(x, H)$. We provide the two examples for such nonlinearities from Example 2.1 in [35].

Example 4.1.1. We consider nonlinearities of the form

$$\varepsilon(x, E) = \varepsilon_{\text{lin}}(x) + \varepsilon_{\text{nl}}(x, E)$$

where $\varepsilon_{\text{lin}} \in C^3(\overline{\Omega}_\pm, \mathbb{R}_{\text{sym}}^{3 \times 3})$ satisfies $\varepsilon_{\text{lin}} \geq 2\eta I$ and the nonlinear term ε_{nl} will be constructed in accordance with (4.7) and (4.8). For a Kerr-type isotropic nonlinearity, ε_{nl} takes the form

$$\varepsilon_{\text{nl}}(x, E) = a(x)\varphi(|E|^2),$$

with scalar functions $a \in C^3(\overline{\Omega})$ and $\varphi \in C^4([0, \infty))$ with $\varphi(0) = 0$, see Chapter 10.1 in [24].

To obtain a non-isotropic example we consider

$$\varepsilon_{\text{nl}}(x, E) = \left(\sum_{j,k=1}^3 \chi_i^{jkl}(x) E_j E_k \right)_{il},$$

for scalar coefficients $\chi_i^{jkl} \in C^3(\overline{\Omega})$. Material laws in the literature are often retarded and allow for an additional frequency dependence in the coefficients. However, on longer time scales it is reasonable to approximate such materials by instantaneous laws, see Equation (2.3.6) and the subsequent paragraph in [2]. Under this assumption the above nonlinearity follows from the material laws considered in Chapter 1.5 in [9] for example.

Since $\varepsilon_{\text{nl}} \in \mathbb{R}_{\text{sym}}^{3 \times 3}$ the coefficients must be symmetric with respect to $\{i, l\}$, i.e., $\chi_i^{jkl} = \chi_l^{jki}$. Moreover, when applying the electric field $\varepsilon_{\text{nl}}(x, E)E = \left(\sum_{j,k,l=1}^3 \chi_i^{jkl}(x) E_j E_k E_l \right)_l$ we realize that $(\chi_{i,j,k,l}^{jkl})$ can be chosen without loss of generality to be symmetric in $\{j, k, l\}$. Combining the two symmetries we can choose 15 independent coefficients χ_i^{jkl} for $1 \leq i \leq j \leq k \leq l \leq 3$.

ASSUMPTIONS ON THE INITIAL VALUES

We take initial fields

$$(E_0, H_0) \in \left(H_\Gamma^3(\Omega)\right)^6,$$

satisfying

$$\mu(H_0)H_0 \in H_{n0}^\Sigma(\operatorname{div} 0) \quad \text{and} \quad \varepsilon(E_0)E_0|_v \in H^\Gamma(\operatorname{div} 0, v), \quad (4.9)$$

as well as the *compatibility conditions* of order 3, see Definition 3.1.2, as outlined below. These arise since Maxwell's equation relate time to space derivatives. In order to obtain solutions of higher regularity the time derivatives at $t = 0$ have to fit to the spatial interface and boundary conditions. Note that these conditions stay invariant under the evolution of (4.1)–(4.4), see also the Definition 3.1.2 preceding text. In order to express the compatibility conditions we let

$$\begin{aligned} \varepsilon_d(x, \xi) &:= \varepsilon(x, \xi) + \left(\sum_{l=1}^3 \partial_{\xi_j} \varepsilon_{il}(x, \xi) \xi_l \right)_{ij} =: \varepsilon(x, \xi) + \varepsilon_{d, \text{nl}}(x, \xi), \\ \mu_d(x, \xi) &:= \mu(x, \xi) + \left(\sum_{l=1}^3 \partial_{\xi_j} \mu_{il}(x, \xi) \xi_l \right)_{ij} =: \mu(x, \xi) + \mu_{d, \text{nl}}(x, \xi), \end{aligned} \quad (4.10)$$

for $x \in \overline{\Omega_\pm}$, $\xi \in \mathbb{R}^3$, i.e., ε_d equals ε up to a perturbation $\varepsilon_{d, \text{nl}}$, which is small for small ξ . We further impose the definiteness assumption

$$\varepsilon(\cdot, E_0), \varepsilon_d(\cdot, E_0), \mu(\cdot, H_0), \mu_d(\cdot, H_0) \geq \eta \quad (4.11)$$

on $\Omega_\pm$ on the initial values. By (4.7) and (4.8) this is true if $\|E_0, H_0\|_{H_\Gamma^3(\Omega)^6} \leq \delta_\eta^0$ for a sufficiently small $\delta_\eta^0 > 0$. We define

$$\begin{aligned} \hat{\varepsilon}_k(t, x) &:= \begin{cases} \varepsilon(x, E(t, x)), & k = 0, \\ \varepsilon_d(x, E(t, x)), & k = 1, 2, 3, \end{cases} & t \geq 0, x \in \Omega_\pm, \\ \hat{\mu}_k(t, x) &:= \begin{cases} \mu(x, E(t, x)), & k = 0, \\ \mu_d(x, E(t, x)), & k = 1, 2, 3, \end{cases} & t \geq 0, x \in \Omega_\pm, \end{aligned}$$

and set

$$\begin{aligned} E_0^1 &:= \varepsilon_d(E_0)^{-1} [\operatorname{curl} H_0 - \sigma E_0], \\ E_0^2 &:= \varepsilon_d(E_0)^{-1} \left[\operatorname{curl} H_0^1 - \sigma E_0^1 - \left(\sum_{\ell=1}^3 \partial_{\xi_\ell} \varepsilon_{d, i, j}(E_0)(E_0^1)_\ell \right)_{i, j} E_0^1 \right], \\ H_0^1 &:= -\mu_d(H_0)^{-1} \operatorname{curl} E_0, \\ H_0^2 &:= -\mu_d(H_0)^{-1} \left[\operatorname{curl} E_0^1 - \left(\sum_{\ell=1}^3 \partial_{\xi_\ell} \mu_{d, i, j}(H_0)(H_0^1)_\ell \right)_{i, j} H_0^1 \right], \end{aligned} \quad (4.12)$$

requiring them to be in $H(\operatorname{curl})$ respectively. We assume

$$\operatorname{tr}_{t, \partial\Omega} E_0 = \operatorname{tr}_{t, \partial\Omega} E_0^1 = \operatorname{tr}_{t, \partial\Omega} E_0^2 = 0, \quad (4.13)$$

and the interface conditions

$$\begin{aligned} \llbracket E_0 \times \nu^\Gamma \rrbracket &= \llbracket E_0^1 \times \nu^\Gamma \rrbracket = \llbracket E_0^2 \times \nu^\Gamma \rrbracket = 0, \\ \llbracket H_0 \times \nu^\Gamma \rrbracket &= \llbracket H_0^1 \times \nu^\Gamma \rrbracket = \llbracket H_0^2 \times \nu^\Gamma \rrbracket = 0, \end{aligned} \quad (4.14)$$

in the absence of surface currents (i.e. $J_\Sigma = 0$ in [56]).

EXISTENCE OF SOLUTIONS

Under these conditions, Theorem 7.3 of [56] yields a unique solution

$$(E, H) \in \mathcal{G}_\Gamma^3((T_{\min}, T_{\max}) \times \Omega)^6, \quad (4.15)$$

to (4.1)–(4.4), where $\bigcap_{j=0}^k C^j((T_{\min}, T_{\max}), H_\Gamma^{3-j}(\Omega)) =: \mathcal{G}_\Gamma^k((T_{\min}, T_{\max}) \times \Omega)$, which takes values in $B_{H_\Gamma^3}(0, \delta_\eta^0)$ and depends continuously on data. Furthermore, for $(E_0, H_0) = 0$ the maximal existence time satisfies $T_{\max} = \infty$. So, for fixed $T \in (0, \infty)$, there is a number $\tilde{r} > 0$ such that for all data $\tilde{u}_0 \in B_{H_\Gamma^3}(0, \tilde{r})$, satisfying the compatibility/interface conditions above, the maximal existence time satisfies $T_{\max}(\tilde{u}_0) > T$. Let $0 < \delta_z \leq \delta_\eta \leq \min\{1, \delta_\eta^0\}$. The bound δ_η will be fixed in (4.26) and (4.27) below.

The flow map

$$\Psi: B_{H_\Gamma^3}(0, \tilde{r}) \rightarrow \mathcal{G}_\Gamma^3((0, T) \times \Omega), \tilde{u}_0 \mapsto u(\cdot; \tilde{u}_0),$$

is continuous, and hence for sufficiently small initial values, i.e.,

$$\|(E_0, H_0)\|_{H_\Gamma^3(\Omega)^6} \leq r(\delta_z) \quad (4.16)$$

for a constant $r(\delta_z) \in (0, \tilde{r})$ depending on δ_z , satisfying the compatibility and interface conditions, there exists a unique solution $(E, H) \in \mathcal{G}_\Gamma^3([0, T_{\max}))$ such that $T_{\max} > T$,

$$\max_{0 \leq j \leq 3} \left(\|\partial_t^j E(t)\|_{H_\Gamma^{3-j}(\Omega)}^2 + \|\partial_t^j H(t)\|_{H_\Gamma^{3-j}(\Omega)}^2 \right) \leq \delta_z^2 \quad \text{for } 0 \leq t \leq T. \quad (4.17)$$

We set

$$T_* := \sup \left\{ \hat{T} \in [0, T_{\max}] \mid (4.17) \text{ holds for } t \in [0, \hat{T}] \right\}.$$

From the blow-up condition in Theorem 7.3 of [56] we infer that $T_* < \infty$ implies

$$\max_{0 \leq j \leq 3} \left(\|\partial_t^j E(T_*)\|_{H_\Gamma^{3-j}(\Omega)}^2 + \|\partial_t^j H(T_*)\|_{H_\Gamma^{3-j}(\Omega)}^2 \right) = \delta_z^2. \quad (4.18)$$

Furthermore, the condition (4.9) on the initial values is invariant under the evolution of the system, which follows from (4.1) and Proposition IX.1.3 in [16] on v for the electric field and Proposition IX.1.4 in [16] for the magnetic field.

Lemma 4.1.2. *Let $(E_0, H_0) \in (H_\Gamma^3(\Omega))^6$ satisfy (4.9). Then the solution (E, H) of (4.15) satisfies*

$$\varepsilon E|_v \in H^\Gamma(\operatorname{div} 0, v) \quad \text{and} \quad \mu H \in H_{n0}^\Sigma(\operatorname{div} 0) \quad \text{for all } t \in (T_{\min}, T_{\max}).$$

NON-TRAPPING CONDITIONS

In order to show decay, we assume the following *non-trapping condition* in the non-dissipative region and on a neighborhood of Γ . Let $x_0 \in \mathbb{R}^3$ and $m(x) := x - x_0$. Then,

$$\begin{aligned} \tilde{\varepsilon}(x, \xi)|_{\xi=0} &:= (\varepsilon + (m(x) \cdot \nabla_x) \varepsilon)(x, \xi)|_{\xi=0} \geq 2\tilde{\eta} \varepsilon(x, 0), \\ \tilde{\mu}(x, \xi)|_{\xi=0} &:= (\mu + (m(x) \cdot \nabla_x) \mu)(x, \xi)|_{\xi=0} \geq 2\tilde{\eta} \mu(x, 0), \end{aligned} \quad (4.19)$$

for $x \in v \cup \mathcal{N}_b(\Gamma)$ and some constants $\tilde{\eta}, b > 0$. This condition conceptually limits back reflection of the solutions towards x_0 , forcing the fields towards the interior of ω , where we assumed dissipation to occur. We transfer (4.19) to ε_d and μ_d using

$$\begin{aligned} \tilde{\varepsilon}_d(x, \xi) &:= (\varepsilon_d + m \cdot \nabla_x \varepsilon_d)(x, \xi), \\ \tilde{\mu}_d(x, \xi) &:= (\mu_d + m \cdot \nabla_x \mu_d)(x, \xi). \end{aligned} \quad (4.20)$$

The non-trapping condition (4.19) implies

$$\begin{aligned}\tilde{\varepsilon}_d(x, 0) &= (\tilde{\varepsilon} + \varepsilon_{d, \text{nl}} + m \cdot \nabla_x \varepsilon_{d, \text{nl}})(x, 0) = \tilde{\varepsilon}(x, 0) \geq 2\tilde{\eta}\varepsilon_d(x, 0), \\ \tilde{\mu}_d(x, 0) &= (\tilde{\mu} + \mu_{d, \text{nl}} + m \cdot \nabla_x \mu_{d, \text{nl}})(x, 0) = \tilde{\mu}(x, 0) \geq 2\tilde{\eta}\mu_d(x, 0),\end{aligned}$$

at $(x, 0)$ for $x \in v \cup \mathcal{N}_b(\Gamma)$.

Furthermore, we require the *non-trapping interface condition*

$$\begin{aligned}(m \cdot \nu^\Gamma) \llbracket \varepsilon_d \rrbracket &\leq 0, \\ (m \cdot \nu^\Gamma) \llbracket \mu_d \rrbracket &\leq 0,\end{aligned} \quad \text{on } \Gamma, \tag{4.21}$$

to also limit reflections at the interface. Note that this condition involves the solution and might therefore be hard to verify directly. However, this condition is for example satisfied for small solutions if we assume

$$\begin{aligned}(m \cdot \nu^\Gamma) \llbracket \varepsilon(x, 0) \rrbracket &< 0, \\ (m \cdot \nu^\Gamma) \llbracket \mu(x, 0) \rrbracket &< 0,\end{aligned} \tag{4.22}$$

for $x \in \Gamma$.

Let us quickly illustrate what happens on the (unbounded) half-space if this condition is violated.

Example 4.1.3. Let $\omega = \mathbb{R}_+^3 := \{x_3 > 0\}$, $v = \mathbb{R}_-^3$ and $\Omega = v \cup \Gamma \cup \omega$ with $\nu^\Gamma = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Set $m(x) = x - \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ and therefore $m \cdot \nu^\Gamma = 1$ on Γ . Assume that $\varepsilon|_v > \varepsilon|_\omega$ is piecewise constant (violating (4.22)), μ is constant, and $\sigma = 0$ on $\mathbb{R}^3$. Then, an easy calculation show that the plane waves

$$(E, H) = \left(a^E e^{i(k \cdot x - \omega t)}, a^H e^{i(k \cdot x - \omega t)} \right) \tag{4.23}$$

for the wave vector $k \in \mathbb{R}^3$, angular frequency $\omega = \frac{|k|}{\sqrt{\mu\varepsilon}}$, and amplitudes $a^E, a^H \in \mathbb{R}^3$ satisfying $a^E \perp a^H \perp k$ with $k \times a^E = \mu\omega a^H$ or equivalently $k \times a^H = -\varepsilon\omega a^E$, solve (4.1) and (4.2). The behavior at the interface $\Gamma = \{x_3 = 0\}$ is given by *Snell's law*. It can be derived from the interface conditions (4.6)

$$\llbracket E \times \nu^\Gamma \rrbracket = 0, \quad \llbracket H \times \nu^\Gamma \rrbracket = 0,$$

as follows. We assume that an incident plane wave $(E_i, H_i) \neq 0$ in v gets reflected as a plane wave (E_r, H_r) at the interface Γ and transmitted as another plane wave (E_t, H_t) into ω . Thus, the interface condition implies

$$\begin{aligned}\text{tr}_{t,v}(a_i^E e^{i(k_i \cdot x - \omega_i t)} + a_r^E e^{i(k_r \cdot x - \omega_r t)}) &= \text{tr}_{t,\omega}(a_t^E e^{i(k_t \cdot x - \omega_t t)}), \\ \text{tr}_{t,v}(a_i^H e^{i(k_i \cdot x - \omega_i t)} + a_r^H e^{i(k_r \cdot x - \omega_r t)}) &= \text{tr}_{t,\omega}(a_t^H e^{i(k_t \cdot x - \omega_t t)}),\end{aligned}$$

where $k_s, \omega_s, a_s^E, a_s^H$ for $s \in \{i, r, t\}$ are the parameters of the waves as in (4.23). In order for these equations to be satisfied for all $x \in \Gamma$ and $t \geq 0$ one sees that the exponents have to agree on Γ , i.e.,

$$\begin{aligned}k_i \cdot x - \omega_i t &= k_r \cdot x - \omega_r t = k_t \cdot x - \omega_t t, \\ k_i \cdot x - \omega_i t &= k_r \cdot x - \omega_r t = k_t \cdot x - \omega_t t.\end{aligned}$$

In particular $\omega_i = \omega_r = \omega_t$. Likewise,

$$k_i \cdot x = k_r \cdot x = k_t \cdot x \tag{4.24}$$

for $x \in \Gamma$.

Without loss of generality assume that k_i lies in the x_1x_3 -plane, i.e., $k_i \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = k_{i,x_2} = 0$. Otherwise, rotate the coordinate system around ν^Γ accordingly. Thus, by (4.24) we also see that $k_{t,x_2} = 0$. If we denote θ_i and θ_t to be the angle between k_i respectively k_t and $\nu^\Gamma = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, then (4.24) leads to Snell's law

$$\sin(\theta_i) |k_i| = k_{i,x_1} = k_{t,x_1} = \sin(\theta_t) |k_t|, \quad (4.25)$$

see Figure 4.2 below for an illustration of this relationship. Now our assumption that $\varepsilon|_v > \varepsilon|_\omega$ and $\frac{|k_i|}{\sqrt{\mu\varepsilon|_v}} = \omega_i = \omega_t = \frac{|k_t|}{\sqrt{\mu\varepsilon|_\omega}}$ lead to $|k_i| > |k_t|$. This however means that there exists a critical angle

$$\theta_c = \arcsin \left(\sqrt{\frac{\varepsilon|_\omega}{\varepsilon|_v}} \right) \in (0, \frac{\pi}{2})$$

such that for $\theta_i > \theta_c$ Equation (4.25) can no longer be satisfied, i.e., the incident wave cannot be transmitted as a plane wave into ω . It turns out that in this case the wave gets completely reflected back into v .

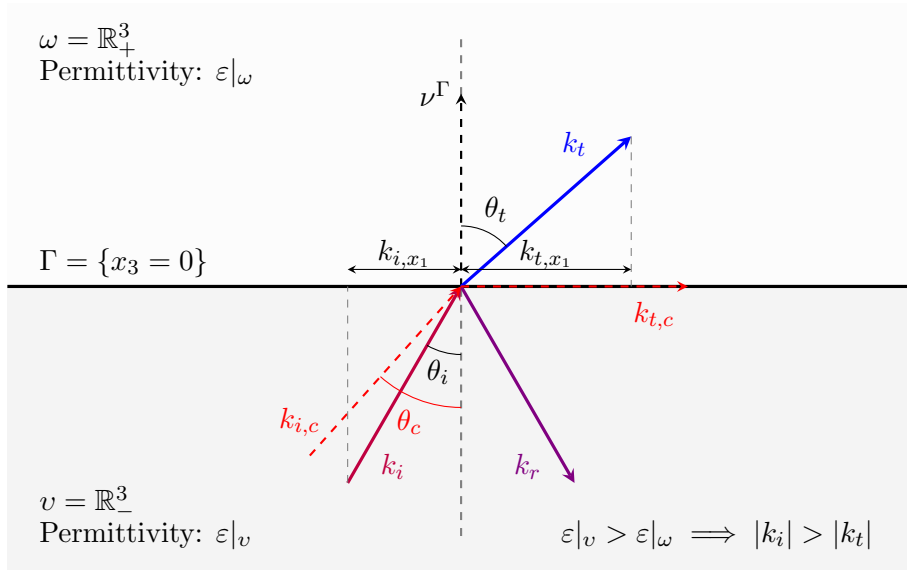


Figure 4.2.: Note, for θ_i approaching θ_c the angle of the transmitted wave θ_t goes to $\frac{\pi}{2}$.

Heuristically, the non trapping interface condition (4.21) therefore ensures that our solution is never fully back reflected at the interface Γ .

We assume without loss of generality that $\eta \leq \tilde{\eta}$ and fix $\delta_\eta \in (0, \min\{1, \delta_\eta^0\})$ (see (4.17)) such that

$$\hat{\varepsilon}_k \geq \eta, \quad \hat{\mu}_k \geq \eta, \quad \text{for } x \in \Omega_\pm, \quad (4.26)$$

and

$$\begin{aligned} \tilde{\varepsilon}(E) &\geq \eta \varepsilon(E), & \tilde{\varepsilon}_d(E) &\geq \eta \varepsilon_d(E), \\ \tilde{\mu}(H) &\geq \eta \mu(H), & \tilde{\mu}_d(H) &\geq \eta \mu_d(H), \end{aligned} \quad \text{for } x \in v \cup \mathcal{N}_b(\Gamma), \quad (4.27)$$

is satisfied for all $t \in [0, T_*)$.

So, to summarize we assume the following:

Let $\Omega \subset \mathbb{R}^3$ be open, bounded and multiply connected, and $\partial\Omega \in C^5$ have connected components $\partial\Omega_0, \dots, \partial\Omega_K$. Let $\omega \subseteq \Omega$ with C^5 -boundary such that the *conductivity* σ is supported on ω and satisfies (4.5). We assume the *permittivity* ε and *permeability* μ to satisfy (4.7) and (4.8). Let the initial values $(E_0, H_0) \in (H_q^3(\Omega))^6$ satisfy (4.9) as well as the *compatibility* and *smallness conditions* (4.13), (4.14), and (4.16). In (4.16) we assume that $0 < \delta_z \leq \delta_\eta \leq 1$ so that the coefficients are positive definite, see (4.26), and that the *non-trapping condition* (4.27) on $v \cup \mathcal{N}_b(\Gamma)$ is satisfied. In addition, we assume the *non-trapping interface condition* (4.21).

4.2. NOTATION, ASSUMPTIONS AND AUXILIARY RESULTS

REFORMULATIONS OF MAXWELL EQUATIONS

Let $k \in \{0, 1, 2, 3\}$. We will study the time derivatives of the solutions $\partial_t^k(E, H) \in \mathcal{G}_\Gamma^{3-k}$ of (4.1)–(4.4) from Section 4.1 separately. To this end, we define the commutator terms

$$\begin{aligned} f_0 = f_1 = 0, \quad f_2 &= (\partial_t \varepsilon_d(\cdot, E)) \partial_t E, \quad f_3 = (\partial_t^2 \varepsilon_d(\cdot, E)) \partial_t E + 2(\partial_t \varepsilon_d(\cdot, E)) \partial_t^2 E, \\ g_0 = g_1 = 0, \quad g_2 &= (\partial_t \mu_d(\cdot, H)) \partial_t H, \quad g_3 = (\partial_t^2 \mu_d(\cdot, H)) \partial_t H + 2(\partial_t \mu_d(\cdot, H)) \partial_t^2 H. \end{aligned}$$

Then, with the definitions above the time-derived solutions satisfy

$$\partial_t^k(\varepsilon E) = \hat{\varepsilon}_k \partial_t^k E + f_k, \quad \partial_t^k(\mu H) = \hat{\mu}_k \partial_t^k H + g_k \quad \text{on } \Omega, \quad (4.28)$$

and solve for $k \in \{0, 1, 2\}$ the Maxwell system

$$\begin{aligned} \partial_t(\hat{\varepsilon}_k \partial_t^k E) &= \text{curl } \partial_t^k H - \sigma \partial_t^k E - \partial_t f_k, \\ \partial_t(\hat{\mu}_k \partial_t^k H) &= -\text{curl } \partial_t^k E - \partial_t g_k, \end{aligned} \quad t \geq 0, x \in \Omega, \quad (4.29)$$

$$\llbracket \partial_t^k E \times \nu^\Gamma \rrbracket = 0, \quad \llbracket \partial_t^k H \times \nu^\Gamma \rrbracket = 0, \quad t \geq 0, x \in \Gamma, \quad (4.30)$$

$$\text{tr}_t \partial_t^k E = 0, \quad \text{tr}_n \hat{\mu}_k \partial_t^k H = -\text{tr}_n g_k, \quad t \geq 0, x \in \partial\Omega, \quad (4.31)$$

in the sense of Sobolev spaces.

Using the commutator terms

$$\begin{aligned} \tilde{f}_0 &:= 0, & \tilde{f}_k &:= \sum_{j=1}^k \binom{k}{j} \partial_t^j \varepsilon_d \partial_t^{k+1-j} E, \\ \tilde{g}_0 &:= 0, & \tilde{g}_k &:= \sum_{j=1}^k \binom{k}{j} \partial_t^j \mu_d \partial_t^{k+1-j} H, \end{aligned}$$

one can also derive from (4.1)–(4.4) the system

$$\varepsilon_d \partial_t(\partial_t^k E) = \text{curl } \partial_t^k H - \sigma \partial_t^k E - \tilde{f}_k, \quad t \geq 0, x \in \Omega, \quad (4.32)$$

$$\begin{aligned} \mu_d \partial_t(\partial_t^k H) &= -\text{curl } \partial_t^k E - \tilde{g}_k, \\ \llbracket \partial_t^k E \times \nu^\Gamma \rrbracket &= 0, \quad \llbracket \partial_t^k H \times \nu^\Gamma \rrbracket = 0, \end{aligned} \quad t \geq 0, x \in \Gamma, \quad (4.33)$$

$$\text{tr}_t \partial_t^k E = 0, \quad t \geq 0, x \in \partial\Omega. \quad (4.34)$$

Both are needed below. We apply the divergence to (4.1) and (4.32), obtaining

$$\begin{aligned} \text{div}(\varepsilon E) &= \text{div}(\varepsilon(E_0)E_0) - \int_0^t \text{div}(\sigma E) \, d\tau, & \text{div}(\mu H) &= 0, \\ \text{div}(\varepsilon_d \partial_t^k E) &= -\text{div}(\sigma \partial_t^{k-1} E + \tilde{f}_{k-1}), & \text{div}(\mu_d \partial_t^k H) &= -\text{div } \tilde{g}_{k-1}, \end{aligned}$$

on $\Omega_{\pm}$ for $k \neq 0$. We observe that for $k = 3$ the right-hand side of (4.35) lies in $L^2(\Omega)$. Therefore, for $k \in \{0, 1, 2, 3\}$ the term $(\hat{\varepsilon}_k \partial_t^k E, \hat{\mu}_k \partial_t^k H)$ lies in $H(\operatorname{div}, \Omega_{\pm})^2$. By the trace theorem, see for example Theorem 2.2.18 in [5], its normal trace on either side of the interface exists in $H^{-1/2}(\Gamma)$, allowing us to derive interface conditions for the normal jump for $k \in \{1, 2, 3\}$ in addition to (4.6). For $k \in \{1, 2, 3\}$ and $\psi \in C_c^\infty(\Omega)$, the divergence theorem, (4.32), and (4.30) yields

$$\begin{aligned} & \left\langle \llbracket (\varepsilon_d \partial_t^k E + \sigma \partial_t^{k-1} E + \tilde{f}_{k-1}) \cdot \nu^\Gamma \rrbracket, \psi \right\rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)} \\ &= \int_{\Omega_{\pm}} \left(\psi \operatorname{div}(\varepsilon_d \partial_t^k E + \sigma \partial_t^{k-1} E + \tilde{f}_{k-1}) + \nabla \psi \cdot (\varepsilon_d \partial_t^k E + \sigma \partial_t^{k-1} E + \tilde{f}_{k-1}) \right) dx \\ &= \int_{\Omega_{\pm}} \left(\psi \operatorname{div}(\operatorname{curl} \partial_t^{k-1} H) + \nabla \psi \cdot (\operatorname{curl} \partial_t^{k-1} H) \right) dx = \int_{\Omega_{\pm}} \nabla \psi \cdot (\operatorname{curl} \partial_t^{k-1} H) dx \\ &= \langle \nabla \psi, \llbracket \nu^\Gamma \times \partial_t^{k-1} H \rrbracket \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)} = 0. \end{aligned}$$

Therefore,

$$\llbracket (\varepsilon_d \partial_t^k E) \cdot \nu^\Gamma \rrbracket = -\operatorname{tr}_{\partial\omega} \left((\sigma \partial_t^{k-1} E) \cdot \nu^\Gamma \right) - \llbracket \tilde{f}_{k-1} \cdot \nu^\Gamma \rrbracket \quad \text{for } k \geq 1, \quad (4.35)$$

where the equality holds in $H^{1/2}(\Gamma)$ due to the regularity of the right-hand side. Likewise, one can show the analogous interface condition for the jump in normal direction of the magnetic field

$$\llbracket (\mu_d \partial_t^k H) \cdot \nu^\Gamma \rrbracket = -\llbracket \tilde{g}_{k-1} \cdot \nu^\Gamma \rrbracket \quad \text{for } k \geq 1.$$

Equations (4.31), (4.32), and Lemma A.1.2 thus result in

$$\mu_d \partial_t^k H + \tilde{g}_{k-1} \in H_{n0}(\operatorname{div} 0) \quad \text{for } k \geq 1. \quad (4.36)$$

ENERGIES AND ERROR TERMS

We set

$$\begin{aligned} e_k(t) &:= \frac{1}{2} \max_{0 \leq j \leq k} \left(\left\| \hat{\varepsilon}_j^{1/2} \partial_t^j E(t) \right\|_{L^2(\Omega)}^2 + \left\| \hat{\mu}_j^{1/2} \partial_t^j H(t) \right\|_{L^2(\Omega)}^2 \right), & e &:= e_3, \\ d_k(t) &:= \max_{0 \leq j \leq k} \left\| \sigma^{1/2} \partial_t^j E(t) \right\|_{L^2(\Omega)}^2, & d &:= d_3, \\ z_k(t) &:= \max_{0 \leq j \leq k} \left(\left\| \partial_t^j E(t) \right\|_{H_\Gamma^{k-j}(\Omega)}^2 + \left\| \partial_t^j H(t) \right\|_{H_\Gamma^{k-j}(\Omega)}^2 \right), & z &:= z_3. \end{aligned} \quad (4.37)$$

One can show the following estimates as in [35]. We will need them to estimate error and commutator terms. We occasionally apply the lemma without referencing it explicitly.

Lemma 4.2.1 (Error estimates). *We have*

$$\begin{aligned} & \|\hat{\varepsilon}_k(E)(t)\|_\infty, \|\hat{\mu}_k(t)\|_\infty, \|\hat{\varepsilon}_k(E)^{-1}(t)\|_\infty, \|\hat{\mu}_k(H)^{-1}(t)\|_\infty \leq c, \\ & \|\partial^\alpha \hat{\varepsilon}_k(E)(t)\|_{L^2(\Omega)}, \|\partial^\alpha \hat{\mu}_k(H)(t)\|_{L^2(\Omega)} \leq c \left(z_k^{1/2}(t) + \delta_{\alpha_0=0} \right), \\ & \max_{k \in \{0,1,2,3\}, j \in \{0,1\}} \left(\left\| \partial_t^j f_k(t) \right\|_{H_\Gamma^{4-j-k}} + \left\| \partial_t^j g_k(t) \right\|_{H_\Gamma^{4-j-k}} \right) \leq cz(t), \\ & \max_{k \in \{0,1,2,3\}} \left(\left\| \tilde{f}_k(t) \right\|_{H_\Gamma^{3-k}} + \left\| \tilde{g}_k(t) \right\|_{H_\Gamma^{3-k}} \right) \leq cz(t), \\ & \max_{k \in \{0,1,2\}} \left(\left\| \partial_t \tilde{f}_k(t) \right\|_{H_\Gamma^{2-k}} + \left\| \partial_t \tilde{g}_k(t) \right\|_{H_\Gamma^{2-k}} \right) \leq cz(t). \end{aligned}$$

Here the multiindex $\alpha \in \mathbb{N}_0 \times \mathbb{N}_0^3$ satisfies $|\alpha| = k \leq 3$ with $\alpha_0 \in \mathbb{N}_0$ corresponding to the derivative in time $\partial^{(\alpha_0, 0, 0, 0)} = \partial_t^{\alpha_0}$, and $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$ denoting the spatial derivatives.

Sketch of proof for Lemma 4.2.1. We carry out the proof for f_k , since for g_k the argument can be repeated. For $k \in \{0, 1\}$ the commutator term f_k vanishes and there is nothing to show. Notice that $H_{\Gamma}^2(\Omega) \hookrightarrow L^\infty(\Omega)$ and therefore E and $\partial_t E$ are bounded. Since ε_d is assumed to belong to $C^3(\overline{\Omega_\pm} \times \mathbb{R}^3)$, we can estimate the time derivative of

$$\begin{aligned} f_3 &= \left[\sum_{i,j=1}^3 \partial_{\xi_i, \xi_j} \varepsilon_d(\cdot, E) \partial_t E_i \partial_t E_j \right] \partial_t E + \left[\sum_{i=1}^3 \partial_{\xi_i} \varepsilon_d(\cdot, E) \partial_t^2 E_i \right] \partial_t E \\ &\quad + 2 \left[\sum_{i=1}^3 \partial_{\xi_i} \varepsilon_d(\cdot, E) \partial_t E_i \right] \partial_t^2 E =: f_{3,1} + f_{3,2} + 2f_{3,3} \end{aligned}$$

for instance by

$$\begin{aligned} \|\partial_t f_{3,1}\|_{L^2(\Omega)} &= \left\| \partial_t \left[\left(\sum_{i,j=1}^3 \partial_{\xi_i, \xi_j} \varepsilon_d(\cdot, E) \partial_t E_i \partial_t E_j \right) \partial_t E \right] \right\|_{L^2(\Omega)} \\ &\leq \left\| \left(\sum_{i,j,k=1}^3 \partial_{\xi_i, \xi_j, \xi_k} \varepsilon_d(\cdot, E) \partial_t E_i \partial_t E_j \partial_t E_k \right) \partial_t E \right\|_{L^2(\Omega)} \\ &\quad + \left\| 2 \left(\sum_{i,j=1}^3 \partial_{\xi_i, \xi_j} \varepsilon_d(\cdot, E) \partial_t^2 E_i \partial_t E_j \right) \partial_t E \right\|_{L^2(\Omega)} \\ &\quad + \left\| \left(\sum_{i,j=1}^3 \partial_{\xi_i, \xi_j} \varepsilon_d(\cdot, E) \partial_t E_i \partial_t E_j \right) \partial_t^2 E \right\|_{L^2(\Omega)} \\ &\lesssim \sum_{i,j=1}^3 \left(\|\partial_t E_i\|_{L^\infty(\Omega)} \|\partial_t E_j\|_{L^2(\Omega)} + \|\partial_t^2 E_i\|_{L^2(\Omega)} \|\partial_t E_j\|_{L^\infty(\Omega)} \right. \\ &\quad \left. + \|\partial_t E_i\|_{L^\infty(\Omega)} \|\partial_t^2 E_j\|_{L^2(\Omega)} \right) \\ &\lesssim \sum_{i,j=1}^3 \left(\|\partial_t E_i\|_{H_{\Gamma}^2(\Omega)} \|\partial_t E_j\|_{L^2(\Omega)} + \|\partial_t^2 E_i\|_{L^2(\Omega)} \|\partial_t E_j\|_{H_{\Gamma}^2(\Omega)} \right. \\ &\quad \left. + \|\partial_t E_i\|_{H_{\Gamma}^2(\Omega)} \|\partial_t^2 E_j\|_{L^2(\Omega)} \right) \\ &\lesssim z(t). \end{aligned}$$

Analogous estimates for $\|\partial_t f_{3,2}\|_{L^2(\Omega)}$ and $\|\partial_t f_{3,3}\|_{L^2(\Omega)}$ can be shown, using also the embedding $H_{\Gamma}^1(\Omega) \hookrightarrow L^6(\Omega)$. In a similar way one obtains bounds on $\|f_2\|_{H_{\Gamma}^2(\Omega)}$, $\|\partial_t f_2\|_{H_{\Gamma}^1(\Omega)}$, and $\|f_3\|_{H_{\Gamma}^1(\Omega)}$. $\square$

The error estimates are often applied implicitly in combination with the following lemma, see Lemma 2.1 in [56] and the proof of Lemma 2.22 in [58]. It is shown by combining the usual Sobolev embeddings with Hölder's inequality.

Lemma 4.2.2. *Let $m_1, m_2 \in \mathbb{N}_0$ with $m_1 \geq m_2$ and $m_1 \geq 2$.*

a) Let $k \in \{0, \dots, m_1\}$, $f \in H_{\Gamma}^{m_1-k}(\Omega)$ and $g \in H_{\Gamma}^k(\Omega)$. Then,

$$fg \in L^2(\Omega) \quad \text{with} \quad \|fg\|_{L^2(\Omega)} \lesssim \|f\|_{H_{\Gamma}^{m_1-k}} \|g\|_{H_{\Gamma}^k}.$$

b) Let $f \in H_{\Gamma}^{m_1}(\Omega)$ and $g \in H_{\Gamma}^{m_2}(\Omega)$. Then

$$fg \in H_{\Gamma}^{m_2}(\Omega) \quad \text{with} \quad \|fg\|_{H_{\Gamma}^{m_2}} \lesssim \|f\|_{H_{\Gamma}^{m_1}} \|g\|_{H_{\Gamma}^{m_2}}.$$

Integration by parts shows the following energy estimate (for $k = 3$ an additional approximation argument is required, see Theorem 3.3.5), cf. Section 4 in [35].

Lemma 4.2.3 (Energy estimates). *The energies e_k are decreasing up to an error term, namely for $k \in \{0, 1, 2, 3\}$*

$$e_k(t) + \int_s^t d_k(\tau) \, d\tau \leq e_k(s) + c \int_s^t z^{3/2}(\tau) \, d\tau.$$

Because of the time integral in (4.35), for instance, we cannot treat (E, H) directly. Instead we use the next result, where we estimate the usual energy through that of the time derivatives. Here we make use of our assumption that σ either vanishes or is uniformly positive.

Lemma 4.2.4. *For all $0 < t < T_*$ we have*

$$e_0(t) \lesssim e_1(t).$$

The proof uses ideas from Theorem 5.4 of [49] for the linear, constant coefficients case and is similar to the proof of Lemma 2.3.3.

Proof. Condition (H_q) , Maxwell equations (4.1), and (4.3), integration by parts and (4.28) lead to

$$\begin{aligned} \sigma_0 \int_{\Omega} |E(t)|^2 \, dx &\leq \int_{\Omega} \sigma E(t) \cdot E(t) = \int_{\Omega} (-\partial_t(\varepsilon E(t)) + \operatorname{curl} H(t)) \cdot E(t) \, dx \\ &= - \int_{\Omega} \varepsilon_d \partial_t E(t) \cdot E(t) - \int_{\Omega} H(t) \cdot \mu_d \partial_t H(t) \, dx \\ &\lesssim \sqrt{e_0(t)} \sqrt{e_1(t)}. \end{aligned} \tag{4.38}$$

Using Lemma 4.1.2, Lemma A.1.3 b) and (4.32), we then estimate the magnetic field by

$$\begin{aligned} \|H(t)\|_{L^2(\Omega)}^2 &\lesssim \|\operatorname{curl} H(t)\|_{L^2(\Omega)}^2 \lesssim \|\varepsilon_d \partial_t E(t)\|_{L^2(\Omega)}^2 + \|\sigma E(t)\|_{L^2(\Omega)}^2 \\ &\lesssim e_1(t) + \sqrt{e_0(t) e_1(t)}. \end{aligned} \tag{4.39}$$

It remains to control E on v . Using (4.9), Lemma 2.3.2 yields

$$\|E(t)\|_{L^2(v)} \lesssim \|\operatorname{curl} E(t)\|_{L^2(v)}^2 + \|\operatorname{tr}_t E(t)\|_{H^{-1/2}(\partial v)}.$$

Since $E \in H(\operatorname{curl})$, Lemma A.1.2 shows that $\operatorname{tr}_{t, \partial v} E = \operatorname{tr}_{t, \partial \omega} E$ on ∂v . From the trace estimate and applying Equation (4.32) twice we thus deduce

$$\begin{aligned} \|E(t)\|_{L^2(v)} &\lesssim \|\operatorname{curl} E(t)\|_{L^2(v)} + \|\operatorname{tr}_t E(t)\|_{H^{-\frac{1}{2}}(\partial \omega)} \lesssim \|\mu_d \partial_t H(t)\|_{L^2(v)} + \|E(t)\|_{H(\operatorname{curl}, \omega)} \\ &\lesssim \|\mu_d \partial_t H(t)\|_{L^2(\Omega)} + \|E(t)\|_{L^2(\omega)}. \end{aligned}$$

Combined with (4.38) and (4.39), we arrive at

$$e_0(t) \leq c_1 \sqrt{e_0(t)} \sqrt{e_1(t)} + c_2 e_1(t) \leq \frac{1}{2} e_0(t) + C e_1(t),$$

which yields the assertion. $\square$

4.3. HELMHOLTZ DECOMPOSITION

We split our solution $\partial_t^k(E, H) \in \mathcal{G}_\Gamma^{3-k}$ into a $\operatorname{div}_{\varepsilon_d}$ -free and a curl-free part, adjusting the interface and boundary conditions in such a way that the curl operator behaves well with the fields. Since we only show the observability estimate for $k \geq 1$, it suffices to consider $k \in \{1, 2, 3\}$, simplifying the notations slightly. For the case of $k = 0$, we then rely on Lemma 4.2.4 in the proof of exponential decay.

ELECTRIC HELMHOLTZ DECOMPOSITION

To construct the curl-free part, we consider the elliptic interface problem

$$\operatorname{div}(\varepsilon_d \nabla p^k) = \operatorname{div}(\varepsilon_d \partial_t^k E) = -\operatorname{div}(\sigma \partial_t^{k-1} E + \tilde{f}_{k-1}) \quad \text{on } \Omega_\pm, \quad (4.40)$$

$$[[p^k]] = 0 \quad \text{on } \Gamma, \quad (4.41)$$

$$[[\operatorname{tr}_n(\varepsilon_d \nabla p^k)]] = [[\operatorname{tr}_n(\varepsilon_d \partial_t^k E)]] \quad \text{on } \Gamma, \quad (4.42)$$

$$p^k = 0 \quad \text{on } \partial\Omega_0, \quad (4.43)$$

$$\begin{aligned} p^k &\text{ constant on } \partial\Omega_\ell, \text{ and} \\ 0 &= \langle \operatorname{tr}_n \varepsilon_d (\nabla p^k - \partial_t^k E), \mathbb{1} \rangle_{H^{-\frac{1}{2}}(\partial\Omega_\ell)}, \end{aligned} \quad \forall \ell \in \{1, \dots, K\}. \quad (4.44)$$

In (4.40) we use (4.32) for $k-1$. Observe that the right-hand side of (4.40) belongs to $L^2(\Omega)$, so that the first-order conditions on Γ and $\partial\Omega_\ell$ in (4.42) and (4.44) are well-defined in $H^{-1/2}$. For the weak formulation, we use the space

$$\mathcal{H}_c := \{\varphi \in H^1(\Omega) \mid \varphi = 0 \text{ on } \partial\Omega_0, \varphi \text{ constant on } \partial\Omega_\ell, \forall \ell \in \{1, \dots, K\}\},$$

endowed with the norm of $H^1(\Omega)$. We introduce the form

$$B[u, v] = B_{E,k}[u, v] := \int_\Omega \nabla u \cdot \varepsilon_d \nabla v \, dx \quad \text{on } \mathcal{H}_c$$

and set

$$\ell(v) = \ell_{E,k}(v) := - \int_\Omega (\sigma \partial_t^{k-1} E + \tilde{f}_{k-1}) \cdot \nabla v \, dx$$

for $v \in \mathcal{H}_c$. Then the weak formulation of (4.40)–(4.44) reads as

$$B_{E,k}[p^k, \varphi] = \ell_{E,k}(\varphi) \quad \text{for all } \varphi \in \mathcal{H}_c. \quad (4.45)$$

Clearly, B and ℓ are bounded on $\mathcal{H}_c$, and B is coercive by Theorem 13.6.9 in [64]. The Lax–Milgram lemma thus yields a unique solution $p^k = p^k(t) \in \mathcal{H}_c$ of (4.45) with

$$\|p^k\|_{H^1(\Omega)} \lesssim \|\ell\|_{\mathcal{H}_c^*} = \sup_{\|\varphi\|_{\mathcal{H}_c}=1} |\ell_{E,k}(\varphi)| \lesssim \|\sigma \partial_t^{k-1} E + \tilde{f}_{k-1}\|_{L^2(\Omega)} \lesssim d^{1/2} + z, \quad (4.46)$$

where we also used Lemma 4.2.1. Since the map $\mathcal{G}^3 \rightarrow \mathcal{H}_c^*; E \mapsto \ell_{E,k}$ is continuous and bounded, we obtain a continuous and bounded mapping $\mathcal{G}^3 \rightarrow \mathcal{H}_c; E \mapsto p^k$.

We verify that the solution to (4.45) solves (4.40)–(4.44). Since $p^k \in \mathcal{H}_c$, it satisfies (4.43) and the first part of (4.44) in $H^{1/2}$, as well as (4.41) by Lemma A.1.2. Note that $\varepsilon_d \partial_t^k E - \varepsilon_d \nabla p^k$ belongs to $H(\operatorname{div} 0, \Omega)$, since (4.45) holds in particular for $H_0^1(\Omega) \subseteq \mathcal{H}_c$. Therefore, again by Lemma A.1.2, p^k also satisfies (4.42) in $H^{-1/2}$. In particular, the last boundary condition in (4.44) is well-defined. It follows from (4.45) by inserting $\varphi_\ell \in \mathcal{H}_c$ with $\varphi_\ell|_{\partial\Omega_j} = \delta_{\ell j}$ for $\ell \neq 0$ via

$$0 = \int_\Omega (\varepsilon_d \partial_t^k E - \varepsilon_d \nabla p^k) \cdot \nabla \varphi_\ell \, dx = \langle \operatorname{tr}_n(\varepsilon_d \partial_t^k E - \varepsilon_d \nabla p^k), \mathbb{1} \rangle_{H^{-\frac{1}{2}}(\partial\Omega_\ell)}.$$

Based on elliptic regularity, Lemma 4.3.1 below furthermore shows that $p^k \in H_\Gamma^2(\Omega)$.

MAGNETIC HELMHOLTZ DECOMPOSITION

For H we solve the following Neumann problem to correct for the jump over the interface

$$\begin{aligned} \operatorname{div}(\mu_d \nabla q^k) &= \operatorname{div}(\mu_d \partial_t^k H) = -\operatorname{div}(\tilde{g}_{k-1}) && \text{on } \Omega, \\ \llbracket q^k \rrbracket &= 0 && \text{on } \Gamma, \\ \llbracket \operatorname{tr}_n(\mu_d \nabla q^k) \rrbracket &= \llbracket \operatorname{tr}_n(\mu_d \partial_t^k H) \rrbracket && \text{on } \Gamma, \\ \operatorname{tr}_n(\mu_d \nabla q^k) &= \operatorname{tr}_n(\mu_d \partial_t^k H) = -\operatorname{tr}_n(\tilde{g}_{k-1}) && \text{on } \partial\Omega. \end{aligned} \quad (4.47)$$

We define the space $\mathcal{H}_n(\Omega) := \{\varphi \in H^1(\Omega) \mid \int_\Omega \varphi \, dx = 0\}$ of functions with zero mean. As for (4.40)–(4.44), we introduce the form

$$B[u, v] = B_{H,k}[u, v] := \int_\Omega \nabla u \cdot \mu_d \nabla v \, dx \quad \text{on } H^1(\Omega)$$

and set

$$\ell(v) = \ell_{H,k}(v) := - \int_\Omega \tilde{g}_{k-1} \cdot \nabla v \, dx$$

for $v \in H^1(\Omega)$. Then the weak formulation of (4.47) reads as

$$B_{H,k}[q^k, \varphi] = \ell_{H,k}(\varphi) \quad \text{for all } \varphi \in H^1(\Omega). \quad (4.48)$$

By the Poincaré inequality, see for example Section 5.8.1 in [23], the form B is coercive on $\mathcal{H}_n(\Omega)$. Thus, as before Lax–Milgram provides a unique weak solution $q^k \in \mathcal{H}_n(\Omega)$ satisfying (4.48) with

$$\|q^k\|_{H^1(\Omega)} \lesssim \|\ell\|_{(\mathcal{H}_n)^*} = \sup_{\|\varphi\|_{\mathcal{H}_n}=1} |\ell_{H,k}(\varphi)| \lesssim z. \quad (4.49)$$

As above, the map $\mathcal{G}^3 \rightarrow \mathcal{H}_n$; $H \mapsto q^k$ is continuous and bounded. Moreover, q^k satisfies (4.47).

ESTIMATES FOR p^k AND q^k

The following Lemma guarantees that the correction terms belong to $H_1^2(\Omega)$, since $\operatorname{div}(\varepsilon_d \partial_t^k E) \in L^2(\Omega)$ by Maxwell.

Lemma 4.3.1. *Let $k \in \{1, 2, 3\}$. The weak solutions (p^k, q^k) of (4.45) respectively (4.48) lie in $H_1^2(\Omega)$.*

Proof. 1) We focus on p^k as the proof for q^k is similar. Let $\chi_j \in C^\infty(\overline{\Omega})$ be smooth cutoffs satisfying $\chi_j|_{(\partial\Omega)_\ell} = \delta_{j\ell}$ and $\chi_j|_\Gamma = 0$ for $j \in \{0, \dots, K\}$. We set

$$h := \sum_{j=0}^K \chi_j c_j \in C^\infty(\overline{\Omega}) \subseteq H^2(\Omega),$$

where $p^k|_{(\partial\Omega)_j} = c_j$. (By Theorem 1.5.1.2 in [28] also general inhomogeneous Dirichlet data h in $H^{3/2}(\partial\Omega)$ could be considered.) Let $\tilde{p}^k := p^k - h$. Then $\tilde{p}^k$ is a weak solution of

$$\begin{aligned} \operatorname{div}(\varepsilon_d \nabla \tilde{p}^k) &= \operatorname{div}(\varepsilon_d (\partial_t^k E - \nabla h)) = -\operatorname{div}(\sigma \partial_t^{k-1} E + \tilde{f}_{k-1} + \nabla h) =: f && \text{on } \Omega, \\ \llbracket \tilde{p}^k \rrbracket &= 0 && \text{on } \Gamma, \\ \llbracket \operatorname{tr}_n(\varepsilon_d \nabla \tilde{p}^k) \rrbracket &= \llbracket \operatorname{tr}_n(\varepsilon_d \partial_t^k E) \rrbracket = -\operatorname{tr}_{\partial\Omega} \left(\sigma \partial_t^{k-1} E \cdot \nu^\Gamma \right) - \llbracket \operatorname{tr}_n \tilde{f}_{k-1} \rrbracket =: g && \text{on } \Gamma, \\ \tilde{p}^k &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where we used (4.32) as well as (4.35). By Lemma 4.2.1 we have

$$f \in L^2(\Omega) \quad \text{and} \quad g \in H^{1/2}(\Gamma).$$

By an elliptic regularity result for the transmission problem above, see Theorem 5.3.7 in [13] or Theorem 1.4 in [10], the weak solution $\tilde{p}^k$ lies in $H_{\Gamma}^2(\Omega)$ and satisfies

$$\|\tilde{p}^k\|_{H_{\Gamma}^2} \lesssim \|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)} + \|\tilde{p}^k\|_{H_{\Gamma}^1}.$$

Therefore, also $p^k = \tilde{p}^k + h$ belongs to $H_{\Gamma}^2(\Omega)$.

2) We can apply Theorem 5.3.7 in [13] directly to q^k as the solution of the transmission problem (4.48) with Neumann boundary conditions. Then q^k also lies in $H_{\Gamma}^2(\Omega)$ and satisfies a similar estimate. $\square$

We can bound the curl-free part of the decomposition as follows, cf. Proposition 5.2 in [49] for constant, scalar ε and μ .

Lemma 4.3.2. *Let $k \in \{1, 2, 3\}$ and $\tau \in [0, T_*)$. Then $(\nabla p^k, \nabla q^k) \in C^1([0, T_*), L^2(\Omega))$ with*

$$\|\partial_t^j \nabla p^k(\tau)\|_{L^2(\Omega)}^2, \|\partial_t^j \nabla q^k(\tau)\|_{L^2(\Omega)}^2 \lesssim d(\tau) + z^2(\tau) \quad \text{for } j \in \{0, 1\}.$$

Proof. We only consider the case for p^k , since the other case follows similarly. First, we show that $\varepsilon_d \nabla p^k + \tilde{f}_{k-1}$ is differentiable. Let $h \in L^2(\Omega)$. We decompose h into

$$h = \varepsilon_d^{1/2} \nabla \varphi \oplus \varepsilon_d^{-1/2} \Psi$$

for $\varphi \in \mathcal{H}_c$ and $\Psi \in H^{\partial\Omega}(\text{div } 0)$ satisfying $[\nu^{\Gamma} \cdot \Psi] = 0$ on Γ . To prove this decomposition, we solve the elliptic problem

$$\int_{\Omega} \varepsilon_d \nabla \varphi \cdot \nabla v \, dx = \int_{\Omega} \varepsilon_d^{1/2} h \cdot \nabla v \, dx \quad \text{for all } v \in \mathcal{H}_c, \quad (4.50)$$

which by the Lax–Milgram lemma yields a unique solution $\varphi \in \mathcal{H}_c$ with $\|\varepsilon_d^{1/2} \nabla \varphi\|_{L^2} \lesssim \|h\|_{L^2}$, analogous to the weak formulation for p^k . Defining $\Psi := \varepsilon_d^{1/2} h - \varepsilon_d \nabla \varphi$, we immediately see that $\int_{\Omega} \Psi \cdot \nabla v \, dx = 0$ for all $v \in \mathcal{H}_c$, which implies the stated divergence and trace properties of Ψ as above. Furthermore, the difference satisfies $\|\varepsilon_d^{-1/2} \Psi\|_{L^2} \lesssim \|h\|_{L^2}$. By setting $v = \varphi$ in (4.50), the orthogonality in $L^2(\Omega)$ follows. This decomposition leads to

$$\begin{aligned} & \int_{\Omega} (\varepsilon_d^{1/2} \nabla p^k + \varepsilon_d^{-1/2} \tilde{f}_{k-1}) \cdot h \, dx \\ &= \int_{\Omega} (\varepsilon_d \nabla p^k + \tilde{f}_{k-1}) \cdot \nabla \varphi \, dx + \int_{\Omega} (\varepsilon_d^{-1} \tilde{f}_{k-1}) \cdot \Psi \, dx \\ &= - \int_{\Omega} \text{div}(\varepsilon_d \nabla p^k + \tilde{f}_{k-1}) \varphi \, dx + \oint_{\partial\Omega} \nu \cdot (\varepsilon_d \nabla p^k + \tilde{f}_{k-1}) \varphi \, d\varsigma \\ & \quad + \oint_{\Gamma} [\nu^{\Gamma} \cdot (\varepsilon_d \nabla p^k + \tilde{f}_{k-1})] \varphi \, d\varsigma + \int_{\Omega} (\varepsilon_d^{-1} \tilde{f}_{k-1}) \cdot \Psi \, dx. \end{aligned} \quad (4.51)$$

We take a closer look at the integral at the boundary. Recall that $\varphi \in \mathcal{H}_c$, i.e., it is zero on $\partial\Omega_0$ and constant, say c_{ℓ} , on each $\partial\Omega_{\ell}$ for $\ell \in \{1, \dots, K\}$. Hence, by (4.44) yields

$$\oint_{\partial\Omega_{\ell}} \nu \cdot \varepsilon_d \nabla p^k \, d\varsigma = \oint_{\partial\Omega_{\ell}} \nu \cdot \varepsilon_d \partial_t^k E \, d\varsigma.$$

Moreover, Stokes' theorem provides $\oint_{\partial\Omega_\ell} \nu \cdot \operatorname{curl}(\partial_t^{k-1} H) \, d\varsigma = 0$ as $\partial\Omega_\ell$ has no boundary in $\partial\Omega$. So, (4.32) yields

$$\begin{aligned} & \oint_{\partial\Omega} \nu \cdot (\varepsilon_d \nabla p^k + \tilde{f}_{k-1}) \varphi \, d\varsigma \\ &= \sum_{\ell=1}^K c_\ell \oint_{\partial\Omega_\ell} \nu \cdot (\varepsilon_d \partial_t^k E + \tilde{f}_{k-1}) \, d\varsigma \\ &= \sum_{\ell=1}^K c_\ell \oint_{\partial\Omega_\ell} \nu \cdot (\operatorname{curl}(\partial_t^{k-1} H) - \sigma \partial_t^{k-1} E - \tilde{f}_{k-1} + \tilde{f}_{k-1}) \, d\varsigma \\ &= - \oint_{\partial\Omega} \nu \cdot (\sigma \partial_t^{k-1} E) \varphi \, d\varsigma. \end{aligned}$$

Inserting this, (4.42) and (4.35) into (4.51), we derive

$$\begin{aligned} & \int_{\Omega} (\varepsilon_d^{1/2} \nabla p^k + \varepsilon_d^{-1/2} \tilde{f}_{k-1}) \cdot h \, dx \\ &= \int_{\Omega} \operatorname{div}(\sigma \partial_t^{k-1} E) \varphi \, dx - \oint_{\partial\Omega} \nu \cdot (\sigma \partial_t^{k-1} E) \varphi \, d\varsigma - \oint_{\Gamma} [\nu^\Gamma \cdot (\sigma \partial_t^{k-1} E)] \varphi \, d\varsigma \\ & \quad + \int_{\Omega} (\varepsilon_d^{-1} \tilde{f}_{k-1}) \cdot \Psi \, dx \\ &= - \int_{\Omega} (\sigma \partial_t^{k-1} E) \cdot \nabla \varphi \, dx + \int_{\Omega} (\varepsilon_d^{-1} \tilde{f}_{k-1}) \cdot \Psi \, dx, \end{aligned} \tag{4.52}$$

by means of the divergence theorem on v and ω . Since the right-hand side is continuously differentiable in time, the differences and Cauchy sequences of difference quotients in time of the left-hand side converge in $L^2(\Omega)^* \simeq L^2(\Omega)$. (Recall that $\|\nabla \varphi\|_{L^2}, \|\Psi\|_{L^2} \lesssim \|h\|_{L^2}$.) It easily follows that $\varepsilon_d^{1/2} \nabla p^k + \varepsilon_d^{-1/2} \tilde{f}_{k-1}$ belongs to $C^1([0, T_*], L^2(\Omega))$ and thus also ∇p^k by the properties of ε_d and $\tilde{f}_{k-1}$. The differentiated version of (4.52) further yields

$$\begin{aligned} \left\| \partial_t (\varepsilon_d \nabla p^k) \right\|_{L^2(\Omega)} &\lesssim \left\| \partial_t (\varepsilon_d^{1/2} \nabla p^k + \varepsilon_d^{-1/2} \tilde{f}_{k-1}) \right\|_{L^2(\Omega)} + \left\| (\partial_t \varepsilon_d^{1/2}) \varepsilon_d^{1/2} \nabla p^k \right\|_{L^2(\Omega)} \\ &\quad + \left\| \partial_t (\varepsilon_d^{-1/2} \tilde{f}_{k-1}) \right\|_{L^2(\Omega)} \\ &\lesssim \left\| \sigma \partial_t^k E \right\|_{L^2(\Omega)} + \left\| \partial_t (\varepsilon_d^{-1} \tilde{f}_{k-1}) \right\|_{L^2(\Omega)} \\ &\quad + \left\| (\partial_t \varepsilon_d^{1/2}) \varepsilon_d^{1/2} \nabla p^k \right\|_{L^2(\Omega)} + \left\| \partial_t (\varepsilon_d^{-1/2} \tilde{f}_{k-1}) \right\|_{L^2(\Omega)} \\ &\lesssim d^{1/2}(\tau) + z(\tau), \end{aligned}$$

where we used (4.46) and $\left\| \partial_t \varepsilon_d^{1/2} \right\|_{H_\Gamma^2} \lesssim z^{1/2}(\tau)$ in the final inequality. $\square$

Next, we subtract (p^k, q^k) from $\partial_t^k(E, H)$, obtaining the divergence-free part of our Helmholtz decomposition.

Definition 4.3.3. For $k \in \{1, 2, 3\}$ we set

$$\begin{aligned} V^k &:= \partial_t^k E - \nabla p^k \in C([0, T_*], H(\operatorname{div}_{\varepsilon_d} 0)), \\ W^k &:= \partial_t^k H - \nabla q^k \in C([0, T_*], H_{n0}(\operatorname{div}_{\mu_d} 0)). \end{aligned} \tag{4.53}$$

Remark 4.3.4. In the following, for $k = 3$ several trace terms might not exist in L^2 . To justify the calculations we therefore, using Theorem 3.3.5, first approximate the data and coefficients in the Maxwell system (4.29)–(4.31) for $k = 3$ to obtain solutions $(E_n^3, H_n^3) \in \mathcal{G}_\Gamma^1([0, T_*])$.

By Lemma 4.3.1 the corresponding weak solutions (p_n^3, q_n^3) of (4.45) and (4.48) already lie in $H_\Gamma^2(\Omega)$ for $t \in [0, T_*)$. Thus, also $(V_n^3, W_n^3) := (E_n^3, H_n^3) - (\nabla p_n^3, \nabla q_n^3)$ belongs to $H_\Gamma^1(\Omega)$ for $t \in [0, T_*)$. The observability estimate in Proposition 4.4.1 for $k = 3$ only depends on (V_n^3, W_n^3) in the $\mathcal{G}^0$ -norm and therefore also holds for the original solution $(V^3, W^3) \in \mathcal{G}^0$ by taking the limit $n \rightarrow \infty$. Here we use Theorem 3.3.5 and the above observations on $\nabla(p^k, q^k)$. In the following we drop the index to simplify the notation.

By the properties of p^k and q^k , the functions satisfy

$$\begin{aligned} \operatorname{curl} V^k &= \operatorname{curl} \partial_t^k E, & \varepsilon_d V^k &\in H^{\partial\Omega}(\operatorname{div} 0), \\ \operatorname{curl} W^k &= \operatorname{curl} \partial_t^k H, & \mu_d W^k &\in H_{n0}(\operatorname{div} 0), \end{aligned} \quad (4.54)$$

For the last assertion on $\varepsilon_d V^k$, the condition $\langle \operatorname{tr}_n \varepsilon_d V^k, \mathbb{1} \rangle_{H^{-\frac{1}{2}}(\partial\Omega_i)} = 0$ is clear by (4.44) for $i \neq 0$. For $i = 0$ it then follows from $\operatorname{div}(\varepsilon_d V^k) = 0$ which yields $\langle \operatorname{tr}_n \varepsilon_d V^k, \mathbb{1} \rangle_{H^{-\frac{1}{2}}(\partial\Omega)} = 0$.

Crucially, (p^k, q^k) were constructed in such a way that

$$\begin{aligned} \operatorname{tr}_{t,\partial\Omega} V^k &= \llbracket V^k \times \nu^\Gamma \rrbracket = \llbracket W^k \times \nu^\Gamma \rrbracket = 0, \\ \operatorname{tr}_{n,\partial\Omega}(\mu_d W^k) &= \llbracket \varepsilon_d V^k \cdot \nu^\Gamma \rrbracket = \llbracket \mu_d W^k \cdot \nu^\Gamma \rrbracket = 0, \end{aligned} \quad (4.55)$$

in $H^{1/2}$ preserving (4.30), see also Remark 4.3.4 for $k = 3$.

We prove two results needed in the observability estimate. Lemmas 4.2.3 and 4.3.2 yield the following energy estimate for (V^k, W^k) .

Lemma 4.3.5. *For $0 \leq s \leq t < T_*$ we have*

$$\begin{aligned} \|(V^k, W^k)(t)\|_{L^2(\Omega)}^2 &\lesssim e(t) + \|(\nabla p^k, \nabla q^k)(t)\|_{L^2(\Omega)}^2 \\ &\lesssim e(t) + z^2(t) \\ &\lesssim e(s) + \int_s^t z^{3/2}(\tau) \, d\tau + z^2(t). \end{aligned}$$

Next, we compare (V^k, W^k) to $\partial_t(V^{k-1}, W^{k-1})$ for $k \geq 1$.

Lemma 4.3.6. *Let $k \in \{1, 2, 3\}$. Then for $0 \leq \tau < T_*$*

$$\begin{aligned} \|(V^k - \partial_t V^{k-1})(\tau)\|_{L^2(\Omega)} &\lesssim \|((\partial_t \varepsilon_d) \nabla p^{k-1})(\tau)\|_{L^2(\Omega)} \lesssim z^{3/2}(\tau), \\ \|(W^k - \partial_t W^{k-1})(\tau)\|_{L^2(\Omega)} &\lesssim \|((\partial_t \mu_d) \nabla q^{k-1})(\tau)\|_{L^2(\Omega)} \lesssim z^{3/2}(\tau). \end{aligned}$$

Proof. We only consider V^k , since for W^k the argument works in a similar way. Note that $V^k - \partial_t V^{k-1} = \nabla p^k - \partial_t \nabla p^{k-1}$ solves

$$\operatorname{div}_{\varepsilon_d}(\nabla p^k - \partial_t \nabla p^{k-1}) = \operatorname{div}_{\varepsilon_d} \partial_t^k E - \operatorname{div}_{\varepsilon_d} \partial_t^k E - \operatorname{div}_{\partial_t \varepsilon_d} \nabla p^{k-1}.$$

As in (4.46) we then deduce

$$\begin{aligned} \|(V^k - \partial_t V^{k-1})(\tau)\|_{L^2(\Omega)} &\lesssim \|((\partial_t \varepsilon_d) \nabla p^{k-1})(\tau)\|_{L^2(\Omega)} \\ &\lesssim \|(\partial_t \varepsilon_d)(\tau)\|_{H^2} \|p^{k-1}(\tau)\|_{H^1} \lesssim z^{3/2}(\tau). \end{aligned} \quad \square$$

4.4. OBSERVABILITY-TYPE ESTIMATE FOR $\partial_t(V^k, W^k)$

The observability-type estimate in Proposition 4.4.1 only works for charge-free fields up to small errors. At first we pass to the fields (V_k, W_k) from Definition 4.3.3. Lemma 4.3.2 allows us to go back to $\partial_t^k(E, H)$ in Theorem 4.4.5. So, we insert the Helmholtz decomposition $(V^k, W^k) = (\partial_t^k E - \nabla p^k, \partial_t^k H - \nabla q^k)$ from (4.53) into the Maxwell Equations (4.29) for $k \in \{1, 2, 3\}$. (The case $k = 0$ is treated by means of Lemma 4.2.4.) Together with Equation (4.54), this results in the system

$$\begin{aligned} \partial_t(\varepsilon_d V^k) &= \operatorname{curl} W^k - \sigma \partial_t^k E - \partial_t(\varepsilon_d \nabla p^k) - \partial_t f_k, & t \geq 0, x \in \Omega, \\ \partial_t(\mu_d W^k) &= -\operatorname{curl} V^k - \partial_t(\mu_d \nabla q^k) - \partial_t g_k, & (4.56) \\ \llbracket V^k \times \nu^\Gamma \rrbracket &= \llbracket W^k \times \nu^\Gamma \rrbracket = 0, & t \geq 0, x \in \Gamma, \\ \llbracket \varepsilon_d V^k \cdot \nu^\Gamma \rrbracket &= \llbracket \mu_d W^k \cdot \nu^\Gamma \rrbracket = 0, \\ V^k \times \nu &= 0, & t \geq 0, x \in \partial\Omega, \\ V^k(0) &= \partial_t^k E(0) - \nabla p^k(0), \quad W^k(0) = \partial_t^k H(0) - \nabla q^k(0), & x \in \Omega, \end{aligned}$$

for $k \in \{1, 2, 3\}$.

Our goal is to control space-time L^2 -norms of V^k and W^k for $k > 0$ by the initial data, dissipation terms supported on $\mathcal{N}_a$ and various small error terms. We start with the following estimate. For now, this includes the L^2 -norm of W^k on the right, which is not damped through Maxwell. So, we still have to treat this term afterwards. (One could establish a similar estimate for $k = 0$, however here one lacks control over the magnetic term on the right-hand side.) For $k = 3$ we approximate (V^k, W^k) as in Remark 4.3.4, so that the trace terms appearing in the subsequent proof are well-defined.

Proposition 4.4.1 (Observability-type Estimate I). *Assume that (H_q) holds. For $k \in \{1, 2, 3\}$ and $0 \leq s \leq t < T_*$, we have*

$$\begin{aligned} \int_s^t \int_{v \cup \mathcal{N}_b^\Gamma} \left(|V^k|^2 + |W^k|^2 \right) dx d\tau &\lesssim e(s) + z^2(s) + z^2(t) + \int_s^t \left(d(\tau) + z^{3/2}(\tau) \right) d\tau \\ &\quad + \int_s^t \int_{\omega \setminus \mathcal{N}_b^\Gamma} |W^k|^2 dx d\tau. \end{aligned}$$

Proof. We first want to give a brief overview of the structure of the proof. It is subdivided into several steps that we shortly summarize below using terms we have yet to define. As such the subsequent list serves as a guideline throughout the proof to not lose sight of the result we want to prove. Our reasoning is an involved energy-type estimate using a *Morawetz multiplier*, also called *multiplier technique*. Starting from Equation (4.56), we obtain

$$0 = \int_s^t \int_\Omega \left(\partial_t(\varepsilon_d V^k) - \operatorname{curl} W^k + \sigma \partial_t^k E + \partial_t(\varepsilon_d \nabla p^k) + \partial_t f_k \right) \cdot ((\mu_d W^k) \times \chi m) dx d\tau, \quad (4.57)$$

where $k \in \{1, 2, 3\}$, $m(x) = x - x_0$ for $x_0 \in \mathbb{R}^3$ from (4.19) and χ is a cutoff. It turns out, that integrating by parts and several lengthy calculations lead to the desired observability estimate. We freely use standard formulas from vector analysis.

Step 1) We start with the Morawetz multiplier in Equation (4.57) above.

Step 2) After integrating by parts (see Equation (20) of Chapter 2.1 in [12] or Equation (3.1) in [5]), Equation (4.57) leads to several terms we deal with individually. Namely,

Step 2.1) I_t – a term, where the time derivative falls upon $\operatorname{curl} V^k$ and

Step 2.2) I_c – a term where curl is applied to $\chi m \times \operatorname{curl} V^k$.

Step 2.3) We combine the previous two steps to one estimate.

Step 2.4) Letting $\chi = 1$ on Γ we show that under the non-trapping condition on the interface (4.21), the interface terms have a negative sign and can be neglected.

Step 2.5) We estimate several error terms.

Step 2.6) For $\chi \equiv 1$ we obtain our desired observability estimate up to boundary terms, which we treat in

Step 2.7) to finally complete the estimate using a cutoff with $\chi = 1$ on $\partial\Omega$.

Let $0 \leq \chi \in C^1(\overline{\Omega})$ be a cutoff, which we will vary at various places throughout the proof. In particular, it will be used to show bounds for terms at the boundary and interface.

1) Starting from Equation (4.57) and using (4.56) for $\partial_t(\mu_d W^k)$, we calculate

$$\begin{aligned}
0 &= \int_s^t \int_{\Omega} \left(\partial_t(\varepsilon_d V^k) - \operatorname{curl} W^k + \sigma \partial_t^k E + \partial_t(\varepsilon_d \nabla p^k) + \partial_t f_k \right) \cdot ((\mu_d W^k) \times \chi m) \, dx \, d\tau \\
&= \int_s^t \int_{\Omega} \partial_t \left(\varepsilon_d V^k \cdot ((\mu_d W^k) \times \chi m) \right) \, dx \, d\tau \\
&\quad + \int_s^t \int_{\Omega} \left(\varepsilon_d V^k \cdot (\operatorname{curl} V^k \times \chi m) \right) \, dx \, d\tau + \int_s^t \int_{\Omega} \left(\varepsilon_d V^k \cdot (\partial_t(\mu_d \nabla q^k) \times \chi m) \right) \, dx \, d\tau \\
&\quad + \int_s^t \int_{\Omega} \left(\varepsilon_d V^k \cdot (\partial_t g_k \times \chi m) \right) \, dx \, d\tau \\
&\quad - \int_s^t \int_{\Omega} \operatorname{curl} W^k \cdot ((\mu_d W^k) \times \chi m) \, dx \, d\tau \\
&\quad + \int_s^t \int_{\Omega} \sigma \partial_t^k E \cdot ((\mu_d W^k) \times \chi m) \, dx \, d\tau + \int_s^t \int_{\Omega} \partial_t(\varepsilon_d \nabla p^k) \cdot ((\mu_d W^k) \times \chi m) \, dx \, d\tau \\
&\quad + \int_s^t \int_{\Omega} \partial_t f_k \cdot ((\mu_d W^k) \times \chi m) \, dx \, d\tau \\
&=: \left[\int_{\Omega} \varepsilon_d V^k \cdot ((\mu_d W^k) \times \chi m) \, dx \right]_s^t + I_t + I_q + I_g + I_c + I_{\sigma} + I_p + I_f.
\end{aligned}$$

2) We consider the terms above individually.

2.1) By the interface and boundary conditions (4.55), the field $V^k \times \nu$ vanishes at $\partial\Omega$ and $V^k \times \nu$ does not jump at the interface Γ . We integrate by parts on v and ω separately. Then, I_t becomes

$$\begin{aligned}
I_t &= \int_s^t \int_{\Omega} \operatorname{curl} V^k \cdot (\chi m \times (\varepsilon_d V^k)) \, dx \, d\tau \\
&= \int_s^t \int_{\Omega} V^k \cdot \operatorname{curl} (\chi m \times (\varepsilon_d V^k)) \, dx \, d\tau \\
&\quad + \int_s^t \oint_{\Gamma} \left[(\nu^{\Gamma} \times V^k) \cdot (\chi m \times (\varepsilon_d V^k)) \right] \, dx \, d\tau \\
&= \int_s^t \int_{\Omega} V^k \cdot \operatorname{curl} (\chi m \times (\varepsilon_d V^k)) \, dx \, d\tau \\
&\quad + \int_s^t \oint_{\Gamma} \left[\chi m \cdot ((\varepsilon_d V^k) \times (\nu^{\Gamma} \times V^k)) \right] \, dx \, d\tau \\
&= \int_s^t \int_{\Omega} V^k \cdot \operatorname{curl} (\chi m \times (\varepsilon_d V^k)) \, dx \, d\tau \\
&\quad + \int_s^t \oint_{\Gamma} \left[\chi (m \cdot \nu^{\Gamma}) (\varepsilon_d V^k \cdot V^k) - (\chi m \cdot V^k) (\varepsilon_d V^k \cdot \nu^{\Gamma}) \right] \, dx \, d\tau.
\end{aligned}$$

Using the curl vector product rule

$$\operatorname{curl}(a \times b) = (b \cdot \nabla)a - (a \cdot \nabla)b + a \operatorname{div} b - b \operatorname{div} a, \quad (4.58)$$

we obtain the pointwise identity

$$\begin{aligned}
\operatorname{curl}(\chi m \times (\varepsilon_d V^k)) &= ((\varepsilon_d V^k) \cdot \nabla)(\chi m) - \chi(m \cdot \nabla)(\varepsilon_d V^k) \\
&\quad + \underbrace{\chi m \operatorname{div}(\varepsilon_d V^k)}_{=0} - (\varepsilon_d V^k) \operatorname{div}(\chi m) \\
&= ((\varepsilon_d V^k) \cdot \nabla \chi) m + \chi(\varepsilon_d V^k) - \chi(m \cdot \nabla)(\varepsilon_d V^k) \\
&\quad - (\nabla \chi \cdot m)(\varepsilon_d V^k) - 3\chi(\varepsilon_d V^k) \\
&= ((\varepsilon_d V^k) \cdot \nabla \chi) m - 2\chi(\varepsilon_d V^k) - \chi(m \cdot \nabla)(\varepsilon_d V^k) \\
&\quad - (\nabla \chi \cdot m)(\varepsilon_d V^k)
\end{aligned}$$

almost everywhere on $v \cup \omega$. It follows that

$$\begin{aligned}
I_t &= \int_s^t \int_{\Omega} \left((V^k \cdot m)((\varepsilon_d V^k) \cdot \nabla \chi) - 2\chi \varepsilon_d V^k \cdot V^k \right. \\
&\quad \left. - (\nabla \chi \cdot m)(\varepsilon_d V^k \cdot V^k) - \left(\chi(m \cdot \nabla)(\varepsilon_d V^k) \right) \cdot V^k \right) dx d\tau \\
&\quad + \int_s^t \oint_{\Gamma} [\chi(m \cdot \nu^\Gamma)(\varepsilon_d V^k \cdot V^k) - (\chi m \cdot V^k)(\varepsilon_d V^k \cdot \nu^\Gamma)] d\varsigma d\tau.
\end{aligned}$$

Notice that

$$\left(\chi(m \cdot \nabla)(\varepsilon_d V^k) \right) \cdot V^k = \frac{1}{2} \chi(m \cdot \nabla) (\varepsilon_d V^k \cdot V^k) + \frac{1}{2} (\chi(m \cdot \nabla) \varepsilon_d) V^k \cdot V^k.$$

The divergence theorem on $v \cup \omega$ further yields

$$\begin{aligned}
& - \frac{1}{2} \int_s^t \int_{\Omega} \chi(m \cdot \nabla) (\varepsilon_d V^k \cdot V^k) dx d\tau \\
&= \int_s^t \int_{\Omega} \left(\frac{1}{2} \nabla \chi \cdot m (\varepsilon_d V^k \cdot V^k) + \frac{3}{2} \chi(\varepsilon_d V^k \cdot V^k) \right) dx d\tau \\
&\quad - \int_s^t \oint_{\Gamma} \frac{1}{2} (m \cdot \nu^\Gamma) [\chi \varepsilon_d V^k \cdot V^k] d\varsigma d\tau - \int_s^t \oint_{\partial\Omega} \frac{1}{2} \chi(m \cdot \nu) (\varepsilon_d V^k \cdot V^k) d\varsigma d\tau.
\end{aligned}$$

Involving $\tilde{\varepsilon}_d = \varepsilon_d + (m \cdot \nabla) \varepsilon_d$, which is positive definite on $v \cup \mathcal{N}_b^\Gamma$ by Equation (4.20), we thus derive

$$\begin{aligned}
I_t &= \int_s^t \int_{\Omega} \left((V^k \cdot m)((\varepsilon_d V^k) \cdot \nabla \chi) - 2\chi \varepsilon_d V^k \cdot V^k - (\nabla \chi \cdot m)(\varepsilon_d V^k \cdot V^k) \right. \\
&\quad \left. + \frac{1}{2} (\nabla \chi \cdot m)(\varepsilon_d V^k \cdot V^k) + \frac{3}{2} \chi(\varepsilon_d V^k \cdot V^k) - \frac{1}{2} (\chi(m \cdot \nabla) \varepsilon_d) V^k \cdot V^k \right) dx d\tau \\
&\quad + \int_s^t \oint_{\Gamma} \left[\frac{1}{2} \chi(m \cdot \nu^\Gamma)(\varepsilon_d V^k \cdot V^k) - (\chi m \cdot V^k)(\varepsilon_d V^k \cdot \nu^\Gamma) \right] dx d\tau \\
&\quad - \int_s^t \oint_{\partial\Omega} \frac{1}{2} \chi(m \cdot \nu) (\varepsilon_d V^k \cdot V^k) d\varsigma d\tau \\
&= \int_s^t \int_{\Omega} \left((V^k \cdot m)((\varepsilon_d V^k) \cdot \nabla \chi) - \frac{1}{2} \chi \varepsilon_d V^k \cdot V^k - \frac{1}{2} (\nabla \chi \cdot m)(\varepsilon_d V^k \cdot V^k) \right. \\
&\quad \left. - \frac{1}{2} (\chi(m \cdot \nabla) \varepsilon_d) V^k \cdot V^k \right) dx d\tau \\
&\quad + \int_s^t \oint_{\Gamma} \left[\frac{1}{2} \chi(m \cdot \nu^\Gamma)(\varepsilon_d V^k \cdot V^k) - (\chi m \cdot V^k)(\varepsilon_d V^k \cdot \nu^\Gamma) \right] dx d\tau \\
&\quad - \int_s^t \oint_{\partial\Omega} \frac{1}{2} \chi(m \cdot \nu) (\varepsilon_d V^k \cdot V^k) d\varsigma d\tau
\end{aligned}$$

$$\begin{aligned}
&= \int_s^t \int_{\Omega} \left((V^k \cdot m)(\varepsilon_d V^k) \cdot \nabla \chi - \frac{1}{2}(\nabla \chi \cdot m)(\varepsilon_d V^k \cdot V^k) \right) dx d\tau \\
&\quad + \int_s^t \oint_{\Gamma} (m \cdot \nu^{\Gamma}) \left[\frac{1}{2} \chi(\varepsilon_d V^k \cdot V^k) - (\chi m \cdot V^k)(\varepsilon_d V^k \cdot \nu^{\Gamma}) \right] dx d\tau \\
&\quad - \int_s^t \oint_{\partial\Omega} \frac{1}{2} \chi(m \cdot \nu)(\varepsilon_d V^k \cdot V^k) d\varsigma d\tau \\
&\quad - \int_s^t \int_{v \cup \mathcal{N}_b^{\Gamma}} \frac{1}{2} \chi \tilde{\varepsilon}_d V^k \cdot V^k dx d\tau - \int_s^t \int_{\omega \setminus \mathcal{N}_b^{\Gamma}} \frac{1}{2} \chi \tilde{\varepsilon}_d V^k \cdot V^k dx d\tau.
\end{aligned}$$

We denote the interface term $I_t^{\Gamma} := I_{t,1}^{\Gamma} + I_{t,2}^{\Gamma}$ and boundary term by

$$\begin{aligned}
I_{t,1}^{\Gamma} &:= \int_s^t \oint_{\Gamma} (m \cdot \nu^{\Gamma}) \left[\frac{1}{2} \chi(\varepsilon_d V^k \cdot V^k) \right] dx d\tau, \\
I_{t,2}^{\Gamma} &:= \int_s^t \oint_{\Gamma} \left[-(\chi m \cdot V^k)(\varepsilon_d V^k \cdot \nu^{\Gamma}) \right] dx d\tau, \\
I_t^{\partial\Omega} &:= - \int_s^t \oint_{\partial\Omega} \frac{1}{2} \chi(m \cdot \nu)(\varepsilon_d V^k \cdot V^k) d\varsigma d\tau.
\end{aligned}$$

2.2) Now, let us turn our attention towards I_c . Using the curl product rule (4.58) from above, we get

$$\begin{aligned}
I_c &= \int_s^t \int_{\Omega} W^k \cdot \operatorname{curl}(\chi m \times (\mu_d W^k)) dx d\tau + I_{c,1}^{\partial\Omega} + I_{c,1}^{\Gamma} \\
&= \int_s^t \int_{\Omega} W^k \cdot \left(\underbrace{\chi m \operatorname{div}(\mu_d W^k)}_{=0} - \operatorname{div}(\chi m)(\mu_d W^k) \right. \\
&\quad \left. + ((\mu_d W^k) \cdot \nabla)(\chi m) - (\chi m \cdot \nabla)(\mu_d W^k) \right) dx d\tau + I_{c,1}^{\partial\Omega} + I_{c,1}^{\Gamma} \\
&= \int_s^t \int_{\Omega} W^k \cdot \left(-\operatorname{div}(\chi m)(\mu_d W^k) \right. \\
&\quad \left. + ((\mu_d W^k) \cdot \nabla)(\chi m) - (\chi m \cdot \nabla \mu_d) W^k - \mu_d (\chi m \cdot \nabla) W^k \right) dx d\tau + I_{c,1}^{\partial\Omega} + I_{c,1}^{\Gamma},
\end{aligned}$$

with

$$\begin{aligned}
I_{c,1}^{\partial\Omega} &:= \int_s^t \oint_{\partial\Omega} W^k \cdot \left(\nu \times ((\mu_d W^k) \times \chi m) \right) d\varsigma d\tau, \\
I_{c,1}^{\Gamma} &:= \int_s^t \oint_{\Gamma} \left[W^k \cdot \left(\nu^{\Gamma} \times ((\mu_d W^k) \times \chi m) \right) \right] d\varsigma d\tau.
\end{aligned}$$

The last term on Ω is treated by the divergence theorem on $v \cup \omega$, resulting in

$$\begin{aligned}
&- \int_s^t \int_{\Omega} \mu_d W^k \cdot ((\chi m \cdot \nabla) W^k) dx d\tau \\
&= - \sum_{j=1}^3 \int_s^t \int_{\Omega} \chi (\mu_d W^k)_j m \cdot \nabla (W^k)_j dx d\tau \\
&= \sum_{j=1}^3 \int_s^t \int_{\Omega} \operatorname{div} \left(\chi (\mu_d W^k)_j m \right) (W^k)_j dx d\tau \\
&\quad - \int_s^t \oint_{\Gamma} (m \cdot \nu^{\Gamma}) \left[\chi \mu_d W^k \cdot W^k \right] d\varsigma d\tau \\
&\quad - \int_s^t \oint_{\partial\Omega} \chi (m \cdot \nu) (\mu_d W^k \cdot W^k) d\varsigma d\tau
\end{aligned}$$

$$\begin{aligned}
&= \int_s^t \int_{\Omega} \left[\operatorname{div}(\chi m) \mu_d W^k \cdot W^k + (\chi m \cdot \nabla)(\mu_d W^k) \cdot (W^k) \right] dx \, d\tau \\
&\quad - \int_s^t \oint_{\Gamma} (m \cdot \nu^{\Gamma}) [\chi \mu_d W^k \cdot W^k] \, d\varsigma \, d\tau \\
&\quad - \int_s^t \oint_{\partial\Omega} \chi(m \cdot \nu)(\mu_d W^k \cdot W^k) \, d\varsigma \, d\tau \\
&= \int_s^t \int_{\Omega} \left[\operatorname{div}(\chi m) \mu_d W^k \cdot W^k + (\chi m \cdot \nabla \mu_d) W^k \cdot W^k \right. \\
&\quad \left. + \mu_d W^k \cdot ((\chi m \cdot \nabla) W^k) \right] dx \, d\tau \\
&\quad - \int_s^t \oint_{\Gamma} (m \cdot \nu^{\Gamma}) [\chi \mu_d W^k \cdot W^k] \, d\varsigma \, d\tau \\
&\quad - \int_s^t \oint_{\partial\Omega} \chi(m \cdot \nu)(\mu_d W^k \cdot W^k) \, d\varsigma \, d\tau.
\end{aligned}$$

Bringing the $\int_s^t \int_{\Omega} \mu_d W^k \cdot ((\chi m \cdot \nabla) W^k) \, dx \, d\tau$ term to the left-hand side, we infer

$$\begin{aligned}
&- \int_s^t \int_{\Omega} \mu_d W^k \cdot ((\chi m \cdot \nabla) W^k) \, dx \, d\tau \\
&= \frac{1}{2} \int_s^t \int_{\Omega} \left[\operatorname{div}(\chi m) \mu_d W^k \cdot W^k + (\chi m \cdot \nabla \mu_d) W^k \cdot W^k \right] dx \, d\tau \\
&\quad - \frac{1}{2} \int_s^t \oint_{\Gamma} (m \cdot \nu^{\Gamma}) [\chi \mu_d W^k \cdot W^k] \, d\varsigma \, d\tau \\
&\quad - \frac{1}{2} \int_s^t \oint_{\partial\Omega} \chi(m \cdot \nu)(\mu_d W^k \cdot W^k) \, d\varsigma \, d\tau.
\end{aligned}$$

These identities lead to

$$\begin{aligned}
I_c &= \int_s^t \int_{\Omega} W^k \cdot \left(-\frac{1}{2} \operatorname{div}(\chi m)(\mu_d W^k) \right. \\
&\quad \left. + ((\mu_d W^k) \cdot \nabla)(\chi m) - \frac{1}{2}(\chi m \cdot \nabla \mu_d) W^k \right) dx \, d\tau \\
&\quad - \frac{1}{2} \int_s^t \oint_{\Gamma} (m \cdot \nu^{\Gamma}) [\chi \mu_d W^k \cdot W^k] \, d\varsigma \, d\tau \\
&\quad - \frac{1}{2} \int_s^t \oint_{\partial\Omega} \chi(m \cdot \nu)(\mu_d W^k \cdot W^k) \, d\varsigma \, d\tau + I_{c,1}^{\partial\Omega} + I_{c,1}^{\Gamma} \\
&= \int_s^t \int_{\Omega} W^k \cdot \left(-\frac{1}{2}(m \cdot \nabla \chi + 3\chi)(\mu_d W^k) \right. \\
&\quad \left. + (\mu_d W^k \nabla \chi \cdot m + \mu_d W^k \chi - \frac{1}{2}(\chi m \cdot \nabla \mu_d) W^k) \right) dx \, d\tau \\
&\quad - \frac{1}{2} \int_s^t \oint_{\Gamma} (m \cdot \nu^{\Gamma}) [\chi \mu_d W^k \cdot W^k] \, d\varsigma \, d\tau \\
&\quad - \frac{1}{2} \int_s^t \oint_{\partial\Omega} \chi(m \cdot \nu)(\mu_d W^k \cdot W^k) \, d\varsigma \, d\tau + I_{c,1}^{\partial\Omega} + I_{c,1}^{\Gamma} \\
&= \int_s^t \int_{\Omega} W^k \cdot \left(-\frac{1}{2}(m \cdot \nabla \chi)(\mu_d W^k) + (\mu_d W^k \cdot \nabla \chi) m \right) dx \, d\tau \\
&\quad + I_{c,1}^{\Gamma} - \frac{1}{2} \int_s^t \oint_{\Gamma} (m \cdot \nu^{\Gamma}) [\chi \mu_d W^k \cdot W^k] \, d\varsigma \, d\tau \\
&\quad + I_{c,1}^{\partial\Omega} - \frac{1}{2} \int_s^t \oint_{\partial\Omega} \chi(m \cdot \nu)(\mu_d W^k \cdot W^k) \, d\varsigma \, d\tau \\
&\quad - \int_s^t \int_{\nu \cup \mathcal{N}_b^{\Gamma}} \frac{1}{2} \chi \mu_d W^k \cdot W^k \, dx \, d\tau - \int_s^t \int_{\omega \setminus \mathcal{N}_b^{\Gamma}} \frac{1}{2} \chi \tilde{\mu}_d W^k \cdot W^k \, dx \, d\tau,
\end{aligned}$$

where $\tilde{\mu}_d := \mu_d + m \cdot \nabla \mu_d \gtrsim \eta \mu_d$, see Equation (4.20). The interface and boundary terms above are denoted by $I_{c,2}^\Gamma$ and $I_{c,2}^{\partial\Omega}$, so that

$$\begin{aligned} I_c^\Gamma &:= I_{c,1}^\Gamma + I_{c,2}^\Gamma = \int_s^t \oint_\Gamma \left[W^k \cdot (\nu^\Gamma \times ((\mu_d W^k) \times \chi m)) \right] d\varsigma d\tau \\ &\quad - \frac{1}{2} \int_s^t \oint_\Gamma \chi(m \cdot \nu^\Gamma) [\mu_d W^k \cdot W^k] d\varsigma d\tau, \\ I_c^{\partial\Omega} &:= I_{c,1}^{\partial\Omega} + I_{c,2}^{\partial\Omega} = \int_s^t \oint_{\partial\Omega} W^k \cdot (\nu \times ((\mu_d W^k) \times \chi m)) d\varsigma d\tau \\ &\quad - \frac{1}{2} \int_s^t \oint_{\partial\Omega} \chi(m \cdot \nu) (\mu_d W^k \cdot W^k) d\varsigma d\tau. \end{aligned} \tag{4.59}$$

2.3) The Steps 2.1) and 2.2) yield

$$\begin{aligned} &\left[\int_\Omega (\varepsilon_d V^k)(\chi m \times -W^k) \right]_s^t = -I_t - I_c - I_\sigma - I_p - I_f - I_q - I_g \\ &= \int_s^t \int_{v \cup \mathcal{N}_b^\Gamma} \frac{\chi}{2} (\tilde{\varepsilon}_d V^k \cdot V^k + \tilde{\mu}_d W^k \cdot W^k) dx d\tau \\ &\quad + \int_s^t \int_{\omega \setminus \mathcal{N}_b^\Gamma} \frac{\chi}{2} (\tilde{\varepsilon}_d V^k \cdot V^k + \tilde{\mu}_d W^k \cdot W^k) dx d\tau \\ &\quad + \int_s^t \int_\Omega \left(\frac{1}{2} (\nabla \chi \cdot m) (\varepsilon_d V^k \cdot V^k + \mu_d W^k \cdot W^k) \right. \\ &\quad \quad \left. - (V^k \cdot m) ((\varepsilon_d V^k) \cdot \nabla \chi) - (m \cdot W^k) (\mu_d W^k \cdot \nabla \chi) \right) dx d\tau \\ &\quad - I_t^\Gamma - I_t^{\partial\Omega} - I_c^\Gamma - I_c^{\partial\Omega} - I_\sigma - I_p - I_f - I_q - I_g. \end{aligned}$$

We rearrange the equation into

$$\begin{aligned} &\int_s^t \int_{v \cup \mathcal{N}_b^\Gamma} \frac{\chi}{2} (\tilde{\varepsilon}_d V^k \cdot V^k + \tilde{\mu}_d W^k \cdot W^k) dx d\tau \\ &= \left[\int_\Omega (\varepsilon_d V^k)(\chi m \times -W^k) \right]_s^t \\ &\quad - \int_s^t \int_\Omega \left(\frac{1}{2} (\nabla \chi \cdot m) (\varepsilon_d V^k \cdot V^k + \mu_d W^k \cdot W^k) \right. \\ &\quad \quad \left. - (V^k \cdot m) ((\varepsilon_d V^k) \cdot \nabla \chi) - (m \cdot W^k) (\mu_d W^k \cdot \nabla \chi) \right) dx d\tau \\ &\quad + I_t^\Gamma + I_t^{\partial\Omega} + I_c^\Gamma + I_c^{\partial\Omega} + I_\sigma + I_p + I_f + I_q + I_g \\ &\quad - \int_s^t \int_{\omega \setminus \mathcal{N}_b^\Gamma} \frac{\chi}{2} (\tilde{\varepsilon}_d V^k \cdot V^k + \tilde{\mu}_d W^k \cdot W^k) dx d\tau. \end{aligned} \tag{4.60}$$

2.4) The non-trapping condition at the interface $(m \cdot \nu^\Gamma) [\varepsilon_d] \leq 0$ yields

$$\begin{aligned} &[\varepsilon_d] > 0 \text{ and } (m \cdot \nu^\Gamma) \leq 0, \\ \text{or } &[\varepsilon_d] = 0, \quad \text{or } (m \cdot \nu^\Gamma) = 0, \\ \text{or } &[\varepsilon_d] < 0 \text{ and } (m \cdot \nu^\Gamma) \geq 0, \end{aligned}$$

at each $x \in \Gamma$, see (4.21). We next show that it the interface integrals to be non-positive, which allows us to drop these terms. Recall that $I_t^\Gamma = I_{t,1}^\Gamma + I_{t,2}^\Gamma$ and $I_c^\Gamma = I_{c,1}^\Gamma + I_{c,2}^\Gamma$ were given by

$$I_{t,1}^\Gamma = \frac{1}{2} \int_s^t \oint_\Gamma (m \cdot \nu^\Gamma) [\chi (\varepsilon_d V^k \cdot V^k)] dx d\tau,$$

$$\begin{aligned}
I_{t,2}^\Gamma &= - \int_s^t \oint_\Gamma [(\chi m \cdot V^k)(\varepsilon_d V^k \cdot \nu^\Gamma)] dx d\tau, \\
I_{c,1}^\Gamma &= \int_s^t \oint_\Gamma \left[W^k \cdot \left(\nu^\Gamma \times ((\mu_d W^k) \times \chi m) \right) \right] d\varsigma d\tau, \\
I_{c,2}^\Gamma &= - \frac{1}{2} \int_s^t \oint_\Gamma (m \cdot \nu^\Gamma) [\chi \mu_d W^k \cdot W^k] d\varsigma d\tau.
\end{aligned}$$

(If we had $m = \nu^\Gamma$ the term in $I_{c,1}^\Gamma$ would correspond to the tangential projection.)

We take χ with $\chi = 1$ on the interface Γ .

Hence, the integrand in I_t^Γ becomes

$$\frac{1}{2}(m \cdot \nu^\Gamma) [V^k \cdot \varepsilon_d V^k] - [(m \cdot V^k)(\nu^\Gamma \cdot \varepsilon_d V^k)]. \quad (4.61)$$

Since $[\text{tr}_\tau V^k] = 0$ by (4.55), the jump across the interface can only occur in normal direction, i.e.,

$$V^k|_v = V^k|_\omega + \alpha \nu^\Gamma$$

for some $\alpha(x) \in \mathbb{R}$. Furthermore, using also the continuity $[\nu^\Gamma \cdot \varepsilon_d V^k] = 0$ of the jump of the ‘displacement’ field from (4.55), we can determine α as

$$\nu^\Gamma \cdot \varepsilon_d|_v (V^k|_\omega + \alpha \nu^\Gamma) = \nu^\Gamma \cdot \varepsilon_d|_\omega V^k|_\omega \implies \alpha = - \frac{\nu^\Gamma \cdot [\varepsilon_d] V^k|_\omega}{\nu^\Gamma \cdot \varepsilon_d|_v \nu^\Gamma}.$$

Next, we insert this decomposition into (4.61). The first term is given by

$$\begin{aligned}
[V^k \cdot \varepsilon_d V^k] &= (V^k|_\omega + \alpha \nu^\Gamma) \cdot \varepsilon_d|_v (V^k|_\omega + \alpha \nu^\Gamma) - (V^k \cdot \varepsilon_d V^k)|_\omega \\
&= V^k|_\omega \cdot \varepsilon_d|_v V^k|_\omega + 2\alpha(\nu^\Gamma \cdot \varepsilon_d|_v V^k|_\omega) + \alpha^2(\nu^\Gamma \cdot \varepsilon_d|_v \nu^\Gamma) \\
&\quad - (V^k \cdot \varepsilon_d V^k)|_\omega \\
&= V^k|_\omega \cdot [\varepsilon_d] V^k|_\omega + 2\alpha(\nu^\Gamma \cdot \varepsilon_d|_v V^k|_\omega) + \alpha^2(\nu^\Gamma \cdot \varepsilon_d|_v \nu^\Gamma).
\end{aligned}$$

Because of (4.55), the second term becomes

$$\begin{aligned}
-[\nu^\Gamma \cdot (\varepsilon_d V^k)(m \cdot V^k)] &= -\text{tr}_{\partial v}(\nu^\Gamma \cdot \varepsilon_d V^k)(m \cdot (V^k|_v - V^k|_\omega)) \\
&= -\alpha \text{tr}_{\partial v}(\nu^\Gamma \cdot \varepsilon_d V^k)(m \cdot \nu^\Gamma) \\
&= -\alpha \nu^\Gamma \cdot \varepsilon_d|_v (V^k|_\omega + \alpha \nu^\Gamma)(m \cdot \nu^\Gamma) \\
&= -(m \cdot \nu^\Gamma) (\alpha \nu^\Gamma \cdot \varepsilon_d|_v V^k|_\omega + \alpha^2 \nu^\Gamma \cdot \varepsilon_d|_v \nu^\Gamma).
\end{aligned}$$

We factor out $\frac{1}{2}(m \cdot \nu^\Gamma)$, combine these terms and insert $\alpha = - \frac{\nu^\Gamma \cdot [\varepsilon_d] V^k|_\omega}{\nu^\Gamma \cdot \varepsilon_d|_v \nu^\Gamma}$, resulting in

$$\begin{aligned}
I_t^\Gamma &= \int_s^t \oint_\Gamma \frac{1}{2}(m \cdot \nu^\Gamma) ([V^k \cdot \varepsilon_d V^k] - 2\alpha \nu^\Gamma \cdot \varepsilon_d|_v V^k|_\omega - 2\alpha^2 \nu^\Gamma \cdot \varepsilon_d|_v \nu^\Gamma) d\varsigma d\tau \\
&= \int_s^t \oint_\Gamma \frac{1}{2}(m \cdot \nu^\Gamma) (V^k|_\omega \cdot [\varepsilon_d] V^k|_\omega - \alpha^2(\nu^\Gamma \cdot \varepsilon_d|_v \nu^\Gamma)) d\varsigma d\tau \\
&= \int_s^t \oint_\Gamma \frac{1}{2}(m \cdot \nu^\Gamma) \left(V^k|_\omega \cdot [\varepsilon_d] V^k|_\omega - \frac{(\nu^\Gamma \cdot [\varepsilon_d] V^k|_\omega)^2}{\nu^\Gamma \cdot \varepsilon_d|_v \nu^\Gamma} \right) d\varsigma d\tau.
\end{aligned}$$

Next, let us discuss the different cases of the non-trapping condition at the interface. First, if $[\varepsilon_d] = 0$ or $m \cdot \nu^\Gamma = 0$ then $I_t^\Gamma = 0$.

Second, let $\llbracket \varepsilon_d \rrbracket > 0$ and $m \cdot \nu^\Gamma \leq 0$. Since

$$(\nu^\Gamma \cdot \llbracket \varepsilon_d \rrbracket V^k|_\omega)^2 \leq (\nu^\Gamma \cdot \llbracket \varepsilon_d \rrbracket \nu^\Gamma) \left(V^k|_\omega \cdot \llbracket \varepsilon_d \rrbracket V^k|_\omega \right).$$

by Cauchy-Schwarz, we estimate

$$\begin{aligned} & V^k|_\omega \cdot \llbracket \varepsilon_d \rrbracket V^k|_\omega - \frac{(\nu^\Gamma \cdot \llbracket \varepsilon_d \rrbracket V^k|_\omega)^2}{\nu^\Gamma \cdot \varepsilon_d|_v \nu^\Gamma} \\ & \geq \left(V^k|_\omega \cdot \llbracket \varepsilon_d \rrbracket V^k|_\omega \right) \left[1 - \frac{\nu^\Gamma \cdot \llbracket \varepsilon_d \rrbracket \nu^\Gamma}{\nu^\Gamma \cdot \varepsilon_d|_v \nu^\Gamma} \right] \\ & = \left(V^k|_\omega \cdot \llbracket \varepsilon_d \rrbracket V^k|_\omega \right) \frac{\nu^\Gamma \cdot \varepsilon_d|_\omega \nu^\Gamma}{\nu^\Gamma \cdot \varepsilon_d|_v \nu^\Gamma} > 0. \end{aligned}$$

As $m \cdot \nu^\Gamma \leq 0$, it follows

$$I_t^\Gamma \leq \frac{1}{2} \int_s^t \oint_\Gamma (m \cdot \nu^\Gamma) \left(V^k|_\omega \cdot \llbracket \varepsilon_d \rrbracket V^k|_\omega \right) \frac{\nu^\Gamma \cdot \varepsilon_d|_\omega \nu^\Gamma}{\nu^\Gamma \cdot \varepsilon_d|_v \nu^\Gamma} d\varsigma d\tau \leq 0.$$

Finally, let $\llbracket \varepsilon_d \rrbracket < 0$ and $m \cdot \nu^\Gamma \geq 0$. As above, we obtain

$$\left(V^k|_\omega \cdot \llbracket \varepsilon_d \rrbracket V^k|_\omega - \frac{(\nu^\Gamma \cdot \llbracket \varepsilon_d \rrbracket V^k|_\omega)^2}{\nu^\Gamma \cdot \varepsilon_d|_v \nu^\Gamma} \right) \leq 0$$

and therefore again

$$I_t^\Gamma \leq 0.$$

Next, we consider the integrand of I_c^Γ given by

$$\left[W^k \cdot \left(\nu^\Gamma \times ((\mu_d W^k) \times m) \right) \right] - \frac{1}{2} (m \cdot \nu^\Gamma) \llbracket \mu_d W^k \cdot W^k \rrbracket.$$

We rearrange the terms so that we can repeat the above arguments. Indeed, the bac-cab-identity yields

$$\begin{aligned} & \left[W^k \cdot \left(\nu^\Gamma \times ((\mu_d W^k) \times m) \right) \right] \\ & = \left[\mu_d W^k \cdot W^k (\nu^\Gamma \cdot m) - (W^k \cdot m) (\nu^\Gamma \cdot \mu_d W^k) \right]. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} & \left[W^k \cdot \left(\nu^\Gamma \times ((\mu_d W^k) \times m) \right) \right] - \frac{1}{2} (m \cdot \nu^\Gamma) \llbracket \mu_d W^k \cdot W^k \rrbracket \\ & = \frac{1}{2} \left[(\nu^\Gamma \cdot m) \mu_d W^k \cdot W^k \right] - \left[(W^k \cdot m) (\nu^\Gamma \cdot \mu_d W^k) \right], \end{aligned}$$

which we can treat as in (4.61) by means of the magnetic non-trapping condition $(m \cdot \nu^\Gamma) \llbracket \mu_d \rrbracket \leq 0$.

Hence the terms I_t^Γ and I_c^Γ are non-positive and can be dropped when estimating (4.60) below.

2.5) Next we take a look at the other terms not involving $\nabla \chi$. Using Lemma 4.2.1 and Lemma 4.3.2, for any $\delta > 0$ we get

$$|I_\sigma| = \left| \int_s^t \int_\Omega \sigma \partial_t^k E \cdot ((\mu_d W^k) \times \chi m) dx d\tau \right| \lesssim c_\delta \int_s^t d(\tau) d\tau + \delta \int_s^t \int_\Omega |W^k|^2 dx d\tau,$$

$$\begin{aligned}
|I_p| &= \left| \int_s^t \int_{\Omega} \partial_t(\varepsilon_d \nabla p^k) \cdot ((\mu_d W^k) \times \chi m) \, dx \, d\tau \right| \\
&\lesssim c_\delta \int_s^t \int_{\Omega} (d(\tau) + z^2(\tau)) \, dx \, d\tau + \delta \int_s^t \int_{\Omega} |W^k|^2 \, dx \, d\tau, \\
|I_q| &= \left| \int_s^t \int_{\Omega} \varepsilon_d V^k \cdot (\partial_t(\varepsilon_d \nabla q^k) \times \chi m) \, dx \, d\tau \right| \\
&\lesssim c_\delta \int_s^t \int_{\Omega} (d(\tau) + z^2(\tau)) \, dx \, d\tau + \delta \int_s^t \int_{\Omega} |V^k|^2 \, dx \, d\tau, \\
|I_f| &= \left| \int_s^t \int_{\Omega} \partial_t f_k \cdot ((\mu_d W^k) \times \chi m) \, dx \, d\tau \right| \lesssim c_\delta \int_s^t z^2(\tau) \, d\tau + \delta \int_s^t \int_{\Omega} |W^k|^2 \, dx \, d\tau, \\
|I_g| &= \left| \int_s^t \int_{\Omega} \varepsilon_d V^k \cdot (\partial_t g_k \times \chi m) \, dx \, d\tau \right| \lesssim c_\delta \int_s^t z^2(\tau) \, d\tau + \delta \int_s^t \int_{\Omega} |V^k|^2 \, dx \, d\tau.
\end{aligned}$$

2.6) Let $\chi = 1$. In (4.60) the terms involving $\nabla \chi$ thus vanish and we have seen that we can drop I_t^Γ and I_c^Γ . We then use the non-trapping condition (4.27) and absorb the integrals of $|V^k|^2$ and $|W^k|^2$ on $v \cup \mathcal{N}_b^\Gamma$ by the right-hand side fixing a small $\delta > 0$ in Step 2.5). This results in the inequality

$$\begin{aligned}
&\eta \int_s^t \int_{v \cup \mathcal{N}_b^\Gamma} (|V^k|^2 + |W^k|^2) \, dx \, d\tau \\
&\lesssim \int_{\Omega} |(\varepsilon_d V^k)(t)| |W^k(t)| \, dx + \int_{\Omega} |(\varepsilon_d V^k)(s)| |W^k(s)| \, dx \\
&\quad + \int_s^t \int_{\omega \setminus \mathcal{N}_b^\Gamma} (|V^k|^2 + |W^k|^2) \, dx \, d\tau + |I_t^{\partial\Omega} + I_c^{\partial\Omega}| + \int_s^t (d(\tau) + z^2(\tau)) \, d\tau.
\end{aligned} \tag{4.62}$$

Thus, it remains to estimate the trace terms on $\partial\Omega$.

2.7) To do so, we choose χ with $\text{supp } \chi \subseteq \omega \setminus \mathcal{N}_b^\Gamma$ and $\chi = 1$ on $\partial\Omega$. In particular, interface terms do not appear in (4.60). Equation (4.60) and Step 2.5) then imply

$$\begin{aligned}
|I_t^{\partial\Omega} + I_c^{\partial\Omega}| &\lesssim \left| \left[\int_{\omega \setminus \mathcal{N}_b^\Gamma} (\varepsilon_d V^k)(\chi m \times \mu_d W^k) \right]_s^t \right| \\
&\quad + \int_s^t \int_{\omega \setminus \mathcal{N}_b^\Gamma} (|V^k|^2 + |W^k|^2) \, dx \, d\tau + |I_\sigma + I_p + I_f + I_q + I_g| \\
&\lesssim \int_{\Omega} |(\varepsilon_d V^k)(t)| |W^k(t)| \, dx + \int_{\Omega} |(\varepsilon_d V^k)(s)| |W^k(s)| \, dx \\
&\quad + \int_s^t \int_{\omega \setminus \mathcal{N}_b^\Gamma} (|V^k|^2 + |W^k|^2) \, dx \, d\tau \\
&\quad + \delta \int_s^t \int_{\Omega} (|V^k|^2 + |W^k|^2) \, dx \, d\tau + c_\delta \int_s^t (d(\tau) + z^2(\tau)) \, d\tau.
\end{aligned} \tag{4.63}$$

We now combine (4.62) and (4.63), again absorbing terms for a small, fixed $\delta > 0$. The

space integral at times t and s are handled via Lemma 4.3.5. Also inserting (4.53), we derive

$$\begin{aligned}
& \eta \int_s^t \int_{\omega \cup \mathcal{N}_b^\Gamma} \left(|V^k|^2 + |W^k|^2 \right) dx d\tau \lesssim \int_\Omega |V^k(t)| |W^k(t)| dx + \int_\Omega |V^k(s)| |W^k(s)| dx \\
& \quad + \int_s^t \int_{\omega \setminus \mathcal{N}_b^\Gamma} \left(|V^k|^2 + |W^k|^2 \right) dx d\tau + c_\delta \int_s^t \left(d(\tau) + z^2(\tau) \right) d\tau \\
& \lesssim e(s) + z^2(s) + z^2(t) + (1 + c_\delta) \int_s^t \left(d(\tau) + z^{3/2}(\tau) \right) d\tau \\
& \quad + \int_s^t \int_{\omega \setminus \mathcal{N}_b^\Gamma} \left(|\partial_t^k E|^2 + |\nabla p^k|^2 + |W^k|^2 \right) dx d\tau \\
& \lesssim e(s) + z^2(s) + z^2(t) + (1 + c_\delta) \int_s^t \left(d(\tau) + z^{3/2}(\tau) \right) d\tau + \int_s^t \int_{\omega \setminus \mathcal{N}_b^\Gamma} |W^k|^2 dx d\tau,
\end{aligned}$$

where we use the Lemma 4.3.2. $\square$

Next, we have to get rid of the W^k term on $\omega \setminus \mathcal{N}_b^\Gamma$, since the dissipation only acts on the electric field. This is done for $k \geq 1$ in a similar way as in (2.38), however there we used a second-order formulation of Maxwell which was then integrated by parts in time. It seems that in the present quasilinear case a second-order approach fails because of the resulting interface terms. Therefore, we keep the first-order system and apply a time derived multiplier.

Proposition 4.4.2 (Observability-type Estimate II). *Assume that (H_q) holds. Let $\delta \in (0, 1]$. For $k \in \{1, 2, 3\}$ and $0 \leq s \leq t < T_*$, we have*

$$\begin{aligned}
\int_s^t \|W^k\|_{L^2(\omega \setminus \mathcal{N}_b^\Gamma)}^2 dx & \lesssim \int_s^t \left(\delta z(\tau) + c_\delta(z^{3/2}(\tau) + d(\tau)) \right) d\tau \\
& \quad + \delta(z(s) + z(t)) + c_\delta(e(s) + z^2(s) + z^2(t)).
\end{aligned}$$

Proof. Let $k \in \{1, 2, 3\}$. Equation (4.53), the Maxwell system (4.32), and formula (4.54) imply

$$\begin{aligned}
\mu_d W^k &= \mu_d \partial_t W^{k-1} + \mu_d (W^k - \partial_t W^{k-1}) \\
&= -\operatorname{curl} V^{k-1} - \mu_d \partial_t \nabla q^{k-1} - \tilde{g}_{k-1} + \mu_d (W^k - \partial_t W^{k-1}).
\end{aligned}$$

Lemmas 4.2.1, 4.3.2, and 4.3.6 then lead to

$$\int_s^t \|W^k\|_{L^2(\omega \setminus \mathcal{N}_b^\Gamma)}^2 dx \lesssim \int_s^t \|\operatorname{curl} V^{k-1}\|_{L^2(\omega \setminus \mathcal{N}_b^\Gamma)}^2 d\tau + \int_s^t \left(d(\tau) + z^{3/2}(\tau) \right) d\tau. \quad (4.64)$$

It remains to estimate the curl term. To this end, take $0 \leq \vartheta \in C^\infty(\bar{\omega})$ with $\vartheta = 1$ on $\omega \setminus \mathcal{N}_b^\Gamma$ and $\vartheta = 0$ on Γ . Using this and also (4.55), we start from (4.56). Integrating by parts, we compute

$$\begin{aligned}
0 &= \int_s^t \int_\omega \left(\partial_t(\varepsilon_d V^{k-1}) - \operatorname{curl} W^{k-1} + \sigma \partial_t^{k-1} E + \partial_t(\varepsilon_d \nabla p^{k-1}) + \partial_t f_{k-1} \right) \cdot \vartheta \partial_t V^{k-1} dx d\tau \\
&= \int_s^t \int_\omega \left(\partial_t(\varepsilon_d V^{k-1}) + \sigma \partial_t^{k-1} E + \partial_t(\varepsilon_d \nabla p^{k-1}) + \partial_t f_{k-1} \right) \cdot \vartheta \partial_t V^{k-1} dx d\tau \\
& \quad + \int_s^t \int_\omega \partial_t(\mu_d^{-1} \mu_d W^{k-1}) \cdot \operatorname{curl}(\vartheta V^{k-1}) dx d\tau - \left[\int_\omega \mu_d W^{k-1} \cdot \operatorname{curl}(\vartheta V^{k-1}) dx \right]_s^t \\
&= \int_s^t \int_\omega \left(\partial_t(\varepsilon_d V^{k-1}) + \sigma \partial_t^{k-1} E + \partial_t(\varepsilon_d \nabla p^{k-1}) + \partial_t f_{k-1} \right) \cdot \vartheta \partial_t V^{k-1} dx d\tau
\end{aligned}$$

$$\begin{aligned}
& + \int_s^t \int_\omega (\partial_t(\mu_d^{-1})\mu_d W^{k-1} + \mu_d^{-1} \partial_t(\mu_d W^{k-1})) \cdot \operatorname{curl}(\vartheta V^{k-1}) \, dx \, d\tau \\
& - \left[\int_\omega \mu_d W^{k-1} \cdot \operatorname{curl}(\vartheta V^{k-1}) \, dx \right]_s^t \\
& = \int_s^t \int_\omega (\partial_t(\varepsilon_d V^{k-1}) + \sigma \partial_t^{k-1} E + \partial_t(\varepsilon_d \nabla p^{k-1}) + \partial_t f_{k-1}) \cdot \vartheta \partial_t V^{k-1} \, dx \, d\tau \\
& - \int_s^t \int_\omega \mu_d^{-1} (\operatorname{curl} V^{k-1} + \partial_t(\mu_d \nabla q^k) + \partial_t g_k) \cdot \operatorname{curl}(\vartheta V^{k-1}) \, dx \, d\tau \\
& + \int_s^t \int_\omega (\partial_t(\mu_d^{-1})\mu_d W^{k-1}) \cdot \operatorname{curl}(\vartheta V^{k-1}) \, dx \, d\tau - \left[\int_\omega \mu_d W^{k-1} \cdot \operatorname{curl}(\vartheta V^{k-1}) \, dx \right]_s^t \\
& = \int_s^t \int_\omega (\partial_t(\varepsilon_d V^{k-1}) + \sigma \partial_t^{k-1} E + \partial_t(\varepsilon_d \nabla p^{k-1}) + \partial_t f_{k-1}) \cdot \vartheta \partial_t V^{k-1} \, dx \, d\tau \\
& - \int_s^t \int_\omega \mu_d^{-1} \operatorname{curl} V^{k-1} \cdot \vartheta \operatorname{curl} V^{k-1} \, dx \, d\tau - \int_s^t \int_\omega \mu_d^{-1} \operatorname{curl} V^{k-1} \cdot (\nabla \vartheta \times V^{k-1}) \, dx \, d\tau \\
& - \int_s^t \int_\omega \mu_d^{-1} (\partial_t(\mu_d \nabla q^k) + \partial_t g_k) \cdot \operatorname{curl}(\vartheta V^{k-1}) \, dx \, d\tau \\
& + \int_s^t \int_\omega (\partial_t(\mu_d^{-1})\mu_d W^{k-1}) \cdot \operatorname{curl}(\vartheta V^{k-1}) \, dx \, d\tau - \left[\int_\omega \mu_d W^{k-1} \cdot \operatorname{curl}(\vartheta V^{k-1}) \, dx \right]_s^t.
\end{aligned}$$

To justify the integration by parts for $k = 3$ in the second line above, we use Theorem 3.3.5 to approximate the solution (V^2, W^2) of the system (4.56) in $\mathcal{G}^1$ by $(V_n^2, W_n^2) \in \mathcal{G}^2$ with $\operatorname{tr}_t V_n^2 = 0$. Taking the limit $n \rightarrow \infty$ then shows the equality.

It follows

$$\begin{aligned}
& \int_s^t \int_{\omega \setminus \mathcal{N}_b^\Gamma} |\operatorname{curl} V^{k-1}|^2 \, dx \, d\tau \\
& \lesssim \int_s^t \int_\omega \left| \partial_t(\varepsilon_d) V^{k-1} + \varepsilon_d \partial_t V^{k-1} + \sigma \partial_t^{k-1} E + \partial_t(\varepsilon_d \nabla p^{k-1}) + \partial_t f_{k-1} \right| |\partial_t V^{k-1}| \, dx \, d\tau \\
& + \int_s^t \int_{\mathcal{N}_b^\Gamma \cap \omega} |\operatorname{curl} V^{k-1}| |V^{k-1}| \, dx \, d\tau + \int_s^t \int_\omega \left| \partial_t(\mu_d \nabla q^k) + \partial_t g_k \right| |\operatorname{curl}(\vartheta V^{k-1})| \, dx \, d\tau \\
& + \int_s^t \int_\omega \left| \partial_t(\mu_d^{-1})\mu_d W^{k-1} \right| |\operatorname{curl}(\vartheta V^{k-1})| \, dx \, d\tau - \left[\int_\omega \mu_d W^{k-1} \cdot \operatorname{curl}(\vartheta V^{k-1}) \, dx \right]_s^t \\
& =: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

We now insert Definition 4.3.3, (4.54) and

$$\operatorname{curl}(\vartheta V^{k-1}) = \nabla \vartheta \times (\partial_t^{k-1} E - \nabla p^{k-1}) + \vartheta \operatorname{curl}(\partial_t^{k-1} E)$$

Lemmas 4.2.1 and 4.3.2 then imply

$$\begin{aligned}
I_1 & \lesssim \int_s^t (d^{1/2}(\tau) + z(\tau))(d^{1/2}(\tau) + z(\tau)) \, d\tau \lesssim \int_s^t (d(\tau) + z^2(\tau)) \, d\tau, \\
I_2 & \lesssim \int_s^t z^{1/2}(\tau)(d^{1/2}(\tau) + z(\tau)) \, d\tau \lesssim \int_s^t (\delta z(\tau) + c_\delta d(\tau) + z^{3/2}(\tau)) \, d\tau, \\
I_3 & \lesssim \int_s^t (d^{1/2}(\tau) + z(\tau))(d^{1/2}(\tau) + z^{1/2}(\tau) + z(\tau)) \, d\tau \\
& \lesssim \int_s^t (\delta z(\tau) + c_\delta d(\tau) + z^{3/2}(\tau)) \, d\tau, \\
I_4 & \lesssim \int_s^t z^{1/2}(\tau)(d^{1/2}(\tau) + z^{1/2}(\tau) + z(\tau))(d^{1/2}(\tau) + z^{1/2}(\tau) + z(\tau)) \, d\tau
\end{aligned}$$

$$\begin{aligned}
&\lesssim \int_s^t (d(\tau) + z^{3/2}(\tau)) \, d\tau, \\
I_5 &\lesssim \sum_{r=s,t} (e^{1/2}(r) + d^{1/2}(r) + z(r))(e^{1/2}(r) + z^{1/2}(r) + z(r)) \\
&\lesssim \sum_{r=s,t} \delta z(r) + c_\delta(e(r) + d(r) + z^2(r)),
\end{aligned}$$

for any $\delta > 0$. To remove $e(t)$, we invoke the energy estimate Lemma 4.2.3 arriving at

$$\begin{aligned}
\int_s^t \int_{\omega \setminus \mathcal{N}_b^\Gamma} |\operatorname{curl} V^{k-1}|^2 \, dx \, d\tau &\lesssim \int_s^t (\delta z(\tau) + c_\delta(z^{3/2}(\tau) + d(\tau))) \, dx \, d\tau \\
&\quad + \delta(z(s) + z(t)) + c_\delta(e(s) + z^2(s) + z^2(t)).
\end{aligned}$$

The claim follows by inserting this estimate into (4.64). $\square$

Proposition 4.4.1 and Proposition 4.4.2 then imply the core observability estimate.

Corollary 4.4.3 (Observability-type Estimate III). *Assume that (H_q) holds. Let $\delta \in (0, 1]$. For $k \in \{1, 2, 3\}$ and $0 \leq s \leq t < T_*$, we have*

$$\begin{aligned}
\int_s^t \int_{v \cup \mathcal{N}_b^\Gamma} (|V^k|^2 + |W^k|^2) \, dx \, d\tau &\lesssim \int_s^t (\delta z(\tau) + c_\delta(z^{3/2}(\tau) + d(\tau))) \, dx \, d\tau \\
&\quad + \delta(z(s) + z(t)) + c_\delta(e(s) + z^2(s) + z^2(t)).
\end{aligned}$$

Before we derive an observability estimate for $\partial_t^k(E, H)$, we first control the magnetic contribution to $\int e \, d\tau$.

Lemma 4.4.4. *Let (H_q) hold. For $k \in \{0, 1, 2, 3\}$ and $0 \leq s \leq t < T_*$ we have*

$$\int_s^t \|\partial_t^k H\|_{L^2(\Omega)}^2 \, d\tau \lesssim (e(s) + e(t)) + \int_s^t \|\partial_t^k E\|_{L^2(\Omega)}^2 \, d\tau + \int_s^t z^{3/2}(\tau) \, d\tau.$$

This inequality can be shown as in Lemma 4.3 and Proposition 4.4 in [35] (cf. Equation (4.13) therein), where simply connected Ω without an interface were treated.

Proof. Using the positive definiteness of μ and Lemma 4.2.1 we start by writing

$$\int_s^t \|\partial_t^k H\|_{L^2(\Omega)}^2 \, d\tau \lesssim \int_s^t \int_{\Omega} \partial_t^k(\mu H) \cdot \partial_t^k H \, dx \, d\tau + \int_s^t z^{3/2}(\tau) \, d\tau. \quad (4.65)$$

Observe that $\partial_t^k(\mu H)$ belongs to $H_{n0}^\Sigma(\operatorname{div} 0) = \operatorname{ran}(\operatorname{curl}_E)$, by (4.1), (4.9), and Lemma A.1.3. Therefore there exists a unique vector potential $u^k \in C([0, T_*], H_{t0}(\operatorname{curl}) \cap H^{\partial\Omega}(\operatorname{div} 0))$ such that

$$\operatorname{curl} u^k = \partial_t^k(\mu H) \quad \text{and} \quad \|u^k\|_{L^2} \leq \|\partial_t^k(\mu H)\|_{L^2}. \quad (4.66)$$

By (4.1) we have $\partial_t(\mu H) = -\operatorname{curl} E$. For $k = 0$, taking the the difference quotient, we see that $u^0 \in C^1([0, T_*], H_{t0}(\operatorname{curl}) \cap H^{\partial\Omega}(\operatorname{div} 0))$. Moreover, the time derivative of (4.66) yields $\operatorname{curl}(\partial_t u^0) = \partial_t(\mu H) = -\operatorname{curl} E$ and hence $\partial_t u^0 + E$ belongs to $H_{t0}(\operatorname{curl} 0)$. From Equation (2.107) and Theorem 2.8' of [12] thus follows the orthogonal decomposition

$$\partial_t u^0 + E = \nabla \phi_0^0 + h_2^0$$

for unique $\phi_0^0 \in H_0^1(\Omega)$ and $h_2^0 \in \mathbb{H}_2$. Furthermore, by Theorems 2.9 and 2.10' in [12] the space $L^2(\Omega)$ has the orthogonal decomposition

$$L^2(\Omega) = H^{\partial\Omega}(\operatorname{div} 0) \oplus \nabla H_0^1(\Omega) \oplus \mathbb{H}_2,$$

which provides the decomposition

$$E = -\partial_t u^0 + \nabla \phi_0^0 + h_2^0$$

and thus

$$\|\partial_t u^0\|_{L^2} \leq \|E\|_{L^2} \quad (4.67)$$

by orthogonality.

For $k \in \{1, 2, 3\}$, we obtain $\operatorname{curl} u^k = \partial_t^k(\mu H) = -\operatorname{curl}(\partial_t^{k-1} E)$ and therefore $u^k + \partial_t^{k-1} E \in H_{t0}(\operatorname{curl} 0)$. Again, Equation (2.107), Theorem 2.8', Theorems 2.9, and 2.10' of [12] give the orthogonal decomposition

$$\partial_t^{k-1} E = -u^k + \nabla \phi_0^k + h_2^k,$$

for unique $\phi_0^k \in H_0^1(\Omega)$ and $h_2^k \in \mathbb{H}_2$. In particular the regularity of the left-hand side and the decomposition imply that $u^k \in C^1([0, T_*], H_{t0}(\operatorname{curl}) \cap H^{\partial\Omega}(\operatorname{div} 0))$ and

$$\|\partial_t u^k\|_{L^2} \leq \|\partial_t^k E\|_{L^2}. \quad (4.68)$$

In all cases, for $k \in \{0, 1, 2, 3\}$, integration by parts, (4.29), and (4.66) lead to

$$\begin{aligned} \int_s^t \int_\Omega \partial_t^k(\mu H) \cdot \partial_t^k H \, dx \, d\tau &= \int_s^t \int_\Omega \operatorname{curl} u^k \cdot \partial_t^k H \, dx = \int_s^t \langle u^k, \operatorname{curl} \partial_t^k H \rangle_{H_{t0}(\operatorname{curl})} \, d\tau \\ &= \int_s^t \langle u^k, \partial_t(\hat{\varepsilon}_k \partial_t^k E) + \sigma \partial_t^k E + \partial_t f_k \rangle_{H_{t0}(\operatorname{curl})} \, d\tau \\ &\lesssim \int_s^t \langle u^k, \partial_t(\hat{\varepsilon}_k \partial_t^k E) \rangle_{H_{t0}(\operatorname{curl})} \, d\tau + \delta \int_s^t \|\partial_t^k H\|_{L^2(\Omega)}^2 \, d\tau \\ &\quad + c_\delta \int_s^t \left(\|\partial_t^k E\|_{L^2(\Omega)}^2 + z^2(\tau) \right) \, d\tau. \end{aligned} \quad (4.69)$$

The first term on the right is rewritten as

$$\int_s^t \langle u^k, \partial_t(\hat{\varepsilon}_k \partial_t^k E) \rangle_{H_{t0}(\operatorname{curl})} \, d\tau = \left[\int_\Omega u^k \cdot (\hat{\varepsilon}_k \partial_t^k E) \, dx \right]_s^t - \int_s^t \int_\Omega \partial_t u^k \cdot (\hat{\varepsilon}_k \partial_t^k E) \, dx \, d\tau. \quad (4.70)$$

Inserting (4.66), (4.67), and (4.68) into (4.70), we derive

$$\int_s^t \langle u^k, \partial_t(\hat{\varepsilon}_k \partial_t^k E) \rangle_{H_{t0}(\operatorname{curl})} \, d\tau \lesssim (e_k(s) + e_k(t)) + \int_s^t \|\partial_t^k E\|_{L^2(\Omega)}^2 \, d\tau.$$

For small $\delta > 0$, this estimate, (4.65), and (4.69) imply the claim. $\square$

Next, we establish an observability-type estimate for $\partial_t^k(E, H)$, up to small and higher order error terms.

Theorem 4.4.5. *Let (H_q) hold. For $0 \leq s \leq t < T_*$ and $\delta \in (0, 1]$, we obtain*

$$\begin{aligned} \int_s^t e(\tau) \, d\tau &\lesssim e(t) + c_\delta \left(z(s) + z^2(t) + \int_s^t (d(\tau) + z^{3/2}(\tau)) \, d\tau \right) \\ &\quad + \delta \left(z(t) + \int_s^t z(\tau) \, d\tau \right). \end{aligned}$$

Proof. By definition we have

$$\int_s^t e(\tau) \, d\tau \lesssim \max_{k \in \{0, 1, 2, 3\}} \int_s^t \left(\|\hat{\varepsilon}_k \partial_t^k E\|_{L^2}^2 + \|\hat{\mu}_k \partial_t^k H\|_{L^2}^2 \right) \, d\tau.$$

Lemma 4.2.4 now allows us to restrict ourselves to $k \in \{1, 2, 3\}$, namely

$$\int_s^t e(\tau) d\tau \lesssim \max_{k \in \{1, 2, 3\}} \int_s^t \left(\|\hat{\varepsilon}_k \partial_t^k E\|_{L^2}^2 + \|\hat{\mu}_k \partial_t^k H\|_{L^2}^2 \right) d\tau.$$

Lemma 4.4.4, Definition 4.3.3 and Corollary 4.4.3 then yield

$$\begin{aligned} \int_s^t e(\tau) d\tau &\lesssim \max_{k \in \{1, 2, 3\}} \int_s^t \|\hat{\varepsilon}_k \partial_t^k E\|_{L^2}^2 d\tau + e(s) + e(t) + \int_s^t z^{3/2}(\tau) d\tau \\ &\lesssim \max_{k \in \{1, 2, 3\}} \int_s^t \left(\|V^k\|_{L^2}^2 + \|\nabla p^k\|_{L^2}^2 \right) d\tau \\ &\quad + e(s) + e(t) + \int_s^t z^{3/2}(\tau) d\tau \\ &\lesssim c_\delta \max_{k \in \{1, 2, 3\}} \int_s^t \int_\omega |V^k|^2 dx d\tau + \max_{k \in \{1, 2, 3\}} \int_s^t \|\nabla p^k\|_{L^2}^2 d\tau \\ &\quad + e(t) + \delta \left(z(t) + \int_s^t z(\tau) d\tau \right) \\ &\quad + c_\delta \left(z(s) + z^2(t) + \int_s^t (d(\tau) + z^{3/2}(\tau)) d\tau \right), \end{aligned}$$

for any $\delta \in (0, 1]$. Since σ is uniformly positive on ω , Lemma 4.3.2 and Definition 4.3.3 lead to

$$\begin{aligned} \int_s^t e(\tau) d\tau &\lesssim \max_{k \in \{1, 2, 3\}} \int_s^t \int_\omega |\sigma \partial_t^k E|^2 dx d\tau + \max_{k \in \{1, 2, 3\}} \int_s^t \|\nabla p^k\|_{L^2}^2 d\tau \\ &\quad + e(t) + \delta \left(z(t) + \int_s^t z(\tau) d\tau \right) \\ &\quad + c_\delta \left(z(s) + z^2(t) + \int_s^t (d(\tau) + z^{3/2}(\tau)) d\tau \right) \\ &\lesssim e(t) + c_\delta \left(z(s) + z^2(t) + \int_s^t (d(\tau) + z^{3/2}(\tau)) d\tau \right) \\ &\quad + \delta \left(z(t) + \int_s^t z(\tau) d\tau \right). \end{aligned} \quad \square$$

4.5. REGULARITY ANALYSIS

Our proof of exponential stability will be based on Theorem 4.4.5 and Lemma 4.2.3. These results allow us to control time derivatives of (E, H) on L^2 , but so far the estimates include small error terms also involving space derivatives. We thus also need a version of Theorem 4.4.5 for higher-order spatial derivatives, i.e., we have to control z on the left by e and higher order terms on the right. Here it is crucial that the constants do not depend on time and that we bound z uniformly and in L^1 .

For uniformly positive conductivity this has been carried out in [35]. However, the jump of the conductivity leads to the interface Γ , which requires sophisticated additional arguments.

Theorem 4.5.1 (Regularity analysis). *Let (H_q) hold. For $0 \leq s \leq t < T_*$ the solutions (E, H) of (4.1)–(4.4) satisfy*

$$z(t) + \int_s^t z(\tau) d\tau \lesssim z(s) + z^2(t) + e(t) + \int_s^t (e(\tau) + z^{3/2}(\tau)) d\tau.$$

To show this result we split the solutions in different spatial regions using cutoffs, iterated over the spatial regularity, and reduce the estimate to lower spatial norms as illustrated in Figure 4.3.

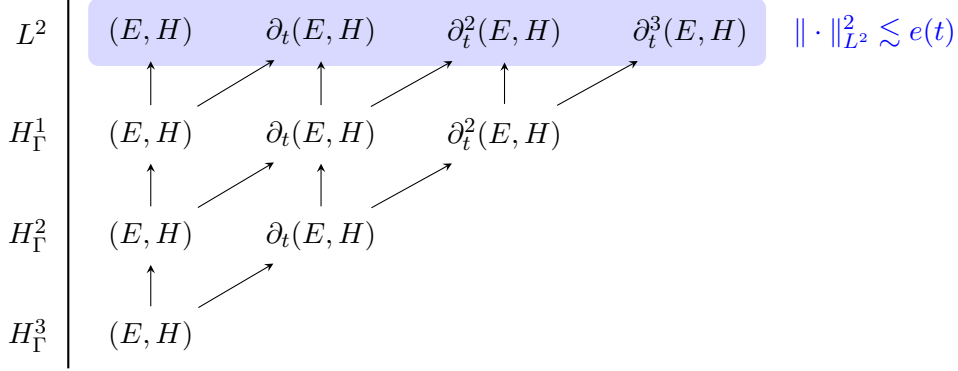


Figure 4.3.: We estimate the H_Γ^j -norm by terms in the H_Γ^{j-1} -norm, finally reaching the L^2 -level. These terms are then bound by $e(t)$.

We use the collars $\mathcal{N}_a = \{x \in \bar{\Omega} \mid \text{dist}(x, \partial\Omega) < a\}$ and $\mathcal{N}_a^\Gamma = \{x \in \bar{\Omega} \mid \text{dist}(x, \Gamma) < a\}$ around $\partial\Omega$ and Γ . Choose $a > 0$ such that

- $\mathcal{N}_a \cap \mathcal{N}_a^\Gamma = \emptyset$,
- $\partial\mathcal{N}_a$ and $\partial\mathcal{N}_a^\Gamma$ retain the C^5 regularity of $\partial\omega$, see Lemma 14.16 in [26],
- and the basis $\{\tau_1, \tau_2, \nu\}$ of normal and tangential vectors can be extended from $\partial\Omega$, respectively Γ , to a basis B also denoted by $B = \{\tau_1, \tau_2, \nu\}$ on $\mathcal{N}_a$, respectively $\mathcal{N}_a^\Gamma$.

We introduce 4 smooth cutoffs $\chi_{\partial\Omega}, \chi_\omega, \chi_\Gamma$ and $\chi_\nu \geq 0$, satisfying $\sum_{\Xi \in \{\partial\Omega, \omega, \Gamma, \nu\}} \chi_\Xi = 1$ on Ω . To this end, we take $\chi_{\partial\Omega}$ with $\chi_{\partial\Omega} = 1$ on $\mathcal{N}_{a/2}$ and $\text{supp } \chi_{\partial\Omega} \subseteq \mathcal{N}_a$, and χ_Γ with $\chi_\Gamma = 1$ on $\mathcal{N}_{a/2}^\Gamma$ and $\text{supp } \chi_\Gamma \subseteq \mathcal{N}_a^\Gamma$. Then we set $\chi_\omega = \mathbb{1}_\omega(1 - \chi_{\partial\Omega} - \chi_\Gamma)$ and $\chi_\nu = \mathbb{1}_\nu(1 - \chi_\Gamma)$.

We define the quantities

$$y_\Xi^{k,j}(t) := \left\| \partial_t^k [\chi_\Xi(E, H)] \right\|_{H_\Gamma^j(\Omega)}^2$$

for $\Xi \in \{\nu, \Gamma, \omega, \partial\Omega\}$, $j, k \in \{0, 1, 2, 3\}$, $j \leq 3 - k$. We have to show

$$\begin{aligned} y_\Xi^{k,j}(t) + \int_s^t y_\Xi^{k,j}(\tau) \, d\tau &\lesssim (1 + c_\theta) \left[z(s) + e(t) + z^2(t) + \int_s^t (e(\tau) + z^{3/2}(\tau)) \, d\tau \right] \\ &\quad + \theta \int_s^t z(\tau) \, d\tau \end{aligned}$$

for small $\theta > 0$. Note that for $j = 0$ this is true for $\theta = 0$, since $y_\Xi^{k,0}(t) \lesssim e(t)$. In particular for $k = 3$ there is nothing to show. In the case $\Xi \neq \Gamma$ the proof is essentially contained in [51] and [35]. First, we recall these results and adapt their proofs. We deviate in the structure of the proof in [51] and [35] to prepare for the treatment of the interface terms.

First, we derive an estimate for the magnetic field on $\Omega_\pm$ from the div-curl inequality of Theorem A.2.2. Here we trade a space derivative of H to a time derivative of E which we can control in our iteration scheme.

Lemma 4.5.2. *Let (H_q) hold. For $0 \leq s \leq t < T_*$, $j, k \in \mathbb{N}_0$ with $j + k \leq 2$ the solutions (E, H) of (4.1)–(4.4) satisfy*

$$\left\| \partial_t^k H \right\|_{H_\Gamma^{j+1}} \lesssim e^{1/2}(t) + \left\| \partial_t^k E \right\|_{H_\Gamma^j} + \left\| \partial_t^{k+1} E \right\|_{H_\Gamma^j} + z(t).$$

Proof. This is a direct result of the div-curl estimate in Theorem A.2.2. Let $k \leq 2$. By (4.2), (4.4), (4.30), (4.36), and Lemma A.1.2 the fields

$$\partial_t^k H \in H(\text{curl}), \quad \hat{\mu}_k \partial_t^k H + g_k = \partial_t^k(\mu H) \in H_{n0}(\text{div } 0)$$

satisfy the assumptions of Theorem A.2.2. Together with (4.32) and Lemma 4.2.1, Theorem A.2.2 implies

$$\begin{aligned} \left\| \partial_t^k H \right\|_{H_\Gamma^{j+1}} &\lesssim \left\| \partial_t^k H \right\|_{L^2} + \left\| \text{curl } \partial_t^k H \right\|_{H_\Gamma^j} + \|g_k\|_{H_\Gamma^{j+1}} \\ &\lesssim e^{1/2}(t) + \left\| \partial_t^k E \right\|_{H_\Gamma^j} + \left\| \partial_t^{k+1} E \right\|_{H_\Gamma^j} + z(t). \end{aligned} \quad \square$$

The next result on v can be shown as in Section 6 of [51] by differentiating and applying a div-curl estimate. We present the details since here an additional term near the interface Γ appears. Its space regularity is controlled by time derivatives, lower-order and error terms, and additional contributions at the boundary.

Lemma 4.5.3. *Let (H_q) hold. For $0 \leq s \leq t < T_*$, $j, k \in \mathbb{N}_0$ with $j + k \leq 2$ the solutions (E, H) of (4.1)–(4.4) satisfy*

$$y_v^{k,j+1}(t) \lesssim z^2(t) + y_v^{k+1,j}(t) + y_v^{k,j}(t) + y_\Gamma^{k,j}(t).$$

Proof. Let $t \geq 0, k \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^3$ with $k + |\alpha| \leq 2$. We multiply (4.1) by χ_v and apply $\partial_x^\alpha \partial_t^k$ to obtain

$$\begin{aligned} \text{curl}(\partial_x^\alpha \partial_t^k \chi_v H) &= \partial_t^{k+1} \partial_x^\alpha (\varepsilon(x, E) \chi_v E) - \partial_x^\alpha \partial_t^k (\nabla \chi_v \times H), \\ \text{curl}(\partial_x^\alpha \partial_t^k \chi_v E) &= -\partial_t^{k+1} \partial_x^\alpha (\mu(x, H) \chi_v H) - \partial_x^\alpha \partial_t^k (\nabla \chi_v \times E), \end{aligned}$$

using that the conductivity σ vanishes on $v \subseteq \Omega$. Equation (4.28) and Lemma 4.2.1 thus imply

$$\begin{aligned} &\left\| \text{curl}(\partial_x^\alpha \partial_t^k (\chi_v H)) \right\|_{L^2}, \left\| \text{curl}(\partial_x^\alpha \partial_t^k (\chi_v E)) \right\|_{L^2} \\ &\lesssim \left\| \partial_t^k (E, H) \right\|_{H^{|\alpha|}(v)} + \left\| \partial_t^{k+1} \chi_v (E, H) \right\|_{H^{|\alpha|}} + z(t), \end{aligned} \quad (4.71)$$

where we also used the Sobolev embeddings $H^1 \hookrightarrow L^6$ and $H^2 \hookrightarrow L^\infty$, see also Lemma 4.2.2. The z term appears if a time derivative hits ε or μ , or if a space derivative falls on the fields in ε or μ . Lemma 4.1.2 and the Maxwell equation (4.32) give

$$\begin{aligned} \text{div}(\varepsilon(E) \chi_v E) &= \nabla \chi_v \cdot \varepsilon(E) E, \quad \text{div}(\mu(H) \chi_v H) = \nabla \chi_v \cdot \mu(H) H, \\ \text{div}(\varepsilon_d \partial_x^\alpha \partial_t^k (\chi_v E)) &= - \sum_{0 \leq a \leq \alpha} \binom{\alpha}{a} \partial_x^{\alpha-a} ((\nabla \chi_v)^\top \varepsilon_d) \partial_t^k \partial_x^a E - \partial_x^\alpha (\chi_v \text{div } \tilde{f}_{k-1}) \\ &\quad - \sum_{0 \leq a < \alpha} \binom{\alpha}{a} \text{div}(\partial_x^{\alpha-a} \varepsilon_d \partial_t^k \partial_x^a (\chi_v E)), \\ \text{div}(\mu_d \partial_x^\alpha \partial_t^k (\chi_v H)) &= - \sum_{0 \leq a \leq \alpha} \binom{\alpha}{a} \partial_x^{\alpha-a} ((\nabla \chi_v)^\top \mu_d) \partial_t^k \partial_x^a H - \partial_x^\alpha (\chi_v \text{div } \tilde{g}_{k-1}) \\ &\quad - \sum_{0 \leq a < \alpha} \binom{\alpha}{a} \text{div}(\partial_x^{\alpha-a} \mu_d \partial_t^k \partial_x^a (\chi_v H)), \end{aligned}$$

for $k \geq 1$. Therefore,

$$\left\| \text{div}(\varepsilon_d \partial_x^\alpha \partial_t^k (\chi_v E)) \right\|_{L^2}, \left\| \text{div}(\mu_d \partial_x^\alpha \partial_t^k (\chi_v H)) \right\| \lesssim \left\| \partial_t^k (E, H) \right\|_{H^{|\alpha|}(v)} + z(t). \quad (4.72)$$

Since $\chi_v(E, H)|_{\partial v} = 0$ we can apply the div-curl estimate of Theorem A.2.9 a). In combination with the estimates (4.71) and (4.72) yield

$$\begin{aligned} \left\| \partial_x^\alpha \partial_t^k \chi_v(E, H) \right\|_{H^1} &\lesssim \left\| \partial_t^{k+1} \chi_v(E, H) \right\|_{H^{|\alpha|}} + \left\| \partial_t^k \chi_v(E, H) \right\|_{H^{|\alpha|}} + \left\| \partial_t^k \chi_\Gamma(E, H) \right\|_{H^{|\alpha|}} + z(t), \\ y_v^{k, |\alpha|+1}(t) &\lesssim z^2(t) + y_v^{k+1, |\alpha|}(t) + y_v^{k, |\alpha|}(t) + y_\Gamma^{k, |\alpha|}(t). \end{aligned} \quad \square$$

The part of ω away from $\partial\Omega$ and Γ can be handled as in [35], see Equation (7.6) therein. The magnetic field is estimated via Lemma 4.5.2 whereas the electric one has to be treated differently due to the presence of the conductivity. Here we use an energy estimate instead.

Lemma 4.5.4. *Let (H_q) hold. For $0 \leq s \leq t < T_*$ and $j, k \in \mathbb{N}_0$ with $j + k \leq 2$, the solutions (E, H) of (4.1)–(4.4) satisfy*

$$\begin{aligned} &y_\omega^{k, j+1}(t) + \sigma_0 \int_s^t y_\omega^{k, j+1}(\tau) d\tau \\ &\lesssim z(s) + c_\theta \int_s^t \left(e(\tau) + z^{3/2}(\tau) + \left\| \partial_t^k E \right\|_{H_\Gamma^j}^2 + \left\| \partial_t^{k+1}(E, H) \right\|_{H_\Gamma^j}^2 \right) d\tau + \theta \int_s^t z(\tau) d\tau \end{aligned}$$

for any $\theta > 0$.

Proof. Let $k \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^3$ with $|\alpha| \geq 1$ and $k + |\alpha| \leq 3$. The differentiated and localized version of (4.32) in H_Γ^{-1} is given by

$$\begin{aligned} \varepsilon_d \partial_t \partial_x^\alpha \partial_t^k (\chi_\omega E) &= \text{curl} \partial_x^\alpha \partial_t^k (\chi_\omega H) - \sigma \partial_x^\alpha \partial_t^k (\chi_\omega E) + \partial_x^\alpha ([\chi_\omega, \text{curl}] \partial_t^k H) \\ &\quad - \sum_{0 \leq a < \alpha} \binom{\alpha}{a} \left[\partial_x^{\alpha-a} \sigma \partial_x^a \partial_t^k (\chi_\omega E) + \partial_x^{\alpha-a} \varepsilon_d \partial_x^a \partial_t^{k+1} (\chi_\omega E) \right] - \partial_x^\alpha (\chi_\omega \tilde{f}_k), \\ &=: \text{curl} \partial_x^\alpha \partial_t^k (\chi_\omega H) - \sigma \partial_x^\alpha \partial_t^k (\chi_\omega E) - \mathfrak{f}_{k, \alpha} \\ \mu_d \partial_t \partial_x^\alpha \partial_t^k (\chi_\omega H) &= -\text{curl} \partial_x^\alpha \partial_t^k (\chi_\omega E) - \partial_x^\alpha ([\chi_\omega, \text{curl}] \partial_t^k E) \\ &\quad - \sum_{0 \leq a < \alpha} \binom{\alpha}{a} \partial_x^{\alpha-a} \mu_d \partial_x^a \partial_t^{k+1} (\chi_\omega H) - \partial_x^\alpha (\chi_\omega \tilde{g}_k), \\ &=: -\text{curl} \partial_x^\alpha \partial_t^k (\chi_\omega E) - \mathfrak{g}_{k, \alpha} \\ \text{tr}_t \partial_x^\alpha \partial_t^k (\chi_\omega E) &= 0, \quad \text{tr}_n \partial_x^\alpha \partial_t^k (\chi_\omega H) = 0. \end{aligned} \quad (4.73)$$

One derives an energy equality as in (4.97) below. Here the situation is considerably simpler due to the lack of boundary and interface terms, allowing for an approximation converging in $L^2(\Omega)$. Then, since $\sigma \geq \sigma_0 > 0$ on ω , we obtain

$$\begin{aligned} &y_\omega^{k, |\alpha|}(t) + \sigma_0 \int_s^t \left\| \partial_t^k (\chi_\omega E) \right\|_{H^{|\alpha|}}^2 d\tau \\ &\lesssim \int_\Omega \left(\varepsilon_d \partial_x^\alpha \partial_t^k (\chi_\omega E) \cdot \partial_x^\alpha \partial_t^k (\chi_\omega E) + \mu_d \partial_x^\alpha \partial_t^k (\chi_\omega H) \cdot \partial_x^\alpha \partial_t^k (\chi_\omega H) \right) (s) dx \\ &\quad + \int_s^t \int_\Omega \left(\partial_t \varepsilon_d \partial_x^\alpha \partial_t^k (\chi_\omega E) \cdot \partial_x^\alpha \partial_t^k (\chi_\omega E) + \partial_t \mu_d \partial_x^\alpha \partial_t^k (\chi_\omega H) \cdot \partial_x^\alpha \partial_t^k (\chi_\omega H) \right) dx d\tau \\ &\quad + \int_s^t \int_\Omega \left(\mathfrak{f}_{k, \alpha} \cdot \partial_x^\alpha \partial_t^k (\chi_\omega E) + \mathfrak{g}_{k, \alpha} \cdot \partial_x^\alpha \partial_t^k (\chi_\omega H) \right) dx d\tau \\ &\lesssim z(s) + \int_s^t z^{3/2}(\tau) d\tau \\ &\quad + \int_s^t \int_\Omega \left(\mathfrak{f}_{k, \alpha} \cdot \partial_x^\alpha \partial_t^k (\chi_\omega E) + \mathfrak{g}_{k, \alpha} \cdot \partial_x^\alpha \partial_t^k (\chi_\omega H) \right) dx d\tau, \end{aligned} \quad (4.74)$$

where we also used Lemma 4.2.1. Let us take a closer look at the last term. Note that the commutator $[\chi_\omega, \text{curl}]$ is simply a multiplication operator. We thus get

$$\begin{aligned} \int_s^t \int_\Omega \mathfrak{f}_{k,\alpha} \cdot \partial_x^\alpha \partial_t^k (\chi_\omega E) \, dx \, d\tau &\lesssim \theta \int_s^t \left\| \partial_t^k (\chi_\omega E) \right\|_{H^{|\alpha|}}^2 \, d\tau + c_\theta \left(\int_s^t \left\| \partial_t^k H \right\|_{H^{|\alpha|}(\omega)}^2 \, d\tau \right. \\ &\quad + \int_s^t \left(\left\| \partial_t^k (\chi_\omega E) \right\|_{H^{|\alpha|-1}}^2 + \left\| \partial_t^{k+1} (\chi_\omega E) \right\|_{H^{|\alpha|-1}}^2 \right) \, d\tau \\ &\quad \left. + \int_s^t z^2(\tau) \, d\tau \right), \\ \int_s^t \int_\Omega \mathfrak{g}_{k,\alpha} \cdot \partial_x^\alpha \partial_t^k (\chi_\omega H) \, dx \, d\tau &\lesssim \theta \int_s^t \left\| \partial_t^k E \right\|_{H^{|\alpha|}(\omega)}^2 \, d\tau + c_\theta \int_s^t \left\| \partial_t^k (\chi_\omega H) \right\|_{H^{|\alpha|}}^2 \, d\tau \\ &\quad + \int_s^t \left\| \partial_t^{k+1} (\chi_\omega H) \right\|_{H^{|\alpha|-1}}^2 \, d\tau + \int_s^t z^2(\tau) \, d\tau, \end{aligned}$$

for $\theta > 0$ using Young's inequality. The $H^{|\alpha|}$ -norms of the electric field (with θ in front) will be simply estimated by z , whereas we keep the $H^{|\alpha|}$ -norm the magnetic field for now. Inserting this estimate into (4.74), we obtain

$$\begin{aligned} y_\omega^{k,|\alpha|}(t) + \sigma_0 \int_s^t \left\| \partial_t^k (\chi_\omega E) \right\|_{H^{|\alpha|}}^2 \, d\tau \\ \lesssim z(s) + \int_s^t z^{3/2}(\tau) \, d\tau + c_\theta \int_s^t \left\| \partial_t^k H \right\|_{H^{|\alpha|}(\omega)}^2 \, d\tau \\ + c_\theta \int_s^t \left(\left\| \partial_t^k (\chi_\omega E) \right\|_{H^{|\alpha|-1}}^2 + \left\| \chi_\omega \partial_t^{k+1} (E, H) \right\|_{H^{|\alpha|-1}}^2 \right) \, d\tau + \theta \int_s^t z(\tau) \, d\tau. \end{aligned}$$

Hence, we have reduced the estimate of the electric field in $H^{|\alpha|}$ to one on the magnetic field. Lemma 4.5.2 thus shows that

$$\begin{aligned} y_\omega^{k,|\alpha|}(t) + \sigma_0 \int_s^t y_\omega^{k,|\alpha|}(\tau) \, d\tau \\ \lesssim z(s) + c_\theta \int_s^t \left(e(\tau) + z^{3/2}(\tau) + \left\| \partial_t^k E \right\|_{H_\Gamma^{|\alpha|-1}}^2 + \left\| \partial_t^{k+1} (E, H) \right\|_{H_\Gamma^{|\alpha|-1}}^2 \right) \, d\tau + \theta \int_s^t z(\tau) \, d\tau. \square \end{aligned}$$

Up to this point differentiating in space did not change the boundary or interface conditions since the cutoffs vanished there. This is different for $\chi_{\partial\Omega}$ and χ_Γ . In these cases we may still differentiate in tangential direction picking up lower-order terms, however the derivatives in normal direction require an additional argument based on the structure of the Maxwell system. We start with the cutoff at the boundary $\partial\Omega$. Again, the reasoning is adapted from [35].

Lemma 4.5.5. *Let (H_q) hold. For $0 \leq s \leq t < T_*$ and $j, k \in \mathbb{N}_0$ with $j + k \leq 2$, the solutions (E, H) of (4.1)–(4.4) satisfy*

$$\begin{aligned} y_{\partial\Omega}^{k,j+1}(t) + \int_s^t y_{\partial\Omega}^{k,j+1}(\tau) \, d\tau \\ \lesssim z(s) + e(t) + z^2(t) + \theta \int_s^t z(\tau) \, d\tau + (1 + c_\theta) \left[\int_s^t (e(\tau) + z^{3/2}(\tau)) \, d\tau \right. \\ + \left\| \partial_t^k E(t) \right\|_{H_\Gamma^j}^2 + \left\| \partial_t^{k+1} E(t) \right\|_{H_\Gamma^j}^2 + \int_s^t \left(\left\| \partial_t^k E(\tau) \right\|_{H_\Gamma^j}^2 + \left\| \partial_t^{k+1} E(\tau) \right\|_{H_\Gamma^j}^2 \right) \, d\tau \\ \left. + y_{\partial\Omega}^{k+1,j}(t) + \int_s^t y_{\partial\Omega}^{k+1,j}(\tau) \, d\tau \right] \end{aligned}$$

for any $\theta > 0$.

Proof. 1) We start by applying $\partial_t^k \partial_\tau^\beta \chi_{\partial\Omega}$ for $\beta \in \mathbb{N}_0^2$ with $|\beta| + k \leq 3$ to (4.32)–(4.34) resulting in

$$\begin{aligned} \varepsilon_d \partial_t \partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} E) &= \operatorname{curl} \partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} H) - \sigma \partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} E) - \mathfrak{f}_{k,\beta} \\ \mathfrak{f}_{k,\beta} &:= -[\partial_\tau^\beta, \operatorname{curl}] \partial_t^k (\chi_{\partial\Omega} H) - \partial_\tau^\beta ([\chi_{\partial\Omega}, \operatorname{curl}] \partial_t^k H) + \partial_\tau^\beta (\chi_{\partial\Omega} \tilde{f}_k) \\ &\quad + \sum_{0 \leq b < \beta} \binom{\beta}{b} \left[\partial_\tau^{\beta-b} \sigma \partial_\tau^b \partial_t^k (\chi_{\partial\Omega} E) + \partial_\tau^{\beta-b} \varepsilon_d \partial_\tau^b \partial_t^{k+1} (\chi_{\partial\Omega} E) \right], \\ \mu_d \partial_t \partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} H) &= -\operatorname{curl} \partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} E) - \mathfrak{g}_{k,\beta} \end{aligned} \quad (4.75)$$

$$\begin{aligned} \mathfrak{g}_{k,\beta} &:= \partial_\tau^\beta ([\chi_{\partial\Omega}, \operatorname{curl}] \partial_t^k E) + [\partial_\tau^\beta, \operatorname{curl}] \partial_t^k (\chi_{\partial\Omega} E) \\ &\quad + \sum_{0 \leq b < \beta} \binom{\beta}{b} \partial_\tau^{\beta-b} \mu_d \partial_\tau^b \partial_t^{k+1} (\chi_{\partial\Omega} H) + \partial_\tau^\beta (\chi_{\partial\Omega} \tilde{g}_k), \end{aligned}$$

$$\begin{aligned} \operatorname{tr}_t \partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} E) &= [\operatorname{tr}_t, \partial_\tau^\beta] \partial_t^k (\chi_{\partial\Omega} E) = - \sum_{0 \leq b < \beta} \binom{\beta}{b} \operatorname{tr} \partial_\tau^b \partial_t^k (\chi_{\partial\Omega} E) \times \partial_\tau^{\beta-b} \nu \\ &=: \tau_{k,\beta}^E, \end{aligned} \quad (4.76)$$

Note that $\tau_{k,0}^E = 0$ and that the commutator $[\partial_\tau^\beta, \operatorname{curl}]$ is a differential operator of order $|\beta|$, $[\chi_{\partial\Omega}, \operatorname{curl}]$ is again a multiplication operator and $[\operatorname{tr}_t, \partial_\tau^\beta]$ is of order $|\beta| - 1$. As in (4.97) below, one derives an energy equality which then yields

$$\begin{aligned} &\left\| \partial_t^k \partial_\tau^\beta (\chi_{\partial\Omega} E)(t) \right\|_{L^2}^2 + \sigma_0 \int_s^t \left\| \partial_t^k \partial_\tau^\beta (\chi_{\partial\Omega} E) \right\|_{L^2}^2 d\tau \\ &\lesssim z(s) + \int_s^t z^{3/2}(\tau) d\tau + \int_s^t \oint_{\partial\Omega} |\tau_{k,\beta}^E \cdot \operatorname{tr}_\tau \partial_t^k \partial_\tau^\beta H| d\varsigma d\tau \\ &\quad + \int_s^t \int_\Omega \left| \mathfrak{f}_{k,\beta} \cdot \partial_x^\beta \partial_t^k (\chi_{\partial\Omega} E) + \mathfrak{g}_{k,\beta} \cdot \partial_x^\beta \partial_t^k (\chi_{\partial\Omega} H) \right| dx d\tau, \end{aligned}$$

where we now get an additional term at the boundary. This term is treated in the sense of the dual pairing $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ using that $[\partial_\tau^\beta, \operatorname{tr}_\tau]$ is of lower order and that ∂_{τ_i} belongs to $\mathcal{B}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$. For $|\beta| > 0$ we thus get

$$\begin{aligned} \oint_{\partial\Omega} |\tau_{k,\beta}^E \cdot \operatorname{tr}_\tau \partial_t^k \partial_\tau^\beta H| d\varsigma &\lesssim \left\| \tau_{k,\beta}^E \right\|_{H^{1/2}(\partial\Omega)} \left\| \operatorname{tr}_\tau \partial_t^k \partial_\tau^\beta H \right\|_{H^{-1/2}(\partial\Omega)} \\ &\lesssim \theta z(t) + c_\theta \left\| \partial_t^k H \right\|_{H_\Gamma^{|\beta|}}^2. \end{aligned}$$

Next, the inhomogeneities are estimated by

$$\begin{aligned} &\int_s^t \int_\Omega \left| \mathfrak{f}_{k,\beta} \cdot \partial_x^\beta \partial_t^k (\chi_{\partial\Omega} E) + \mathfrak{g}_{k,\beta} \cdot \partial_x^\beta \partial_t^k (\chi_{\partial\Omega} H) \right| dx d\tau \\ &\lesssim \theta \int_s^t z(\tau) d\tau + c_\theta \int_s^t \left(\left\| \partial_t^k H \right\|_{H_\Gamma^{|\beta|}}^2 + \left\| \partial_t^k (\chi_{\partial\Omega} E) \right\|_{H^{|\beta|-1}}^2 \right. \\ &\quad \left. + \left\| \partial_t^{k+1} \chi_{\partial\Omega}(E, H) \right\|_{H^{|\beta|-1}}^2 + z^2 \right) d\tau. \end{aligned}$$

Thus,

$$\begin{aligned} &\left\| \partial_t^k \partial_\tau^\beta (\chi_{\partial\Omega} E)(t) \right\|_{L^2}^2 + \sigma_0 \int_s^t \left\| \partial_t^k \partial_\tau^\beta (\chi_{\partial\Omega} E) \right\|_{L^2}^2 d\tau \\ &\lesssim z(s) + \theta \int_s^t z(\tau) d\tau + c_\theta \left[\int_s^t z^{3/2}(\tau) d\tau + \left\| \partial_t^k H \right\|_{H_\Gamma^{|\beta|}}^2 + y_{\partial\Omega}^{k,|\beta|-1} + y_{\partial\Omega}^{k+1,|\beta|-1} \right], \end{aligned} \quad (4.77)$$

where the magnetic term is again treated by means of Lemma 4.5.2 later on.

2) a) To estimate the full Sobolev norm, we next consider normal and mixed derivatives. Let $\alpha = (\gamma, \beta) \in \mathbb{N}_0 \times \mathbb{N}_0^2$ with $\gamma \neq 0$ and $|\alpha| \leq 3 - k$.

We furthermore split the terms with normal derivatives into

$$\left\| \partial_\nu^\gamma \partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} E) \right\|_{L^2} \lesssim \left\| \partial_\nu^\gamma \left(\partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} E) \right)_\nu \right\|_{L^2} + \left\| \partial_\nu^\gamma \left(\partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} E) \right)_\tau \right\|_{L^2}, \quad (4.78)$$

where for

$$u := \partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} E)$$

we write

$$(u)_\xi = (u \cdot \xi) \xi \text{ for } \xi \in \{\tau_1, \tau_2, \nu\} \text{ and } (u)_\tau = (u)_{\tau_1} + (u)_{\tau_2}. \quad (4.79)$$

We start with the tangential part, which is recovered from the curl as follows. Let us first introduce some notation from Chapter 6 of [35], namely

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$J(\xi) = \sum_j \xi_j J_j.$$

It follows

$$\text{curl} = \sum_j J_j \partial_j = \sum_{\substack{j \in \{1,2,3\} \\ \xi \in \{\nu, \tau_1, \tau_2\}}} J_j \xi_j \partial_\xi = \sum_{\xi \in \{\nu, \tau_1, \tau_2\}} J(\xi) \partial_\xi.$$

Note that the $\ker(J(\nu)) = \text{span}\{\nu\}$. Hence, $J(\nu)u = J(\nu)u_\tau$, and its restriction to $\text{span}\{\tau_1, \tau_2\}$ has an inverse $R(\nu)$. This allows us to recover the tangential part of the normal derivative via

$$\partial_\nu u_\tau = \sum_i (\partial_\nu \tau_i u_{\tau_i} + \tau_i \partial_\nu \tau_i \cdot u) + R(\nu) \left(\text{curl} u - \sum_i J(\tau_i) \partial_{\tau_i} u \right). \quad (4.80)$$

We thus obtain

$$\begin{aligned} \left\| \partial_\nu^\gamma \left(\partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} E) \right)_\tau \right\|_{L^2} &= \left\| \partial_\nu^{\gamma-1} \partial_\nu \left(\partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} E) \right)_\tau \right\|_{L^2} \\ &\lesssim \sum_{0 \leq \delta < \gamma} \left\| \partial_\nu^\delta \left(\partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} E) \right)_\tau \right\|_{L^2} \\ &\quad + \left\| \partial_\nu^{\gamma-1} \text{curl}(\partial_\tau^\beta \partial_t^k \chi_{\partial\Omega} E) \right\|_{L^2} + \left\| \partial_\nu^{\gamma-1} \partial_\tau^{\beta+(1,0)} \partial_t^k (\chi_{\partial\Omega} E) \right\|_{L^2} + \left\| \partial_\nu^{\gamma-1} \partial_\tau^{\beta+(0,1)} \partial_t^k (\chi_{\partial\Omega} E) \right\|_{L^2}. \end{aligned}$$

In this way, we trade normal derivatives of the tangential part for a curl term and tangential derivatives. We now insert Equation (4.75) and iterate this estimate to move all derivatives onto tangential ones, which we already treated, i.e.,

$$\begin{aligned} \left\| \partial_\nu^\gamma \left(\partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} E) \right)_\tau \right\|_{L^2}^2 &\lesssim \left\| \partial_\nu^{\gamma-1} (\mu_d \partial_t \partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} H)) \right\|_{L^2}^2 + \left\| \partial_\nu^{\gamma-1} \mathbf{g}_{k,\beta} \right\|_{L^2}^2 \\ &\quad + \sum_{0 \leq \delta < \gamma} \left\| \partial_\nu^\delta \left(\partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} E) \right)_\tau \right\|_{L^2}^2 + \left\| \partial_\nu^{\gamma-1} \partial_\tau^{\beta+(1,0)} \partial_t^k (\chi_{\partial\Omega} E) \right\|_{L^2}^2 + \left\| \partial_\nu^{\gamma-1} \partial_\tau^{\beta+(0,1)} \partial_t^k (\chi_{\partial\Omega} E) \right\|_{L^2}^2 \\ &\lesssim \left\| \partial_t^k (\chi_{\partial\Omega} E) \right\|_{H^{|\alpha|-1}}^2 + \left\| \partial_t^{k+1} (\chi_{\partial\Omega} H) \right\|_{H^{|\alpha|-1}}^2 \\ &\quad + \sum_{b \in \mathbb{N}_0^2: |b| \leq |\alpha|-1} \left\| \mathbf{g}_{k,b} \right\|_{H^{|\alpha|-1-|b|}}^2 + \sum_{b \in \mathbb{N}_0^2: |b| \leq |\alpha|} \left\| \partial_\tau^b \partial_t^k (\chi_{\partial\Omega} E) \right\|_{L^2}^2. \end{aligned}$$

Since

$$\left\| \mathbf{g}_{k,b} \right\|_{H^{|\alpha|-1-|b|}}^2 \lesssim \left\| \partial_t^k E \right\|_{H^{|\alpha|-1}(\omega)}^2 + y_{\partial\Omega}^{k+1, |\alpha|-1} + z^2,$$

we conclude

$$\begin{aligned} \left\| \partial_\nu^\gamma \left(\partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} E) \right) \right\|_{L^2}^2 &\lesssim \left\| \partial_t^k E \right\|_{H^{|\alpha|-1}(\omega)}^2 + y_{\partial\Omega}^{k+1, |\alpha|-1} + z^2 \\ &+ \sum_{b \in \mathbb{N}_0^2: |b| \leq |\alpha|} \left\| \partial_\tau^b \partial_t^k (\chi_{\partial\Omega} E) \right\|_{L^2}^2, \end{aligned} \quad (4.81)$$

where the last term was treated in (4.77).

b) Next, we recover the normal part via the divergence as in Chapter 6 of [35]. One has

$$\operatorname{div} u = \sum_j \partial_j \sum_\xi u_\xi \xi_j = \sum_\xi (\partial_\xi u_\xi + \operatorname{div}(\xi) u_\xi).$$

Hence, for a matrix $\varkappa \in \{\hat{\varepsilon}_k, \sigma\}$, we obtain

$$\begin{aligned} \operatorname{div}(\varkappa u) &= \sum_{\xi, \zeta} \partial_\xi (\xi^\top \varkappa \zeta u_\zeta) + \sum_\xi \operatorname{div}(\xi) \xi^\top \varkappa u \\ &= \sum_{\xi, \zeta} (\varkappa_{\xi\zeta} \partial_\xi u_\zeta + \partial_\xi \varkappa_{\xi\zeta} u_\zeta) + \sum_\xi \operatorname{div}(\xi) \xi^\top \varkappa u, \\ \varkappa_{\nu\nu} \partial_\nu u_\nu &= \operatorname{div}(\varkappa u) - \sum_{(\xi, \zeta) \neq (\nu, \nu)} \varkappa_{\xi\zeta} \partial_\xi u_\zeta - \sum_{\xi, \zeta} \partial_\xi \varkappa_{\xi\zeta} u_\zeta - \sum_\xi \operatorname{div}(\xi) \xi^\top \varkappa u \\ &=: \operatorname{div}(\varkappa u) - D(\varkappa)u, \end{aligned} \quad (4.82)$$

where again ξ, ζ run over $\{\nu, \tau_1, \tau_2\}$ and $\varkappa_{\xi, \zeta} := \varkappa \xi \cdot \zeta$. Thus, for $\varkappa = \hat{\varepsilon}_k$ and $u = \partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} E)$ we obtain

$$(\hat{\varepsilon}_k)_{\nu\nu} \partial_\nu u_\nu = \operatorname{div}(\hat{\varepsilon}_k u) - D(\hat{\varepsilon}_k)u.$$

We next derive an equation for the divergence. Taking the tangential derivative and divergence of (4.29) yields

$$\begin{aligned} \partial_t(\hat{\varepsilon}_k u) &= \operatorname{curl} \partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} H) - \sigma u \\ &\quad - \partial_\tau^\beta (\chi_{\partial\Omega} \partial_t f_k) - [\operatorname{curl}, \partial_\tau^\beta] (\chi_{\partial\Omega} \partial_t^k H) - \partial_\tau^\beta ([\operatorname{curl}, \chi_{\partial\Omega}] \partial_t^k H) \\ &\quad - \sum_{0 \leq b < \beta} \binom{\beta}{b} \left[\partial_\tau^{\beta-b} \sigma \partial_\tau^b \partial_t^k (\chi_{\partial\Omega} E) + \partial_t \left(\partial_\tau^{\beta-b} \hat{\varepsilon}_k \partial_\tau^b \partial_t^k (\chi_{\partial\Omega} E) \right) \right] \\ &=: \operatorname{curl} \partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} H) - \sigma u - \varphi, \end{aligned} \quad (4.83)$$

$$\operatorname{div}(\hat{\varepsilon}_k u)(t) = \operatorname{div}(\hat{\varepsilon}_k u)(s) - \int_s^t (\operatorname{div}(\sigma u) + \operatorname{div} \varphi)(\tau) d\tau. \quad (4.84)$$

Next, we apply (4.82) to $\operatorname{div}(\sigma u)$ and get

$$\operatorname{div}(\hat{\varepsilon}_k u)(t) = \operatorname{div}(\hat{\varepsilon}_k u)(s) - \int_s^t (\sigma_{\nu\nu} \partial_\nu u_\nu + D(\sigma)u + \operatorname{div} \varphi)(\tau) d\tau. \quad (4.85)$$

Set $\pi(\tau) := \frac{\sigma_{\nu\nu}}{(\hat{\varepsilon}_k)_{\nu\nu}} \geq \frac{\sigma_0}{\|\hat{\varepsilon}_k\|_\infty} =: \pi_0 > 0$ on ω . Equation (4.82) also leads to the integral equation

$$\begin{aligned} (\hat{\varepsilon}_k)_{\nu\nu} \partial_\nu u_\nu &= \operatorname{div}(\hat{\varepsilon}_k u) - D(\hat{\varepsilon}_k)u \\ &= \operatorname{div}(\hat{\varepsilon}_k u)(s) - D(\hat{\varepsilon}_k)u - \int_s^t \left(\pi(\tau) (\hat{\varepsilon}_k)_{\nu\nu} \partial_\nu u_\nu + D(\sigma)u + \operatorname{div} \varphi \right)(\tau) d\tau \end{aligned}$$

for $(\hat{\varepsilon}_k)_{\nu\nu} \partial_\nu u_\nu$. Defining

$$k(t) := -D(\hat{\varepsilon}_k)u \quad \text{and} \quad \ell(t) := D(\sigma)u + \operatorname{div} \varphi,$$

we rewrite the integral equation as

$$(\hat{\varepsilon}_k)_{\nu\nu} \partial_\nu u_\nu(t) = f(t) + y(t), \quad (4.86)$$

where $f(t) := \operatorname{div}(\hat{\varepsilon}_k u)(s) + k(t)$ and $y(t) := -\int_s^t (\pi(\tau)(\hat{\varepsilon}_k)_{\nu\nu} \partial_\nu u_\nu(\tau) + \ell(\tau)) \, d\tau$. Differentiating y in time results in the ODE

$$y'(t) + \pi(t)y(t) = -\pi(t)f(t) - \ell(t), \quad y(s) = 0,$$

which is solved by $y(t) = -\int_s^t (\pi(\tau)f(\tau) + \ell(\tau)) e^{-\int_\tau^t \pi(r) \, dr} \, d\tau$. Therefore, (4.86) becomes

$$\begin{aligned} (\hat{\varepsilon}_k)_{\nu\nu} \partial_\nu u_\nu(t) &= f(t) - \int_s^t (\pi(\tau)f(\tau) + \ell(\tau)) e^{-\int_\tau^t \pi(r) \, dr} \, d\tau \\ &= \operatorname{div}(\hat{\varepsilon}_k u)(s) + k(t) \\ &\quad - \int_s^t (\pi(\tau)(\operatorname{div}(\hat{\varepsilon}_k u)(s) + k(\tau)) + \ell(\tau)) e^{-\int_\tau^t \pi(r) \, dr} \, d\tau \\ &= e^{-\int_s^t \pi(r) \, dr} \operatorname{div}(\hat{\varepsilon}_k u)(s) + k(t) - \int_s^t (\pi(\tau)k(\tau) + \ell(\tau)) e^{-\int_\tau^t \pi(r) \, dr} \, d\tau. \end{aligned} \quad (4.87)$$

In order to bound the normal part in (4.78), we apply $\partial_\nu^{\gamma-1}$ to (4.87) and take the L^2 -norm. The first term becomes

$$\begin{aligned} \partial_\nu^{\gamma-1} \left(e^{-\int_s^t \pi(r) \, dr} \operatorname{div}(\hat{\varepsilon}_k u)(s) \right) &= e^{-\int_s^t \pi(r) \, dr} \partial_\nu^{\gamma-1} \operatorname{div}(\hat{\varepsilon}_k u)(s) \\ &\quad + \sum_{i=1}^{\gamma-1} \binom{\gamma-1}{i} \partial_\nu^i \left(e^{-\int_s^t \pi(r) \, dr} \right) \partial_\nu^{\gamma-1-i} \operatorname{div}(\hat{\varepsilon}_k u)(s). \end{aligned}$$

Note that by the chain rule for $1 \leq i \leq \gamma-1 \leq 2$,

$$\begin{aligned} \partial_\nu \left(e^{-\int_s^t \pi(r) \, dr} \right) &= \left(-\int_s^t \partial_\nu \pi(r) \, dr \right) e^{-\int_s^t \pi(r) \, dr} && \text{for } i = 1, \\ \partial_\nu^2 \left(e^{-\int_s^t \pi(r) \, dr} \right) &= \left[\left(-\int_s^t \partial_\nu \pi(r) \, dr \right)^2 - \int_s^t \partial_\nu^2 \pi(r) \, dr \right] e^{-\int_s^t \pi(r) \, dr} && \text{for } i = 2. \end{aligned}$$

By Lemma 4.2.2, we can therefore bound each summand by

$$\left\| \partial_\nu^i \left(e^{-\int_s^t \pi(r) \, dr} \right) \partial_\nu^{\gamma-1-i} \operatorname{div}(\hat{\varepsilon}_k u)(s) \right\|_{L^2} \lesssim \sup_{\tau \in [s, t]} \|\pi(\tau)\|_{H_\Gamma^2} ((t-s)^2 + (t-s)) e^{-\pi_0(t-s)} \|u(s)\|_{H_\Gamma^{\gamma-1}}$$

for $1 \leq i \leq \gamma-1 \leq 2$. The norm $\|\pi(\tau)\|_{H_\Gamma^2}$ is controlled uniformly via (4.5), (4.26), and Lemma 4.2.1. It follows

$$\left\| \partial_\nu^{\gamma-1} \left(e^{-\int_s^t \pi(r) \, dr} \operatorname{div}(\hat{\varepsilon}_k u)(s) \right) \right\|_{L^2}^2 \lesssim e^{-\pi_0(t-s)} \left(\left\| \partial_\nu^{\gamma-1} \operatorname{div}(\hat{\varepsilon}_k u)(s) \right\|_{L^2}^2 + \|u(s)\|_{H_\Gamma^{\gamma-1}}^2 \right).$$

Applying the same reasoning to the integrand $\pi(\tau)k(\tau)$ in (4.87) then yields

$$\begin{aligned} \|\partial_\nu^\gamma u_\nu(t)\|_{L^2}^2 &\lesssim \left\| \partial_\nu^{\gamma-1} ((\hat{\varepsilon}_k)_{\nu\nu} \partial_\nu u_\nu(t)) \right\|_{L^2}^2 + \|u(t)\|_{H_\Gamma^{\gamma-1}}^2 \\ &\lesssim e^{-\pi_0(t-s)} \left(\left\| \partial_\nu^{\gamma-1} \operatorname{div}(\hat{\varepsilon}_k u)(s) \right\|_{L^2}^2 + \|u(s)\|_{H_\Gamma^{\gamma-1}}^2 \right) + \left\| \partial_\nu^{\gamma-1} k(t) \right\|_{L^2}^2 + \|u(t)\|_{H_\Gamma^{\gamma-1}}^2 \\ &\quad + \left(\int_s^t e^{-\frac{1}{2}\pi_0(t-\tau)} \cdot e^{-\frac{1}{2}\pi_0(t-\tau)} \left(\left\| \partial_\nu^{\gamma-1} k(\tau) \right\|_{L^2} + \left\| \partial_\nu^{\gamma-1} \ell(\tau) \right\|_{L^2} + \|u(\tau)\|_{H_\Gamma^{\gamma-1}} \right) d\tau \right)^2 \\ &\lesssim e^{-\pi_0(t-s)} \left(\left\| \partial_\nu^{\gamma-1} \operatorname{div}(\hat{\varepsilon}_k u)(s) \right\|_{L^2}^2 + \|u(s)\|_{H_\Gamma^{\gamma-1}}^2 \right) + \left\| \partial_\nu^{\gamma-1} k(t) \right\|_{L^2}^2 + \|u(t)\|_{H_\Gamma^{\gamma-1}}^2 \\ &\quad + \int_s^t e^{-\pi_0(t-\tau)} \, d\tau \int_s^t e^{-\pi_0(t-\tau)} \left(\left\| \partial_\nu^{\gamma-1} k(\tau) \right\|_{L^2} + \left\| \partial_\nu^{\gamma-1} \ell(\tau) \right\|_{L^2} + \|u(\tau)\|_{H_\Gamma^{\gamma-1}} \right)^2 d\tau \end{aligned}$$

$$\begin{aligned} &\lesssim e^{-\pi_0(t-s)} \left(\left\| \partial_\nu^{\gamma-1} \operatorname{div}(\hat{\varepsilon}_k u)(s) \right\|_{L^2}^2 + \|u(s)\|_{H_\Gamma^{\gamma-1}}^2 \right) + \left\| \partial_\nu^{\gamma-1} k(t) \right\|_{L^2}^2 + \|u(t)\|_{H_\Gamma^{\gamma-1}}^2 \\ &\quad + \frac{1}{\pi_0} \int_s^t e^{-\pi_0(t-\tau)} \left(\left\| \partial_\nu^{\gamma-1} k(\tau) \right\|_{L^2}^2 + \left\| \partial_\nu^{\gamma-1} \ell(\tau) \right\|_{L^2}^2 + \|u(\tau)\|_{H_\Gamma^{\gamma-1}}^2 \right) d\tau, \end{aligned} \quad (4.88)$$

where we used Cauchy-Schwarz in time and Young's inequality.

Integrating over time results in

$$\begin{aligned} \int_s^t \left\| \partial_\nu^\gamma u_\nu(\tau) \right\|_{L^2}^2 d\tau &\lesssim \frac{1}{\pi_0} \left(\left\| \partial_\nu^{\gamma-1} \operatorname{div}(\hat{\varepsilon}_k u)(s) \right\|_{L^2}^2 + \|u(s)\|_{H_\Gamma^{\gamma-1}}^2 \right) \\ &\quad + \left(1 + \frac{1}{\pi_0^2} \right) \int_s^t \left(\left\| \partial_\nu^{\gamma-1} k(\tau) \right\|_{L^2}^2 + \left\| \partial_\nu^{\gamma-1} \ell(\tau) \right\|_{L^2}^2 + \|u(\tau)\|_{H_\Gamma^{\gamma-1}}^2 \right) d\tau. \end{aligned} \quad (4.89)$$

Note that $\left\| \partial_\nu^{\gamma-1} \operatorname{div}(\hat{\varepsilon}_k u)(s) \right\|_{L^2}^2 \lesssim \left\| \partial_t^k (\chi_{\partial\Omega} E)(s) \right\|_{H^{|\alpha|}}^2 \lesssim z(s)$. As above Lemmas 4.2.1 and 4.2.2 yield

$$\begin{aligned} \left\| \partial_\nu^{\gamma-1} k(\tau) \right\|_{L^2}^2 &\lesssim \left\| \partial_\nu^{\gamma-1} \left(\sum_{(\xi, \zeta) \neq (\nu, \nu)} \partial_\xi u_\zeta \right) \right\|_{L^2}^2 + \|u(\tau)\|_{H_\Gamma^{\gamma-1}}^2 \\ &\lesssim \left\| \partial_\nu^\gamma (u)_\tau \right\|_{L^2}^2 + \left\| \partial_\nu^{\gamma-1} (\partial_\tau^{(1,0)} u)_\nu \right\|_{L^2}^2 + \left\| \partial_\nu^{\gamma-1} (\partial_\tau^{(0,1)} u)_\nu \right\|_{L^2}^2 + y_{\partial\Omega}^{k, |\alpha|-1}. \end{aligned} \quad (4.90)$$

Here we treat the first term by (4.81) and the second and third terms have fewer normal derivatives, so we can iterate this estimate in the end. Moreover,

$$\begin{aligned} \left\| \partial_\nu^{\gamma-1} \ell(\tau) \right\|_{L^2}^2 &\lesssim \left\| \partial_\nu^{\gamma-1} \left(\sum_{(\xi, \zeta) \neq (\nu, \nu)} \partial_\xi u_\zeta \right) \right\|_{L^2}^2 + \|u(\tau)\|_{H_\Gamma^{\gamma-1}}^2 + \left\| \partial_\nu^\gamma \varphi \right\|_{L^2}^2 \\ &\lesssim \left\| \partial_\nu^\gamma (u)_\tau \right\|_{L^2}^2 + \left\| \partial_\nu^{\gamma-1} (\partial_\tau^{(1,0)} u)_\nu \right\|_{L^2}^2 + \left\| \partial_\nu^{\gamma-1} (\partial_\tau^{(0,1)} u)_\nu \right\|_{L^2}^2 + y_{\partial\Omega}^{k, |\alpha|-1} + \|\varphi\|_{H_\Gamma^\gamma}^2, \\ \|\varphi\|_{H_\Gamma^\gamma}^2 &\lesssim \left\| \partial_t f_k \right\|_{H_\Gamma^{3-k}}^2 + \left\| \partial_t^k H \right\|_{H_\Gamma^{|\alpha|}}^2 + \left\| \chi_{\partial\Omega} \partial_t^k E \right\|_{H_\Gamma^{|\alpha|-1}}^2 + \left\| \chi_{\partial\Omega} \partial_t^{k+1} E \right\|_{H_\Gamma^{|\alpha|-1}}^2 \\ &\lesssim z^2(\tau) + \left\| \partial_t^k H \right\|_{H_\Gamma^{|\alpha|}}^2 + y_{\partial\Omega}^{k, |\alpha|-1} + y_{\partial\Omega}^{k+1, |\alpha|-1}. \end{aligned} \quad (4.92)$$

Equations (4.88), (4.89), (4.90), (4.91), and (4.92) lead to

$$\begin{aligned} &\left\| \partial_\nu^\gamma \left(\partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} E) \right)_\nu(t) \right\|_{L^2}^2 + \int_s^t \left\| \partial_\nu^\gamma \left(\partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} E) \right)_\nu \right\|_{L^2}^2 d\tau \\ &\lesssim \sum_{\substack{b \in \mathbb{N}_0^2 \\ |b| \leq 1}} \left[\left\| \partial_\nu^{\gamma-1} \left(\partial_\tau^{\beta+b} \partial_t^k (\chi_{\partial\Omega} E) \right)_\nu(t) \right\|_{L^2}^2 + \int_s^t \left\| \partial_\nu^{\gamma-1} \left(\partial_\tau^{\beta+b} \partial_t^k (\chi_{\partial\Omega} E) \right)_\nu \right\|_{L^2}^2 d\tau \right] \\ &\quad + \left\| \partial_\nu^\gamma \left(\partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} E) \right)_\tau(t) \right\|_{L^2}^2 + \int_s^t \left\| \partial_\nu^\gamma \left(\partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} E) \right)_\tau \right\|_{L^2}^2 d\tau \\ &\quad + z(s) + z^2(t) + \left\| \partial_t^k H \right\|_{H_\Gamma^{|\alpha|}}^2 + y_{\partial\Omega}^{k, |\alpha|-1} + y_{\partial\Omega}^{k+1, |\alpha|-1} \\ &\quad + \int_s^t \left(z^2(\tau) + \left\| \partial_t^k H \right\|_{H_\Gamma^{|\alpha|}}^2 + y_{\partial\Omega}^{k, |\alpha|-1} + y_{\partial\Omega}^{k+1, |\alpha|-1} \right) d\tau. \end{aligned}$$

Iteratively, we then obtain

$$\left\| \partial_\nu^\gamma \left(\partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} E) \right)_\nu(t) \right\|_{L^2}^2 + \int_s^t \left\| \partial_\nu^\gamma \left(\partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} E) \right)_\nu \right\|_{L^2}^2 d\tau \quad (4.93)$$

$$\begin{aligned}
&\lesssim \sum_{\substack{b \in \mathbb{N}_0^2 \\ |b| \leq \gamma}} \left[\left\| \partial_t^k \partial_\tau^{\beta+b} (\chi_{\partial\Omega} E)(t) \right\|_{L^2}^2 + \int_s^t \left\| \partial_t^k \partial_\tau^{\beta+b} (\chi_{\partial\Omega} E) \right\|_{L^2}^2 d\tau \right] \\
&\quad + \sum_{m=1}^{\gamma} \sum_{\substack{b \in \mathbb{N}_0^2 \\ |b| \leq \gamma-m}} \left[\left\| \partial_\nu^m \left(\partial_\tau^{\beta+b} \partial_t^k (\chi_{\partial\Omega} E) \right)_\tau(t) \right\|_{L^2}^2 + \int_s^t \left\| \partial_\nu^m \left(\partial_\tau^{\beta+b} \partial_t^k (\chi_{\partial\Omega} E) \right)_\tau \right\|_{L^2}^2 d\tau \right] \\
&\quad + z(s) + z^2(t) + \left\| \partial_t^k H \right\|_{H_\Gamma^{|\alpha|}}^2 + y_{\partial\Omega}^{k,|\alpha|-1} + y_{\partial\Omega}^{k+1,|\alpha|-1} \\
&\quad + \int_s^t \left(z^2(\tau) + \left\| \partial_t^k H \right\|_{H_\Gamma^{|\alpha|}}^2 + y_{\partial\Omega}^{k,|\alpha|-1} + y_{\partial\Omega}^{k+1,|\alpha|-1} \right) d\tau,
\end{aligned}$$

where the tangential and mixed terms were already estimated in (4.77) and (4.81).

By means of (4.77), (4.78), (4.81), (4.93), and since $y_{\partial\Omega}^{k,|\alpha|-1} \lesssim \left\| \partial_t^k (E, H) \right\|_{H_\Gamma^{|\alpha|-1}}^2$, we finally recover the full Sobolev norm via

$$\begin{aligned}
&\left\| \partial_t^k (\chi_{\partial\Omega} E)(t) \right\|_{H^{|\alpha|}}^2 + \int_s^t \left\| \partial_t^k (\chi_{\partial\Omega} E) \right\|_{H^{|\alpha|}}^2 d\tau \\
&\lesssim \sum_{\substack{(\gamma, \beta) \in \mathbb{N}_0 \times \mathbb{N}_0^2 \\ \gamma + |\beta| \leq |\alpha|}} \left[\left\| \partial_\nu^\gamma \partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} E)(t) \right\|_{L^2}^2 + \int_s^t \left\| \partial_\nu^\gamma \partial_\tau^\beta \partial_t^k (\chi_{\partial\Omega} E) \right\|_{L^2}^2 d\tau \right] \\
&\lesssim z(s) + z^2(t) + \theta \int_s^t z(\tau) d\tau + (1 + c_\theta) \left[\left\| \partial_t^k H(t) \right\|_{H_\Gamma^{|\alpha|}}^2 + \int_s^t \left(z^{3/2}(\tau) + \left\| \partial_t^k H \right\|_{H_\Gamma^{|\alpha|}}^2 \right) d\tau \right. \\
&\quad \left. + \left\| \partial_t^k E(t) \right\|_{H_\Gamma^{|\alpha|-1}}^2 + \int_s^t \left\| \partial_t^k E \right\|_{H_\Gamma^{|\alpha|-1}}^2 d\tau + y_{\partial\Omega}^{k+1,|\alpha|-1}(t) + \int_s^t y_{\partial\Omega}^{k+1,|\alpha|-1} d\tau \right].
\end{aligned} \tag{4.94}$$

Lemma 4.5.2 then shows the desired estimate. $\square$

At last, we have to bound the Sobolev norms of $\partial_t^k (E, H)$ near the interface Γ . We first consider each side of Γ individually. The proof of the inequality on ω is similar as in the proof of Lemma 4.5.5. The tangential derivative of $\chi_\Gamma (E, H)$ on ω can be estimated using the energy identity exploiting that $\sigma \geq \sigma_0 > 0$ on ω . The normal derivatives are then reduced to the tangential case a before via the Maxwell system. On ν we bound the fields using a div-curl lemma, where the traces can be controlled via the interface conditions and lower-order estimates.

Lemma 4.5.6. *Let (H_q) hold. For $0 \leq s \leq t < T_*$ and $k, j \in \mathbb{N}_0$ with $k + j \leq 2$, the solutions (E, H) of (4.1)–(4.4) satisfy*

$$\begin{aligned}
&y_\Gamma^{k,j+1} + \int_s^t y_\Gamma^{k,j+1} d\tau \\
&\lesssim e(t) + z(s) + z^2(t) + \theta \int_s^t z(\tau) d\tau + (1 + c_\theta) \left[\int_s^t \left(e(\tau) + z^{3/2}(\tau) \right) d\tau \right. \\
&\quad \left. + \left\| \partial_t^k (E, H) \right\|_{H_\Gamma^j}^2 + \left\| \partial_t^{k+1} (E, H) \right\|_{H_\Gamma^j}^2 + \int_s^t \left(\left\| \partial_t^k (E, H) \right\|_{H_\Gamma^j}^2 + \left\| \partial_t^{k+1} (E, H) \right\|_{H_\Gamma^j}^2 \right) d\tau \right]
\end{aligned}$$

for any $\theta > 0$.

Proof. We consider a neighborhood of the interface Γ . To simplify the notation slightly we drop the superscript of $\nu^\Gamma = \nu$ and $\tau_i^\Gamma = \tau_i$ when used in a subscript. From (4.6) we recall the interface conditions

$$\llbracket \partial_t^k E \times \nu^\Gamma \rrbracket(t) = 0, \quad \llbracket \partial_t^k H \times \nu^\Gamma \rrbracket(t) = 0. \tag{4.95}$$

1) For $0 \neq \alpha = (\gamma, \beta) \in \mathbb{N}_0^3 \cong \mathbb{N}_0 \times \mathbb{N}_0^2$ we write $\partial_\nu^\gamma \partial_\tau^\beta = \partial_\nu^{\gamma_1} \partial_{\tau_1}^{\beta_1} \partial_{\tau_2}^{\beta_2}$

1.1) The tangential derivative of the localized solution $\partial_\tau^\beta \partial_t^k(\chi_\Gamma(E, H))$, $|\alpha| \leq 3 - k$, solves the system

$$\begin{aligned} \varepsilon_d \partial_t \left[\partial_\tau^\beta \partial_t^k(\chi_\Gamma E) \right] &= \operatorname{curl} \left(\partial_\tau^\beta \partial_t^k(\chi_\Gamma H) \right) - \sigma \left[\partial_\tau^\beta \partial_t^k(\chi_\Gamma E) \right] - \mathfrak{f}_{\beta,k}, \\ \mathfrak{f}_{\beta,k} &:= -\partial_\tau^\beta \left([\chi_\Gamma, \operatorname{curl}] \partial_t^k H \right) - [\partial_\tau^\beta, \operatorname{curl}] (\chi_\Gamma \partial_t^k H) + \partial_\tau^\beta (\chi_\Gamma \tilde{f}_k) \\ &\quad + \sum_{0 \leq b < \beta} \binom{\beta}{b} \left(\partial_\tau^{\beta-b} \sigma \partial_\tau^b \partial_t^k(\chi_\Gamma E) + \partial_\tau^{\beta-b} \varepsilon_d \partial_\tau^b \partial_t^{k+1}(\chi_\Gamma E) \right), \\ \mu_d \partial_t \left[\partial_\tau^\beta \partial_t^k(\chi_\Gamma H) \right] &= -\operatorname{curl} \left(\partial_\tau^\beta \partial_t^k(\chi_\Gamma E) \right) - \mathfrak{g}_{\beta,k}, \\ \mathfrak{g}_{\beta,k} &:= \partial_\tau^\beta \left([\chi_\Gamma, \operatorname{curl}] \partial_t^k E \right) + [\partial_\tau^\beta, \operatorname{curl}] (\chi_\Gamma \partial_t^k E) + \partial_\tau^\beta (\chi_\Gamma \tilde{g}_k) \\ &\quad + \sum_{0 \leq b < \beta} \binom{\beta}{b} \partial_\tau^{\beta-b} \mu_d \partial_\tau^b \partial_t^{k+1}(\chi_\Gamma H), \end{aligned} \quad (4.96)$$

on both sides of Γ , cf. Equation (4.75). Again, note that $[\chi_\Gamma, \operatorname{curl}]$ is just a multiplier and $[\partial_\tau^\beta, \operatorname{curl}]$ is of order $|\beta|$. They also satisfy the boundary conditions

$$\operatorname{tr}_t \partial_\tau^\beta \partial_t^k(\chi_\Gamma E) = 0, \quad (\operatorname{tr}_n \partial_\tau^\beta \partial_t^k \mu_d \chi_\Gamma H = 0) \quad \text{on } \partial \mathcal{N}_a^\Gamma.$$

The tangential derivative commutes with the trace. Indeed, one can first establish this result for smooth functions and then apply the continuity of the operators. Hence, $\partial_\tau^\beta \partial_t^k(\chi_\Gamma E, \chi_\Gamma H)$ satisfies

$$\begin{aligned} \llbracket \operatorname{tr}_t \partial_\tau^\beta \partial_t^k(\chi_\Gamma E) \rrbracket &= - \sum_{0 \leq b < \beta} \binom{\beta}{b} \llbracket \operatorname{tr} \partial_\tau^b \partial_t^k(\chi_\Gamma E) \rrbracket \times \partial_\tau^{\beta-b} \nu^\Gamma =: \varphi_{\beta,k} \quad \text{on } \Gamma, \\ \llbracket \operatorname{tr}_t \partial_\tau^\beta \partial_t^k(\chi_\Gamma H) \rrbracket &= - \sum_{0 \leq b < \beta} \binom{\beta}{b} \llbracket \operatorname{tr} \partial_\tau^b \partial_t^k(\chi_\Gamma H) \rrbracket \times \partial_\tau^{\beta-b} \nu^\Gamma =: \psi_{\beta,k} \quad \text{on } \Gamma, \end{aligned}$$

where only terms of order strictly smaller than $|\beta|$ appear on the right-hand side.

1.2) We start by deriving the energy equality

$$\begin{aligned} &\frac{1}{2} \int_{\mathcal{N}_a^\Gamma} \left[\varepsilon_d \partial_\tau^\beta \partial_t^k(\chi_\Gamma E(t)) \cdot \partial_\tau^\beta \partial_t^k(\chi_\Gamma E(t)) + \mu_d \partial_\tau^\beta \partial_t^k(\chi_\Gamma H(t)) \cdot \partial_\tau^\beta \partial_t^k(\chi_\Gamma H(t)) \right] dx \\ &+ \int_s^t \int_{\mathcal{N}_a^\Gamma} \sigma \partial_\tau^\beta \partial_t^k(\chi_\Gamma E(\tau)) \cdot \partial_\tau^\beta \partial_t^k(\chi_\Gamma E(\tau)) dx d\tau \\ &= \frac{1}{2} \int_{\mathcal{N}_a^\Gamma} \left[\varepsilon_d \partial_\tau^\beta \partial_t^k(\chi_\Gamma E(s)) \cdot \partial_\tau^\beta \partial_t^k(\chi_\Gamma E(s)) + \mu_d \partial_\tau^\beta \partial_t^k(\chi_\Gamma H(s)) \cdot \partial_\tau^\beta \partial_t^k(\chi_\Gamma H(s)) \right] dx \\ &+ \frac{1}{2} \int_s^t \int_{\mathcal{N}_a^\Gamma} \left[(\partial_t \varepsilon_d) \partial_\tau^\beta \partial_t^k(\chi_\Gamma E) \cdot \partial_\tau^\beta \partial_t^k(\chi_\Gamma E) + (\partial_t \mu_d) \partial_\tau^\beta \partial_t^k(\chi_\Gamma H) \cdot \partial_\tau^\beta \partial_t^k(\chi_\Gamma H) \right] dx d\tau \\ &- \int_s^t \int_{\mathcal{N}_a^\Gamma} \left[\partial_\tau^\beta \partial_t^k(\chi_\Gamma E) \cdot \mathfrak{f}_{\beta,k} + \partial_\tau^\beta \partial_t^k(\chi_\Gamma H) \cdot \mathfrak{g}_{\beta,k} \right] dx d\tau \\ &+ \int_s^t \oint_\Gamma \left(\varphi_{\beta,k} \cdot \operatorname{tr}_{\tau, \partial \nu} \partial_\tau^\beta \partial_t^k(\chi_\Gamma H) - \operatorname{tr}_{\tau, \partial \omega} \partial_\tau^\beta \partial_t^k(\chi_\Gamma E) \cdot \psi_{\beta,k} \right) d\varsigma d\tau. \end{aligned} \quad (4.97)$$

For $\partial_\tau^\beta \partial_t^k(\chi_\Gamma E, \chi_\Gamma H) \in \mathcal{G}_\Gamma^1$ this follows from (4.96) and an integration by parts argument. However, if $k + |\beta| = 3$, we have to approximate the solutions suitably. Since $k \leq 2$ there is at least one tangential derivative $\partial_\tau^{(\beta_1, \beta_2)}$ with $\beta_i \geq 1$ for $i \in \{1, 2\}$. Using Theorem 3.3.5 we

approximate $(u, v) = \partial_t^k \partial_\tau^{\beta-e_i}(\chi_\Gamma E, \chi_\Gamma H) \in \mathcal{G}_\Gamma^1$ on $\mathcal{N}_a^\Gamma$ by $(u_n, v_n) \in \mathcal{G}_\Gamma^2$ in $\mathcal{G}_\Gamma^1$ solving

$$\begin{aligned} \varepsilon_d \partial_t u_n &= \operatorname{curl} v_n - \sigma u_n - \mathfrak{f}_{\beta-e_i, k, n}, & t \geq 0, x \in \mathcal{N}_a^\Gamma \setminus \Gamma, \\ \mu_d \partial_t v_n &= -\operatorname{curl} u_n - \mathfrak{g}_{\beta-e_i, k, n}, \\ \llbracket u \times \nu^\Gamma \rrbracket &= \varphi_{\beta-e_i, k, n}, \quad \llbracket v \times \nu^\Gamma \rrbracket = \psi_{\beta-e_i, k, n}, & t \geq 0, x \in \Gamma, \\ u \times \nu &= 0, & t \geq 0, x \in \partial \mathcal{N}_a^\Gamma, \\ u(0, x) &= u_{0, n}, \quad v(0, x) = v_{0, n}, & x \in \mathcal{N}_a^\Gamma \setminus \Gamma, \end{aligned}$$

where the approximations $\mathfrak{f}_{\beta-e_i, k, n}$, $\mathfrak{g}_{\beta-e_i, k, n}$, $\varphi_{\beta-e_i, k, n}$, and $\psi_{\beta-e_i, k, n}$ are chosen in accordance with Theorem 3.3.5. We need convergence in $\mathcal{G}_\Gamma^1$ to ensure the convergence of interface terms in a moment. Then $\partial_\tau^{e_i}(u_n, v_n)$ belongs to $\mathcal{G}_\Gamma^1$, and one derives the energy equality for the approximation

$$\begin{aligned} & \frac{1}{2} \int_{\mathcal{N}_a^\Gamma} [\varepsilon_d \partial_\tau^{e_i} u_n(t) \cdot \partial_\tau^{e_i} u_n(t) + \mu_d \partial_\tau^{e_i} v_n(t) \cdot \partial_\tau^{e_i} v_n(t)] dx \\ & + \int_s^t \int_{\mathcal{N}_a^\Gamma} \sigma \partial_\tau^{e_i} u_n(\tau) \cdot \partial_\tau^{e_i} u_n(\tau) dx d\tau \\ & = \frac{1}{2} \int_{\mathcal{N}_a^\Gamma} [\varepsilon_d \partial_\tau^{e_i} u_n(s) \cdot \partial_\tau^{e_i} u_n(s) + \mu_d \partial_\tau^{e_i} v_n(s) \cdot \partial_\tau^{e_i} v_n(s)] dx \\ & + \frac{1}{2} \int_s^t \int_{\mathcal{N}_a^\Gamma} [(\partial_t \varepsilon_d) \partial_\tau^{e_i} u_n \cdot \partial_\tau^{e_i} u_n + (\partial_t \mu_d) \partial_\tau^{e_i} v_n \cdot \partial_\tau^{e_i} v_n] dx d\tau \\ & - \int_s^t \int_{\mathcal{N}_a^\Gamma} [\partial_\tau^{e_i} u_n \cdot (\partial_\tau^{e_i} \mathfrak{f}_{\beta-e_i, k, n} - [\partial_\tau^{e_i}, \operatorname{curl}] v_n + (\partial_\tau^{e_i} \varepsilon_d) \partial_t u_n + (\partial_\tau^{e_i} \sigma) u_n) \\ & \quad + \partial_\tau^{e_i} v_n \cdot (\partial_\tau^{e_i} \mathfrak{g}_{\beta-e_i, k, n} + [\partial_\tau^{e_i}, \operatorname{curl}] u_n + (\partial_\tau^{e_i} \mu_d) \partial_t v_n)] dx d\tau \\ & + \int_s^t \oint_\Gamma (\partial_\tau^{e_i} \varphi_{\beta-e_i, k, n} \cdot \operatorname{tr}_{\tau, \partial v} \partial_\tau^{e_i} v_n - \operatorname{tr}_{\tau, \partial \omega} \partial_\tau^{e_i} u_n \cdot \partial_\tau^{e_i} \psi_{\beta-e_i, k, n}) d\varsigma d\tau. \end{aligned} \tag{4.98}$$

The convergence of $\partial_\tau^{e_i}(u_n, v_n) \rightarrow \partial_\tau^\beta(u, v)$ in $L^2(\mathcal{N}_a^\Gamma)$ shows that

$$\begin{aligned} \int_{\mathcal{N}_a^\Gamma} \varepsilon_d \partial_\tau^{e_i} u_n(r) \cdot \partial_\tau^{e_i} u_n(r) dx &\rightarrow \int_{\mathcal{N}_a^\Gamma} \varepsilon_d \partial_\tau^\beta \partial_t^k(\chi_\Gamma E(r)) \cdot \partial_\tau^\beta \partial_t^k(\chi_\Gamma E(r)) dx, \\ \int_{\mathcal{N}_a^\Gamma} \mu_d \partial_\tau^{e_i} v_n(r) \cdot \partial_\tau^{e_i} v_n(r) dx &\rightarrow \int_{\mathcal{N}_a^\Gamma} \mu_d \partial_\tau^\beta \partial_t^k(\chi_\Gamma H(r)) \cdot \partial_\tau^\beta \partial_t^k(\chi_\Gamma H(r)) dx, \end{aligned}$$

for $r \in \{s, t\}$. Similarly, the terms involving σ , $\partial_t \varepsilon_d$, and $\partial_t \mu_d$ converge to the expressions without n , and likewise the inhomogeneities and commutator terms. To reconstruct $\mathfrak{f}_{\beta, k}$ we use the commutator identity $[AB, C] = A[B, C] + [A, C]B$ and calculate

$$\begin{aligned} & \partial_\tau^{e_i} \mathfrak{f}_{\beta-e_i, k} - [\partial_\tau^{e_i}, \operatorname{curl}] v + (\partial_\tau^{e_i} \varepsilon_d) \partial_t u + (\partial_\tau^{e_i} \sigma) u \\ & = \partial_\tau^{e_i} \left(-\partial_\tau^{\beta-e_i} ([\chi_\Gamma, \operatorname{curl}] \partial_t^k H) - [\partial_\tau^{\beta-e_i}, \operatorname{curl}] (\chi_\Gamma \partial_t^k H) + \partial_\tau^{\beta-e_i} (\chi_\Gamma \tilde{f}_k) \right. \\ & \quad \left. + \sum_{0 \leq b < \beta-e_i} \binom{\beta-e_i}{b} \left(\partial_\tau^{\beta-e_i-b} \sigma \partial_\tau^b \partial_t^k(\chi_\Gamma E) + \partial_\tau^{\beta-e_i-b} \varepsilon_d \partial_\tau^b \partial_t^{k+1}(\chi_\Gamma E) \right) \right) \\ & \quad - [\partial_\tau^{e_i}, \operatorname{curl}] v + [\partial_\tau^{e_i}, \varepsilon_d] \partial_t u + [\partial_\tau^{e_i}, \sigma] u \\ & = -\partial_\tau^\beta ([\chi_\Gamma, \operatorname{curl}] \partial_t^k H) + \partial_\tau^\beta (\chi_\Gamma \tilde{f}_k) \\ & \quad - \left(\partial_\tau^{e_i} [\partial_\tau^{\beta-e_i}, \operatorname{curl}] (\chi_\Gamma \partial_t^k H) + [\partial_\tau^{e_i}, \operatorname{curl}] v \right) \end{aligned}$$

$$\begin{aligned}
& + \left(\partial_\tau^{e_i} [\partial_\tau^{\beta-e_i}, \sigma] \partial_t^k (\chi_\Gamma E) + [\partial_\tau^{e_i}, \sigma] \partial_\tau^{\beta-e_i} \partial_t^k (\chi_\Gamma E) \right) \\
& + \left(\partial_\tau^{e_i} [\partial_\tau^{\beta-e_i}, \varepsilon_d] \partial_t^{k+1} (\chi_\Gamma E) + [\partial_\tau^{e_i}, \varepsilon_d] \partial_\tau^{\beta-e_i} \partial_t^{k+1} (\chi_\Gamma E) \right) \\
& = -\partial_\tau^\beta \left([\chi_\Gamma, \text{curl}] \partial_t^k H \right) + \partial_\tau^\beta (\chi_\Gamma \tilde{f}_k) - [\partial_\tau^\beta, \text{curl}] (\chi_\Gamma \partial_t^k H) \\
& + [\partial_\tau^\beta, \sigma] \partial_t^k (\chi_\Gamma E) + [\partial_\tau^\beta, \varepsilon_d] \partial_t^{k+1} (\chi_\Gamma E) \\
& = -\partial_\tau^\beta \left([\chi_\Gamma, \text{curl}] \partial_t^k H \right) - [\partial_\tau^\beta, \text{curl}] (\chi_\Gamma \partial_t^k H) + \partial_\tau^\beta (\chi_\Gamma \tilde{f}_k) \\
& + \sum_{0 \leq b < \beta} \binom{\beta}{b} \left(\partial_\tau^{\beta-b} \sigma \partial_\tau^b \partial_t^k (\chi_\Gamma E) + \partial_\tau^{\beta-b} \varepsilon_d \partial_\tau^b \partial_t^{k+1} (\chi_\Gamma E) \right) \\
& = \mathfrak{f}_{\beta,k},
\end{aligned}$$

see Equation (4.96). Likewise, one can show the convergence of

$$\partial_\tau^{e_i} \mathfrak{g}_{\beta-e_i,k,n} + [\partial_\tau^{e_i}, \text{curl}] u_n + (\partial_\tau^{e_i} \mu_d) \partial_t v_n \rightarrow \mathfrak{g}_{\beta,k},$$

in $L^2((s, t), L^2(\mathcal{N}_a^\Gamma))$, which yields the convergence of all interior integrals over $\mathcal{N}_a^\Gamma$ in (4.98) to the interior terms of (4.97). Next, we consider the interface terms. We see, using the continuity of $\partial_\tau^{e_i} : H^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$, see Chapter 2.3 in [12], that

$$\begin{aligned}
& \int_s^t \oint_\Gamma \left(\partial_\tau^{e_i} \varphi_{\beta-e_i,k,n} \cdot \text{tr}_{\tau,\partial v} \partial_\tau^{e_i} v_n - \varphi_{\beta,k} \cdot \text{tr}_{\tau,\partial v} \partial_\tau^\beta \partial_t^k (\chi_\Gamma H) \right) d\varsigma d\tau \quad (4.99) \\
& \lesssim \int_s^t \left(\left\| \partial_\tau^{e_i} \varphi_{\beta-e_i,k,n} - \varphi_{\beta,k} \right\|_{H^{1/2}(\Gamma)} \left\| \text{tr}_{\tau,\partial v} \partial_\tau^{e_i} v_n \right\|_{H^{-1/2}(\Gamma)} \right. \\
& \quad \left. + \left\| \varphi_{\beta,k} \right\|_{H^{1/2}(\Gamma)} \left\| \text{tr}_{\tau,\partial v} \left(\partial_\tau^{e_i} v_n - \partial_\tau^\beta \partial_t^k (\chi_\Gamma H) \right) \right\|_{H^{-1/2}(\Gamma)} \right) d\tau \\
& \lesssim \int_s^t \left(\left\| \partial_\tau^{e_i} \varphi_{\beta-e_i,k,n} - \varphi_{\beta,k} \right\|_{H^{1/2}(\Gamma)} \left\| \text{tr}_{\tau,\partial v} \partial_\tau^{e_i} v_n \right\|_{H^{-1/2}(\Gamma)} \right. \\
& \quad + \left\| \varphi_{\beta,k} \right\|_{H^{1/2}(\Gamma)} \cdot \left(\left\| \text{tr}_{\tau,\partial v} \left(v_n - \partial_\tau^{\beta-e_i} \partial_t^k (\chi_\Gamma H) \right) \right\|_{H^{1/2}(\Gamma)} \right. \\
& \quad \left. + \left\| [\text{tr}_{\tau,\partial v}, \partial_\tau^{e_i}] \left(v_n - \partial_\tau^{\beta-e_i} \partial_t^k (\chi_\Gamma H) \right) \right\|_{H^{-1/2}(\Gamma)} \right) d\tau \\
& \lesssim \int_s^t \left(\left\| \partial_\tau^{e_i} \varphi_{\beta-e_i,k,n} - \varphi_{\beta,k} \right\|_{H^{1/2}(\Gamma)} \left\| \text{tr}_{\tau,\partial v} \partial_\tau^{e_i} v_n \right\|_{H^{-1/2}(\Gamma)} \right. \\
& \quad \left. + \left\| \varphi_{\beta,k} \right\|_{H^{1/2}(\Gamma)} \cdot \left\| v_n - \partial_\tau^{\beta-e_i} \partial_t^k (\chi_\Gamma H) \right\|_{H_\Gamma^1} \right) d\tau
\end{aligned}$$

where the commutator $[\text{tr}_{\tau,\partial v}, \partial_\tau^{e_i}]$ is of order 0. The first term vanishes because $\partial_\tau^{e_i} \varphi_{\beta-e_i,k,n} \rightarrow \varphi_{\beta,k}$ in $L^2((s, t), H^{1/2}(\Gamma))$ and $\text{tr}_{\tau,\partial v} \partial_\tau^{e_i} v_n$ is bounded in $L^2((s, t), H^{-1/2}(\Gamma))$. The second term vanishes since $v_n \rightarrow \partial_\tau^{\beta-e_i} \partial_t^k (\chi_\Gamma H)$ in $\mathcal{G}_\Gamma^1$ and thus the difference in (4.99) converges to 0. The same argument applies to the second interface term, leading to (4.97).

1.3) The trace integrals in (4.97) can be estimated by

$$\begin{aligned}
& \left| \int_s^t \oint_\Gamma \left(\varphi_{\beta,k} \cdot \text{tr}_{\tau,\partial v} \partial_\tau^\beta \partial_t^k (\chi_\Gamma H) - \text{tr}_{\tau,\partial \omega} \partial_\tau^\beta \partial_t^k (\chi_\Gamma E) \cdot \psi_{\beta,k} \right) d\varsigma d\tau \right| \\
& \leq \int_s^t \left(\left\| \varphi_{\beta,k} \right\|_{H^{1/2}(\Gamma)} \left\| \text{tr}_{\tau,\partial v} \partial_\tau^\beta \partial_t^k (\chi_\Gamma H) \right\|_{H^{-1/2}(\Gamma)} \right. \\
& \quad \left. + \left\| \psi_{\beta,k} \right\|_{H^{1/2}(\Gamma)} \left\| \text{tr}_{\tau,\partial \omega} \partial_\tau^\beta \partial_t^k (\chi_\Gamma E) \right\|_{H^{-1/2}(\Gamma)} \right) d\tau \\
& \leq \|\nu\|_{C^4(\mathcal{N}_a^\Gamma)} \theta \int_s^t \left\| \partial_t^k E \right\|_{H_\Gamma^{|\beta|}(\mathcal{N}_a^\Gamma)}^2 d\tau + c_\theta \int_s^t \left\| \partial_t^k H \right\|_{H_\Gamma^{|\beta|}(\mathcal{N}_a^\Gamma)}^2 d\tau
\end{aligned}$$

$$\lesssim \theta \int_s^t z(\tau) d\tau + c_\theta \int_s^t \left\| \partial_t^k H \right\|_{H_\Gamma^{|\beta|}}^2 d\tau,$$

using that $\varphi_{\beta,k}$ and $\psi_{\beta,k}$ only contain lower-order terms. Furthermore, the inhomogeneities satisfy

$$\begin{aligned} \|\mathbf{f}_{\beta,k}\|_{L^2(\Omega)} &\lesssim \left\| \partial_t^k H \right\|_{H_\Gamma^{|\beta|}(\mathcal{N}_a^\Gamma)} + \left\| \partial_t^k E \right\|_{H_\Gamma^{|\beta|-1}(\mathcal{N}_a^\Gamma)} + \left\| \partial_t^{k+1} E \right\|_{H_\Gamma^{|\beta|-1}(\mathcal{N}_a^\Gamma)} + \|\tilde{f}_k\|_{H_\Gamma^{|\beta|}(\mathcal{N}_a^\Gamma)}, \\ \|\mathbf{g}_{\beta,k}\|_{L^2(\Omega)} &\lesssim \left\| \partial_t^k E \right\|_{H_\Gamma^{|\beta|}(\mathcal{N}_a^\Gamma)} + \left\| \partial_t^{k+1} H \right\|_{H_\Gamma^{|\beta|-1}(\mathcal{N}_a^\Gamma)} + \|\tilde{g}_k\|_{H_\Gamma^{|\beta|}(\mathcal{N}_a^\Gamma)}. \end{aligned}$$

Using also Lemma 4.2.1 we thus obtain the inequality

$$\begin{aligned} &\left\| \partial_\tau^\beta \partial_t^k (\chi_\Gamma E, \chi_\Gamma H)(t) \right\|_{L^2(\Omega)}^2 + \sigma_0 \int_s^t \left\| \partial_\tau^\beta \partial_t^k (\chi_\Gamma E) \right\|_{L^2(\omega)}^2 d\tau \\ &\lesssim z(s) + c_\theta \int_s^t z^{3/2}(\tau) d\tau + \theta \int_s^t z(\tau) d\tau \\ &\quad + c_\theta \int_s^t \left(\left\| \partial_t^k E \right\|_{H_\Gamma^{|\beta|-1}}^2 + \left\| \partial_t^{k+1} (E, H) \right\|_{H_\Gamma^{|\beta|-1}}^2 + \left\| \partial_t^k H \right\|_{H_\Gamma^{|\beta|}}^2 \right) d\tau. \end{aligned} \quad (4.100)$$

We emphasize that the integral-term on the left-hand side unfortunately only contains the L^2 -norm on the support of σ , i.e., the subset $\omega \subseteq \Omega$.

1.4) To estimate the full Sobolev norm on ω , we again consider normal and mixed derivatives next. Let $\alpha = (\gamma, \beta) \in \mathbb{N}_0 \times \mathbb{N}_0^2$ with $\gamma \neq 0$ and $|\alpha| \leq 3 - k$. Set $u := \partial_\tau^\beta \partial_t^k (\chi_\Gamma E)$.

As in (4.78), we split the terms with normal derivatives into

$$\left\| \partial_\nu^\gamma \partial_\tau^\beta \partial_t^k (\chi_\Gamma E) \right\|_{L^2(\omega)} \lesssim \left\| \partial_\nu^\gamma \left(\partial_\tau^\beta \partial_t^k (\chi_\Gamma E) \right)_\nu \right\|_{L^2(\omega)} + \left\| \partial_\nu^\gamma \left(\partial_\tau^\beta \partial_t^k (\chi_\Gamma E) \right)_\tau \right\|_{L^2(\omega)}.$$

The tangential part $\partial_\nu^\gamma \partial_\tau^\beta \partial_t^k (\chi_\Gamma E)_\tau$ can be recovered from the curl as in Equation (4.80) by

$$\partial_\nu(u)_\tau = \sum_{i=1,2} (\partial_\nu \tau_i(u)_{\tau_i} + \tau_i \partial_\nu \tau_i \cdot u) + R(\nu^\Gamma) \left(\operatorname{curl} u - \sum_{i=1,2} J(\tau_i) \partial_{\tau_i} u \right).$$

We insert Equation (4.96) and iterate, which leads to the estimate

$$\left\| \partial_\nu^\gamma \left(\partial_\tau^\beta \partial_t^k (\chi_\Gamma E) \right)_\tau \right\|_{L^2(\omega)}^2 \lesssim \left\| \partial_t^k E \right\|_{H_\Gamma^{|\alpha|-1}}^2 + y_\Gamma^{k+1, |\alpha|-1} + z^2 + \sum_{b \in \mathbb{N}_0^2: |b| \leq |\alpha|} \left\| \partial_\tau^b \partial_t^k (\chi_\Gamma E) \right\|_{L^2(\omega)}^2, \quad (4.101)$$

as in (4.81).

1.5) The normal part $\partial_\nu(\cdot)_\nu$ on ω is again recovered through the divergence. By Equation (4.82), we have

$$\begin{aligned} (\hat{\varepsilon}_k)_{\nu\nu} \partial_\nu(u)_\nu &= \operatorname{div}(\hat{\varepsilon}_k u) - \sum_{(\xi, \zeta) \neq (\nu^\Gamma, \nu^\Gamma)} (\hat{\varepsilon}_k)_{\xi\zeta} \partial_\xi u_\zeta - \sum_{\xi, \zeta} \partial_\xi (\hat{\varepsilon}_k)_{\xi\zeta} u_\zeta - \sum_{\xi} \operatorname{div}(\xi) \xi^\top (\hat{\varepsilon}_k u) \\ &= \operatorname{div}(\hat{\varepsilon}_k u) - D(\hat{\varepsilon}_k) u \end{aligned}$$

Similar to (4.83), u satisfies

$$\begin{aligned} \partial_t(\hat{\varepsilon}_k u) &= \operatorname{curl} \partial_\tau^\beta \partial_t^k (\chi_\Gamma H) - \sigma u \\ &\quad - \partial_\tau^\beta (\chi_\Gamma \partial_t f_k) - [\operatorname{curl}, \partial_\tau^\beta] (\chi_\Gamma \partial_t^k H) - \partial_\tau^\beta ([\operatorname{curl}, \chi_\Gamma] \partial_t^k H) \\ &\quad - \sum_{0 \leq b < \beta} \binom{\beta}{b} \left[\partial_\tau^{\beta-b} \sigma \partial_\tau^b \partial_t^k (\chi_\Gamma E) + \partial_t \left(\partial_\tau^{\beta-b} \hat{\varepsilon}_k \partial_\tau^b \partial_t^k (\chi_\Gamma E) \right) \right] \end{aligned}$$

$$=: \operatorname{curl} \partial_\tau^\beta \partial_t^k (\chi_\Gamma H) - \sigma u - \varphi. \quad (4.102)$$

As in (4.84) and (4.85), these formulas lead to the integral equation

$$\begin{aligned} (\hat{\varepsilon}_k)_{\nu\nu} \partial_\nu u_\nu &= \operatorname{div}(\hat{\varepsilon}_k u) - D(\hat{\varepsilon}_k)u \\ &= \operatorname{div}(\hat{\varepsilon}_k u)(s) - D(\hat{\varepsilon}_k)u - \int_s^t (\pi(\tau)(\hat{\varepsilon}_k)_{\nu\nu} \partial_\nu u_\nu + D(\sigma)u + \operatorname{div} \varphi)(\tau) d\tau. \end{aligned}$$

Here we have again $\pi(\tau) = \frac{\sigma_{\nu\nu}}{(\hat{\varepsilon}_k)_{\nu\nu}} \geq \frac{\sigma_0}{\eta} =: \pi_0$ on ω , but $\pi(\tau) = 0$ and $D(\sigma) = 0$ on v . This equation is solved like in (4.86). We can estimate the resulting terms following (4.88), (4.89), (4.90), (4.91), and (4.92), which results in

$$\begin{aligned} & \left\| \partial_\nu^\gamma \left(\partial_\tau^\beta \partial_t^k (\chi_\Gamma E) \right)_\nu(t) \right\|_{L^2(\omega)}^2 + \int_s^t \left\| \partial_\nu^\gamma \left(\partial_\tau^\beta \partial_t^k (\chi_\Gamma E) \right)_\nu \right\|_{L^2(\omega)}^2 d\tau \\ & \lesssim \sum_{\substack{b \in \mathbb{N}_0^2 \\ |b| \leq \gamma}} \left[\left\| \partial_t^k \partial_\tau^{\beta+b} (\chi_\Gamma E)(t) \right\|_{L^2(\omega)}^2 + \int_s^t \left\| \partial_t^k \partial_\tau^{\beta+b} (\chi_\Gamma E) \right\|_{L^2(\omega)}^2 d\tau \right] \\ & \quad + \sum_{m=1}^\gamma \sum_{\substack{b \in \mathbb{N}_0^2 \\ |b| \leq \gamma-m}} \left[\left\| \partial_\nu^m \left(\partial_\tau^{\beta+b} \partial_t^k (\chi_\Gamma E) \right)_\tau(t) \right\|_{L^2(\omega)}^2 + \int_s^t \left\| \partial_\nu^m \left(\partial_\tau^{\beta+b} \partial_t^k (\chi_\Gamma E) \right)_\tau \right\|_{L^2(\omega)}^2 d\tau \right] \\ & \quad + z(s) + z^2(t) + \left\| \partial_t^k H \right\|_{H_\Gamma^{|\alpha|}}^2 + y_\Gamma^{k,|\alpha|-1} + y_\Gamma^{k+1,|\alpha|-1} \\ & \quad + \int_s^t \left(z^2(\tau) + \left\| \partial_t^k H \right\|_{H_\Gamma^{|\alpha|}}^2 + y_\Gamma^{k,|\alpha|-1} + y_\Gamma^{k+1,|\alpha|-1} \right) d\tau. \end{aligned} \quad (4.103)$$

As in (4.94), the inequalities (4.100), (4.101), (4.103), and Lemma 4.5.2 then imply

$$\begin{aligned} & \left\| \partial_t^k (\chi_\Gamma E)(t) \right\|_{H^{|\alpha|}(\omega)}^2 + \int_s^t \left\| \partial_t^k (\chi_\Gamma E) \right\|_{H^{|\alpha|}(\omega)}^2 d\tau \\ & \lesssim z(s) + z^2(t) + \theta \int_s^t z(\tau) d\tau + (1 + c_\theta) \left[\left\| \partial_t^k H(t) \right\|_{H_\Gamma^{|\alpha|}}^2 + \int_s^t \left(z^{3/2}(\tau) + \left\| \partial_t^k H \right\|_{H_\Gamma^{|\alpha|}}^2 \right) d\tau \right. \\ & \quad \left. + \left\| \partial_t^k E(t) \right\|_{H_\Gamma^{|\alpha|-1}}^2 + \int_s^t \left\| \partial_t^k E \right\|_{H_\Gamma^{|\alpha|-1}}^2 d\tau + y_\Gamma^{k+1,|\alpha|-1}(t) + \int_s^t y_\Gamma^{k+1,|\alpha|-1} d\tau \right]. \end{aligned} \quad (4.104)$$

2) It remains to control $\left\| \partial_t^k (\chi_\Gamma E(t)) \right\|_{H^{|\alpha|}(v)} + \int_s^t \left\| \partial_t^k (\chi_\Gamma E) \right\|_{H^{|\alpha|}(v)} d\tau$. Here we use the standard div-curl Lemma A.2.1

$$\begin{aligned} \left\| \partial_t^k (\chi_\Gamma E) \right\|_{H^{|\alpha|}(v)}^2 & \lesssim \left\| \partial_t^k (\chi_\Gamma E) \right\|_{L^2(v)}^2 + \left\| \operatorname{curl} \partial_t^k (\chi_\Gamma E) \right\|_{H^{|\alpha|-1}(v)}^2 \\ & \quad + \left\| \operatorname{div}_{\hat{\varepsilon}_k} \partial_t^k (\chi_\Gamma E) \right\|_{H^{|\alpha|-1}(v)}^2 + \left\| \operatorname{tr}_{t,\partial v} \partial_t^k (\chi_\Gamma E) \right\|_{H^{|\alpha|-1/2}(\Gamma)}^2. \end{aligned} \quad (4.105)$$

Note that $\partial_t^k (\chi_\Gamma E)$ vanishes at $\partial \mathcal{N}_a^\Gamma \cap v$. The Maxwell system (4.1) implies

$$\begin{aligned} \operatorname{curl}(\chi_\Gamma E) &= \nabla \chi_\Gamma \times E - \chi_\Gamma \partial_t(\mu H), \\ \operatorname{div}_\varepsilon(\chi_\Gamma E) &= \nabla \chi_\Gamma \cdot \varepsilon E, \end{aligned} \quad \text{on } v,$$

since $\operatorname{div}(\varepsilon(E_0)E_0)|_v = 0$ and $\sigma|_v = 0$. For $k \in \{1, 2\}$ Equation (4.32) yields

$$\begin{aligned} \operatorname{curl} \partial_t^k (\chi_\Gamma E) &= -\mu_d \partial_t^{k+1} (\chi_\Gamma H) - [\chi_\Gamma, \operatorname{curl}] \partial_t^k E - \chi_\Gamma \tilde{g}_k, \\ \operatorname{div}_{\varepsilon_d} \partial_t^k (\chi_\Gamma E) &= \nabla \chi_\Gamma \cdot \varepsilon_d \partial_t^k E - \chi_\Gamma \operatorname{div} \tilde{f}_{k-1}, \end{aligned} \quad \text{on } v,$$

where the commutator $[\chi_\Gamma, \text{curl}]$ is a multiplication operator. We then obtain

$$\begin{aligned} \left\| \text{curl } \partial_t^k(\chi_\Gamma E) \right\|_{H^{|\alpha|-1}(v)}^2 &\lesssim \left\| \partial_t^{k+1} H \right\|_{H^{|\alpha|-1}(v)}^2 + \left\| \partial_t^k E \right\|_{H^{|\alpha|-1}(v)}^2 + z^2(t), \\ \left\| \text{div}_{\hat{e}_k} \partial_t^k(\chi_\Gamma E) \right\|_{H^{|\alpha|-1}(v)}^2 &\lesssim \left\| \partial_t^k E \right\|_{H^{|\alpha|-1}(v)}^2 + z^2(t). \end{aligned}$$

The interface condition (4.95) yields

$$\left\| \text{tr}_{t,\partial v} \partial_t^k(\chi_\Gamma E) \right\|_{H^{|\alpha|-1/2}(\Gamma)}^2 = \left\| \text{tr}_{t,\partial \omega} \partial_t^k(\chi_\Gamma E) \right\|_{H^{|\alpha|-1/2}(\Gamma)}^2 \lesssim \left\| \partial_t^k(\chi_\Gamma E) \right\|_{H^{|\alpha|}(\omega)}^2.$$

From (4.105) we thus infer

$$\begin{aligned} \left\| \partial_t^k(\chi_\Gamma E) \right\|_{H^{|\alpha|}(v)}^2 &\lesssim e(t) + z^2(t) + \left\| \partial_t^k E \right\|_{H_\Gamma^{|\alpha|-1}}^2 + \left\| \partial_t^{k+1} H \right\|_{H_\Gamma^{|\alpha|-1}}^2 \\ &\quad + \left\| \partial_t^k(\chi_\Gamma E) \right\|_{H^{|\alpha|}(\omega)}^2. \end{aligned} \quad (4.106)$$

Estimates (4.104) and (4.106) result in

$$\begin{aligned} &\left\| \partial_t^k(\chi_\Gamma E)(t) \right\|_{H_\Gamma^{|\alpha|}}^2 + \int_s^t \left\| \partial_t^k(\chi_\Gamma E) \right\|_{H_\Gamma^{|\alpha|}(\Omega)}^2 d\tau \\ &\lesssim e(t) + z^2(t) + \left\| \partial_t^k E(t) \right\|_{H_\Gamma^{|\alpha|-1}}^2 + \left\| \partial_t^{k+1} H(t) \right\|_{H_\Gamma^{|\alpha|-1}}^2 \\ &\quad + \int_s^t \left(e(\tau) + z^2(\tau) + \left\| \partial_t^k E \right\|_{H_\Gamma^{|\alpha|-1}}^2 + \left\| \partial_t^{k+1} H \right\|_{H_\Gamma^{|\alpha|-1}}^2 \right) d\tau \\ &\quad + \left\| \partial_t^k(\chi_\Gamma E)(t) \right\|_{H^{|\alpha|}(\omega)}^2 + \int_s^t \left\| \partial_t^k(\chi_\Gamma E) \right\|_{H^{|\alpha|}(\omega)}^2 d\tau \\ &\lesssim e(t) + z(s) + z^2(t) + \theta \int_s^t z(\tau) d\tau + (1 + c_\theta) \left[\int_s^t \left(e(\tau) + z^{3/2}(\tau) \right) d\tau \right. \\ &\quad + \left\| \partial_t^k(E, H)(t) \right\|_{H_\Gamma^{|\alpha|-1}}^2 + \left\| \partial_t^{k+1}(E, H)(t) \right\|_{H_\Gamma^{|\alpha|-1}}^2 + \left\| \partial_t^k H(t) \right\|_{H_\Gamma^{|\alpha|}}^2 + \int_s^t \left\| \partial_t^k H \right\|_{H_\Gamma^{|\alpha|}}^2 d\tau \\ &\quad \left. + \int_s^t \left(\left\| \partial_t^k(E, H) \right\|_{H_\Gamma^{|\alpha|-1}}^2 + \left\| \partial_t^{k+1}(E, H) \right\|_{H_\Gamma^{|\alpha|-1}}^2 \right) d\tau \right]. \end{aligned}$$

3) Finally, Lemma 4.5.2 yields our result

$$\begin{aligned} &y_\Gamma^{k,|\alpha|} + \int_s^t y_\Gamma^{k,|\alpha|} d\tau \\ &\lesssim e(t) + z(s) + z^2(t) + \theta \int_s^t z(\tau) d\tau + (1 + c_\theta) \left[\int_s^t \left(e(\tau) + z^{3/2}(\tau) \right) d\tau \right. \\ &\quad + \left\| \partial_t^k(E, H)(t) \right\|_{H_\Gamma^{|\alpha|-1}}^2 + \left\| \partial_t^{k+1}(E, H)(t) \right\|_{H_\Gamma^{|\alpha|-1}}^2 \\ &\quad \left. + \int_s^t \left(\left\| \partial_t^k(E, H) \right\|_{H_\Gamma^{|\alpha|-1}}^2 + \left\| \partial_t^{k+1}(E, H) \right\|_{H_\Gamma^{|\alpha|-1}}^2 \right) d\tau \right]. \end{aligned} \quad \square$$

All that remains is to piece together the estimates above.

Proof of Theorem 4.5.1. For $k \in \mathbb{N}_0$, $j \in \mathbb{N}_0$ with $k + j \leq 2$ we have

$$\left\| \partial_t^k(E, H)(t) \right\|_{H_\Gamma^{j+1}}^2 + \int_s^t \left\| \partial_t^k(E, H)(\tau) \right\|_{H_\Gamma^{j+1}}^2 d\tau \lesssim \sum_{\Xi \in \{v, \Gamma, \omega, \partial\Omega\}} y_\Xi^{k,j+1}(t) + \int_s^t y_\Xi^{k,j+1}(\tau) d\tau.$$

By the Lemmas 4.5.3, 4.5.4, 4.5.5 and 4.5.6 the right-hand side can be estimated by

$$\begin{aligned} & e(t) + z(s) + z^2(t) + \theta \int_s^t z(\tau) \, d\tau + (1 + c_\theta) \left[\int_s^t \left(e(\tau) + z^{3/2}(\tau) \right) \, d\tau \right. \\ & \quad + \left\| \partial_t^k(E, H) \right\|_{H_\Gamma^j}^2 + \left\| \partial_t^{k+1}(E, H) \right\|_{H_\Gamma^j}^2 \\ & \quad \left. + \int_s^t \left(\left\| \partial_t^k(E, H) \right\|_{H_\Gamma^j}^2 + \left\| \partial_t^{k+1}(E, H) \right\|_{H_\Gamma^j}^2 \right) \, d\tau \right]. \end{aligned}$$

By iterating over j and k , see Figure 4.3, we thus obtain

$$\begin{aligned} & \left\| \partial_t^k(E, H)(t) \right\|_{H_\Gamma^{j+1}} + \int_s^t \left\| \partial_t^k(E, H)(\tau) \right\|_{H_\Gamma^{j+1}} \, d\tau \\ & \lesssim e(t) + z(s) + z^2(t) + \theta \int_s^t z(\tau) \, d\tau + (1 + c_\theta) \left[\int_s^t \left(e(\tau) + z^{3/2}(\tau) \right) \, d\tau \right]. \end{aligned} \tag{4.107}$$

Hence,

$$\begin{aligned} z(t) + \int_s^t z(\tau) \, d\tau & \lesssim e(t) + \int_s^t e(\tau) \, d\tau \\ & \quad + \sum_{j+k \leq 2} \left(\left\| \partial_t^k(E, H)(t) \right\|_{H_\Gamma^{j+1}} + \int_s^t \left\| \partial_t^k(E, H)(\tau) \right\|_{H_\Gamma^{j+1}} \, d\tau \right) \\ & \lesssim e(t) + z(s) + z^2(t) + \int_s^t \left(e(\tau) + z^{3/2}(\tau) \right) \, d\tau, \end{aligned}$$

where we absorbed the term prefaced by θ in (4.107) by the left-hand side. $\square$

4.6. EXPONENTIAL DECAY

We now combine the observability-type estimate of Theorem 4.4.5, the regularity analysis of Theorem 4.5.1, and the energy inequality in Lemma 4.2.3 to show the main decay result of this chapter.

Theorem 4.6.1 (Exponential decay). *Assume that (H_q) holds. Let the initial values $(E_0, H_0) \in (H_\Gamma^3(\Omega))^6$ be sufficiently small. Then the corresponding solution (E, H) to (4.1)–(4.4) exists for all $t \geq 0$. Moreover, there exist constants $M \geq 1$ and $\omega > 0$ such that*

$$z(t) \leq M e^{-\omega t} z(0), \quad t \geq 0.$$

Proof of Theorem 4.6.1. Theorem 4.4.5, Theorem 4.5.1 and Lemma 4.2.3 imply

$$\begin{aligned} z(t) + \int_s^t z(\tau) \, d\tau & \lesssim z(s) + z^2(t) + e(t) + \int_s^t \left(e(\tau) + z^{3/2}(\tau) \right) \, d\tau \\ & \lesssim e(t) + c_\delta \left(z(s) + z^2(t) + \int_s^t \left(d(\tau) + z^{3/2}(\tau) \right) \, d\tau \right) \\ & \quad + \delta \left(z(t) + \int_s^t z(\tau) \, d\tau \right) \\ & \lesssim c_\delta \left(z(s) + z^2(t) + \int_s^t z^{3/2}(\tau) \, d\tau \right) + \delta \left(z(t) + \int_s^t z(\tau) \, d\tau \right), \end{aligned}$$

for $\delta \in (0, 1]$. We first fix a sufficiently small $\delta > 0$ to absorb the final terms by the left-hand side. Then fix $\delta_z > 0$ such that the superlinear terms can be put on the left as $z^{1/2}(t) \leq \delta_z$ for $t \in [0, T_*)$. So, we arrive at

$$z(t) + \int_s^t z(\tau) d\tau \leq Cz(s), \quad (4.108)$$

for $0 \leq s \leq t < T_*$ and a constant $C \geq 1$. Now, global existence and exponential decay follow as in the proof of Theorem 2.2 in [35] (given after Remark 5.2 in [35]).

Suppose that $T_* < \infty$. It follows $z(T_*) = \delta_z^2$ by (4.18). On the other hand, inequality (4.108) yields $z(t) \leq Cz(s)$. For initial fields with $\|(E_0, H_0)\|_{H_1^3} \leq r$, one can bound $z(0) \leq c_0 r^2$ using its definition. As a result for a fixed, sufficiently small radius $\bar{r} \in (0, r(\delta_z)]$ we infer the contradiction $z(T_*) \leq \frac{1}{2}\delta_z^2$. Therefore, the solution exists and (4.108) is valid for all times. For $T = C(2C - 1)$ we deduce

$$\begin{aligned} z(T) + \frac{T}{C}z(T) &\leq z(T) + \int_0^T z(\tau) d\tau \leq Cz(0), \\ z(T) &\leq \frac{C}{1 + \frac{T}{C}}z(0) = \frac{1}{2}z(0). \end{aligned}$$

The assertion follows by iteration. □

PART II

DECAY CAUSED BY ABSORBING BOUNDARY CONDITIONS

OBSERVABILITY AND NORMAL TRACE ESTIMATES

Whereas the first part of the thesis considered internal damping caused by the conductivity, we now replace the perfectly conducting boundary condition by an absorbing one. We first study the linear non-autonomous system with a generalized inhomogeneous absorbing boundary condition

$$\begin{aligned}
 \partial_t(\alpha(t, x)u(t, x)) &= \operatorname{curl} v(t, x) + \partial_t \varphi(t, x), & t \geq 0, x \in \Omega, \\
 \partial_t(\beta(t, x)v(t, x)) &= -\operatorname{curl} u(t, x) + \partial_t \psi(t, x), & t \geq 0, x \in \Omega, \\
 v(t, x) \times \nu + (\gamma(t, x)(u(t, x) \times \nu)) \times \nu &= h(t, x), & t \geq 0, x \in \partial\Omega, \\
 u(0, x) &= u_0(x), \quad v(0, x) = v_0(x), & x \in \Omega,
 \end{aligned} \tag{5.1}$$

similar to the system in Chapter 3. We will use this system to derive trace estimates, which are required to first show a preliminary observability-type estimate. This is then used to derive exponential stability for both the linear and quasilinear systems, see Chapters 6 and 7.

We consider a bounded, multiply connected domain $\Omega \subset \mathbb{R}^3$ with $C^{2,1}$ -boundary $\partial\Omega$. In order to define our coefficient on the boundary we introduce the space

$$C_\tau^k([0, T] \times \partial\Omega) := \left\{ f \in C^k([0, T] \times \partial\Omega) \mid f(t) \cdot \nu = 0 \text{ on } \partial\Omega \text{ for } t \in [0, T] \right\},$$

of which we will also use a time-independent version $C^k(\partial\Omega)$ later on. We assume that

$$\begin{aligned}
 \alpha, \beta &\in C^1([0, T] \times \overline{\Omega}, \mathbb{R}_{\text{sym}}^{3 \times 3}), \quad \gamma \in C_\tau^1([0, T] \times \partial\Omega, \mathbb{R}_{\text{sym}}^{3 \times 3}), \quad \text{with } \alpha, \beta, \gamma \geq \eta, \\
 \varphi, \psi &\in \mathcal{G}^1 = C^1([0, T], L^2(\Omega)^3) \cap C([0, T], H^1(\Omega)^3) \\
 h &\in C([0, T], H^{1/2}(\partial\Omega)) \quad \text{with } \nu \cdot h = 0,
 \end{aligned} \tag{5.2}$$

and choose initial values $u_0, v_0 \in (L^2(\Omega))^3$ such that

$$\operatorname{div}(\alpha(0)u_0) = \operatorname{div} \varphi(0) \quad \text{and} \quad \operatorname{div}(\beta(0)v_0) = \operatorname{div} \psi(0). \tag{5.3}$$

By Proposition 3.1 of [55] there exists a unique weak solution $(u, v) \in C([0, T], (L^2(\Omega))^6)$ of (5.1) with tangential trace $(\operatorname{tr}_\tau u, \operatorname{tr}_\tau v) \in L^2(\Omega_T, \mathbb{R}^6)$. Observe that

$$\operatorname{div}(\alpha u) = \operatorname{div}(\varphi) \quad \text{and} \quad \operatorname{div}(\beta v) = \operatorname{div}(\psi) \tag{5.4}$$

belong to $C([0, T], L^2(\Omega))$.

As in (7.24), we assume the non-trapping condition

$$\tilde{\alpha} := \alpha + (m \cdot \nabla)\alpha \geq \tilde{\eta}\alpha, \quad \tilde{\beta} := \beta + (m \cdot \nabla)\beta \geq \tilde{\eta}\beta, \tag{5.5}$$

for some $\tilde{\eta} > 0$.

5.1. NORMAL TRACE ESTIMATES

The main goal of this section is to establish new bounds on the normal traces of the electric and magnetic fields and their derivatives, which are also needed to show the observability-like estimate for the linear problem in Lemma 6.1.1 and for the quasilinear problem in Proposition 7.2.3.

THE PSEUDODIFFERENTIAL COLLAR OPERATOR X

To obtain the desired trace regularity, we introduce the pseudodifferential operator X as defined in Definition 2.1 of [18]. In the context of wave equations a similar operator was introduced in Section 7 of [37]. This operator has the advantageous property that it allows us to trade time regularity for spatial regularity on the boundary $\partial\Omega$ of our domain, as shown in Lemma 2.2 of [18]. We restate its definition.

Definition 5.1.1. *Since our domain Ω has in particular a C^2 boundary there exists a tubular neighborhood U of Ω , and a partition $\Omega \cup U = \bigcup_{i=0}^m U_i$ by open $U_i \subseteq \mathbb{R}^3$, such that*

- $U_0 \subseteq \Omega$,
- $U_i \subseteq U$ with $U_i \cap \partial\Omega \neq \emptyset$ for $i \geq 1$, and
- there exists a partition of unity $(\phi_i)_{i=0,\dots,m}$ on Ω such that $\phi_i \in C_c^\infty(U_i)$ and $\sum_i \phi_i(x) = 1$ for $x \in \Omega$.

Furthermore, we choose a neighborhood $V_i \subseteq \mathbb{R}^3$ around the origin and coordinate mappings $\Phi_i: V_i \rightarrow U_i \in C^2(V_i)$ such that

- $\Phi_i(V_i \cap \{(x, y, z) \mid z \geq 0\}) = U_i \cap \Omega$,
- $\Phi_i(V_i \cap \{(x, y, z) \mid z = 0\}) = U_i \cap \partial\Omega$.

We begin constructing X locally on $\mathbb{R} \times U_i$. Let $\varkappa > 0$, $\tilde{\chi} \in C^\infty(\mathbb{R}^3 \setminus \{0\})$ with $0 \leq \tilde{\chi} \leq 1$, and

$$\tilde{\chi}(\xi_0, \eta_1, \eta_2) = \begin{cases} 1, & |\xi_0| \leq \varkappa |(\eta_1, \eta_2)| / 2, \\ 0, & |\xi_0| \geq \varkappa |(\eta_1, \eta_2)|, \end{cases}$$

and $0 \leq \rho \leq 1$ be a cutoff at the origin with

$$\rho(\xi_0, \eta_1, \eta_2) = \begin{cases} 1, & |(\xi_0, \eta_1, \eta_2)|^2 \geq 2, \\ 0, & |(\xi_0, \eta_1, \eta_2)|^2 \leq 1. \end{cases}$$

We set $\chi(\xi_0, \eta_1, \eta_2, \eta_3) = (\rho\tilde{\chi})(\xi_0, \eta_1, \eta_2) \in C^\infty(\mathbb{R}^4)$. For $u \in L^2(\mathbb{R}, H^s(\Omega))$ define $u_i = \phi_i u$ and $v_i(\xi_0, \eta) = u_i(\xi_0, \Phi_i(\eta))$ for $(\xi_0, \eta) \in \mathbb{R} \times V_i$. Take $\sigma_i \in C_0^\infty(V_i)$ such that $\sigma_i|_{\text{supp}(\phi_i \circ \Phi_i)} = 1$. The operator X is then given by

$$X(x, D)u = \sum_{i=1}^m [\sigma_i \chi(D_t, D_{y_1}, D_{y_2}) v_i] \circ \Phi_i^{-1}.$$

Its symbol is of class $C^1 S_{cl}^0$, cf. [62] and [18].

The restriction of X to the boundary $\partial\Omega$ (as a pseudodifferential operator on a manifold) is also denoted by X .

We recall the central properties of this operator as stated and proven in Lemmas 2.2 and 2.3 of [18], setting $\partial\Omega_\infty := \mathbb{R} \times \partial\Omega$. Observe that in the next Lemma the constant K_1 can be chosen arbitrarily small by fixing a small $\varkappa > 0$ in Definition 5.1.1.

Lemma 5.1.2. *Let $u \in L^2(\mathbb{R}, H^s(\Omega))$. Then there exists a constant $K_1 > 0$ such that*

$$\int_{\mathbb{R}} \|X \partial_t u\|_{H^{s-1}(\Omega)}^2 d\tau \leq K_1 \left[\int_{\mathbb{R}} \|Xu\|_{H^s(\Omega)}^2 d\tau + \int_{\mathbb{R}} \|u\|_{H^{s-1}(\Omega)}^2 d\tau \right]$$

for $0 \leq s \leq 1$. The constant $K_1 = K_1(\varkappa)$ belongs to $\mathcal{O}(\varkappa^2)$ for $\varkappa \rightarrow 0$.

Lemma 5.1.3. *Let $u \in L^2(\partial\Omega_\infty)$. Then there exists a constant $K_2 = K_2(\varkappa) > 0$ with*

$$\|(1 - X)u\|_{L^2(\partial\Omega_\infty)}^2 \leq K_2 \left[\|u\|_{H^{-1}(\partial\Omega_\infty)}^2 + \|\partial_t u\|_{H^{-1}(\partial\Omega_\infty)}^2 \right].$$

Using the fact that X is essentially a Fourier-multiplier at least locally and Chapter 4 of [62], we obtain the following regularity result.

Lemma 5.1.4. *Let $-1 \leq s \leq 1$. Then the operators $X: L^2(\mathbb{R}, H^s(\Omega)) \rightarrow L^2(\mathbb{R}, H^s(\Omega))$ and $X: L^2(\mathbb{R}, H^s(\partial\Omega)) \rightarrow L^2(\mathbb{R}, H^s(\partial\Omega))$ are continuous.*

TRACE INEQUALITY

We consider the nonautonomous Maxwell system with inhomogeneities $\xi, \zeta \in L^2([0, T] \times \Omega)$, i.e.,

$$\begin{aligned} \partial_t(\alpha u) &= \operatorname{curl} v + \xi, & t \geq 0, x \in \Omega, \\ \partial_t(\beta v) &= -\operatorname{curl} u + \zeta, & t \geq 0, x \in \Omega, \\ v \times \nu + (\gamma(u \times \nu)) \times \nu &= h, & t \geq 0, x \in \partial\Omega, \\ u(0) &= u_0, \quad v(0) = v_0, & x \in \Omega, \end{aligned} \quad (5.6)$$

with coefficients and data as in (5.2) and (5.3) above.

Then Proposition 3.1 of [55] yields a unique weak solution $(u, v) \in C([0, T], (L^2(\Omega))^6)$ with $(\operatorname{tr}_\tau u, \operatorname{tr}_\tau v) \in L^2([0, T] \times \partial\Omega)^6$.

In order to apply the pseudodifferential operator X from Definition 5.1.1 to the solutions (u, v) defined on $[0, T]$, we have to extend (u, v) by 0 to $L^2(\mathbb{R}, L^2(\Omega))$. We still denote this extension by u respectively v , and proceed analogously with ξ, ζ and h . The following div-curl estimate allows us to bound the image of the pseudodifferential operator in H^1 . Initially we assume that (u, v) belong to $\mathcal{G}^1 = \mathcal{G}^1([0, T])$. This assumption will be removed in Corollary 5.1.9. We note that below the case $\theta = 0$ suffices for the proof of Proposition 7.2.3. Negative θ are needed in Chapter 6 for improved results on the autonomous linear problem.

Lemma 5.1.5. *Under the above assumptions, let $(u, v) \in \mathcal{G}^1$ be solutions to the system (5.6) and $\theta \in [0, \frac{1}{2})$. We then have*

$$\begin{aligned} \int_{\mathbb{R}} \|X(u, v)\|_{H^{1-\theta}(\Omega)}^2 d\tau &\lesssim \int_{\mathbb{R}} \left(\|X \operatorname{div}(\alpha u, \beta v)\|_{H^{-\theta}(\Omega)}^2 + \|X \operatorname{curl}(u, v)\|_{H^{-\theta}(\Omega)}^2 \right. \\ &\quad \left. + \|(u, v)\|_{H^{-\theta}(\Omega)}^2 + \|(u, v)\|_{H^{-1/2-\theta}(\partial\Omega)}^2 + \|h\|_{H^{1/2-\theta}(\partial\Omega)}^2 \right) d\tau. \end{aligned}$$

Proof. By Lemma 5.1.4 the operator $X: L^2(\mathbb{R}, H^s(\Omega)) \rightarrow L^2(\mathbb{R}, H^s(\Omega))$ is continuous for $-1 \leq s \leq 1$. We apply Theorem A.2.9 a) to Xu and Xv , obtaining

$$\begin{aligned} \|X(u, v)\|_{H^{1-\theta}(\Omega)} &\lesssim \|\operatorname{div}_\alpha Xu\|_{H^{-\theta}(\Omega)} + \|\operatorname{div}_\beta Xv\|_{H^{-\theta}(\Omega)} + \|X(u, v)\|_{H^{-\theta}(\Omega)} \\ &\quad + \|\operatorname{curl} X(u, v)\|_{H^{-\theta}(\Omega)} \\ &\quad + \|Xv \times \nu + (\gamma((Xu) \times \nu)) \times \nu\|_{H^{1/2-\theta}(\partial\Omega)}. \end{aligned} \quad (5.7)$$

To estimate the last term, we note that the cross product with ν can be written as a multiplication with a matrix B whose entries and their derivatives are bounded. The same holds for γ and thus for the entries of $B\gamma B$. We write

$$(Xv) \times \nu + (\gamma((Xu) \times \nu)) \times \nu = B(Xv) + B\gamma B(Xu).$$

The commutators $[X, B], [X, B\gamma B]: L^2(\mathbb{R}, H^{-s}(\partial\Omega)) \rightarrow L^2(\mathbb{R}, H^{1-s}(\partial\Omega))$ are continuous for $s \in [0, 1]$ (see Proposition 4.1.E of [62]). We then compute

$$\begin{aligned} &\int_{\mathbb{R}} \|Xv \times \nu + (\gamma((Xu) \times \nu)) \times \nu\|_{H^{1/2-\theta}(\partial\Omega)}^2 d\tau \\ &= \int_{\mathbb{R}} \|X(v \times \nu + \gamma(u \times \nu) \times \nu) + [X, B]v + [X, B\gamma B]u\|_{H^{1/2-\theta}(\partial\Omega)}^2 d\tau \\ &\lesssim \int_{\mathbb{R}} \left(\|h\|_{H^{1/2-\theta}(\partial\Omega)}^2 + \|(u, v)\|_{H^{-1/2-\theta}(\partial\Omega)}^2 \right) d\tau, \end{aligned}$$

using the boundary condition $v \times \nu + \gamma(u \times \nu) \times \nu = w$ of (5.6) and the continuity of X on $\partial\Omega$. Since curl , $\operatorname{div}_\alpha$ and div_β have $C^1 S_{cl}^1$ -symbols, the commutators with X are bounded on $H^{-\theta}$ which can be shown as in Proposition 4.1.E of [62]. The assertion now easily follows. $\square$

Exploiting the smallness of $K_1(\varkappa)$ in Lemma 5.1.2, we can simplify the estimate above.

Corollary 5.1.6. *In the situation of Lemma 5.1.5 there is a number $\varkappa_0 > 0$ so that for all $\varkappa \in (0, \varkappa_0]$ in Definition 5.1.1 we have*

$$\begin{aligned} \int_{\mathbb{R}} \|X(u, v)\|_{H^{1-\theta}(\Omega)}^2 d\tau &\lesssim \int_{\mathbb{R}} \left(\|(u, v)\|_{H^{-\theta}(\operatorname{div}_{\alpha}) \times H^{-\theta}(\operatorname{div}_{\beta})}^2 \right. \\ &\quad \left. + \|(u, v)\|_{H^{-1/2-\theta}(\partial\Omega)}^2 + \|X(\xi, \zeta)\|_{H^{-\theta}(\Omega)}^2 + \|h\|_{H^{1/2-\theta}(\partial\Omega)}^2 \right) d\tau. \end{aligned}$$

Proof. Via (5.6) we rewrite the curl-terms in Lemma 5.1.5 as

$$\|X \operatorname{curl} u\|_{H^{-\theta}}^2 + \|X \operatorname{curl} v\|_{H^{-\theta}}^2 = \|X \partial_t(\alpha u) - X \xi\|_{H^{-\theta}}^2 + \|X \partial_t(\beta v) - X \zeta\|_{H^{-\theta}}^2.$$

By Proposition 4.1.E in [62] the commutator $[X, \alpha]: L^2(\mathbb{R}, H^{-\theta}(\Omega)) \rightarrow L^2(\mathbb{R}, H^{1-\theta}(\Omega))$ is bounded. This fact and Lemmas 5.1.2, 5.1.4, and 5.1.5 lead to

$$\begin{aligned} \int_{\mathbb{R}} \|X(u, v)\|_{H^{1-\theta}}^2 d\tau &\lesssim \int_{\mathbb{R}} \left(\|\operatorname{div}(\alpha u, \beta v)\|_{H^{-\theta}}^2 + K_1(\varkappa) \|X(\alpha u, \beta v)\|_{H^{1-\theta}}^2 \right. \\ &\quad \left. + \|X(\xi, \zeta)\|_{H^{-\theta}}^2 + \|(u, v)\|_{H^{-\theta}}^2 + \|(u, v)\|_{H^{-1/2-\theta}(\partial\Omega)}^2 + \|h\|_{H^{1/2-\theta}(\partial\Omega)}^2 \right) d\tau \\ &\lesssim \int_{\mathbb{R}} \left(\|\operatorname{div}(\alpha u, \beta v)\|_{H^{-\theta}}^2 + K_1(\varkappa) \|X(u, v)\|_{H^{1-\theta}}^2 + \|(u, v)\|_{H^{-\theta}}^2 \right. \\ &\quad \left. + \|(u, v)\|_{H^{-1/2-\theta}(\partial\Omega)}^2 + \|X(\xi, \zeta)\|_{H^{-\theta}}^2 + \|h\|_{H^{1/2-\theta}(\partial\Omega)}^2 \right) d\tau. \end{aligned}$$

For small $\varkappa$ and hence $K_1(\varkappa)$, we derive

$$\begin{aligned} \int_{\mathbb{R}} \|X(u, v)\|_{H^{1-\theta}}^2 d\tau &\lesssim \int_{\mathbb{R}} \left(\|u\|_{H^{-\theta}(\operatorname{div}_{\alpha}, \Omega)}^2 + \|v\|_{H^{-\theta}(\operatorname{div}_{\beta}, \Omega)}^2 + \|(u, v)\|_{H^{-1/2-\theta}(\partial\Omega)}^2 \right. \\ &\quad \left. + \|X(\xi, \zeta)\|_{H^{-\theta}}^2 + \|h\|_{H^{1/2-\theta}(\partial\Omega)}^2 \right) d\tau. \end{aligned} \quad \square$$

From now on $\varkappa = \varkappa_0$ is fixed in the definition of X . We will use this inequality in a moment to establish an estimate for the normal trace of αu and βv . For this we will have to split the trace term $\|(u, v)\|_{L^2(\partial\Omega)}^2 \lesssim \oint_{\partial\Omega} (\alpha u \cdot u + \beta v \cdot v) d\varsigma$ related to the right-hand side of Corollary 5.1.6 into a normal and tangential component.

In the following, $u_{\nu} = (u \cdot \nu)\nu$ denotes the part of u in normal direction, whereas $u_{\tau} = \nu \times (u \times \nu)$ is the tangential one, see (4.79). We can thus decompose u as $u = u_{\nu} + u_{\tau}$.

Lemma 5.1.7. *In the situation of Lemma 5.1.5 above we have*

$$\int_s^t \oint_{\partial\Omega} (\alpha u \cdot u + v \cdot \beta v) d\varsigma d\tau \lesssim \int_s^t \oint_{\partial\Omega} (|\nu \cdot \alpha u|^2 + |\nu \cdot \beta v|^2 + |u_{\tau}|^2 + |v_{\tau}|^2) d\varsigma d\tau.$$

Proof. We only treat $\operatorname{tr}(u \cdot \alpha u)$. The estimate for v is shown analogously. Since the matrix operator α is assumed to be bounded and uniformly positive definite, we can compute

$$\begin{aligned} \int_s^t \oint_{\partial\Omega} \alpha u \cdot u d\varsigma d\tau &= \int_s^t \oint_{\partial\Omega} (u_{\nu} \cdot \alpha u_{\nu} + 2u_{\tau} \alpha u_{\nu} + u_{\tau} \cdot \alpha u_{\tau}) d\varsigma d\tau \\ &\lesssim \int_s^t \oint_{\partial\Omega} \left(\sqrt{\delta} |u \cdot \nu| \frac{1}{\sqrt{\delta}} |\nu \cdot \alpha u_{\nu}| + |u_{\tau}|^2 \right) d\varsigma d\tau \\ &\lesssim \int_s^t \oint_{\partial\Omega} \left(\delta |u \cdot \nu|^2 + \frac{1}{\delta} |\nu \cdot \alpha u_{\nu}|^2 + |u_{\tau}|^2 \right) d\varsigma d\tau \\ &\lesssim \int_s^t \oint_{\partial\Omega} \left(\delta |u|^2 + \frac{1}{\delta} (|\nu \cdot \alpha u|^2 + |\nu \cdot \alpha u_{\tau}|^2) + |u_{\tau}|^2 \right) d\varsigma d\tau \\ &\lesssim \int_s^t \oint_{\partial\Omega} \left(\delta |u|^2 + \frac{1}{\delta} (|\nu \cdot \alpha u|^2 + |u_{\tau}|^2) + |u_{\tau}|^2 \right) d\varsigma d\tau. \end{aligned}$$

Fixing a sufficiently small $\delta > 0$ we thus obtain the assertion. $\square$

The main difficulty in the proof of Proposition 7.2.3 is that one has to bound trace terms of the electric and magnetic field by the tangential trace of $\partial_t^k E$. The boundary condition directly connects this trace to the tangential trace of the magnetic field. We thus inspect the normal boundary terms in detail, where we gain small constants in front of the divergence terms, but obtain a (large) error in $H^{-1}(\partial\Omega)$. In the next lemma, one can remove the operator X on the right-hand side by continuity. We keep it in the statement in view of Corollary 5.1.9.

Lemma 5.1.8. *Let $(u, v) \in \mathcal{G}^1$ be a solution to the system (5.6) and $\theta \in [0, \frac{1}{2})$. Then there exists a constant $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0]$ and a constant c_δ depending on δ we get*

$$\begin{aligned} \int_s^t \left(\|\nu \cdot \alpha u\|_{L^2(\partial\Omega)}^2 + \|\nu \cdot \beta v\|_{L^2(\partial\Omega)}^2 \right) d\tau &\lesssim \int_s^t \left(\delta \|u\|_{H^{-\theta}(\operatorname{div}_\alpha)}^2 + \delta \|v\|_{H^{-\theta}(\operatorname{div}_\beta)}^2 \right. \\ &\quad \left. + c_\delta \|\nu \cdot (\alpha u, \beta v)\|_{H^{-1}(\partial\Omega)}^2 + \|\nu \times u\|_{L^2(\partial\Omega)}^2 + \|h\|_{H^{1/2-\theta}(\partial\Omega)}^2 \right) d\tau \\ &\quad + \int_{\mathbb{R}} \delta \|X(\xi, \zeta)\|_{H^{-\theta}(\Omega)}^2 d\tau + \|(\nu \cdot \xi, \nu \cdot \zeta)\|_{H^{-1}(\partial\Omega_\infty)}^2 \end{aligned} \quad (5.8)$$

for $0 \leq s \leq t \leq T$.

Proof. We show the inequality only for $s = 0$ and $t = T$ since general times can be treated analogously.

As $\alpha u \in \mathcal{G}^1([0, T])$, the normal trace belongs to $C([0, T], L^2(\partial\Omega))$. Extending αu by 0 to $\mathbb{R}$, we obtain

$$\begin{aligned} \int_0^T \|\nu \cdot \alpha u\|_{L^2(\partial\Omega)}^2 d\tau &= \|\nu \cdot \alpha u\|_{L^2(\partial\Omega_\infty)}^2 \\ &\lesssim \|X(\nu \cdot \alpha u)\|_{L^2(\partial\Omega_\infty)}^2 + \|(1 - X)(\nu \cdot \alpha u)\|_{L^2(\partial\Omega_\infty)}^2. \end{aligned} \quad (5.9)$$

We first estimate the second term on the right-hand side. Lemma 5.1.3 yields

$$\|(1 - X)(\nu \cdot \alpha u)\|_{L^2(\partial\Omega_\infty)}^2 \lesssim \|\nu \cdot \alpha u\|_{H^{-1}(\partial\Omega_\infty)}^2 + \|\partial_t(\nu \cdot \alpha u)\|_{H^{-1}(\partial\Omega_\infty)}^2.$$

We have $\nu \cdot \partial_t(\alpha u) = \nu \cdot (\operatorname{curl} v) + \nu \cdot \xi$ by (5.6) and

$$\|\nu \cdot (\operatorname{curl} v)\|_{H^{-1}(\partial\Omega_\infty)} \lesssim \|\nu \times v\|_{L^2(\partial\Omega_\infty)} < \infty$$

by Lemma A.1.6. Hence, the difference $\nu \cdot \xi$ belongs to $H^{-1}(\partial\Omega_\infty)$, and we can infer

$$\begin{aligned} \|(1 - X)\nu \cdot (\alpha u)\|_{L^2(\partial\Omega_\infty)}^2 &\lesssim \int_0^T \left(\|\nu \cdot \alpha u\|_{H^{-1}(\partial\Omega)}^2 + \|\nu \times v\|_{L^2(\partial\Omega)}^2 \right) d\tau + \|\nu \cdot \xi\|_{H^{-1}(\partial\Omega_\infty)}^2 \end{aligned} \quad (5.10)$$

By interpolation (see Theorem 2.7.7 in [38] for smooth $\partial\Omega$), the first term of the right-hand side of (5.9) can be bounded by

$$\begin{aligned} \|X(\nu \cdot \alpha u)\|_{L^2(\partial\Omega_\infty)}^2 &\lesssim \int_{\mathbb{R}} \left(\hat{\delta} \|X(\nu \cdot \alpha u)\|_{H^{1/2-\theta}(\partial\Omega)}^{2/(3-2\theta)} \frac{1}{\hat{\delta}} \|X(\nu \cdot \alpha u)\|_{H^{-1}(\partial\Omega)}^{(1-2\theta)/(3-2\theta)} \right)^2 d\tau \\ &\lesssim \int_{\mathbb{R}} \left(\hat{\delta}^{3-2\theta} \|X\nu \cdot \alpha u\|_{H^{1/2-\theta}(\partial\Omega)}^2 \right. \\ &\quad \left. + \hat{\delta}^{-2(3-2\theta)/(1-2\theta)} \|X\nu \cdot \alpha u\|_{H^{-1}(\partial\Omega)}^2 \right) d\tau \end{aligned}$$

for an arbitrary $\hat{\delta} := \hat{\delta}^{3-2\theta} \in (0, 1]$. Since $\partial\Omega \in C^5$ and $\alpha \in C^1$, the commutators $[X, \nu]$ and $[X, \alpha]$ are bounded from $L^2(\mathbb{R}, H^{-1/2-\theta}(\partial\Omega))$ to $L^2(\mathbb{R}, H^{1/2-\theta}(\partial\Omega))$, cf. Proposition 4.1.E in [62]. It follows that

$$J := \int_{\mathbb{R}} \left(\delta \|X\nu \cdot \alpha u\|_{H^{1/2-\theta}(\partial\Omega)}^2 + c_\delta \|X\nu \cdot \alpha u\|_{H^{-1}(\partial\Omega)}^2 \right) d\tau$$

$$\begin{aligned}
&\lesssim \int_{\mathbb{R}} \left(\delta (\|Xu\|_{H^{1/2-\theta}(\partial\Omega)}^2 + \|u\|_{H^{-1/2-\theta}(\partial\Omega)}^2) + c_\delta \|\nu \cdot \alpha u\|_{H^{-1}(\partial\Omega)}^2 \right) d\tau \\
&\lesssim \int_{\mathbb{R}} \left(\delta (\|Xu\|_{H^{1-\theta}(\Omega)}^2 + \|u\|_{L^2(\partial\Omega)}^2) + c_\delta \|\nu \cdot \alpha u\|_{H^{-1}(\partial\Omega)}^2 \right) d\tau. \\
&\lesssim \int_{\mathbb{R}} \left(\delta (\|u\|_{H^{-\theta}(\operatorname{div}_\alpha)}^2 + \|v\|_{H^{-\theta}(\operatorname{div}_\beta)}^2 + \|(u, v)\|_{L^2(\partial\Omega)}^2 + \|X(\xi, \zeta)\|_{H^{-\theta}(\Omega)}^2 \right. \\
&\quad \left. + \|h\|_{H^{1/2-\theta}(\partial\Omega)}^2) + c_\delta \|\nu \cdot \alpha u\|_{H^{-1}(\partial\Omega)}^2 \right) d\tau,
\end{aligned}$$

where we also used Corollary 5.1.6. Lemma 5.1.7 now allows us to split the trace term into normal and tangential parts, obtaining

$$\begin{aligned}
J &\lesssim \delta \int_0^T \|\nu \cdot (\alpha u, \beta v)\|_{L^2(\partial\Omega)}^2 d\tau + \delta \int_{\mathbb{R}} \|X(\xi, \zeta)\|_{H^{-\theta}(\Omega)}^2 d\tau + \int_0^T \left(\delta (\|u\|_{H^{-\theta}(\operatorname{div}_\alpha)}^2 \right. \\
&\quad \left. + \|v\|_{H^{-\theta}(\operatorname{div}_\beta)}^2 + \|\nu \times (u, v)\|_{L^2(\partial\Omega)}^2 + \|h\|_{H^{1/2-\theta}(\partial\Omega)}^2) + c_\delta \|\nu \cdot \alpha u\|_{H^{-1}(\partial\Omega)}^2 \right) d\tau.
\end{aligned}$$

We have the same estimate for $\nu \cdot \beta v$ in (5.9). By choosing $\delta > 0$ sufficiently small, we absorb the normal trace terms by the left-hand side of (5.9). We infer

$$\begin{aligned}
&\int_0^T \left(\|\nu \cdot \alpha u\|_{L^2(\partial\Omega)}^2 + \|\nu \cdot \beta v\|_{L^2(\partial\Omega)}^2 \right) d\tau \lesssim \int_0^T \left(\delta \|u\|_{H^{-\theta}(\operatorname{div}_\alpha)}^2 + \delta \|v\|_{H^{-\theta}(\operatorname{div}_\beta)}^2 \right. \\
&\quad \left. + c_\delta \|\nu \cdot (\alpha u, \beta v)\|_{H^{-1}(\partial\Omega)}^2 + \|\nu \times (u, v)\|_{L^2(\partial\Omega)}^2 + \delta \|h\|_{H^{1/2-\theta}(\partial\Omega)}^2 \right) d\tau \\
&\quad + \int_{\mathbb{R}} \delta \|X(\xi, \zeta)\|_{H^{-\theta}(\Omega)}^2 d\tau + \|(\nu \cdot \xi, \nu \cdot \zeta)\|_{H^{-1}(\partial\Omega_\infty)}^2,
\end{aligned}$$

using also (5.10). Finally, we note that the boundary condition in (5.6) yields

$$\int_0^T \|\nu \times v\|_{L^2(\partial\Omega)}^2 d\tau \lesssim \int_0^T \left(\|\nu \times u\|_{L^2(\partial\Omega)}^2 + \|h\|_{L^2(\partial\Omega)}^2 \right) d\tau, \quad (5.11)$$

from which the claim follows. $\square$

We now rewrite the above result for the problem (5.1) with $\theta = 0$ where $\xi = \partial_t \varphi$ and $\zeta = \partial_t \psi$. Here we can apply Lemma 5.1.2 to the inhomogeneities in (5.8).

Corollary 5.1.9. *Let $\alpha, \beta, \varphi, \psi, h, u_0$, and v_0 satisfy (5.2) and (5.3). Then the solution $(u, v) \in C([0, T], L^2(\Omega))$ to (5.1) possesses a trace in $L^2([0, T], L^2(\partial\Omega))$ satisfying*

$$\begin{aligned}
&\int_0^T \left(\|\nu \cdot \alpha u\|_{L^2(\partial\Omega)}^2 + \|\nu \cdot \beta v\|_{L^2(\partial\Omega)}^2 \right) d\tau \\
&\lesssim \int_0^T \left(\delta \|u\|_{H(\operatorname{div}_\alpha)}^2 + \delta \|v\|_{H(\operatorname{div}_\beta)}^2 + \|(\varphi, \psi)\|_{H^1(\Omega)}^2 \right. \\
&\quad \left. + c_\delta \|\nu \cdot (\alpha u, \beta v)\|_{H^{-1}(\partial\Omega)}^2 + \|\nu \times u\|_{L^2(\partial\Omega)}^2 + \|h\|_{H^{1/2}(\partial\Omega)}^2 \right) d\tau \\
&\quad + \|(\nu \cdot \partial_t \varphi, \nu \cdot \partial_t \psi)\|_{H^{-1}(\partial\Omega_\infty)}^2.
\end{aligned} \quad (5.12)$$

Proof. For $(u, v) \in \mathcal{G}^1$ the assertion follows from Lemma 5.1.8 with $\xi = \partial_t \varphi$ and $\zeta = \partial_t \psi$, where we apply Lemma 5.1.2 to the term with X on the right-hand side. It remains to remove the assumption $(u, v) \in \mathcal{G}^1$. This can be done as in the proof of Lemma 3.4 in [51], see also Chapter 3. One approximates the initial data in $H(\operatorname{div}_{\alpha(0)}) \times H(\operatorname{div}_{\beta(0)})$ by $(u_{0,n}, v_{0,n}) \in H^1(\Omega)$ satisfying the compatibility condition $v_{0,n} \times \nu + (\gamma(u_{0,n} \times \nu))\nu = 0$, (φ, ψ) in $\mathcal{G}^1$ by $(\varphi_n, \psi_n) \in \mathcal{G}^2$, and h in $L^2([0, T], H^{1/2}(\partial\Omega))$ by $h_n \in H^1([0, T] \times \partial\Omega)$ with $h \cdot \nu = 0$ and $h_n(0) = 0$. By Theorem 1.3 and Remark 2.1 in [11] there are solutions

$(u_n, v_n) \in \mathcal{G}^1$. According to Proposition 1.1 in [11] these solutions converge to (u, v) in $C([0, T], L^2(\Omega))$ and $(\text{tr}_t u_n, \text{tr}_t v_n)$ to $(\text{tr}_t u, \text{tr}_t v)$ in $L^2([0, T] \times \Omega)$. Moreover, we have the limits $\text{div}(\alpha u_n) = \text{div}(\varphi_n) \rightarrow \text{div}(\alpha u)$ and $\text{div}(\beta v_n) = \text{div}(\psi_n) \rightarrow \text{div}(\beta v)$ in $C([0, T], L^2(\Omega))$. As a result the traces $\text{tr}_n(\alpha u_n)$ and $\text{tr}_n(\beta v_n)$ converge in $C([0, T], H^{-1/2}(\partial\Omega))$. The assertion for (u, v) now follows from the claim for (u_n, v_n) . $\square$

Remark 5.1.10. In Corollary 5.1.9 we can insert $\text{div}(\alpha u) = \text{div}(\varphi)$ and $\text{div}(\beta v) = \text{div}(\psi)$ and estimate

$$\|(\nu \cdot \partial_t \varphi, \nu \cdot \partial_t \psi)\|_{H^{-1}(\partial\Omega_\infty)}^2 \lesssim \|(\nu \cdot \varphi, \nu \cdot \psi)\|_{L^2([0, T], L^2(\partial\Omega))}^2 \lesssim \|(\varphi, \psi)\|_{L^2([0, T], H^1(\Omega))}^2.$$

Moreover, we can bound $c_\delta \|\nu \cdot (\alpha u, \beta v)\|_{H^{-1}(\partial\Omega)}^2$ by

$$c_\delta (\|u\|_{H(\text{div}_\alpha)}^2 + \|v\|_{H(\text{div}_\beta)}^2).$$

The estimate (5.12) has time-independent constants and can therefore be used below to show Proposition 7.2.3 and thus Theorem 7.2.1. Concluding this section, we also prove a variant only having norms of data on the right-hand side but with time dependent constants. In this way we give a different proof of Theorem 1.2 in [11]. We will revisit this trace estimate in Chapter 6 in the linear, autonomous and homogeneous case and prove a version with time-independent constants there.

In view of Remark 5.1.10 we have to remove the L^2 -norm of u and v and the tangential trace term on the right-hand side of (5.8). We control these terms by means of the energy estimate for (5.1). It was shown in Lemma 3.2 of [51] for $h = 0$. An obvious modification of this proof gives the equality

$$\begin{aligned} & \|\alpha(t)^{1/2} u(t)\|_{L^2}^2 + \|\beta(t)^{1/2} v(t)\|_{L^2}^2 + 2 \int_0^t \|\gamma^{1/2} \text{tr}_t u(\tau)\|_{L^2(\partial\Omega)}^2 d\tau \\ &= \|\alpha(0)^{1/2} u(0)\|_{L^2}^2 + \|\beta(0)^{1/2} v(0)\|_{L^2}^2 + 2 \int_0^t \int_\Omega (u \cdot \partial_t \varphi + v \cdot \partial_t \psi) dx d\tau \\ & \quad - 2 \int_0^t \oint_{\partial\Omega} \text{tr}_t u \cdot h d\varsigma d\tau - \int_0^t \int_\Omega (u \cdot (\partial_t \alpha) u + v \cdot (\partial_t \beta) v) dx d\tau \end{aligned} \quad (5.13)$$

for $0 \leq t \leq T$. To remove the trace term on the right-hand side, we estimate

$$\left| \int_0^t \oint_{\partial\Omega} \text{tr}_t u \cdot h d\varsigma d\tau \right| \leq \delta \int_0^t \|\text{tr}_t u\|_{L^2(\partial\Omega)}^2 d\tau + c_\delta \int_0^t \|h\|_{L^2(\partial\Omega)}^2 d\tau. \quad (5.14)$$

Since $\gamma \geq \eta$, setting $\delta = \eta$ we arrive at

$$\begin{aligned} \|(u(t), v(t))\|_{L^2}^2 + \int_0^t \|\text{tr}_t u(\tau)\|_{L^2(\partial\Omega)}^2 d\tau &\leq c \|(u(0), v(0))\|_{L^2}^2 \\ & \quad + \bar{c} \int_0^t \|(u, v)\|_{L^2}^2 d\tau + c \int_0^t (\|(\partial_t \varphi, \partial_t \psi)\|_{L^2}^2 + \|h\|_{L^2(\partial\Omega)}^2) d\tau \end{aligned} \quad (5.15)$$

with $\bar{c} := \tilde{c} + \max\{\|\partial_t \alpha\|_\infty, \|\partial_t \beta\|_\infty\}$ where $\tilde{c} = 0$ if $\partial_t \varphi = \partial_t \psi = 0$ and $\tilde{c} = 1$ otherwise. Gronwall's inequality now implies

$$\begin{aligned} \|(u(t), v(t))\|_{L^2}^2 + \int_0^t \|\text{tr}_t u(\tau)\|_{L^2(\partial\Omega)}^2 d\tau \\ \lesssim \left(\|(u(0), v(0))\|_{L^2}^2 + \int_0^t (\|(\partial_t \varphi, \partial_t \psi)\|_{L^2}^2 + \|h\|_{L^2(\partial\Omega)}^2) d\tau \right) e^{\bar{c}t} \end{aligned} \quad (5.16)$$

Using also $\text{div}(\alpha u) = \text{div}(\varphi)$ and $\text{div}(\beta v) = \text{div}(\psi)$, from Corollary 5.1.9 and Remark 5.1.10 we derive a trace estimate for the inhomogeneous, linear Maxwell system (5.1). It can directly be applied to (7.1)–(7.5).

Proposition 5.1.11. *Assume the conditions (5.2), (5.3) to hold, and let $(u, v) \in C([0, T], L^2(\Omega))$ solve (5.1). We obtain*

$$\begin{aligned} \int_0^t \|\operatorname{tr}(u, v)\|_{L^2(\partial\Omega)}^2 d\tau &\lesssim_T \|(u_0, v_0)\|_{L^2(\Omega)}^2 \\ &\quad + \int_0^t \left(\|(\partial_t \varphi, \partial_t \psi)\|_{L^2(\Omega)}^2 + \|(\varphi, \psi)\|_{H^1(\Omega)}^2 + \|h\|_{H^{1/2}(\partial\Omega)}^2 \right) d\tau \end{aligned}$$

for $0 \leq t \leq T$.

As in Theorem 1.2 of [11] the constant depends on T . Thus, we will only use Corollary 5.1.9 in the following.

5.2. OBSERVABILITY-TYPE ESTIMATE

We use the normal trace estimates to improve the observability-type estimate of Lemma 3.4 in [51]. There, these terms were treated by assuming a starshaped-condition on Ω . The proof of Lemma 3.4 in [51] yields the following result.

Lemma 5.2.1. *Let (5.2), (5.3), and (5.5) hold. Then the weak solutions $(u, v) \in C([0, T], L^2(\Omega, \mathbb{R}^6))$ of system (5.1) fulfill*

$$\begin{aligned} &\int_s^t \int_{\Omega} (\alpha u \cdot u + \beta v \cdot v) d\varsigma d\tau \\ &\lesssim \|(u(s), v(s))\|_{L^2}^2 + \|(u(t), v(t))\|_{L^2}^2 \\ &\quad + \int_s^t \int_{\Omega} (|\partial_t \varphi| |v| + |\partial_t \psi| |u| + |\operatorname{div} \varphi| |u| + |\operatorname{div} \psi| |v|) dx d\tau \\ &\quad + \int_s^t \oint_{\partial\Omega} \operatorname{tr}(\alpha u \cdot u + \beta v \cdot v) d\varsigma d\tau + \int_s^t \oint_{\partial\Omega} |h \cdot \operatorname{tr}_{\tau} v| d\varsigma d\tau \end{aligned} \tag{5.17}$$

for $0 \leq s \leq t \leq T$. Note that the last term is bounded by

$$c \int_s^t \oint_{\partial\Omega} \operatorname{tr}(\beta v \cdot v) d\varsigma d\tau + c \int_s^t \|h\|_{L^2(\partial\Omega)}^2 d\tau$$

and that the trace terms are finite by Proposition 5.1.11.

In [51] it was assumed that Ω is strictly starshaped giving control on the full trace of (u, v) which now appears on the right-hand side of (5.17). Moreover, the boundary inhomogeneity h can simply be included in the calculations of [51]. For completeness, we include the relevant parts of the proof.

Proof. First, it can be shown by an approximation argument as in the proof of Lemma 3.4 [51] that it suffices to consider solutions $(u, v) \in \mathcal{G}^1([0, T])$, see also Chapter 3.

Next, (5.1) implies

$$\begin{aligned} J &:= \int_{\Omega} \partial_t ((m \times \alpha u) \cdot \beta v) dx \\ &= \int_{\Omega} \left((m \times (\operatorname{curl} v + \partial_t \varphi)) \cdot \beta v + (m \times \alpha u) \cdot (-\operatorname{curl} u + \partial_t \psi) \right) dx \\ &= - \int_{\Omega} \operatorname{curl} v \cdot (m \times \beta v) dx - \int_{\Omega} \operatorname{curl} u \cdot (m \times \alpha u) dx \\ &\quad + \int_{\Omega} ((m \times \partial_t \varphi) \cdot \beta v + (m \times \alpha u) \cdot \partial_t \psi) dx. \end{aligned} \tag{5.18}$$

Integrating by parts yields

$$\int_{\Omega} \operatorname{curl} v \cdot (m \times \beta v) \, dx = \int_{\Omega} v \cdot \operatorname{curl}(m \times \beta v) \, dx + \oint_{\partial\Omega} (\nu \times v) \cdot (m \times \beta v) \, d\varsigma.$$

Recalling from (4.58)

$$\operatorname{curl}(a \times b) = (b \cdot \nabla)a - (a \cdot \nabla)b + a \operatorname{div} b - b \operatorname{div} a,$$

we compute

$$\begin{aligned} \operatorname{curl}(m \times \beta v) &= (\beta v \cdot \nabla)m - (m \cdot \nabla)(\beta v) + \operatorname{div}(\beta v)m - \operatorname{div}(m)\beta v \\ &= \operatorname{div}(\beta v)m - 2\beta v - (m \cdot \nabla)(\beta v) \\ &= \operatorname{div} \psi m - 2\beta v - (m \cdot \nabla)(\beta v), \end{aligned}$$

by means of (5.3). Notice that

$$-(m \cdot \nabla)(\beta v) = -\frac{1}{2}(m \cdot \nabla)(\beta v \cdot v) - \frac{1}{2}((m \cdot \nabla)\beta)v \cdot v.$$

The divergence theorem then implies

$$\begin{aligned} \int_s^t \int_{\Omega} \operatorname{curl} v \cdot (m \times \beta v) \, dx \, d\tau &= -\frac{1}{2} \int_s^t \int_{\Omega} |\tilde{\beta}^{1/2} v|^2 \, dx \, d\tau - \frac{1}{2} \int_s^t \oint_{\partial\Omega} (\nu \cdot m)(\beta v \cdot v) \, d\varsigma \, d\tau \\ &\quad + \int_s^t \int_{\Omega} m \cdot v \operatorname{div} \psi \, dx \, d\tau + \int_s^t \oint_{\partial\Omega} (\nu \times v) \cdot (m \times \beta v) \, d\varsigma \, d\tau, \end{aligned}$$

using the definition of $(\tilde{\alpha}, \tilde{\beta})$ from (5.5). The integral involving u in (5.18) is treated analogously. Because of the boundary condition from (5.4) we obtain

$$\begin{aligned} &\int_{\Omega} (m \times \alpha(t)u(t))\beta(t)v(t) \, dx - \int_{\Omega} (m \times \alpha(s)u(s))\beta(s)v(s) \, dx = \int_s^t J \, d\tau \\ &= \frac{1}{2} \int_s^t \int_{\Omega} (\tilde{\alpha}u \cdot u + \tilde{\beta}v \cdot v) \, dx \, d\tau + \frac{1}{2} \int_s^t \oint_{\partial\Omega} (\nu \cdot m) \operatorname{tr}(\alpha u \cdot u + \beta v \cdot v) \, d\varsigma \, d\tau \\ &\quad - \int_s^t \oint_{\partial\Omega} ((\nu \times u) \cdot (m \times \alpha u) + (\gamma(u \times \nu) \times \nu - h) \cdot (m \times \beta v)) \, d\varsigma \, d\tau \\ &\quad + \int_s^t \int_{\Omega} ((m \times \partial_t \varphi) \cdot \beta v + (m \times \alpha u) \cdot \partial_t \psi - (m \cdot u \operatorname{div} \varphi + m \cdot v \operatorname{div} \psi)) \, dx \, d\tau. \end{aligned}$$

The non-trapping condition (5.5) then yields

$$\begin{aligned} &\int_s^t \int_{\Omega} (\alpha u \cdot u + \beta v \cdot v) \, d\varsigma \, d\tau \\ &\lesssim \|(u(s), v(s))\|_{L^2}^2 + \|(u(t), v(t))\|_{L^2}^2 \\ &\quad + \int_s^t \int_{\Omega} (|\partial_t \varphi| |v| + |\partial_t \psi| |u| + |\operatorname{div} \varphi| |u| + |\operatorname{div} \psi| |v|) \, dx \, d\tau \\ &\quad + \int_s^t \oint_{\partial\Omega} \operatorname{tr}(\alpha u \cdot u + \beta v \cdot v) \, d\varsigma \, d\tau + \int_s^t \oint_{\partial\Omega} |h \cdot \operatorname{tr} v| \, d\varsigma \, d\tau. \end{aligned} \quad \square$$

Next, we improve the result above to account for boundary terms. We have to estimate the integral $\int_s^t \oint_{\partial\Omega} \operatorname{tr}(\alpha u \cdot u + \beta v \cdot v) \, d\varsigma \, d\tau$ in (5.17) uniformly in T . By Lemma 5.1.7 it can be split into

$$\int_s^t \oint_{\partial\Omega} \operatorname{tr}(\alpha u \cdot u + \beta v \cdot v) \, d\varsigma \, d\tau \lesssim \int_s^t \oint_{\partial\Omega} (|\nu \cdot \alpha u|^2 + |u_{\tau}|^2 + |\nu \cdot \beta v|^2 + |v_{\tau}|^2) \, d\varsigma \, d\tau.$$

The tangential trace of v can be controlled through the boundary condition by

$$\int_s^t \|v_\tau\|_{L^2(\partial\Omega)}^2 d\tau \lesssim \int_s^t \left(\|\nu \times u\|_{L^2(\partial\Omega)}^2 + \|h\|_{L^2(\partial\Omega)}^2 \right) d\tau.$$

Using Corollary 5.1.9 and Remark 5.1.10, we derive

$$\begin{aligned} \int_s^t \oint_{\partial\Omega} \left(|\nu \cdot \alpha u|^2 + |\nu \cdot \beta v|^2 \right) d\varsigma d\tau &\lesssim \int_s^t \left(\delta \|u\|_{L^2(\Omega)}^2 + \delta \|v\|_{L^2(\Omega)}^2 + \|(\varphi, \psi)\|_{H^1(\Omega)}^2 \right. \\ &\quad \left. + \|\nu \times u\|_{L^2(\partial\Omega)}^2 + \|h\|_{H^{1/2}(\partial\Omega)}^2 + c_\delta \|\nu \cdot (\alpha u, \beta v)\|_{H^{-1}(\partial\Omega)}^2 \right) d\tau. \end{aligned}$$

Fixing a sufficiently small $\delta > 0$, the norms in L^2 above can be absorbed by the left-hand side of (5.17). We have shown the following estimate.

Lemma 5.2.2. *Let (5.2), (5.3) and (5.5) be true. Then the weak solutions $(u, v) \in C([0, T], L^2(\Omega, \mathbb{R}^6))$ of system (5.1) fulfill*

$$\begin{aligned} &\int_s^t \int_{\Omega} (\alpha u \cdot u + \beta v \cdot v) dx d\tau + \int_s^t \oint_{\partial\Omega} \text{tr}(\alpha u \cdot u + \beta v \cdot v) \\ &\quad \lesssim \int_s^t \oint_{\partial\Omega} |u \times \nu|^2 d\varsigma d\tau + \|(u(s), v(s))\|_{L^2}^2 + \|(u(t), v(t))\|_{L^2}^2 \\ &\quad + \int_s^t \int_{\Omega} \left(|\partial_t \varphi| |v| + |\partial_t \psi| |u| + |\text{div } \varphi| |u| + |\text{div } \psi| |v| \right) dx d\tau \\ &\quad + \int_s^t \left(\|(\varphi, \psi)\|_{H^1}^2 + \|h\|_{H^{1/2}(\partial\Omega)}^2 + \|\nu \cdot (\alpha u, \beta v)\|_{H^{-1}(\partial\Omega)}^2 \right) d\tau \end{aligned} \tag{5.19}$$

for $0 \leq s \leq t \leq T$.

One could estimate the terms in the second line of the right-hand side by

$$\delta \int_s^t \int_{\Omega} (|u|^2 + |v|^2) dx d\tau + c_\delta \int_s^t \int_{\Omega} (|\partial_t \varphi|^2 + |\partial_t \psi|^2 + |\text{div } \varphi|^2 + |\text{div } \psi|^2) dx d\tau,$$

and absorb the first summand for small $\delta > 0$ by the left-hand side. In the following this is not needed, since these summands can be absorbed into small error terms.

DECAY OF SOLUTIONS TO THE LINEAR MAXWELL SYSTEM WITH ABSORBING BOUNDARY

In this chapter we will improve the trace estimate from [11] or Proposition 5.1.11 in the linear, autonomous and homogeneous case, where we obtain T -independent constants. Moreover, we show exponential stability of this problem. So, we consider

$$\begin{aligned} \partial_t(\varepsilon(x)E(t, x)) &= \operatorname{curl} H(t, x), \\ \partial_t(\mu(x)H(t, x)) &= -\operatorname{curl} E(t, x), & t \geq 0, x \in \Omega, \\ \operatorname{div}(\varepsilon E(t, x)) &= 0 = \operatorname{div}(\mu H(t, x)), & (6.1) \\ H(t, x) \times \nu + (\lambda(x)(E(t, x) \times \nu)) \times \nu &= 0, & t \geq 0, x \in \partial\Omega, \\ E(0, x) &= E_0(x), \quad H(0, x) = H_0(x), & x \in \Omega, \end{aligned}$$

on a bounded, multiply connected domain $\Omega \subset \mathbb{R}^3$ with $C^{2,1}$ -boundary $\partial\Omega$. For the connected components of the boundary $\partial\Omega$ we write $\partial\Omega_0, \dots, \partial\Omega_K$, see Section 1.3. As in (5.2) and (5.3), we assume that

$$\varepsilon, \mu \in C^1(\overline{\Omega}, \mathbb{R}_{\text{sym}}^{3 \times 3}), \quad \lambda \in C_\tau^1(\partial\Omega, \mathbb{R}_{\text{sym}}^{3 \times 3}), \quad \text{with } \varepsilon, \mu, \lambda \geq \eta, \quad (6.2)$$

for some $\eta > 0$ and that

$$E_0, H_0 \in (L^2(\Omega))^3 \quad \text{satisfy} \quad \operatorname{div}(\varepsilon E_0) = \operatorname{div}(\mu H_0) = 0, \quad (6.3)$$

as well as

$$\langle \operatorname{tr}_n(\varepsilon_d E_0), \mathbf{1} \rangle_{H^{-\frac{1}{2}}(\partial\Omega_i)} = 0, \quad \langle \operatorname{tr}_n(\mu H_0), \mathbf{1} \rangle_{H^{-\frac{1}{2}}(\partial\Omega_i)} = 0, \quad i \in \{0, \dots, K\}, \quad (6.4)$$

i.e., $E_0 \in H^{\partial\Omega}(\operatorname{div}_\varepsilon 0)$, $H_0 \in H^{\partial\Omega}(\operatorname{div}_\mu 0)$. Furthermore, as in (4.19), we require

$$\varepsilon + (m \cdot \nabla)\varepsilon \geq \tilde{\eta}\varepsilon, \quad \mu + (m \cdot \nabla)\mu \geq \tilde{\eta}\mu \quad (6.5)$$

for some $\tilde{\eta} > 0$.

On $X_b := \{(E, H) \in L^2(\Omega)^6 \mid \varepsilon E, \mu H \in H^{\partial\Omega}(\operatorname{div} 0)\}$ we define the operator

$$A_b := \begin{pmatrix} 0 & \varepsilon^{-1} \operatorname{curl} \\ -\mu^{-1} \operatorname{curl} & 0 \end{pmatrix}$$

with

$$D(A_b) := \{(E, H) \in X_b \mid \operatorname{curl} E, \operatorname{curl} H \in L^2(\Omega), H \times \nu + (\lambda(E \times \nu)) \times \nu = 0\}.$$

Then (6.1) reduces to the evolution equation $\partial_t \begin{pmatrix} E \\ H \end{pmatrix} = A_b \begin{pmatrix} E \\ H \end{pmatrix}$. As in Lemma 7.2.2.1 of [33] one sees that A_b is maximally dissipative and thus generates a contraction semigroup $(T_b(t))_{t \geq 0}$. We recall the energy equality from equation (7.2.2.12) of [33] or Lemma 3.2 of [51], namely

$$\begin{aligned} & \left\| (\varepsilon^{1/2} E, \mu^{1/2} H)(s) \right\|_{L^2}^2 - 2 \int_s^t \left\| \lambda \operatorname{tr}_t E \cdot \operatorname{tr}_t E \right\|_{L^2(\partial\Omega)}^2 d\tau \\ &= \left\| (\varepsilon^{1/2} E, \mu^{1/2} H)(t) \right\|_{L^2}^2 \end{aligned} \quad (6.6)$$

for $t \geq s \geq 0$. Note that in contrast to [11] we do not consider time dependent coefficients ε and μ , since this would for example introduce the extra term $\int_s^t \int_\Omega E \cdot (\partial_t \varepsilon) E + H \cdot (\partial_t \mu) H \, dx d\tau$ on the left-hand side of (6.6).

6.1. OBSERVABILITY-TYPE ESTIMATE AND NORMAL TRACE ESTIMATE

Lemma 5.2.1 and the above energy equality lead to the following preliminary observability inequality.

Lemma 6.1.1. *Let (6.2), (6.3), (6.5), and (6.4) be true, $(E_0, H_0) \in X_b$, and let $(E, H) \in C([0, \infty), X_b)$ solve (6.1). Then there exists a time $\hat{T} > 0$ not depending on (E_0, H_0) such that for $T \geq \hat{T}$ we have the observability estimate*

$$\|(E, H)(0)\|_{L^2}^2 \lesssim \int_0^T \oint_{\partial\Omega} \operatorname{tr}(\varepsilon E \cdot E + \mu H \cdot H) \, d\varsigma \, d\tau, \quad (6.7)$$

with a constant independent of (E_0, H_0) .

Hence, if E and H vanish on the boundary, the initial values $(E, H)(0)$ were zero and so, since we are in the homogeneous case, the fields E and H vanish as well.

Proof. Lemma 5.2.1 yields the observability-like estimate

$$\begin{aligned} \int_s^t \int_{\Omega} (\varepsilon E \cdot E + \mu H \cdot H) \, dx \, d\tau &\lesssim \|(E, H)(s)\|_{L^2}^2 + \|(E, H)(t)\|_{L^2}^2 \\ &+ \int_s^t \oint_{\partial\Omega} \operatorname{tr}(\varepsilon E \cdot E + \mu H \cdot H) \, d\varsigma \, d\tau \end{aligned} \quad (6.8)$$

for $s, t \geq 0$. We stress that the implicit constant does neither depend on t nor s .

Note that $|\operatorname{tr} E|^2 \geq |\operatorname{tr}_{\tau} E|^2 = |\operatorname{tr}_t E|^2$ by orthogonality. The boundary condition in (6.1) as well as inequalities (6.6) and (6.8) then lead to

$$\begin{aligned} T \left(\|(E, H)(0)\|_{L^2}^2 - \int_0^T \oint_{\partial\Omega} \operatorname{tr}(\varepsilon E \cdot E + \mu H \cdot H) \, d\varsigma \, d\tau \right) \\ \lesssim T \left(\|(E, H)(0)\|_{L^2}^2 - \int_0^T \oint_{\partial\Omega} |\operatorname{tr}_t E|^2 \, d\varsigma \, d\tau \right) \\ \lesssim T \|(E, H)(T)\|_{L^2}^2 \lesssim \int_0^T \|(E, H)(\tau)\|_{L^2}^2 \, d\tau \\ \lesssim \|(E, H)(0)\|_{L^2}^2 + \|(E, H)(T)\|_{L^2}^2 + \int_0^T \oint_{\partial\Omega} \operatorname{tr}(\varepsilon E \cdot E + \mu H \cdot H) \, d\varsigma \, d\tau \\ \lesssim \|(E, H)(0)\|_{L^2}^2 + \int_0^T \oint_{\partial\Omega} \operatorname{tr}(\varepsilon E \cdot E + \mu H \cdot H) \, d\varsigma \, d\tau. \end{aligned}$$

The desired estimate follows for sufficiently large T . $\square$

We now bound the normal traces of E and H solely through the tangential trace of the electric field, where the constants do not depend on T . This improves Lemma 5.1.8 considerably in the autonomous, homogeneous case. We employ a compactness argument which was originally used in the proof of global decay rates for the wave equation, see Lemma 2.2 in [34] (and also [18]). Moreover, we use uniqueness properties both on the dynamical and stationary level which rely on the observability estimate in Lemma 6.1.1 and on the div-curl bound in Theorem A.2.9b). For this reason we have to assume (6.3) and (6.4).

Theorem 6.1.2. *Let (6.2), (6.3), (6.4), and (6.5) hold. Let the time $\hat{T} > 0$ be given by Lemma 6.1.1. Then the solution $(E, H) \in C([0, \infty), L^2(\Omega)^6)$ of (6.1) satisfies*

$$\int_s^t \left(\|\nu \cdot \varepsilon E\|_{L^2(\partial\Omega)}^2 + \|\nu \cdot \mu H\|_{L^2(\partial\Omega)}^2 \right) \, d\tau \lesssim \int_s^t \|\nu \times E\|_{L^2(\partial\Omega)}^2 \, d\tau \quad (6.9)$$

for $0 \leq s \leq s + \hat{T} \leq t$, where the constant is independent of s, t and (E_0, H_0) .

Proof. 1) Let $(E_0, H_0) \in X_b$ and (E, H) be the corresponding solution of (6.1). Take a fixed $T > \hat{T}$ with $\hat{T}$ from Lemma 6.1.1. By Proposition 5.1.11 it suffices to choose $(E_0, H_0) \in D(A_b)$.

We first show the estimate

$$\int_0^T \|\nu \cdot (\varepsilon E, \mu H)\|_{L^2(\partial\Omega)}^2 d\tau \lesssim_T \int_0^T \|\nu \times E\|_{L^2(\partial\Omega)}^2 d\tau. \quad (6.10)$$

Let $\theta \in (0, \frac{1}{2})$. In Lemma 5.1.8 we have already shown

$$\begin{aligned} & \int_0^T \left(\|\nu \cdot \varepsilon E\|_{L^2(\partial\Omega)}^2 + \|\nu \cdot \mu H\|_{L^2(\partial\Omega)}^2 \right) d\tau \\ & \lesssim \int_0^T \left(\delta \|(E, H)\|_{H^{-\theta}}^2 + c \|\nu \times E\|_{L^2(\partial\Omega)}^2 + c_\delta \|\nu \cdot (\varepsilon E, \mu H)\|_{H^{-1}(\partial\Omega)}^2 \right) d\tau. \end{aligned} \quad (6.11)$$

2.a) Let us assume that (6.10) does not hold. Thus, there exist solutions $(\tilde{E}_n, \tilde{H}_n)$ satisfying the divergence condition in (6.1) with initial values $(\tilde{E}_{0,n}, \tilde{H}_{0,n})$ such that

$$\int_0^T \|\nu \cdot (\varepsilon \tilde{E}_n, \mu \tilde{H}_n)\|_{L^2(\partial\Omega)}^2 d\tau > n \int_0^T \|\nu \times \tilde{E}_n\|_{L^2(\partial\Omega)}^2 d\tau. \quad (6.12)$$

We now set $(E_{0,n}, H_{0,n}) := \frac{1}{\alpha_n}(\tilde{E}_{0,n}, \tilde{H}_{0,n})$ with

$$\alpha_n := \int_0^T \left(\delta \|(\tilde{E}_n, \tilde{H}_n)\|_{H^{-\theta}}^2 + c_\delta \|\nu \cdot (\varepsilon \tilde{E}_n, \mu \tilde{H}_n)\|_{H^{-1}(\partial\Omega)}^2 \right) d\tau.$$

Using (6.11), (6.12) and our construction, we obtain classical solutions (E_n, H_n) of (6.1) with initial data $(E_{0,n}, H_{0,n})$ satisfying

$$\begin{aligned} \int_0^T \|\nu \cdot (\varepsilon E_n, \mu H_n)\|_{L^2(\partial\Omega)}^2 d\tau & \leq \int_0^T \left(\delta \|(E_n, H_n)\|_{H^{-\theta}}^2 + c \|\nu \times E_n\|_{L^2(\partial\Omega)}^2 \right. \\ & \quad \left. + c_\delta \|\nu \cdot (\varepsilon E_n, \mu H_n)\|_{H^{-1}(\partial\Omega)}^2 \right) d\tau \\ & = 1 + c \int_0^T \|\nu \times E_n\|_{L^2(\partial\Omega)}^2 d\tau \\ & \leq 1 + \frac{c}{n} \int_0^T \|\nu \cdot (\varepsilon E_n, \mu H_n)\|_{L^2(\partial\Omega)}^2 d\tau. \end{aligned} \quad (6.13)$$

We thus derive the formulas

$$\int_0^T \left(\|\nu \cdot (\varepsilon E_n, \mu H_n)\|_{L^2(\partial\Omega)}^2 \right) d\tau \leq 2, \quad (6.14)$$

$$\int_0^T \left(\delta \|(E_n, H_n)\|_{H^{-\theta}}^2 + c_\delta \|\nu \cdot (\varepsilon E_n, \mu H_n)\|_{H^{-1}(\partial\Omega)}^2 \right) d\tau = 1, \quad (6.15)$$

for all sufficiently large $n \in \mathbb{N}$, and

$$\int_0^T \|\nu \times (E_n, H_n)\|_{L^2(\partial\Omega)}^2 d\tau \rightarrow 0 \quad \text{for } n \rightarrow \infty, \quad (6.16)$$

where we also used the boundary condition in (6.1).

Next, Lemmas 6.1.1 and 5.1.7, as well as (6.1), (6.14), and (6.16) imply

$$\|(E_{0,n}, H_{0,n})\|_{H(\operatorname{div}_\varepsilon) \times H(\operatorname{div}_\mu)} = \|(E_{0,n}, H_{0,n})\|_{L^2} \lesssim 1 \quad \text{for } n \in \mathbb{N}.$$

For that reason, $(E_{0,n}, H_{0,n})$ converges weakly in $L^2(\Omega)$ to some limit (E_0, H_0) . (Here and below we pass to subsequences without mentioning it.) Note that the limit still satisfies

$\operatorname{div}(\varepsilon E_0) = \operatorname{div}(\mu H_0) = 0$. We denote the solution for (E_0, H_0) with $(E, H) \in C([0, T], L^2(\Omega))$. By continuity of the semigroup and the Rellich–Kondrachov theorem we obtain the convergence $(E_n, H_n)(t) \rightarrow (E, H)(t)$ in $H^{-\theta}$ for $t \in [0, T]$. Because of $(E_n, H_n), (E, H) \in \ker(\operatorname{div}_\varepsilon) \times \ker(\operatorname{div}_\mu)$, we also have weak convergence in $H(\operatorname{div}_\varepsilon) \times H(\operatorname{div}_\mu)$, and thus the normal trace converges weakly in $H^{-1/2}(\partial\Omega)$. Therefore, $\operatorname{tr}_n(E_n, H_n)$ tends to $\operatorname{tr}_n(E, H)$ strongly in $H^{-1}(\partial\Omega)$ for $n \rightarrow \infty$.

From (6.15) we conclude that (E, H) also satisfies

$$\int_0^T (\delta \|(E, H)\|_{H^{-\theta}}^2 + c_\delta \|\nu \cdot (\varepsilon E, \mu H)\|_{H^{-1}(\partial\Omega)}^2) d\tau = 1.$$

In particular $(E, H) \neq 0$, which will lead to a contradiction.

2.b) The energy equality Lemma 3.2 of [51] shows that $(E_n \times \nu, H_n \times \nu) \rightharpoonup (E \times \nu, H \times \nu)$ weakly in $L^2([0, T] \times \partial\Omega)$ for $n \rightarrow \infty$ (see also Lemma 7.2.2.2 in [33]). Hence, $(E \times \nu, H \times \nu) = 0$ by (6.16).

We next regularize the fields (E, H) by means of a Friedrich’s mollifier as done in [18], cf. Definition 3.3.1. So we take $0 \leq \eta_{\tau_0} = \eta \in C_c^\infty(-\tau_0, \tau_0)$ with $\int_{-\tau_0}^{\tau_0} \eta dx = 1$ for some $\tau_0 \in (0, \frac{T-\hat{T}}{2})$, and consider

$$\begin{aligned} E^*(t, x) &:= [\eta * E(\cdot, x)](t) = \int_0^T \eta(\tau) E(t - \tau, x) d\tau, \\ H^*(t, x) &:= [\eta * H(\cdot, x)](t) = \int_0^T \eta(\tau) H(t - \tau, x) d\tau, \end{aligned} \quad t \in [\tau_0, T - \tau_0].$$

These are functions $(E^*, H^*) \in C^\infty([\tau_0, T - \tau_0], (L^2(\Omega))^6)$ which solve the partial differential equations in (6.1) and satisfy $(E^* \times \nu, H^* \times \nu) = 0$. In particular, $(E^*, H^*)(t)$ belongs to $H(\operatorname{curl}) \cap X_b$. Applying ∂_t , we see that $(\partial_t E^*, \partial_t H^*)$ fulfill the same properties. Thus, $(E^*, H^*)(t)$ and $(\partial_t E^*, \partial_t H^*)(t)$ belong to $H^1(\Omega)$ for $t \in [\tau_0, T - \tau_0]$ by Theorem A.2.9 a).

Lemma A.1.6 now implies that $\nu \cdot \operatorname{curl} E^* = \nu \cdot \operatorname{curl} H^* = 0$ on $\partial\Omega$. Notice that therefore $(\partial_t E^*, \partial_t H^*)$ are classical solutions of (6.1) with vanishing normal traces $\nu \cdot \varepsilon \partial_t E^* = \nu \cdot \mu \partial_t H^* = 0$. Lemmas 5.1.7 and 6.1.1, since $T - 2\tau_0 \geq \hat{T}$, then show that $(\partial_t E^*, \partial_t H^*)(\tau_0) = 0$ and hence $(\partial_t E^*, \partial_t H^*)$ stays zero for $t \in [\tau_0, T - \tau_0]$ and so $(\operatorname{curl} E^*, \operatorname{curl} H^*)(t) = 0$ by (6.1). Utilizing Theorem A.2.9 b) we see that (E^*, H^*) vanishes on $[\tau_0, T - \tau_0]$. Letting $\tau_0 \rightarrow 0$, it follows that $(E_0, H_0) = 0$, contradicting $(E, H) \neq 0$. Hence, (6.10) holds.

3) Take $\tilde{T} = nT + t' \geq \hat{T}$ for some $n \in \mathbb{N}$ and $t' \in [0, T]$. By means of the previous step and the translation invariance of the problem (6.1), we calculate

$$\begin{aligned} \int_0^{\tilde{T}} \|\nu \cdot (\varepsilon E, \mu H)\|_{L^2(\partial\Omega)}^2 d\tau &= \sum_{k=1}^n \int_{(k-1)T}^{kT} \|\nu \cdot (\varepsilon E, \mu H)\|_{L^2(\partial\Omega)}^2 d\tau \\ &\quad + \int_{nT}^{nT+t'} \|\nu \cdot (\varepsilon E, \mu H)\|_{L^2(\partial\Omega)}^2 d\tau \\ &\lesssim \sum_{k=1}^n \int_{(k-1)T}^{kT} \|\nu \times E\|_{L^2(\partial\Omega)}^2 d\tau + \int_{(n-1)T}^{nT+t'} \|\nu \times E\|_{L^2(\partial\Omega)}^2 d\tau \\ &\lesssim \int_0^{\tilde{T}} \|\nu \times E\|_{L^2(\partial\Omega)}^2 d\tau. \end{aligned} \tag{6.17}$$

For general $t \geq s \geq 0$ the result follows in the same way. $\square$

Lemmas 5.1.7 and 6.1.1, the boundary condition in (6.1), and Theorem 6.1.2 yield the observability estimate

$$\|(E, H)(0)\|_{L^2}^2 \lesssim \int_0^t \|E \times \nu\|_{L^2(\partial\Omega)}^2 d\tau \tag{6.18}$$

for times $t \geq \hat{T}$, where $\hat{T}$ is given by Lemma 6.1.1.

6.2. DECAY RESULT

We further deduce the exponential stability of the semigroup $(T_b(t))_{t \geq 0}$ on X_b . For scalar coefficients this fact was shown in [20], [21] and [43]. For the case of internal damping we refer to [19] or [49]. We note that the next result does not follow from Theorem 7.2.1.

Theorem 6.2.1. *Let (6.2), (6.3), (6.5), and (6.4) hold. For $(E_0, H_0) \in X_b$ there exist constants $M \geq 1$ and $\omega > 0$ such that*

$$\|T_b(t)(E_0, H_0)\|_{L^2(\Omega)} \leq M e^{-\omega t} \|(E_0, H_0)\|_{L^2(\Omega)}$$

for all $t \geq 0$. The constants are independent of (E_0, H_0) .

Proof. Using (6.8), Lemma 5.1.7, Theorem 6.1.2, the boundary condition in (6.1), and (6.6), we obtain the estimate

$$\begin{aligned} \|(E, H)(t)\|_{L^2}^2 &+ \int_0^t \int_{\Omega} (\varepsilon E \cdot E + \mu H \cdot H) \, dx \, d\tau \\ &\lesssim \|(E, H)(t)\|_{L^2}^2 + \|(E, H)(0)\|_{L^2}^2 \\ &\quad + \int_0^t \oint_{\partial\Omega} (|\nu \cdot (\varepsilon E, \mu H)|^2 + |\nu \times (E, H)|^2) \, d\varsigma \, d\tau \\ &\lesssim \|(E, H)(t)\|_{L^2}^2 + \|(E, H)(0)\|_{L^2}^2 + \int_0^t \|E \times \nu\|_{L^2(\partial\Omega)}^2 \, d\tau \\ &\lesssim \|(E, H)(0)\|_{L^2}^2, \end{aligned}$$

for $t \geq \hat{T}$, where $\hat{T}$ is given by Lemma 6.1.1. As in Remark 5.2 of [35], the exponential decay now follows in a standard way. $\square$

Combining Theorem 6.1.2 with (6.6) and Lemma 5.1.7 we derive the following result, allowing us to bound the traces of the fields by their initial values globally in time. (See Theorem 1.2 of [11] or Proposition 5.1.11 for results locally in time.)

Corollary 6.2.2. *Let (6.2), (6.3), (6.4), and (6.5) hold. Then the solution $(E, H) \in C([0, \infty), (L^2(\Omega))^6)$ of (6.1) satisfies*

$$\|(E(t), H(t))\|_{L^2(\Omega)}^2 + \int_s^t \|\operatorname{tr}(E, H)\|_{L^2(\partial\Omega)}^2 \, d\tau \lesssim \|(E(s), H(s))\|_{L^2(\Omega)}^2$$

for $0 \leq s \leq t$.

EXPONENTIAL DECAY OF SOLUTIONS TO THE QUASILINEAR MAXWELL SYSTEM WITH ABSORBING BOUNDARY

Finally, we discuss the long-term behavior of quasilinear system. Similar to Chapter 4 this introduces several challenges. First, the local well-posedness theory is considerably more involved compared to the semigroup approach from the previous chapter. Furthermore, the energy and observability-type estimate introduce additional error terms, which we treat by a similar argument as in the regularity analysis of Section 4.5. Thankfully, the absence of an interface simplifies the proof.

We investigate the quasilinear Maxwell system with generalized inhomogeneous absorbing boundary condition

$$\begin{aligned} \partial_t(\varepsilon(x, E(t, x))E(t, x)) &= \operatorname{curl} H(t, x), \\ \partial_t(\mu(x, H(t, x))H(t, x)) &= -\operatorname{curl} E(t, x), \end{aligned} \quad t \geq 0, x \in \Omega, \quad (7.1)$$

$$\begin{aligned} \operatorname{div}(\varepsilon(x, E(t, x))E(t, x)) &= 0, \\ \operatorname{div}(\mu(x, H(t, x))H(t, x)) &= 0, \end{aligned} \quad t \geq 0, x \in \Omega, \quad (7.2)$$

$$H(t, x) \times \nu + \left(\lambda(x, E(t, x) \times \nu)(E(t, x) \times \nu) \right) \times \nu = 0, \quad t \geq 0, x \in \partial\Omega, \quad (7.3)$$

$$\begin{aligned} \langle \operatorname{tr}_n(\varepsilon(x, E(t, x))E(t, x)), \mathbf{1} \rangle_{H^{-\frac{1}{2}}(\partial\Omega_i)} &= 0, \\ \langle \operatorname{tr}_n(\mu(x, H(t, x))H(t, x)), \mathbf{1} \rangle_{H^{-\frac{1}{2}}(\partial\Omega_i)} &= 0, \end{aligned} \quad \begin{aligned} t \geq 0, x \in \partial\Omega, \\ i \in \{0, \dots, K\}, \end{aligned} \quad (7.4)$$

$$E(0, x) = E_0(x), \quad H(0, x) = H_0(x), \quad x \in \Omega. \quad (7.5)$$

As in Chapter 4, we consider a bounded, multiply connected domain $\Omega \subseteq \mathbb{R}^3$ with C^5 -boundary $\partial\Omega = \cup_{i=0}^K \partial\Omega_i$.

7.1. NOTATION, ASSUMPTIONS AND AUXILIARY RESULTS

In order to define quasilinear coefficients, we introduce the function spaces

$$C_\tau^k(\partial\Omega \times \mathbb{R}^3, \mathbb{R}_{\text{sym}}^{3 \times 3}) := \{A \in C^3(\partial\Omega \times \mathbb{R}^3, \mathbb{R}_{\text{sym}}^{3 \times 3}) \mid A\nu^\perp \subseteq \nu^\perp\}.$$

We assume that the permittivity ε , the permeability μ , and the surface conductivity λ are of class

$$\varepsilon, \mu \in C^3(\overline{\Omega} \times \mathbb{R}^3, \mathbb{R}_{\text{sym}}^{3 \times 3}), \quad \lambda \in C_\tau^3(\partial\Omega \times \mathbb{R}^3, \mathbb{R}_{\text{sym}}^{3 \times 3}) \quad (7.6)$$

and are uniformly positive definite for zero fields, i.e., there is a constant η such that

$$\begin{aligned} \varepsilon(x, 0) &\geq 2\eta, \quad \mu(x, 0) \geq 2\eta \quad \text{for all } x \in \overline{\Omega}, \\ \text{and } \lambda(x, 0) &\geq 2\eta \quad \text{for all } x \in \partial\Omega. \end{aligned} \quad (7.7)$$

In a moment, we will also consider the time derived Maxwell equations for suitably smooth solutions. In order to discuss these systems, we introduce the following matrices as in (4.10)

$$\begin{aligned}
\varepsilon_{dij}(x, \xi) &= \varepsilon_{ij}(x, \xi) + \sum_{l=1}^3 \partial_{\xi_j} \varepsilon_{il}(x, \xi) \xi_l & \text{for } x \in \overline{\Omega}, \xi \in \mathbb{R}^3, \\
\mu_{dij}(x, \xi) &= \mu_{ij}(x, \xi) + \sum_{l=1}^3 \partial_{\xi_j} \mu_{il}(x, \xi) \xi_l & \text{for } x \in \overline{\Omega}, \xi \in \mathbb{R}^3, \\
\lambda_{dij}(x, \xi) &= \lambda_{ij}(x, \xi) + \sum_{l=1}^3 \partial_{\xi_j} \lambda_{il}(x, \xi) \xi_l & \text{for } x \in \partial\Omega, \xi \in \mathbb{R}^3,
\end{aligned} \tag{7.8}$$

where $i, j \in \{1, 2, 3\}$. To unify notation, we likewise set

$$\begin{aligned}
\hat{\varepsilon}_k &= \begin{cases} \varepsilon(\cdot, E), & k = 0, \\ \varepsilon_d(\cdot, E), & k \in \{1, 2, 3\}, \end{cases} & \hat{\mu}_k &= \begin{cases} \mu(\cdot, E), & k = 0, \\ \mu_d(\cdot, E), & k \in \{1, 2, 3\}, \end{cases} \\
\hat{\lambda}_k &= \begin{cases} \lambda(\cdot, E), & k = 0, \\ \lambda_d(\cdot, E), & k \in \{1, 2, 3\}. \end{cases}
\end{aligned}$$

We also assume that

$$\begin{aligned}
\partial_{\xi_j} \varepsilon, \partial_{\xi_j} \mu &\in C^3(\overline{\Omega} \times \mathbb{R}^3, \mathbb{R}^{3 \times 3}), \quad \partial_{\xi_j} \lambda \in C^3(\partial\Omega \times \mathbb{R}^3, \mathbb{R}^{3 \times 3}), \\
\varepsilon_d &= (\varepsilon_d)^\top, \quad \mu_d = (\mu_d)^\top, \quad \lambda_d = (\lambda_d)^\top,
\end{aligned} \tag{7.9}$$

for $i, j \in \{1, 2, 3\}$. Note that $\lambda_d = \lambda$ if $\lambda = \lambda(x)$ is state-independent. We extend λ to a function on $\overline{\Omega} \times \mathbb{R}^3$ satisfying the same conditions as ε and μ . Since the coefficients are continuous, we also have uniform positivity at least for small fields; i.e.,

$$\begin{aligned}
\varepsilon(x, \xi), \varepsilon_d(x, \xi) &\geq \eta I, \quad \mu(x, \xi), \mu_d(x, \xi) \geq \eta I & \text{for all } |\xi| \leq \delta_0, \quad x \in \overline{\Omega}, \\
\lambda(x, \xi), \lambda_d(x, \xi) &\geq \eta I & \text{for all } |\xi| \leq \delta_0, \quad x \in \partial\Omega,
\end{aligned} \tag{7.10}$$

and some $\delta_0 > 0$. An important special case is the Kerr law $\mu = \mu(x)$ and $\varepsilon = \varepsilon(x, \xi) = \varepsilon_{\text{lin}}(x) + \varepsilon_{\text{nl}}(x) |\xi|^2$ for scalar coefficients with $\varepsilon_{\text{lin}} \geq 2\eta$, cf. Example 4.1.1 and also [2] or [24]. Here one has $\varepsilon_d(x, \xi) = \varepsilon(x, \xi) + 2(\varepsilon_{\text{nl}}(x)\xi)\xi^\top$. Anisotropic examples of polynomial type are discussed in Example 2.1 of [35].

Since the solutions of (7.1)–(7.5) are supposed to satisfy the boundary condition at all times, it has to hold also for the initial values, cf. Chapter 3. This leads to so-called “compatibility conditions” (of order 3) on E_0 and H_0 . Namely, for $E_0, H_0 \in H^3(\Omega)$, a solution $(E, H) \in \mathcal{G}^3$ of (7.1) possesses the time derivatives, cf. (4.12)

$$\begin{aligned}
E_0^1 &:= (\varepsilon_d(E_0))^{-1} \operatorname{curl} H_0, \\
H_0^1 &:= -(\mu_d(H_0))^{-1} \operatorname{curl} E_0, \\
E_0^2 &:= (\varepsilon_d(E_0))^{-1} \left[\operatorname{curl} H_0^1 - \left(\sum_{\ell=1}^3 \partial_{\xi_\ell} \varepsilon_{dij}(E_0)(E_0^1)_\ell \right)_{i,j} E_0^1 \right], \\
H_0^2 &:= -(\mu_d(H_0))^{-1} \left[\operatorname{curl} E_0^1 - \left(\sum_{\ell=1}^3 \partial_{\xi_\ell} \mu_{dij}(H_0)(H_0^1)_\ell \right)_{i,j} H_0^1 \right],
\end{aligned} \tag{7.11}$$

at time 0, which leads to the compatibility conditions

$$\begin{aligned}
H_0 \times \nu + \lambda(E_0 \times \nu)(E_0 \times \nu) \times \nu &= 0, \\
H_0^1 \times \nu + \lambda_d(E_0 \times \nu)(E_0^1 \times \nu) \times \nu &= 0, \\
H_0^2 \times \nu + \lambda_d(E_0 \times \nu)(E_0^2 \times \nu) \times \nu &= 0, \\
&= - \left(\left(\sum_{\ell=1}^3 \partial_{\xi_\ell} \lambda_{dij}(E_0 \times \nu)(E_0^1 \times \nu)_\ell \right)_{i,j} E_0^1 \times \nu \right) \times \nu,
\end{aligned} \tag{7.12}$$

on $\partial\Omega$.

Let c_S be the norm of the Sobolev embedding $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$. Set $\tilde{\delta} := \min\{1, \frac{\delta_0}{c_S}\}$, cf. (7.10). Under the above assumptions, in Theorem 6.4 of [55] it was shown that for $\delta \in (0, \tilde{\delta}]$ and initial values satisfying (7.12) that are sufficiently small, i.e.,

$$\|(E_0, H_0)\|_{H^3(\Omega)^6} \leq r(\delta) \quad (7.13)$$

for a constant $r(\delta) > 0$ depending on δ , there exists a unique classical solution $(E, H) \in \mathcal{G}^3([0, T_{\max}))$ of (7.1), (7.2), and (7.5), such that $T_{\max} > 1$ and

$$\max_{0 \leq j \leq 3} \left(\|\partial_t^j E(t)\|_{H^{3-j}(\Omega)}^2 + \|\partial_t^j H(t)\|_{H^{3-j}(\Omega)}^2 \right) \leq \delta^2 \quad \text{for } 0 \leq t \leq 1. \quad (7.14)$$

We set

$$T_* := \sup \left\{ \hat{T} \in [0, T_{\max}] \mid (7.14) \text{ holds for } t \in [0, \hat{T}] \right\} \geq 1. \quad (7.15)$$

From the blow-up condition in Theorem 6.4 of [55] we infer that $T_* < \infty$ implies

$$\max_{0 \leq j \leq 3} \left(\|\partial_t^j E(T_*)\|_{H^{3-j}(\Omega)}^2 + \|\partial_t^j H(T_*)\|_{H^{3-j}(\Omega)}^2 \right) = \delta^2. \quad (7.16)$$

Below we will assume $0 \leq t < T_*$ if we work with the solution (E, H) to (7.1)–(7.5) with data satisfying (7.13).

Remark 7.1.1. If the initial values $(E_0, H_0) \in H^3(\Omega)$ additionally satisfy

$$\operatorname{div}(\varepsilon(E_0)E_0) = 0 = \operatorname{div}(\mu(H_0)H_0) \quad \text{on } \Omega, \quad (7.17)$$

and

$$\langle \operatorname{tr}_n(\varepsilon(E_0)E_0), \mathbf{1} \rangle_{H^{-\frac{1}{2}}(\partial\Omega_i)} = 0, \quad \langle \operatorname{tr}_n(\mu(H_0)H_0), \mathbf{1} \rangle_{H^{-\frac{1}{2}}(\partial\Omega_i)} = 0, \quad i \in \{0, \dots, K\}, \quad (7.18)$$

these conditions are also satisfied by the solution $(E, H) \in \mathcal{G}^3$ for all $t \in [0, T_{\max})$.

Indeed, by the Maxwell equation (7.1) we have

$$\partial_t(\operatorname{div}(\varepsilon(x, E)E)) = \operatorname{div}(\operatorname{curl} H) = 0, \quad \text{and} \quad \partial_t(\operatorname{div}(\mu(x, H)H)) = \operatorname{div}(-\operatorname{curl} E) = 0.$$

Integrating in time from, we thus see that

$$\operatorname{div}(\varepsilon(x, E(t))E(t)) = 0 = \operatorname{div}(\mu(x, H(t))H(t))$$

holds for all $t \in [0, T_{\max})$.

In order to see that the condition (7.18) is preserved under the evolution of (E, H) , we differentiate on each closed component $\partial\Omega_i$ and substitute (7.1) to obtain

$$\partial_t \langle \operatorname{tr}_n(\varepsilon E(t)), \mathbf{1} \rangle_{H^{-\frac{1}{2}}(\partial\Omega_i)} = \langle \operatorname{tr}_n(\operatorname{curl} H), \mathbf{1} \rangle_{H^{-\frac{1}{2}}(\partial\Omega_i)} = 0,$$

where we applied Stokes' theorem in the last equality, using that $\partial(\partial\Omega_i) = \emptyset$. By identical reasoning we also obtain

$$\partial_t \langle \operatorname{tr}_n(\mu H(t)), \mathbf{1} \rangle_{H^{-\frac{1}{2}}(\partial\Omega_i)} = 0$$

Integrating in time then shows that

$$\langle \operatorname{tr}_n(\varepsilon(E(t))E(t)), \mathbf{1} \rangle_{H^{-\frac{1}{2}}(\partial\Omega_i)} = 0, \quad \langle \operatorname{tr}_n(\mu(H(t))H(t)), \mathbf{1} \rangle_{H^{-\frac{1}{2}}(\partial\Omega_i)} = 0, \quad i \in \{0, \dots, K\},$$

is satisfied for $t \in [0, T_{\max})$.

In order to discuss a time differentiated version of equation (7.1), we define the commutator terms

$$\begin{aligned} f_0 = f_1 = 0, \quad f_2 &= (\partial_t \varepsilon_d(\cdot, E)) \partial_t E, \quad f_3 = (\partial_t^2 \varepsilon_d(\cdot, E)) \partial_t E + 2(\partial_t \varepsilon_d(\cdot, E)) \partial_t^2 E, \\ g_0 = g_1 = 0, \quad g_2 &= (\partial_t \mu_d(\cdot, H)) \partial_t H, \quad g_3 = (\partial_t^2 \mu_d(\cdot, H)) \partial_t H + 2(\partial_t \mu_d(\cdot, H)) \partial_t^2 H, \\ h_0 = h_1 = 0, \quad h_2 &= (\partial_t \lambda_d(\cdot, E \times \nu)) \partial_t E \times \nu, \\ h_3 &= (\partial_t^2 \lambda_d(\cdot, E \times \nu)) \partial_t E \times \nu + 2(\partial_t \lambda_d(\cdot, E \times \nu)) \partial_t^2 E \times \nu. \end{aligned}$$

Let $k \in \{0, 1, 2, 3\}$. With the definitions above the time-derived solutions satisfy

$$\begin{aligned} \partial_t^k(\varepsilon E) &= \widehat{\varepsilon}_k \partial_t^k E + f_k, & \partial_t^k(\mu H) &= \widehat{\mu}_k \partial_t^k H + g_k & \text{on } \Omega, \\ \partial_t^k(\lambda E \times \nu) &= \widehat{\lambda}_k \partial_t^k E \times \nu + h_k & & & \text{on } \partial\Omega. \end{aligned} \quad (7.19)$$

We thus arrive at the inhomogeneous system

$$\partial_t(\widehat{\varepsilon}_k \partial_t^k E) = \text{curl } \partial_t^k H - \partial_t f_k, \quad t \geq 0, x \in \Omega, \quad (7.20)$$

$$\begin{aligned} \partial_t(\widehat{\mu}_k \partial_t^k H) &= -\text{curl } \partial_t^k E - \partial_t g_k, \\ \partial_t^k H \times \nu + \widehat{\lambda}_k(\partial_t^k E \times \nu) \times \nu &= -h_k \times \nu, \end{aligned} \quad t \geq 0, x \in \partial\Omega. \quad (7.21)$$

Notice that (7.2) and (7.19) imply

$$\text{div}(\widehat{\varepsilon}_k \partial_t^k E) = -\text{div}(f_k), \quad \text{div}(\widehat{\mu}_k \partial_t^k H) = -\text{div}(g_k). \quad (7.22)$$

Finally, in order to discuss the observability and energy estimates, we introduce the energy, dissipation, and error terms

$$\begin{aligned} e_k(t) &= \frac{1}{2} \max_{0 \leq j \leq k} \left(\left\| \widehat{\varepsilon}_j^{1/2} \partial_t^j E(t) \right\|_{L^2(\Omega)}^2 + \left\| \widehat{\mu}_j^{1/2} \partial_t^j H(t) \right\|_{L^2(\Omega)}^2 \right), & e &:= e_3, \\ d_k(t) &= \max_{0 \leq j \leq k} \left\| \lambda^{1/2} \text{tr}_t \partial_t^j E(t) \right\|_{L^2(\partial\Omega)}^2, & d &:= d_3, \\ z_k(t) &= \max_{0 \leq j \leq k} \left(\left\| \partial_t^j E(t) \right\|_{H^{k-j}(\Omega)}^2 + \left\| \partial_t^j H(t) \right\|_{H^{k-j}(\Omega)}^2 \right), & z &:= z_3, \end{aligned}$$

for the time-derived fields analogously to (4.37). Note that the dissipation term d differs compared to (4.37). One can show the following estimates for the commutator terms

$$\begin{aligned} \max_{2 \leq k \leq 3, 0 \leq j \leq 1} \left\| \partial_t^j f_k \right\|_{H^{4-j-k}(\Omega)} + \left\| \partial_t^j g_k \right\|_{H^{4-j-k}(\Omega)} &\lesssim z(t), \\ \max_{2 \leq k \leq 3} \|h_k\|_{H^{3+1/2-k}(\partial\Omega)} &\lesssim z(t), \end{aligned} \quad (7.23)$$

see equation (2.22) in [51], Lemma 4.2.1 and the proof sketch thereafter. For h_k the argument can be repeated, after extending h_k to $\overline{\Omega}$ using the trace theorem. We stress that z is quadratic in the fields and bounded by $\delta z(t)^{1/2}$ due to (7.14).

We define the function $m(x) = x - x_0$ on Ω for a fixed point $x_0 \in \mathbb{R}^3$ as in Chapter 2. It is also used in the multiplier argument from [51].

7.2. MAIN RESULT FOR THE NONLINEAR PROBLEM

We state our main decay theorem and the core ingredients for its proof. Based on them, we show the theorem at the end of the section. Note that compared to Theorem 2.2 in [51] we no longer require the domain to be strictly starshaped and also permit semilinear boundary

damping. In [51] the strict starshapedness of the domain was used to estimate trace terms, which we will treat here in a more delicate manner, utilizing methods from microlocal analysis as previously introduced in [18].

We require the following *non-trapping condition* on ε and μ , which emanates from the Morawetz multiplier technique used in [51]:

$$\begin{aligned}\varepsilon(x, 0) + (m(x) \cdot \nabla_x) \varepsilon(x, 0) &\geq \tilde{\eta} \varepsilon(x, 0), \\ \mu(x, 0) + (m(x) \cdot \nabla_x) \mu(x, 0) &\geq \tilde{\eta} \mu(x, 0),\end{aligned}\tag{7.24}$$

for a constant $\tilde{\eta} > 0$ and $x \in \bar{\Omega}$. It says that ε and μ do not decay too rapidly in radial directions for small fields. Heuristically this should reduce back reflections preventing the fields to reach the boundary. Similar conditions were used in [15], [18], and [44], for instance.

Theorem 7.2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded, multiply connected domain with C^5 -boundary $\partial\Omega = \cup_{i=0}^K \partial\Omega_i$. Furthermore, assume that the permittivity ε and permeability μ satisfy (7.6), (7.7), (7.9), and (7.24). We require that the initial values $(E_0, H_0) \in (H^3(\Omega))^6$ satisfy the compatibility conditions (7.12), as well as the initial “charge” conditions (7.17) and (7.18), i.e., $E_0 \in H^{\partial\Omega}(\operatorname{div}_\varepsilon 0)$, $H_0 \in H^{\partial\Omega}(\operatorname{div}_\mu 0)$.*

Then there exist constants $M \geq 1$ and $\omega, r > 0$ such that for $\|E_0\|_{H^3(\Omega)}^2 + \|H_0\|_{H^3(\Omega)}^2 \leq r^2$ the solutions $(E, H) \in \mathcal{G}^3$ exist for all times $t \geq 0$, are unique, and decay exponentially, i.e.,

$$\max_{0 \leq j \leq 3} \left\| \left(\partial_t^j E(t), \partial_t^j H(t) \right) \right\|_{H^{3-j}(\Omega)}^2 \leq M e^{-\omega t} \|(E_0, H_0)\|_{H^3(\Omega)}^2.$$

Theorem 7.2.1 is a consequence of the following three propositions on solutions of (7.20) and (7.21): an energy inequality, an observability-type estimate and a regularity result. All contain small error terms of highest order. It is crucial that constants do not depend on time.

Proposition 7.2.2. *Assume the hypotheses of Theorem 7.2.1, except for (7.24), (7.17), and (7.18). Then we have*

$$e(t) + \int_s^t d(\tau) \, d\tau \leq e(s) + c_1 \int_s^t z^{3/2}(\tau) \, d\tau$$

for $0 \leq s \leq t < T_*$.

This proposition controls energy and dissipation by the initial energy plus error terms. The result was shown in Proposition 3.1 of [51] for linear $\lambda = \lambda(x)$ and hence for $h_2 = h_3 = 0$. It is straightforward to extend the proof in [51] to the present situation with an addition $\oint \operatorname{tr}_t \partial_t^k E \cdot h_k \, dx \, d\tau$ term, using (7.23) to estimate the arising terms with h_2 and h_3 by the $z^{3/2}$ -integral and the d -integral to absorb $\delta \oint \operatorname{tr}_t |\partial_t^k E|^2 \, dx \, d\tau$.

In the next proposition we bound the time integral of the energy by the dissipation, the energy of initial and present time, as well as error terms.

Proposition 7.2.3. *Assume that the assumptions of Theorem 7.2.1 are satisfied. For $0 \leq s \leq t < T_*$ we obtain*

$$\int_s^t e(\tau) \, d\tau \leq c_2 \int_s^t d(\tau) \, d\tau + c_3(e(t) + e(s)) + c_4 \int_s^t z^{3/2}(\tau) \, d\tau.$$

Note that system (5.1) covers our system (7.20) and (7.21), if we substitute $\alpha = \hat{\varepsilon}_k$, $\beta = \hat{\mu}_k$, $\gamma = \hat{\lambda}_k$, $\varphi = -f_k$, $\psi = -g_k$ and $h = -h_k \times \nu$ for $k \in \{0, 1, 2, 3\}$. In view of (7.22) and (7.23), it remains to control the H^{-1} -norm of the normal traces in Lemma 5.2.2 to obtain Proposition 7.2.3. To this end, we return to our original problem. Here it does not help to

bound them by the norm of u and v in $H(\operatorname{div}_\alpha)$ and $H(\operatorname{div}_\beta)$, since the time integrals of the squared L^2 -norms of u and v cannot be absorbed by the left-hand side.

We crucially exploit that the time derivatives of solutions still solve a Maxwell system. We begin with $(\partial_t^k E, \partial_t^k H)$ for $k \geq 1$ and obtain estimates with dissipation terms d_{k-1} on the right-hand side. In turn the estimate for $(\partial_t E, \partial_t H)$ from Lemma 7.2.4 allows us to treat (E, H) in Lemma 7.2.5, where we also use the invertibility of curl in our setting to be proved in Theorem A.2.9 b). To this purpose we assume again that our initial fields do not enclose any charges, see (7.18).

Lemma 7.2.4. *Let the assumptions of Theorem 7.2.1 hold. For the solution $(E, H) \in \mathcal{G}^3$ of (7.1)–(7.5) we can estimate the normal trace by*

$$\int_s^t \left\| \nu \cdot (\widehat{\varepsilon}_k \partial_t^k E, \widehat{\mu}_k \partial_t^k H) \right\|_{H^{-1}(\partial\Omega)}^2 d\tau \lesssim \int_s^t (d(\tau) + z(\tau)^2) d\tau$$

for $0 \leq s \leq t < T_*$ and $k \in \{1, 2, 3\}$.

Proof. Let $k \in \{1, 2, 3\}$. Due to (7.19) we can rewrite $\widehat{\varepsilon}_k \partial_t^k E$ as

$$\widehat{\varepsilon}_k \partial_t^k E = \partial_t^k (\varepsilon E) - f_k = \partial_t (\widehat{\varepsilon}_{k-1} \partial_t^{k-1} E + f_{k-1}) - f_k.$$

The Maxwell system (7.20) then leads to

$$\widehat{\varepsilon}_k \partial_t^k E = \operatorname{curl} \partial_t^{k-1} H - \partial_t f_{k-1} + \partial_t f_{k-1} - f_k = \operatorname{curl} \partial_t^{k-1} H - f_k$$

Let $\Phi \in H^1(\partial\Omega)$. The equations above and Lemma A.1.6 yield

$$\begin{aligned} \langle \nu \cdot \widehat{\varepsilon}_k \partial_t^k E, \Phi \rangle_{H^{-1/2}(\partial\Omega)} &= \langle \nu \cdot (\operatorname{curl} \partial_t^{k-1} H), \Phi \rangle_{H^{-1}(\partial\Omega)} - \langle \nu \cdot f_k, \Phi \rangle_{L^2(\partial\Omega)} \\ &\lesssim \left\| \nu \times \partial_t^{k-1} H \right\|_{L^2(\partial\Omega)} \|\Phi\|_{H^1(\partial\Omega)} + \|f_k\|_{L^2(\partial\Omega)} \|\Phi\|_{L^2(\partial\Omega)}. \end{aligned}$$

Dividing by $\|\Phi\|_{H^1(\partial\Omega)}$, we infer

$$\begin{aligned} \left\| \nu \cdot \widehat{\varepsilon}_k \partial_t^k E \right\|_{H^{-1}(\partial\Omega)} &\lesssim \left\| \nu \times \partial_t^{k-1} H \right\|_{L^2(\partial\Omega)} + \|f_k\|_{H^1(\Omega)} \\ &\lesssim \left\| \nu \times \partial_t^{k-1} E \right\|_{L^2(\partial\Omega)} + \|h_{k-1}\|_{L^2(\partial\Omega)} + \|f_k\|_{H^1(\Omega)}, \end{aligned}$$

where we also used the boundary condition (7.21). Analogously we treat $\widehat{\mu}_k \partial_t^k H$. Estimate (7.23) now implies the assertion. $\square$

Lemma 7.2.5. *Let the assumptions of Theorem 7.2.1 hold. For the solution $(E, H) \in \mathcal{G}^3$ of (7.1)–(7.5) we estimate*

$$\int_s^t \left\| \nu \cdot (\widehat{\varepsilon}_0 E, \widehat{\mu}_0 H) \right\|_{H^{-1}(\partial\Omega)}^2 d\tau \lesssim \int_s^t d(\tau) d\tau + (e_1(t) + e_1(s)) + \int_s^t z^2(\tau) d\tau$$

for $0 \leq s \leq t < T_*$.

Proof. The usual normal trace estimate, (7.2), Theorem A.2.9 b), (7.19), and (7.20) yield

$$\begin{aligned} \int_s^t \left\| \nu \cdot (\widehat{\varepsilon}_0 E, \widehat{\mu}_0 H) \right\|_{H^{-1}(\partial\Omega)}^2 d\tau &\lesssim \int_s^t \left\| (\widehat{\varepsilon}_0 E, \widehat{\mu}_0 H) \right\|_{H(\operatorname{div})}^2 d\tau \\ &= \int_s^t \left\| (\widehat{\varepsilon}_0 E, \widehat{\mu}_0 H) \right\|_{L^2(\Omega)}^2 d\tau \end{aligned}$$

$$\begin{aligned}
&\lesssim \int_s^t \|(\operatorname{curl} E, \operatorname{curl} H)\|_{L^2(\Omega)}^2 d\tau \\
&= \int_s^t \|(\widehat{\varepsilon}_1 \partial_t E, \widehat{\mu}_1 \partial_t H)\|_{L^2(\Omega)}^2 d\tau \\
&\lesssim \int_s^t (\partial_t E \cdot \widehat{\varepsilon}_1 \partial_t E + \partial_t H \cdot \widehat{\mu}_1 \partial_t H) d\tau.
\end{aligned}$$

Using Lemma 5.2.2 with $u = \partial_t E$, $v = \partial_t H$, $\alpha = \widehat{\varepsilon}_1$, $\beta = \widehat{\mu}_1$ and $\varphi = \psi = h = 0$, we then obtain

$$\begin{aligned}
\int_s^t \|\nu \cdot (\widehat{\varepsilon}_0 E, \widehat{\mu}_0 H)\|_{H^{-1}(\partial\Omega)}^2 d\tau &\lesssim \int_s^t d(\tau) d\tau + (e_1(t) + e_1(s)) \\
&\quad + \int_s^t \|\nu \cdot (\widehat{\varepsilon}_1 \partial_t E, \widehat{\mu}_1 \partial_t H)\|_{H^{-1}(\partial\Omega)}^2 d\tau.
\end{aligned}$$

By the previous lemma the last term can be estimated by $\int_s^t (d(\tau) + z(\tau)^2) d\tau$ and the claim follows. $\square$

Combining Lemma 5.2.2, 7.2.4, and 7.2.5, as well as (7.23), we have shown Proposition 7.2.3. Exactly as in Corollary 3.5 of [51], Propositions 7.2.2 and 7.2.3 imply the estimate

$$e(t) + \int_s^t e(\tau) d\tau \leq C_1 e(s) + C_2 \int_s^t z^{3/2}(\tau) d\tau \quad (7.25)$$

for $0 \leq s \leq t < T_*$. Without the z -term it is well known how to derive exponential stability from here. However, despite $\sqrt{z} \leq \delta$ being small, we cannot absorb the error term with z by the left-hand side. To achieve this, we have to bound the space derivatives of (E, H) by their time derivatives. This can be done exploiting the structure of the Maxwell systems (7.1)–(7.5), resulting in the following third ingredient of the proof of Theorem 7.2.1.

Proposition 7.2.6. *Under the assumptions of Theorem 7.2.1, with the exception of (7.24) and (7.18), there exist constants $c_5, c_6 > 0$ such that*

$$z(t) \leq c_5 e(t) + c_6 z^2(t)$$

for $0 \leq t < T_*$.

We sketch the proof of this result in Section 7.3, heavily relying on the proof of Proposition 4.1 in [51], cf. the proof of Lemma 4.5.3. Based on the propositions above, we can now show our main result of this chapter.

Proof of Theorem 7.2.1. The reasoning follows the lines of the proof of Theorem 2.2 of [51]. For convenience, we present the arguments briefly. We first combine estimate (7.25) and Proposition 7.2.6. Fixing a sufficiently small $\delta = \bar{\delta} \in (0, \tilde{\delta}]$, we derive

$$z(t) + \int_s^t z(\tau) d\tau \leq C z(s) \quad (7.26)$$

for all $0 \leq s \leq t < T_*$ and a constant $C \geq 1$.

The claim now follows as in the proof of Theorem 4.6.1 after (4.108) therein. $\square$

7.3. REGULARITY ANALYSIS

Due to the nonlinear boundary conditions, one has to account for extra terms compared to the proof of Proposition 4.1 in [51], but we can still follow the reasoning given there. Therefore, we will only outline the proof and focus on the differences to [51] as the rest follows as in the proof of Lemma 4.5.3.

Proof of Proposition 7.2.6. Remember that z is given by

$$z(t) = \max_{0 \leq j \leq 3} \left(\left\| \partial_t^j E(t) \right\|_{H^{3-j}(\Omega)}^2 + \left\| \partial_t^j H(t) \right\|_{H^{3-j}(\Omega)}^2 \right)$$

and e is defined as

$$e(t) = \frac{1}{2} \max_{0 \leq j \leq 3} \left(\left\| \hat{\varepsilon}_j^{1/2} \partial_t^j E(t) \right\|_{L^2(\Omega)}^2 + \left\| \hat{\mu}_j^{1/2} \partial_t^j H(t) \right\|_{L^2(\Omega)}^2 \right).$$

The L^2 -terms of z can thus be trivially bounded by $e(t)$. The other (squared) norms will be estimated by $e(t)$ or $z(t)^2$ or by means of previous steps.

Compared to [51] we only have to modify the arguments that involve boundary conditions of differentiated problems. Here additional error terms appear as λ also depends on E . They enter our reasoning only through the div-curl Theorem A.2.9 a) which controls the H^1 -norm for fields (u, v) by the estimate (A.4). Here we have to control the $H^{1/2}$ -norm of the boundary inhomogeneities on the right-hand side. We follow the steps of the proof given in Section 6 of [51].

1) *H^1 -estimates for $\partial_t^k E$ and $\partial_t^k H$.* The (squared) H^1 -norm of $\partial_t^k(E, H)$ for $k \in \{0, 1, 2\}$ is bounded via the div-curl estimate. Only for $k = 2$ the boundary inhomogeneity h_2 is non-zero, where we have $\|h_2\|_{H^{1/2}(\partial\Omega)} \lesssim z(t)$ by (7.23), as desired.

2) *Interior spatio-temporal estimates for E and H .* One decomposes $(E, H) = (1 - \chi)(E, H) + \chi(E, H)$ for a cutoff function χ , being equal to 1 near the boundary $\partial\Omega$. In the estimates for $(1 - \chi)\partial_t^k(E, H)$ the boundary does not play a role so that the reasoning from [51] has not to be modified, see also the proof of Lemma 4.5.3.

3) *Preparation for the boundary collar estimates for E and H .* To get H^2 - or H^3 - bounds on $\partial_t^k \chi(E, H)$, we employ tangential and normal derivatives ∂_τ and ∂_ν (extended to a neighborhood of $\partial\Omega$). We use the div-curl Theorem A.2.9 a) for $\partial_\tau \partial_t^k(\chi(E, H))$ with $k \in \{0, 1\}$ and for $\partial_\tau^2(\chi(E, H))$. Differentiating (7.21) with $h_0 = 0$ or $h_1 = 0$, we obtain

$$\begin{aligned} \partial_\tau \partial_t^k \chi H \times \nu + \left(\hat{\lambda}_k(\partial_\tau \partial_t^k \chi E \times \nu) \right) \times \nu &= -\partial_t^k \chi H \times \partial_\tau \nu \\ &- \left(\partial_\tau \hat{\lambda}_k(\partial_t^k \chi E \times \nu) \right) \times \nu - \left(\hat{\lambda}_k(\partial_t^k \chi E \times \partial_\tau \nu) \right) \times \nu \\ &- \left(\hat{\lambda}_k(\partial_t^k \chi E \times \nu) \right) \times \partial_\tau \nu, \end{aligned} \tag{7.27}$$

and

$$\begin{aligned} \partial_\tau^2(\chi H) \times \nu + (\lambda \partial_\tau^2(\chi E) \times \nu) \times \nu &= -2\partial_\tau(\chi H) \times \partial_\tau \nu - (\chi H) \times \partial_\tau^2 \nu \\ &- ((\partial_\tau^2 \lambda)(\chi E \times \nu)) \times \nu - (\lambda(\chi E) \times \partial_\tau^2 \nu) \times \nu - (\lambda(\chi E) \times \nu) \times \partial_\tau^2 \nu \\ &- 2((\partial_\tau \lambda)(\partial_\tau(\chi E) \times \nu)) \times \nu - 2((\partial_\tau \lambda)(\chi E \times \partial_\tau \nu)) \times \nu \\ &- 2((\partial_\tau \lambda)(\chi E \times \nu)) \times \partial_\tau \nu - 2(\lambda \partial_\tau(\chi E) \times \partial_\tau \nu) \times \nu \\ &- 2(\lambda(\partial_\tau \chi E) \times \nu) \times \partial_\tau \nu - 2(\lambda \chi E \times \partial_\tau \nu) \times \partial_\tau \nu, \end{aligned}$$

with $\hat{\lambda}_0 = \lambda(\cdot, E)$ and $\hat{\lambda}_1 = \lambda_d(\cdot, E)$. The remaining estimates involving ∂_ν are entirely based on the curl and div equations (7.20) respectively (7.22) and thus carry over from [51].

4) H^2 -estimates for E and H . When estimating (E, H) in H^2 , we need the boundary condition (7.27) with $k = 0$. Compared to [51], the only new term is

$$\tilde{h}_k := \left(\partial_\tau \hat{\lambda}_k (\partial_t^k E \times \nu) \right) \times \nu$$

with $k = 0$, where we note that

$$\partial_j \hat{\lambda}_k(x, t) = (\partial_j \hat{\lambda}_k)(x, E(x, t)) + \sum_{i=1}^3 \partial_{\xi_i} \hat{\lambda}_k(x, E(x, t)) \partial_j E_i(x, t). \quad (7.28)$$

We can thus bound this error term by

$$\begin{aligned} \|\tilde{h}_0\|_{H^{1/2}(\partial\Omega)} &\lesssim \|\partial_\tau \lambda(\cdot, E)E\|_{H^1} \\ &\lesssim \|\partial_x \lambda\|_{W^{1,\infty}} \|E\|_{H^1} + \|\partial_\xi \lambda\|_{W^{1,\infty}} \|E\|_{W^{1,\infty}} \|E\|_{H^1} \\ &\lesssim \|E\|_{H^1} + z(t) \end{aligned} \quad (7.29)$$

as desired, using Sobolev's embedding and $z \leq 1$ by (7.14). (Observe that $\|E\|_{H^1}$ and $\|\partial_t E\|_{H^1}$ were already handled in step 1).)

5) H^2 -estimates for $\partial_t E$ and $\partial_t H$. To bound $\partial_t(E, H)$ in H^2 we use (7.27) with $k = 1$. In (7.29) one only has to replace $\|E\|_{H^1}$ by $\|\partial_t E\|_{H^1}$ in order to show

$$\|\tilde{h}_1\|_{H^{1/2}(\partial\Omega)} \lesssim \|\partial_t E\|_{H^1} + z(t)$$

as before.

6) H^3 -estimates for E and H . We finally treat the H^3 -norm of (E, H) . Compared to steps 4) and 5), in the boundary condition only the term $\hat{h}$ with $\partial_\tau^2 \lambda$ poses new difficulties. To tackle it, we differentiate (7.28) once more in x and employ Sobolev's embedding. It follows

$$\begin{aligned} \|\hat{h}\|_{H^{1/2}(\partial\Omega)} &\lesssim \left\| \partial_\tau^2 \lambda(\cdot, E)E \right\|_{H^1} \lesssim \left\| \partial_x^2 \lambda \right\|_{W^{1,\infty}} \|E\|_{H^1} + \|\partial_{x\xi} \lambda\|_{W^{1,\infty}} \|E\|_{H^2}^2 \\ &\quad + \left\| \partial_\xi^2 \lambda \right\|_{W^{1,\infty}} \|E\|_{H^2}^2 \|E\|_{H^3} + \|\partial_\xi \lambda\|_{W^{1,\infty}} \|E\|_{H^3} \|E\|_{H^2} \\ &\lesssim \|E\|_{H^1} + z(t). \end{aligned} \quad \square$$

APPENDIX

A.1. PROPERTIES OF THE CURL OPERATOR

We first list basic mapping properties of the gradient, divergence and curl. They can be found in Theorem 2.7 of [12], Chapter IX.1.3 of [16], or follow similarly. Unless otherwise specified, let $\Omega \subset \mathbb{R}^3$ be an open, bounded, multiply connected set with a Lipschitz boundary. We assume sufficient boundary regularity as in [46] and [45] for the application of the results discussed below. However, the regularity assumptions may not be optimal.

Lemma A.1.1. *Let $\Omega \subset \mathbb{R}^3$ be open, bounded and multiply connected with Lipschitz boundary $\partial\Omega$. In the following diagram the image of each operator is of finite codimension and contained in the kernel of the succeeding operator:*

$$\begin{aligned} H^1(\Omega) &\xrightarrow{\nabla} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2(\Omega), \\ H_c^1(\Omega) &\xrightarrow{\nabla} H_{t0}(\text{curl}) \xrightarrow{\text{curl}} H_{n0}(\text{div}) \xrightarrow{\text{div}} L^2(\Omega), \end{aligned}$$

where $H_c^1(\Omega) := \{f \in H^1(\Omega) \mid \text{tr}_{\Gamma_k} f \text{ constant for each } k \in \{0, \dots, K\}\} \supset H_0^1(\Omega)$.

Propositions 2.2.30 and 2.2.32 in [5] provide a useful characterization of $H(\text{div})$ and $H(\text{curl})$ in the presence of an interface Γ .

Lemma A.1.2. *Let $\alpha \in L^\infty(\Omega; \mathbb{R}^{3 \times 3})$ invertible with $\alpha^{-1} \in L^\infty(\Omega; \mathbb{R}^{3 \times 3})$ and $\Omega = v \cup \Gamma \cup \omega$. We then have*

$$\begin{aligned} a) \quad H(\text{div}_\alpha, \Omega) &= \{f \in L^2(\Omega) \mid \alpha f|_v \in H(\text{div}, v), \alpha f|_\omega \in H(\text{div}, \omega), \\ &\quad \llbracket \alpha f \cdot \nu^\Gamma \rrbracket = 0 \text{ in } H^{-1/2}(\Gamma)\}, \\ b) \quad H(\text{curl}_\alpha, \Omega) &= \{f \in L^2(\Omega) \mid \alpha f|_v \in H(\text{curl}, v), \alpha f|_\omega \in H(\text{curl}, \omega), \\ &\quad \llbracket \alpha f \times \nu^\Gamma \rrbracket = 0 \text{ in } H^{-1/2}(\Gamma)\}. \end{aligned}$$

The cohomology spaces

$$\begin{aligned} \mathbb{H}_1 &:= \{f \in L^2(\Omega) \mid \text{curl } f = 0, \text{div } f = 0, \text{tr}_n = 0\}, \\ \mathbb{H}_2 &:= \{f \in L^2(\Omega) \mid \text{curl } f = 0, \text{div } f = 0, \text{tr}_t f = 0\}, \end{aligned}$$

contain functions which prevent the invertibility of curl. In our setting they have finite dimensions, which are determined by topological properties of the domain Ω , see Remark A.1.5 and also Sections 3.2, 6.1 and 6.2 in [5], Section 2.9 in [12], or Section IX.1.3 in [16] for more details.

The next lemma collects results on mapping properties of curl from Proposition 6.1.3, Theorem 6.1.4, Corollary 6.2.4 and Theorem 6.2.5 of [5] and from Proposition 3, Proposition 4 and Remark 5 of Section IX.1 in [16] concerning the orthogonal complements of the cohomology spaces.

Lemma A.1.3. *Let $\Omega \subset \mathbb{R}^3$ be open, bounded and multiply connected with Lipschitz boundary $\partial\Omega$. Let $\varepsilon, \mu \in C^1(\overline{\Omega}, \mathbb{R}_{\text{sym}}^{3 \times 3})$ be uniformly positive definite. Then,*

a) the ‘electric’ curl

$$\begin{aligned} \text{curl}_E: H_{t0}(\text{curl}) \cap H(\text{div}_\varepsilon 0) &\rightarrow H_{n0}^\Sigma(\text{div } 0), \quad \text{with} \\ \ker(\text{curl}_E) &= \mathbb{H}_2^\varepsilon := H_{t0}(\text{curl } 0) \cap H(\text{div}_\varepsilon 0) \end{aligned}$$

b) and the ‘magnetic’ curl

$$\begin{aligned} \operatorname{curl}_H: H(\operatorname{curl}) \cap H_{n0}(\operatorname{div}_\mu 0) &\rightarrow H^{\partial\Omega}(\operatorname{div} 0), \quad \text{with} \\ \ker(\operatorname{curl}_H) &= \mathbb{H}_1^\mu := H(\operatorname{curl} 0) \cap H_{n0}(\operatorname{div}_\mu 0), \end{aligned}$$

are surjective. Furthermore, the image sets can be characterized by

$$H_{n0}^\Sigma(\operatorname{div} 0) = \mathbb{H}_1^{\perp_{H_{n0}(\operatorname{div} 0)}}, \quad H^{\partial\Omega}(\operatorname{div} 0) = \mathbb{H}_2^{\perp_{H(\operatorname{div} 0)}}. \quad (\text{A.1})$$

In view of the next result, (A.1) yields the invertibility of

$$\begin{aligned} \operatorname{curl}_E: H_{t0}(\operatorname{curl}) \cap H^{\partial\Omega}(\operatorname{div}_\varepsilon 0) &\rightarrow H_{n0}^\Sigma(\operatorname{div} 0) && \text{in a) and} \\ \operatorname{curl}_H: H(\operatorname{curl}) \cap H_{n0}^\Sigma(\operatorname{div}_\mu 0) &\rightarrow H^{\partial\Omega}(\operatorname{div} 0) && \text{in b).} \end{aligned}$$

Moreover, for $\partial\Omega \in C^2$ Theorem A.2.2 and Remark A.2.3 show that the domains of curl_E and curl_H are subsets of $H^1(\Omega)$.

We have used the following description of the weighted cohomology spaces.

Lemma A.1.4. *Let $\Omega \subset \mathbb{R}^3$ be open, bounded and multiply connected, and $\partial\Omega \in C^2$. Let $\varepsilon, \mu \in C^1(\overline{\Omega}, \mathbb{R}_{\text{sym}}^{3 \times 3})$ be uniformly positive definite. For $u \in H(\operatorname{div}_\varepsilon 0)$ the field εu belongs to $\mathbb{H}_2^{\perp_{L^2}}$ if and only if u is contained in $(\mathbb{H}_2^\varepsilon)^{\perp_{L_\varepsilon^2}}$. Analogously, for $u \in H_{n0}(\operatorname{div}_\mu 0)$ the field μu belongs to $\mathbb{H}_1^{\perp_{L^2}}$ if and only if u is contained in $(\mathbb{H}_1^\mu)^{\perp_{L_\mu^2}}$.*

Proof. For the first equivalence, we assume that $\varepsilon u \in H^{\partial\Omega}(\operatorname{div} 0)$. Equation (IX.1.61) in [16] yields the decomposition

$$L^2(\Omega)^3 = \nabla H_0^1(\Omega) \oplus \mathbb{H}_2 \oplus \operatorname{curl} H^1(\Omega).$$

Since $H(\operatorname{div} 0)^{\perp_{L^2}} = \nabla H_0^1(\Omega)$ by (IX.1.42) in [16], our assumptions and (A.1) imply that εu is contained in $\operatorname{curl} H^1(\Omega)$; i.e., $\varepsilon u = \operatorname{curl} \Phi$ for some $\Phi \in H^1(\Omega)$. Hence, for all $h \in \mathbb{H}_2^\varepsilon$ we derive

$$(u, h)_{L_\varepsilon^2} = (\varepsilon u, h)_{L^2} = (\operatorname{curl} \Phi, h)_{L^2} = 0.$$

The converse implication similarly follows from the decomposition

$$L_\varepsilon^2(\Omega) = \nabla H_0^1(\Omega) \oplus_{L_\varepsilon^2} \mathbb{H}_2^\varepsilon \oplus_{L_\varepsilon^2} \varepsilon^{-1} \operatorname{curl} H^1(\Omega), \quad (\text{A.2})$$

see Propositions 6.1.1 and 6.1.12 in [5] and Proposition IX.1.3 in [16].

The second assertion is shown analogously, using Propositions 6.2.1 and 6.2.12 in [5] and Proposition IX.1.4 and Remark IX.1.5 in [16]. $\square$

Remark A.1.5. Let $\Omega \subset \mathbb{R}^3$ be open, bounded and multiply connected, and $\partial\Omega \in C^2$. The dimension of the first cohomology space equals the ‘cutting number’ L of Ω , i.e.,

$$\dim \mathbb{H}_1 = \dim \ker(\operatorname{curl}_H) = L.$$

For simply connected Ω it is therefore trivial. Moreover, we have

$$\dim \mathbb{H}_2 = \dim \ker(\operatorname{curl}_E) = K.$$

Thus, if $\partial\Omega$ is connected, the second cohomology space equals $\{0\}$. See Proposition 2.8 in [12], as well as Proposition 6.1.1 and Proposition 6.2.1 in [5].

We end this section by proving the following estimates for normal traces of the curl operator.

Lemma A.1.6. For $f \in H^1_\Gamma(\Omega)$ we can estimate

a) the normal surface curl by

$$\|\nu \cdot (\operatorname{curl} f)\|_{H^{-1}(\partial\Omega)} \lesssim \|\nu \times f\|_{L^2(\partial\Omega)},$$

b) and the normal jump at the interface by

$$\left\| \llbracket \nu^\Gamma \cdot (\operatorname{curl} f) \rrbracket \right\|_{H^{-1}(\Gamma)} \lesssim \left\| \llbracket \nu^\Gamma \times f \rrbracket \right\|_{L^2(\Gamma)}.$$

Proof. a) We localize f near $\partial\Omega$ by some cutoff $\chi_{\partial\Omega} \in C^\infty$ and apply Lemma 4.8 from [46]. It is proven as part b) below. See also Chapter 3.1 of [5].

b) Likewise, we choose $\chi_\Gamma \in C_c^\infty(\mathcal{N}_a^\Gamma)$, for a small $a > 0$ such that $\mathcal{N}_a^\Gamma \cap \partial\Omega = \emptyset$, with $0 \leq \chi^\Gamma \leq 1$ and $\chi_\Gamma = 1$ on $\mathcal{N}_{a/2}^\Gamma$ and essentially repeat the proof in [46].

Let $\phi \in H^1(\Gamma)$. This function can be extended in $H^{3/2}(\Omega)$ with $\|\phi\|_{H^{3/2}(\Omega)} \lesssim \|\phi\|_{H^1(\Gamma)}$. Take $\phi_m \in C^\infty(\overline{\Omega})$ with $\phi_m \rightarrow \phi$ in $H^{3/2}(\Omega)$ and $f_n \in C^\infty(\overline{\Omega} \setminus \Gamma)$ with $f_n \rightarrow f$ in $H^1_\Gamma(\Omega)$. Then the divergence theorem and integration by parts yield

$$\begin{aligned} \left| \oint_\Gamma \llbracket \nu^\Gamma \cdot \operatorname{curl} f_n \rrbracket \phi_m \, d\varsigma \right| &= \left| \oint_\Gamma \llbracket \nu^\Gamma \cdot \operatorname{curl}(\chi_\Gamma f_n) \rrbracket \phi_m \, d\varsigma \right| \\ &= \left| (\operatorname{curl}(\chi_\Gamma f_n) | \nabla \phi_m)_{L^2(v)} - (\operatorname{curl}(\chi_\Gamma f_n) | \nabla \phi_m)_{L^2(\omega)} \right| \\ &= \left| (\chi_\Gamma f_n | \operatorname{curl} \nabla \phi_m)_{L^2(v)} - (\chi_\Gamma f_n | \operatorname{curl} \nabla \phi_m)_{L^2(\omega)} \right| \\ &\quad + \left| \oint_\Gamma \llbracket \nu^\Gamma \times (\chi_\Gamma f_n) \rrbracket \cdot \nabla \phi_m \, d\varsigma \right| \\ &\leq 0 + \left\| \llbracket \nu^\Gamma \times (\chi_\Gamma f_n) \rrbracket \right\|_{L^2(\Gamma)} \|\operatorname{tr}_\tau(\nabla \phi_m)\|_{L^2(\Gamma)} \\ &\lesssim \left\| \llbracket \nu^\Gamma \times f_n \rrbracket \right\|_{L^2(\Gamma)} \|\phi_m\|_{H^1(\Gamma)}. \end{aligned}$$

As $m \rightarrow \infty$ we derive

$$\left\| \llbracket \nu^\Gamma \cdot (\operatorname{curl} f_n) \rrbracket \right\|_{H^{-1}(\Gamma)} \lesssim \left\| \llbracket \nu^\Gamma \times f_n \rrbracket \right\|_{L^2(\Gamma)},$$

and thus the claim letting n tend to infinity. $\square$

A.2. DIV-CURL ESTIMATES

Div-curl estimates allow for control of Sobolev norms through the L^2 norms, the divergence and the curl of a given function, if additional information at the boundary is available. Usually this means that either $u \in H_{n0}(\operatorname{div})$ or $u \in H_{t0}(\operatorname{curl})$. For example the following version can be found in literature, cf. Corollary 3.5 in [3].

Lemma A.2.1. Let $k \in \mathbb{N}$, $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$ of class $C^{k,1}$. Assume that $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ with $\overline{\partial\Omega_D} \cap \overline{\partial\Omega_N} = \emptyset$. Let $u \in L^2(\Omega)$ with $\operatorname{div} u \in H^{k-1}(\Omega)$, $\operatorname{curl} u \in H^{k-1}(\Omega)^3$ such that $u \cdot \nu \in H^{k-1/2}(\partial\Omega_N)$ and $u \times \nu \in H^{k-1/2}(\partial\Omega_D)^3$. Then $u \in H^k(\Omega)$ with

$$\begin{aligned} \|u\|_{H^k(\Omega)} &\lesssim \|u\|_{L^2(\Omega)} + \|\operatorname{div} u\|_{H^{k-1}(\Omega)} + \|\operatorname{curl} u\|_{H^{k-1}(\Omega)} \\ &\quad + \|u \cdot \nu\|_{H^{k-1/2}(\partial\Omega_N)} + \|u \times \nu\|_{H^{k-1/2}(\partial\Omega_D)}. \end{aligned}$$

For our analysis, we need more general variants of this result presented below.

A.2.1. HIGHER-ORDER DIV-CURL ESTIMATE FOR INTERFACE PROBLEMS WITH PERFECTLY CONDUCTING BOUNDARY CONDITIONS

We recall the following result, Theorem 1.4 in [10], which is based upon Proposition 2.24 in [17]. We note that the connectedness condition on Ω^c in [17] is not needed.

Theorem A.2.2 (Div-Curl Lemma). *Let $k \in \mathbb{N}$, $\hat{k} := \max\{k, 2\}$, $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2 \subset \mathbb{R}^3$ bounded, where $\Gamma \subset \Omega$ is an interface with $\text{dist}(\Gamma, \partial\Omega) > 0$. Let Γ and $\partial\Omega$ be of class $C^{\hat{k}+1}$ and $\alpha|_{\Omega_i} \in H^k(\Omega_i, \mathbb{R}_{\text{sym}}^{3 \times 3})$ for $i = 1, 2$ be uniformly positive definite, i.e., $\alpha|_{\Omega_i} \geq \eta > 0$. Let $u \in L^2(\Omega)^3$ satisfy either of the cases*

1. $u \in H_{t0}(\text{curl}, \Omega)$ and $\alpha u + v \in H(\text{div}, \Omega)$, or

2. $u \in H(\text{curl}, \Omega)$ and $\alpha u + v \in H_{n0}(\text{div}, \Omega)$

for some $v \in H_\Gamma^k$. Furthermore, let $\text{curl } u \in H_\Gamma^{k-1}(\Omega)$ and $\text{div}(\alpha u) \in H_\Gamma^{k-1}(\Omega)$. Then u is contained in H_Γ^k , and it holds

$$\|u\|_{H_\Gamma^k} \leq C(\|u\|_{L^2(\Omega)} + \|\text{curl } u\|_{H_\Gamma^{k-1}} + \|\text{div}(\alpha u)\|_{H_\Gamma^{k-1}} + \|v\|_{H_\Gamma^k})$$

for a constant $C > 0$ depending only on k , $\|\alpha\|_{H_\Gamma^k}$, η , Ω_1 , and Ω_2 .

Remark A.2.3. The div-curl lemma relies on regularity results for an elliptic transmission problem, see [10], [13], and [65]. The cited results are formulated for $C^k(\overline{\Omega_\pm})$ -coefficients α . However, it is possible to adapt the proofs to H_Γ^k -coefficients, relying on algebra-like bilinear Sobolev estimates as in Lemma 2.1 in [56], see also Lemma 4.2.2.

A.2.2. DIV-CURL ESTIMATE FOR ABSORBING BOUNDARY CONDITIONS

In this section we show a div-curl estimates for Silver-Müller boundary conditions in Theorem A.2.9. To this aim, we have to study the Helmholtz decomposition in $H^{-\theta}$ to some extent.

Remark A.2.4. Let $-\frac{1}{2} < s < \frac{1}{2}$. Here the dual $(H^s(\Omega))^*$ equals $H^{-s}(\Omega)$ since $C_c^\infty(\Omega)$ is dense in $H^s(\Omega)$, see Theorem 11.1 in [38] or Chapter 4.3 in [63]. As in [25], we define the normal trace $\text{tr}_n: H^s(\text{div}, \Omega) \rightarrow H^{s-\frac{1}{2}}(\partial\Omega)$ distributionally by

$$\langle \varphi, \text{tr}_n u \rangle_{H^{-s+\frac{1}{2}}(\partial\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)} = \langle \text{div } u, \Phi \rangle_{H^s(\Omega) \times H^{-s}(\Omega)} + \langle u, \nabla \Phi \rangle_{H^s(\Omega) \times H^{-s}(\Omega)}$$

for $\varphi \in C^\infty(\partial\Omega)$ and $\Phi \in C_c^\infty(U)$ with $\text{tr } \Phi = \varphi$ for a bounded domain $U \supset \overline{\Omega}$. This definition is independent of the choice of the continuation Φ of φ . Furthermore, Theorem 2.3 in [25] yields the continuity of tr_n .

Analogously we can define the tangential trace $\text{tr}_t: H^s(\text{curl}, \Omega) \rightarrow H^{s-\frac{1}{2}}(\partial\Omega)$ by

$$\langle \varphi, \text{tr}_t u \rangle_{H^{-s+\frac{1}{2}}(\partial\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)} = \langle u, \text{curl } \Phi \rangle_{H^s(\Omega) \times H^{-s}(\Omega)} - \langle \text{curl } u, \Phi \rangle_{H^s(\Omega) \times H^{-s}(\Omega)}.$$

We estimate

$$\begin{aligned} \langle \varphi, \text{tr}_t u \rangle_{H^{-s+\frac{1}{2}}(\partial\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)} &\lesssim \|\Phi\|_{H^{-s}(\Omega)} \|\text{curl } u\|_{H^s(\Omega)} + \|u\|_{H^s(\Omega)} \|\text{curl } \Phi\|_{H^{-s}(\Omega)} \\ &\lesssim \|u\|_{H^s(\text{curl})} \|\Phi\|_{H^{1-s}(\Omega)}. \end{aligned}$$

Note that $\|\varphi\|_{H^{\frac{1}{2}-s}(\partial\Omega)}$ is the infimum of $\|\Phi\|_{H^{1-s}(\Omega)}$, where $\Phi \in H^{1-s}(\Omega)$ with $\text{tr } \Phi = \varphi$.

Taking the infimum on the right-hand side of the inequality above, we thus obtain

$$\langle \varphi, \text{tr}_t u \rangle_{H^{-s+\frac{1}{2}}(\partial\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)} \lesssim \|u\|_{H^s(\text{curl})} \|\varphi\|_{H^{\frac{1}{2}-s}(\partial\Omega)}.$$

This yields the continuity of $\text{tr}_t: H^s(\text{curl}) \rightarrow H^{s-1/2}(\partial\Omega)$ of the tangential trace.

Introducing

$$\begin{aligned} H_{n0}^s(\operatorname{div} 0, \Omega) &= H_{n0}^s(\operatorname{div} 0) := \{f \in (H^s(\Omega))^3 \mid \operatorname{div} f = 0, \nu \cdot \operatorname{tr} f = 0\}, \\ G^s &= G^s(\Omega) := \{u \in (H^s(\Omega))^3 \mid \exists \varphi \in H^{s+1}(\Omega) : \nabla \varphi = u\}, \end{aligned}$$

we can recall the following Helmholtz decomposition from Theorem 3.1 in [25].

Lemma A.2.5. *Let $\Omega \subset \mathbb{R}^3$ be bounded with $C^{2,1}$ boundary $\partial\Omega$. Further, suppose that $-\frac{1}{2} < s < \frac{1}{2}$. We then have the topological direct sum*

$$(H^s(\Omega))^3 = H_{n0}^s(\operatorname{div} 0) \oplus G^s(\Omega).$$

THE CURL OPERATOR ON $H^{1-\theta}$

We state the estimate (2.5) from [18]. It is later utilized in an argument involving Peetre's lemma to show that the curl operator in a suitable setting has closed image.

Lemma A.2.6. *Let $\Omega \subset \mathbb{R}^3$ be bounded with C^2 -boundary, $\theta \in [0, \frac{1}{2})$, and $v \in H^{1-\theta}(\Omega)$ with $\operatorname{tr}_n v = 0$. We then have*

$$\|v\|_{H^{1-\theta}(\Omega)} \lesssim \|\operatorname{div} u\|_{H^{-\theta}(\Omega)} + \|\operatorname{curl} v\|_{H^{-\theta}(\Omega)} + \|v\|_{H^{-\theta}(\Omega)}.$$

The following result is well-known for $\theta = 0$, see Remark IX.1.4 of [16] or Lemma A.1.3 above, where variants without connectivity assumptions are proven.

Lemma A.2.7. *Let $\Omega \subset \mathbb{R}^3$ be bounded and simply connected with a connected $C^{2,1}$ boundary $\partial\Omega$. For $\theta \in (0, \frac{1}{2})$ the curl as an operator*

$$\operatorname{curl}: (H_{n0}^{1-\theta}(\operatorname{div} 0), \|\cdot\|_{H^{1-\theta}(\Omega)}) \rightarrow (H^{-\theta}(\operatorname{div} 0), \|\cdot\|_{H^{-\theta}(\Omega)})$$

is an isomorphism, and we have $\operatorname{curl}(H^{1-\theta}(\Omega)) = H^{-\theta}(\operatorname{div} 0)$.

Proof. 1) Let $V := (H_{n0}^{1-\theta}(\operatorname{div} 0), \|\cdot\|_{H^{1-\theta}})$ denote the domain of the curl. Obviously $\operatorname{curl} V$ is a subspace of $\operatorname{curl} H^{1-\theta}(\Omega)$. Let $w \in H^{1-\theta}(\Omega)$. Lemma A.2.5 for $s = 0$ yields functions $v \in H_{n0}^0(\operatorname{div} 0)$ and $\varphi \in H^1(\Omega)$ such that $w = v + \nabla \varphi$. It follows

$$\Delta \varphi = \operatorname{div} \nabla \varphi = \operatorname{div} w \in H^{-\theta}(\Omega) \text{ with } \operatorname{tr}_n \nabla \varphi = \operatorname{tr}_n w \in H^{1/2-\theta}(\partial\Omega).$$

Elliptic regularity implies that φ belongs to $H^{2-\theta}(\Omega)$, so that $v = w - \nabla \varphi \in H^{1-\theta}(\Omega)$. As a result, v is contained in V . Since $\operatorname{curl} w = \operatorname{curl} v$, the images $\operatorname{curl} H^{1-\theta}(\Omega) = \operatorname{curl} V$ agree.

2) Next, we show that $\operatorname{curl} V$ is closed by means of Peetre's lemma (see Lemma IX.1.2 in [16]). To this end, we consider the operators $A_1 = \operatorname{curl}: V \rightarrow H^{-\theta}(\Omega)$ and $A_2 = \operatorname{Id}: V \rightarrow H^{-\theta}(\Omega)$. The Rellich–Kondrachov theorem shows that A_2 is compact. Lemma A.2.6 implies the inequality

$$\|v\|_{H^{1-\theta}(\Omega)} \lesssim \|A_1 v\|_{H^{-\theta}(\Omega)} + \|A_2 v\|_{H^{-\theta}(\Omega)} \quad \text{for } v \in V.$$

Hence, $\operatorname{curl} V$ is closed in $H^{-\theta}(\Omega)$ by Peetre's lemma and, moreover, the kernel of curl on V is finite dimensional. In the following step we show that it is indeed trivial.

3) Let $w \in V$ with $\operatorname{curl} w = 0$ be an element of the kernel. In particular, w belongs to $H(\operatorname{curl} 0)$. Thus, from Proposition IX.1.2 of [16] we obtain $w \in G^0 = \nabla H^1(\Omega)$, using simple connectedness. Since $H_{n0}^{1-\theta}(\operatorname{div} 0) \subseteq H_{n0}^0(\operatorname{div} 0)$, Lemma A.2.5 implies that $w = 0$. Therefore, the kernel of $\operatorname{curl}: H_{n0}^{1-\theta}(\operatorname{div} 0) \rightarrow \operatorname{curl}(H^{1-\theta}(\Omega))$ is trivial.

4) Finally, we establish that the image of the curl operator on $H^{1-\theta}(\Omega)$ is given by $\text{curl}(H^{1-\theta}) = H^{-\theta}(\text{div } 0)$. The result will then follow from the previous steps and the bounded inverse theorem.

4.a) We begin by showing

$$\begin{aligned} {}^{\perp\theta}\text{curl}(H^{1-\theta}(\Omega)) &:= \{u \in H^\theta(\Omega) \mid \forall w \in \text{curl } H^{1-\theta}(\Omega) : \langle u, w \rangle_{H^\theta \times H^{-\theta}} = 0\} \\ &= H_{t0}^\theta(\text{curl } 0) := \{u \in H^\theta(\text{curl}) \mid \text{curl } u = 0, \text{tr}_t u = 0\}. \end{aligned}$$

Let $u \in {}^{\perp\theta}\text{curl}(H^{1-\theta}(\Omega))$. For $v \in C_c^\infty(\Omega) \subseteq H^{1-\theta}(\Omega)$ we compute

$$0 = \langle u, \text{curl } v \rangle_{H^\theta(\Omega) \times H^{-\theta}(\Omega)} = \int_{\Omega} u \text{curl } v \, dx = \langle v, \text{curl } u \rangle_{H_0^{1-\theta}(\Omega) \times H^{\theta-1}(\Omega)}.$$

Hence, $\text{curl } u = 0$ and $u \in H^\theta(\text{curl})$. For $v \in H^1(\Omega)$ we then obtain

$$\begin{aligned} 0 &= \langle u, \text{curl } v \rangle_{H^\theta(\Omega) \times H^{-\theta}(\Omega)} = \langle u, \text{curl } v \rangle_{H^\theta(\Omega) \times H^{-\theta}(\Omega)} - \langle \text{curl } u, v \rangle_{H^\theta(\Omega) \times H^{-\theta}(\Omega)} \\ &= \langle \text{tr}_t u, \text{tr } v \rangle_{H^{\theta-1/2}(\partial\Omega) \times H^{1/2-\theta}(\partial\Omega)}, \end{aligned}$$

and the first inclusion ${}^{\perp\theta}\text{curl}(H^{1-\theta}(\Omega)) \subseteq H_{t0}^\theta(\text{curl } 0)$ follows.

To show the reverse direction, take $u \in H_{t0}^\theta(\text{curl } 0)$. Let $v \in H^{1-\theta}(\Omega)$ and $v_n \in C^\infty(\overline{\Omega}) \cap H^{1-\theta}(\Omega)$ with $v_n \rightarrow v$ in $H^{1-\theta}(\Omega)$. We have

$$\langle u, \text{curl } v_n \rangle_{H^\theta(\Omega) \times H^{-\theta}(\Omega)} = \langle \text{curl } u, v_n \rangle_{H^\theta(\Omega) \times H^{-\theta}(\Omega)} + \langle \text{tr } v_n, \text{tr}_t u \rangle_{H^{1/2-\theta}(\partial\Omega) \times H^{\theta-1/2}(\partial\Omega)} = 0.$$

Since $\text{curl } v_n \rightarrow \text{curl } v$ in $H^{-\theta}(\Omega)$, we deduce $\langle u, \text{curl } v \rangle_{H^\theta(\Omega) \times H^{-\theta}(\Omega)} = 0$. Therefore, u belongs to ${}^{\perp\theta}\text{curl}(H^{1-\theta}(\Omega))$ and (A.3) is shown.

4.b) We finish by proving

$$\text{curl}(H^{1-\theta}) = H^{-\theta}(\text{div } 0). \quad (\text{A.3})$$

Let $u \in \text{curl}(H^{1-\theta})$. Take $\varphi \in H_0^{1+\theta}(\Omega)$ and note that $\nabla \varphi \in H_{t0}^\theta(\text{curl } 0)$, see Lemma A.1.1. Step 4.a) thus yields

$$0 = \langle \nabla \varphi, u \rangle_{H^\theta(\Omega) \times H^{-\theta}(\Omega)} = \langle \varphi, \text{div } u \rangle_{H^{1+\theta}(\Omega) \times H^{-1-\theta}(\Omega)}.$$

Hence, $\text{div } u = 0$ and u lies in $H^{-\theta}(\text{div } 0)$.

Next, let $u \in H^{-\theta}(\text{div } 0)$ and take $v \in H_{t0}^\theta(\text{curl } 0)$. Since $\partial\Omega$ is connected, Proposition IX.1.3 in [16] yields a potential $\varphi \in H^{1+\theta}(\Omega)$ with $\nabla \varphi = v$ and constant trace $\text{tr } \varphi = c$ for some $c \in \mathbb{R}$. By considering $\varphi - c$, we can assume that $\varphi \in H_0^{1+\theta}(\Omega)$. Take test functions φ_k tending to φ in $H^{1+\theta}(\Omega)$. We then calculate

$$\begin{aligned} \langle v, u \rangle_{H^\theta(\Omega) \times H^{-\theta}(\Omega)} &= \langle \nabla \varphi, u \rangle_{H^\theta(\Omega) \times H^{-\theta}(\Omega)} \\ &= \lim_{k \rightarrow \infty} \langle \nabla \varphi_k, u \rangle_{H^\theta(\Omega) \times H^{-\theta}(\Omega)} + \langle \varphi_k, \text{div } u \rangle_{H^\theta(\Omega) \times H^{-\theta}(\Omega)} \\ &= \lim_{k \rightarrow \infty} \langle \text{tr } \varphi_k, \text{tr}_n u \rangle_{H^{\theta+1/2}(\partial\Omega) \times H^{-\theta-1/2}(\partial\Omega)} = 0. \end{aligned}$$

We conclude that

$$\begin{aligned} u \in (H_{t0}^\theta(\text{curl } 0))^{\perp\theta} &:= \{u \in H^{-\theta}(\Omega) \mid \forall v \in H_{t0}^\theta(\text{curl } 0) : \langle v, u \rangle_{H^\theta(\Omega) \times H^{-\theta}(\Omega)} = 0\} \\ &= ({}^{\perp\theta}\text{curl}(H^{1-\theta}(\Omega)))^{\perp\theta} = \overline{\text{curl}(H^{1-\theta}(\Omega))} = \text{curl}(H^{1-\theta}(\Omega)), \end{aligned}$$

using (A.3) and step 2). Therefore, (A.3) is true. $\square$

We next observe that Lemma 2.4 in [12] extends from $\theta = 0$ to $\theta \in (-\frac{1}{2}, 0]$.

Lemma A.2.8. *Let $u \in H^{-\theta}(\text{curl})$ and $\theta \in [0, \frac{1}{2})$. Then*

$$\text{curl}_{\partial\Omega}(\text{tr}_\tau u) = \text{div}_{\partial\Omega}(\text{tr}_t u) \in H^{-\theta-\frac{1}{2}}(\partial\Omega),$$

where we set

$$\text{curl}_{\partial\Omega}(\text{tr}_\tau u) := \text{tr}_n(\text{curl } u)$$

and, for all $\varphi \in C^\infty(\partial\Omega)$ and $\Phi \in C^\infty(U)$ with $\text{tr } \Phi = \varphi$ and open $U \supseteq \bar{\Omega}$,

$$\langle \text{div}_{\partial\Omega}(\text{tr}_t u), \varphi \rangle_{H^{-\theta-\frac{1}{2}}(\partial\Omega) \times H^{\theta+\frac{1}{2}}(\partial\Omega)} := -\langle \text{tr}_t u, \text{tr}_\tau(\nabla \Phi) \rangle_{H^{-\theta-\frac{1}{2}}(\partial\Omega) \times H^{\theta+\frac{1}{2}}(\partial\Omega)}.$$

Proof. First note that $\text{curl } u \in H^{-\theta}(\text{div})$ and thus $\text{tr}_n(\text{curl } u) \in H^{-\theta-\frac{1}{2}}(\partial\Omega)$. Since also $\text{tr}_t u \in H^{-\theta-\frac{1}{2}}(\partial\Omega)$, the definitions and the identity $\text{div } \text{curl } u = 0$ yield

$$\begin{aligned} \langle \text{curl}_{\partial\Omega}(\text{tr}_\tau u), \varphi \rangle_{H^{-\theta-\frac{1}{2}}(\partial\Omega) \times H^{\theta+\frac{1}{2}}(\partial\Omega)} &= \langle \text{tr}_n(\text{curl } u), \varphi \rangle_{H^{-\theta-\frac{1}{2}}(\partial\Omega) \times H^{\theta+\frac{1}{2}}(\partial\Omega)} \\ &= \langle \Phi, \text{div } \text{curl } u \rangle_{H^\theta \times H^{-\theta}} + \langle \nabla \Phi, \text{curl } u \rangle_{H^\theta \times H^{-\theta}} \\ &= 0 - \langle \text{tr}(\nabla \Phi), \text{tr}_t u \rangle_{H^{\theta+\frac{1}{2}}(\partial\Omega) \times H^{-\theta-\frac{1}{2}}(\partial\Omega)} \\ &= -\langle \text{tr}_\tau(\nabla \Phi), \text{tr}_t u \rangle_{H^{\theta+\frac{1}{2}}(\partial\Omega) \times H^{-\theta-\frac{1}{2}}(\partial\Omega)} \\ &= \langle \text{div}_{\partial\Omega}(\text{tr}_t u), \varphi \rangle_{H^{-\theta-\frac{1}{2}}(\partial\Omega) \times H^{\theta+\frac{1}{2}}(\partial\Omega)}. \quad \square \end{aligned}$$

THE DIV-CURL ESTIMATE

We show a corrected and improved version of Lemma 5.1 of [51] and Lemma A.6 in [46]. We note that part a) was shown in Lemma 4.5.5 of [13] for scalar γ and $\theta = 0$ in a similar way. In [51] the L^2 -norms of u and v on the right-hand side of (A.4) were omitted erroneously. The estimate (A.4) for $\theta = 0$ below suffices for [51]. Actually this version does not require additional geometric properties. In part b) we can remove the L^2 -norms on the right-hand side, as needed to obtain global-in-time properties in Lemma 7.2.5 and Theorem 6.1.2, assuming that $\mathbb{R}^3 \setminus \Omega$ is connected. This condition is needed, since without it the operator A in the proof would have a non-trivial kernel of the form $(\nabla \varphi, \nabla \psi)$, where $\text{div}(\alpha \nabla \varphi) = \text{div}(\beta \nabla \psi) = 0$ and φ and ψ are constant on components of $\partial\Omega$.

Theorem A.2.9. *Let $\Omega \subset \mathbb{R}^3$ be bounded with a multiple connected boundary $\partial\Omega = \cup_{i=0}^K \partial\Omega_i \in C^{2,1}$ and $\theta \in [0, \frac{1}{2})$. Assume that $\alpha, \beta \in W^{1,\infty}(\Omega, \mathbb{R}_{\text{sym}}^{3 \times 3})$, $\gamma \in W^{1,\infty}(\partial\Omega, \mathbb{R}_{\text{sym}}^{3 \times 3})$ satisfy $\alpha, \beta, \gamma \geq \eta > 0$ and $\gamma \nu^\perp \subseteq \nu^\perp$. Let $(u, v) \in H^{-\theta}(\text{curl})$ fulfill $\text{div}(\alpha u) \in H^{-\theta}$, $\text{div}(\beta v) \in H^{-\theta}$ and $v \times \nu + \gamma(u \times \nu) \times \nu =: h \in H^{1/2-\theta}(\partial\Omega)^3$.*

a) *Then u and v belong to $H^{1-\theta}(\Omega)^3$ and*

$$\begin{aligned} \|u\|_{H^{1-\theta}(\Omega)} + \|v\|_{H^{1-\theta}(\Omega)} &\lesssim \|u\|_{H^{-\theta}(\text{curl})} + \|v\|_{H^{-\theta}(\text{curl})} + \|\text{div}(\alpha u)\|_{H^{-\theta}(\Omega)} \\ &\quad + \|\text{div}(\beta v)\|_{H^{-\theta}(\Omega)} + \|h\|_{H^{1/2-\theta}(\partial\Omega)} =: N(u, v, h). \end{aligned} \quad (\text{A.4})$$

b) *Let $\theta = 0$. Assume in addition that $u \in (\mathbb{H}_2^\alpha)^\perp{}_{L^\alpha_2}$ and $v \in (\mathbb{H}_2^\beta)^\perp{}_{L^\beta_2}$. We then obtain*

$$\begin{aligned} \|u\|_{H^1(\Omega)} + \|v\|_{H^1(\Omega)} &\lesssim \|\text{curl } u\|_{L^2(\Omega)} + \|\text{curl } v\|_{L^2(\Omega)} + \|\text{div}(\alpha u)\|_{L^2(\Omega)} \\ &\quad + \|\text{div}(\beta v)\|_{L^2(\Omega)} + \|h\|_{H^{1/2}(\partial\Omega)}. \end{aligned} \quad (\text{A.5})$$

Proof. a) As in Lemma 4.5.5 of [13] or Proposition 6.1 of [35], we use a finite partition of unity χ_j for $\bar{\Omega}$ such that $\text{supp } \chi_j \subseteq \bar{\Omega}_j \subseteq \bar{\Omega}$ for starshaped, bounded open sets Ω_j . Following Lemma 5.1 of [51] we will prove that $\chi_j u$ and $\chi_j v$ belong to $H^{1-\theta}(\Omega_j)$ and

$$\|\chi_j u\|_{H^{1-\theta}} + \|\chi_j v\|_{H^{1-\theta}} \leq N(\chi_j u, \chi_j v, \chi_j h) \lesssim N(u, v, h)$$

Summing these pieces, we obtain (A.4).

So let Ω be starshaped. Since $\operatorname{curl} u \in H^{-\theta}(\operatorname{div} 0)$, by Lemma A.2.7 there is a field $w \in H_{n0}^{1-\theta}(\operatorname{div} 0)$ such that $\operatorname{curl} u = \operatorname{curl} w$ and $\|w\|_{H^{1-\theta}} \lesssim \|\operatorname{curl} u\|_{H^{-\theta}}$. So $u - w$ belongs to $H^{-\theta}(\operatorname{curl} 0)$. We show that $H^{-\theta}(\operatorname{curl} 0) = G^{-\theta}$.

Indeed, the inclusion “ $\supseteq$ ” is clear. Conversely, take $f \in H^{-\theta}(\operatorname{curl} 0)$. Lemma A.2.5 provides maps $v_0 \in H_{n0}^{-\theta}(\operatorname{div} 0)$ and $\varphi \in G^{-\theta}$ such that $f = v_0 + \nabla \varphi$. Hence, v_0 is contained in $H_{n0}^{-\theta}(\operatorname{div} 0) \cap H^{-\theta}(\operatorname{curl} 0)$. For an open, simply connected set $U \subseteq \bar{U} \subseteq \Omega$, take a cut-off function $\chi \in C_c^\infty(\Omega)$ with $\chi = 1$ on U and let $\tilde{v}$ be the zero-extension of χv_0 to $\mathbb{R}^3$. Computing

$$\begin{aligned} \|\operatorname{div} \tilde{v}\|_{H^{-\theta}(\mathbb{R}^3)} &= \sup_{\phi \in H^\theta(\mathbb{R}^3)} \left| \frac{\langle \chi \nabla \phi, v_0 \rangle_{H^\theta(\Omega)^3 \times H^{-\theta}(\Omega)^3}}{\|\phi\|_{H^\theta(\mathbb{R}^3)}} \right| \\ &= \sup_{\phi \in H^\theta(\mathbb{R}^3)} \left| \frac{\langle \phi \nabla \chi, v_0 \rangle_{H^\theta(\Omega)^3 \times H^{-\theta}(\Omega)^3}}{\|\phi\|_{H^\theta(\mathbb{R}^3)}} \right| \\ &\lesssim_\chi \sup_{\Psi \in (H^\theta(\Omega))^3} \left| \frac{\langle \Psi, v_0 \rangle_{H^\theta(\Omega)^3 \times H^{-\theta}(\Omega)^3}}{\|\Psi\|_{H^\theta(\Omega)}} \right| = \|v_0\|_{H^{-\theta}(\Omega)}, \\ \|\operatorname{curl} \tilde{v}\|_{H^{-\theta}(\mathbb{R}^3)} &= \sup_{\Phi \in H^\theta(\mathbb{R}^3)} \left| \frac{\langle \chi \operatorname{curl} \Phi, v_0 \rangle_{H^\theta(\Omega)^3 \times H^{-\theta}(\Omega)^3}}{\|\Phi\|_{H^\theta(\mathbb{R}^3)}} \right| \\ &= \sup_{\Phi \in H^\theta(\mathbb{R}^3)} \left| \frac{\langle \Phi \times \nabla \chi, v_0 \rangle_{H^\theta(\Omega)^3 \times H^{-\theta}(\Omega)^3}}{\|\Phi\|_{H^\theta(\mathbb{R}^3)}} \right| \\ &\lesssim_\chi \sup_{\Psi \in (H^\theta(\Omega))^3} \left| \frac{\langle \Psi, v_0 \rangle_{H^\theta(\Omega)^3 \times H^{-\theta}(\Omega)^3}}{\|\Psi\|_{H^\theta(\Omega)}} \right| = \|v_0\|_{H^{-\theta}(\Omega)}, \end{aligned}$$

we see that $\tilde{v}, \operatorname{div} \tilde{v}$ and $\operatorname{curl} \tilde{v}$ belong to $H^{-\theta}(\mathbb{R}^3)$. Hence, $\hat{v} := (1 - \Delta)^{-\theta/2} \tilde{v}$ is an element of $H(\operatorname{div}, \mathbb{R}^3) \cap H(\operatorname{curl}, \mathbb{R}^3) = H^1(\mathbb{R}^3)$. As a result, $\tilde{v}$ is contained in $H^{1-\theta}(\mathbb{R}^3)$ and v_0 in $H_{\operatorname{loc}}^{1-\theta}(\Omega)$, which means that $v_0 \in H(\operatorname{curl} 0, U)$. Proposition IX.1.2 in [16] now yields a map $\Psi_U \in H^1(U)$ with $\nabla \Psi_U = v_0$ on U . Because $\operatorname{div} v_0 = 0$ the map Ψ_U is analytic, and by analytic continuation we find a smooth function Ψ on Ω with $\nabla \Psi = v_0 \in H^{-\theta}$.

Let x_0 be the star center of Ω . We can write $\Psi = \Psi(x_0) + \int_0^1 T_\tau \nabla \Psi \cdot m \, d\tau$, where $T_\tau g(x) = g(x_0 - \tau(x - x_0))$ and $m(x) = x - x_0$. This identity implies that Ψ belongs to $H^{-\theta}(\Omega)$. Since also $\Delta \Psi = 0$ on Ω and $\partial_\nu \Psi = \operatorname{tr}_n v_0 = 0$ on $\partial\Omega$, we deduce $\Psi = 0$ and so $v_0 = 0$, as needed.

Hence, there exists a function $\varphi \in H^{1-\theta}(\Omega)$ such that $\nabla \varphi = u - w$ and therefore

$$\operatorname{div}(\alpha \nabla \varphi) = \operatorname{div}(\alpha u) - \operatorname{div}(\alpha w) \in H^{-\theta}(\Omega). \quad (\text{A.6})$$

Here we can assume that $\int_\Omega \varphi \, dx = 0$. We thus obtain

$$\|\varphi\|_{H^{-\theta}(\Omega)} \lesssim \|\nabla \varphi\|_{H^{-\theta}(\Omega)} \lesssim \|u\|_{H^{-\theta}(\Omega)} + \|w\|_{H^{-\theta}(\Omega)} \lesssim \|u\|_{H^{-\theta}(\operatorname{curl})}$$

by the Poincaré inequality in $H^{-\theta}(\Omega)$ which can be shown as in $L^2(\Omega)$. One also infers

$$\|\operatorname{tr} \varphi\|_{H^{-\theta}(\partial\Omega)} \lesssim \|\varphi\|_{H^{1-\theta}} \lesssim \|u\|_{H^{-\theta}(\operatorname{curl})}.$$

Let $f \times \nu = \begin{pmatrix} 0 & \nu_3 & -\nu_2 \\ -\nu_3 & 0 & \nu_1 \\ \nu_2 & -\nu_1 & 0 \end{pmatrix} f =: Bf$ and $\tilde{\gamma} := B^\top \gamma B$. Note that $\tilde{\gamma}^\perp = \tilde{\gamma} \geq \eta$ on $\partial\Omega$ and $\tilde{\gamma} \nu^\perp \subseteq \nu^\perp$. As in Lemma 5.1 of [51] we now see that

$$\operatorname{div}_{\partial\Omega}(\tilde{\gamma} \operatorname{tr}_\tau(\nabla \varphi)) = \operatorname{div}_{\partial\Omega}(\gamma(w \times \nu) \times \nu) - \operatorname{div}_{\partial\Omega}(\gamma(u \times \nu) \times \nu)$$

$$\begin{aligned}
&= \operatorname{div}_{\partial\Omega}(\gamma(w \times \nu) \times \nu) - \operatorname{div}_{\partial\Omega}(h) + \operatorname{div}_{\partial\Omega}(\operatorname{tr}_t v) \\
&= \operatorname{div}_{\partial\Omega}(\gamma(w \times \nu) \times \nu) - \operatorname{div}_{\partial\Omega}(h) + \operatorname{tr}_n(\operatorname{curl} v)
\end{aligned}$$

belongs to $H^{-\theta-\frac{1}{2}}(\partial\Omega)$, using also Lemma A.2.8 and that $\operatorname{curl} v \in H^{-\theta}(\operatorname{div} 0)$. Elliptic regularity then yields $\operatorname{tr} \varphi \in H^{\frac{3}{2}-\theta}(\partial\Omega)$, and then $\varphi \in H^{2-\theta}$ because of (A.6). Collecting the above relations we can thus bound u in $H^{1-\theta}$ as asserted. One can treat v similarly also using $u \in H^{1-\theta}$ and the boundary condition. (Here one can let $\tilde{\gamma} = I$).

b) Now assume that $\theta = 0$. By a contradiction compactness argument, we show that the L^2 -norms appearing on the right-hand side of (A.4) can be estimated by the div-curl-terms. Assume that

$$\|(u, v)\|_{H^1} \not\lesssim \|\operatorname{curl} u\|_{L^2(\Omega)} + \|\operatorname{curl} v\|_{L^2(\Omega)} + \|\operatorname{div}(\alpha u)\|_{L^2(\Omega)} + \|\operatorname{div}(\beta v)\|_{L^2(\Omega)} + \|h\|_{H^{1/2}(\partial\Omega)}.$$

Hence, there exists a sequence $(u_k, v_k) \in H^1(\Omega) \cap \left((\mathbb{H}_2^\alpha)^\perp_{L^\alpha} \times (\mathbb{H}_2^\beta)^\perp_{L^\beta} \right)$ with $\|(u_k, v_k)\|_{H^1(\Omega)} = 1$ such that the right-hand side tends to zero. By means of the Banach–Alaoglu and Rellich–Kondrachov theorems, we can choose a subsequence, again denoted by (u_k, v_k) , which converges in $L^2(\Omega)$ and weakly in $H^1(\Omega)$ to a limit $(\bar{u}, \bar{v}) \in H^1(\Omega) \cap (\mathbb{H}_2^\alpha)^\perp_{L^\alpha} \times (\mathbb{H}_2^\beta)^\perp_{L^\beta}$ in L^2 .

We employ the operator

$$D(A) := \left\{ (u, v) \in X \mid \operatorname{curl} u, \operatorname{curl} v \in L^2, v \times \nu + \gamma(u \times \nu) \times \nu = 0 \right\}, \quad A = \begin{pmatrix} 0 & \operatorname{curl} \\ -\operatorname{curl} & 0 \end{pmatrix}$$

on

$$X = \left(\left((\mathbb{H}_2^\alpha)^\perp_{L^\alpha} \times (\mathbb{H}_2^\beta)^\perp_{L^\beta} \right) \cap (\ker(\operatorname{div}_\alpha) \times \ker(\operatorname{div}_\beta)), \|\cdot\|_{L^2} \right).$$

Note that $(\bar{u}, \bar{v})$ belongs to the kernel of A .

We will prove below that A is injective. The sequence (u_k, v_k) then converges to $(\bar{u}, \bar{v}) = 0$ in L^2 . The estimate (A.4) with $\theta = 0$ hence shows that $\|(u_k, v_k)\|_{H^1} \rightarrow 0$, contradicting $\|(u_k, v_k)\|_{H^1} = 1$ and the assertion follows.

So take $(u, v) \in D(A)$ with $0 = A\begin{pmatrix} u \\ v \end{pmatrix}$ and consider

$$0 = \left(A\begin{pmatrix} u \\ v \end{pmatrix} \middle| \begin{pmatrix} u \\ v \end{pmatrix} \right) = \int_{\Omega} (\operatorname{curl} v \cdot u - \operatorname{curl} u \cdot v) \, dx.$$

Integrating by parts and using the boundary condition, we compute

$$0 = \oint_{\partial\Omega} (u \times \nu) \cdot v \, d\sigma = - \oint_{\partial\Omega} u \cdot (v \times \nu) \, d\sigma = - \oint_{\partial\Omega} (u \times \nu) \cdot \gamma(u \times \nu) \, d\sigma.$$

The positive definiteness of γ implies that $u \times \nu = 0$ and therefore $v \times \nu = 0$ as well. Thus, $(u, v) \in \ker A$ satisfies

$$\begin{aligned}
\operatorname{curl} u &= \operatorname{curl} v = 0, & \text{on } \Omega, \\
u \times \nu &= v \times \nu = 0, & \text{on } \partial\Omega.
\end{aligned}$$

Because of $(u, v) \in X$ we therefore obtain that $(u, v) \in \mathbb{H}_2^\alpha \times \mathbb{H}_2^\beta$. Again the choice of X implies that $u = v = 0$. Hence, A is injective as desired. $\square$

Let us end this section with a remark on the additional condition that $u \in (\mathbb{H}_2^\alpha)^\perp_{L^\alpha}$ (resp. $v \in (\mathbb{H}_2^\beta)^\perp_{L^\beta}$) in Theorem A.2.9b) above in case $u \in H(\operatorname{div}_\alpha 0)$.

Remark A.2.10. By Lemma A.1.3 and the subsequent Remark A.1.4 for $u \in H(\operatorname{div}_\alpha 0)$ we have

$$u \in (\mathbb{H}_2^\alpha)^\perp_{L_\alpha^2} \iff u \in H^{\partial\Omega}(\operatorname{div}_\alpha 0).$$

Thus, for $(u, v) \in H^{\partial\Omega}(\operatorname{div}_\alpha 0) \times H^{\partial\Omega}(\operatorname{div}_\beta 0)$ the assumptions of Theorem A.2.9 b) are satisfied. Conceptually, this means that Ω does not enclose any regions of its complement containing electric or magnetic charges. This is also trivially satisfied when $\mathbb{R}^3 \setminus \Omega$ is connected, since in this case $\mathbb{H}_2^\alpha = \{0\}$.

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LIST OF SYMBOLS

A	17
A_0	16
A_b	121
$\tilde{A}$	18
B	1
$\llbracket \cdot \rrbracket$	10
$C^3(\bar{\Omega}_{\pm} \times \mathbb{R}^3, \mathbb{R}_{\text{sym}}^{3 \times 3})$	62
$\mathcal{C}^\ell$	36
$\mathcal{C}_\eta^\ell$	36
(CC_m)	37
C_τ^k	111, 127
D	1
d, d_k	68, 130
δ_η	66
δ_z	64
E	1
e, e_k	68, 130
E^m	36
ε	2
ε_d	63, 128
$\varepsilon_{d,\text{nl}}$	63
$\hat{\varepsilon}_k$	63, 128
$\tilde{\varepsilon}$	16, 64
η	36, 62, 111, 121, 127
$\tilde{\eta}$	16, 64, 111, 121, 131
$\mathcal{G}^k, \mathcal{G}_\Gamma^k$	11
G^s	141
H	1
$\mathbb{H}_1, \mathbb{H}_1^\varepsilon$	137
$\mathbb{H}_2, \mathbb{H}_2^\mu$	137
H_c^1	137
$H(\text{curl}), H(\text{curl } 0), H_{t0}(\text{curl}), H_{t0}(\text{curl } 0)$	10
$H(\text{div}), H(\text{div } 0), H_{n0}(\text{div}), H_{n0}(\text{div } 0)$	10
$H^{\partial\Omega}(\text{div}_\alpha 0), H^{\partial\Omega}(\text{div } 0), H^\Gamma(\text{div}_\alpha 0), H^\Gamma(\text{div } 0)$	11
H_Γ^k	11
$H_{n0}^\Sigma(\text{div}_\alpha 0), H_{n0}^\Sigma(\text{div } 0)$	11
$H^\theta(\text{div}_\alpha), H^\theta(\text{curl}_\beta)$	10
J	2
J^m	36
K	9, 121
L	9
λ_d	128
$\hat{\lambda}_k$	128
$\lesssim$	11
m	16, 130

$\hat{m}$	36
$\tilde{m}$	36
μ	2
μ_d	63, 128
$\mu_{d,nl}$	63
$\hat{\mu}_k$	63, 128
$\tilde{\mu}$	16, 64
$\mathcal{N}_a, \mathcal{N}_a^\Gamma$	9, 10
$\Omega_\pm$	9
ρ	1
$S(\cdot)$	17
$S_0(\cdot)$	16
σ	2
σ_0	61
$T_b(\cdot)$	121
T^m	36
$\text{tr}_n, \text{tr}_t, \text{tr}_\tau$	10
T_*	64, 129
X	17
X_0	16
X_b	121
$\tilde{X}$	18
z, z_k	68, 130