



Phase-field peridynamics

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ABSTRACT

Peridynamics formulates the balance of linear momentum as an integro-differential equation, making it naturally suited for fracture modeling without special treatment of discontinuities. The bond-associated correspondence formulation provides a highly accurate peridynamic framework by computing bond-wise deformation gradients that are free of zero-energy modes and yield accurate results even near boundaries. However, the traditional fracture approach based on irreversible bond deletion can compromise this formulation, as the progressive removal of bonds degrades the nonlocal approximation of the deformation gradient and can lead to numerical instabilities. In this work, a novel phase-field peridynamics approach is introduced that avoids these instabilities. Instead of deleting bonds, the energetic contribution of each bond is continuously degraded through a bond phase-field parameter, while a separate kinematic degradation function preserves the accuracy of the nonlocal deformation gradient approximation. The normalization constant ensuring thermodynamic consistency with Griffith's fracture theory is derived analytically for general spherical kernel functions as a ratio of two one-dimensional integrals. Numerical examples including mode I and mode II fracture, the boundary tension test with different kernel functions and horizon ratios, and the Kalthoff-Winkler experiment demonstrate the stability, accuracy, and consistency of the proposed approach.

1. Introduction

Predicting how cracks initiate and propagate through load-bearing structures is one of the central challenges in computational mechanics. Whether fracture is to be prevented in safety-critical applications or deliberately induced in manufacturing processes, its accurate computational prediction at realistic loading rates and deformation levels is of fundamental importance. The foundations for fracture prediction were established by the pioneering work of Griffith [1], who proposed that a crack grows when the energy released by the advancing crack exceeds the energy required to create new crack surfaces. Irwin [2] extended this framework by introducing the stress intensity factor, making fracture mechanics applicable to a broader class of materials. These energy-based approaches remain at the core of modern fracture theory.

Numerous computational approaches have been developed to address the challenges of fracture modeling, each with distinct strengths and limitations. In the standard finite element method (FEM), cracks represent discontinuities in the displacement field that must be explicitly resolved by the mesh, requiring costly remeshing procedures as cracks propagate [3,4]. The extended finite element method (XFEM) [5,6] alleviates this by enriching the approximation with discontinuous functions, though it becomes increasingly complex for crack branching and three-dimensional problems. Cohesive zone models [7–10] describe fracture through

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traction-separation laws governing the progressive decohesion of material surfaces, but distributing cohesive elements throughout the mesh introduces mesh sensitivity in the crack trajectory. The phase-field approach to fracture [11–14] regularizes the sharp crack topology with a continuous damage field, enabling the natural modeling of crack initiation, propagation, and branching without explicit crack tracking. Extensions to dynamic fracture have been developed by Borden et al. [15] and Weinberg and Wiens [16], including approaches combining phase-field methods with peridynamics [17,18]. However, the phase-field method requires a fine mesh resolution around the crack to resolve the diffuse damage band, particularly in three-dimensional dynamic settings. Further comparisons of computational fracture approaches have been provided by Rabczuk [19], Pandolfi et al. [20], and Diehl et al. [21].

All of the above approaches are based on classical continuum mechanics, where the governing equations are formulated as partial differential equations that require spatial derivatives of the displacement field. At a crack, the displacement field is discontinuous and these derivatives are undefined, therefore all classical methods require special techniques to handle the presence of cracks. Peridynamics, introduced by Silling [22], takes a fundamentally different approach by formulating the balance of linear momentum as an integro-differential equation. Internal forces at a material point are computed by integrating the interactions with all neighboring points within a finite distance, called the horizon. While the spherical neighborhood is the standard choice, alternative neighborhood shapes such as cube-shaped horizons have also been explored [23]. Since no spatial derivatives of the displacement field are required, the governing equations remain valid at discontinuities such as cracks, making peridynamics naturally suited for fracture modeling [24,25]. A rigorous variational theory for integral functionals based on finite-horizon nonlocal gradients, including Γ -convergence results that recover classical local models, has been developed by Cueto et al. [26,27]. Energy-based peridynamic fracture models with continuous damage that converge to Griffith fracture in the vanishing-horizon limit have been proposed by Lipton [28–30].

The original bond-based (BB) formulation [22] is simple and robust but restricts the Poisson's ratio. The ordinary state-based (OSB) formulation [31] overcomes this restriction. Independent of the Poisson's ratio restriction, both formulations suffer from the surface effect, a systematic stiffness deviation near free surfaces caused by the truncation of the neighborhood. Several correction methods have been developed to mitigate this error. An overview is given by Le and Bobaru [32], and further approaches are described in [25,33]. The correspondence formulation [31] approximates the deformation gradient through nonlocal weighted averaging, enabling the direct use of classical constitutive models and eliminating the surface effect by construction, without requiring additional correction procedures. However, it introduces stability issues due to zero-energy modes [34,35]. The bond-associated approach [36–39] addresses this instability by computing bond-wise deformation gradients that are inherently free of zero-energy modes. This makes the bond-associated correspondence formulation the most accurate peridynamic formulation that consistently reproduces the elastic behavior of the classical continuum.

Fracture in peridynamics is traditionally modeled by irreversible bond deletion, where a bond is permanently removed when its deformation exceeds a critical threshold derived from Griffith's energy criterion [24]. This approach has been successfully used to simulate dynamic crack propagation and branching with the standard formulations [40–44]. However, in correspondence-based formulations, the progressive deletion of bonds degrades the shape tensor required for approximating the deformation gradient. As more bonds are removed in the vicinity of a propagating crack, the shape tensor can become ill-conditioned or singular, potentially causing a breakdown of the kinematic approximation and numerical instabilities. These stability issues can limit the reliable application of the most accurate peridynamic formulation to fracture problems.

Combinations of phase-field concepts with peridynamics have been proposed before. Roy et al. [45] reformulated the classical phase-field evolution equation of Ginzburg–Landau type as a nonlocal integral equation and employed the resulting point-wise phase field as a criterion for breaking bonds within a correspondence-based framework. Bie et al. [46] developed a unified nonlocal peridynamics-based phase-field damage theory, in which an energy compensation method stabilizes the zero-energy modes of the correspondence formulation and the nonlocal phase field again provides the criterion for bond breakage. In both approaches, the phase field is a point-wise variable governed by an additional nonlocal evolution equation, and fracture is ultimately still realized by the deletion of bonds. The approach presented in this work differs in both respects. The phase field is introduced as a bond-wise quantity with a closed-form local evolution law, and bonds are never deleted, since their energetic contribution is degraded continuously while a separate kinematic degradation function preserves the accuracy and stability of the bond-associated kinematic approximation.

In this work, a novel phase-field peridynamics (PFPD) approach is introduced that avoids these stability issues. Instead of deleting bonds instantaneously when a critical deformation is reached, the energetic contribution of each bond is degraded continuously through a bond phase-field parameter. The damage evolution follows from an energy-based variational principle and ensures thermodynamic consistency with Griffith's fracture theory. A separate kinematic degradation function is introduced to maintain the accuracy of the shape tensor even under significant damage, preventing the numerical instabilities that can arise from bond deletion. The normalization of the damage model is derived analytically for general spherical kernel functions by requiring that the total crack dissipation equals the Griffith energy release rate per unit crack area.

The paper is organized as follows. Section 2 provides a brief introduction to the bond-associated correspondence formulation. In Section 3, the phase-field peridynamics framework is developed, including the bond-associated phase-field approximation, the damage evolution law, and the analytical derivation of the normalization constant. Section 4 presents the discretization and algorithmic implementation, with a detailed comparison to the standard bond-associated approach. Numerical examples are presented in Section 5, including mode I and mode II fracture, the verification of the normalization constant, and the Kalthoff-Winkler experiment. The paper concludes with a summary and discussion of the results in Section 6.

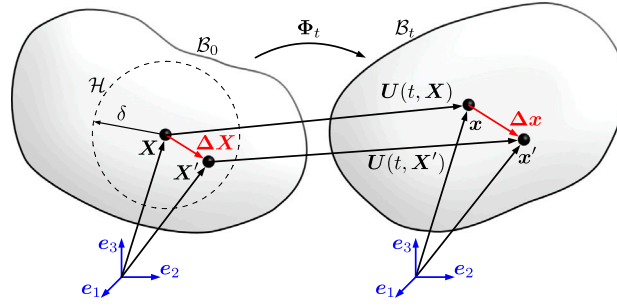


Fig. 1. Deformation of a peridynamic body B from reference configuration B_0 to current configuration B_t . The neighborhood $H(X)$ defines the nonlocal interaction domain around each material point X .

2. Peridynamics theory

This section provides a brief introduction to the peridynamic framework in a continuous setting, focusing on the bond-associated quadrature point (BAQP) correspondence formulation that forms the basis for the phase-field peridynamics model developed in the subsequent sections.

2.1. Nonlocal kinematics and balance of linear momentum

Consider a body B occupying the reference configuration $B_0 \subset \mathbb{R}^3$ and the current configuration $B_t \subset \mathbb{R}^3$. In peridynamics, the body is represented as a continuum of material points $B_0 \subset \mathbb{R}^3$, interacting with each other through nonlocal forces [22]. The interaction between a point X and its neighbor X' is characterized by a *bond* $\Delta X = X' - X$ in the reference configuration and by $\Delta x = x' - x$ in the current configuration. Each point X interacts only with points inside a specified *neighborhood* or *point family*

$$H(X) = \{ X' \in B_0 \mid 0 < |\Delta X| \leq \delta \}, \quad (1)$$

defined as all points within the *horizon* $\delta \in \mathbb{R}^+$. The *bond displacement* is defined as $\Delta U = U(t, X') - U(t, X)$, with the relation $\Delta x = \Delta X + \Delta U$. The peridynamic kinematics are illustrated in Fig. 1.

The balance of linear momentum in peridynamics reads

$$\rho_0(X) \ddot{U}(t, X) = \bar{B}_{\text{int}}(t, X) + \bar{B}_{\text{ext}}(t, X) \quad \forall (t, X) \in [0, \infty) \times B_0, \quad (2)$$

where $\bar{B}_{\text{int}}$ denotes the internal body force density and $\bar{B}_{\text{ext}}$ the external body force density. The initial conditions $U(0, X) = U_0(X)$ and $\dot{U}(0, X) = \dot{U}_0(X)$ specify the displacement and velocity at $t = 0$. Boundary conditions are imposed by prescribing either displacement (Dirichlet) or external force density (Neumann) on a boundary layer of material points with finite thickness, since the integro-differential equation Eq. (2) requires a volumetric treatment rather than classical surface boundary conditions [22].

The internal force density approximates the divergence of the first Piola–Kirchhoff stress tensor with

$$\bar{B}_{\text{int}}(t, X) = \nabla \cdot P(t, X) + \mathcal{O}(\delta^2). \quad (3)$$

Since the governing equation Eq. (2) is an integro-differential equation that does not involve spatial derivatives of the displacement field, it remains valid at discontinuities such as cracks, making peridynamics naturally suited for fracture modeling.

2.2. The correspondence formulation

The correspondence formulation [31] approximates the deformation gradient through nonlocal weighted averaging over the neighborhood, enabling the direct use of classical constitutive models. An approximated deformation gradient $\bar{F}$ is defined as

$$\bar{F}(t, X) = \left(\int_{H(X)} \omega(\Delta X) \Delta x \otimes \Delta X dV' \right) K^{-1}(X), \quad (4)$$

where $\omega : \{\mathbb{R}^3 \setminus \{0\}\} \rightarrow [0, \infty)$ with $\omega(\Delta X) = 0$ for $|\Delta X| > \delta$ is a kernel (or influence) function and the *shape tensor* K is defined as

$$K(X) = \int_{H(X)} \omega(\Delta X) \Delta X \otimes \Delta X dV'. \quad (5)$$

The shape tensor is a symmetric and positive definite matrix for bulk points with full neighborhoods. Due to the multiplication with K^{-1} , the deformation gradient is exactly reproduced for all homogeneous deformations, independent of the neighborhood geometry. This means that the correspondence formulation does not suffer from the surface effect, a shortcoming of the standard bond-based and ordinary state-based formulations [25,32].

With the deformation gradient, the stress tensor of each point can be evaluated from classical constitutive laws. A *bond shape vector* $\mathbf{B} : B_0 \times B_0 \rightarrow \mathbb{R}^3$ is defined as

$$\mathbf{B}(\mathbf{X}, \mathbf{X}') = \omega(\Delta \mathbf{X}) \mathbf{K}^{-1}(\mathbf{X}) \Delta \mathbf{X} . \quad (6)$$

With this, the deformation gradient and the *force vector state* $\mathbf{t}$ can be compactly expressed as

$$\bar{\mathbf{F}}(\mathbf{t}, \mathbf{X}) = \int_{H(\mathbf{X})} \Delta \mathbf{x} \otimes \mathbf{B}(\mathbf{X}, \mathbf{X}') dV' , \quad (7)$$

and, depending on the first Piola–Kirchhoff stress tensor $\mathbf{P}(\mathbf{t}, \mathbf{X}) = \mathbf{P}(\bar{\mathbf{F}}(\mathbf{t}, \mathbf{X}))$,

$$\mathbf{t}(\mathbf{t}, \mathbf{X}, \mathbf{X}') = \mathbf{P}(\mathbf{t}, \mathbf{X}) \mathbf{B}(\mathbf{X}, \mathbf{X}') . \quad (8)$$

The divergence of the first Piola–Kirchhoff stress tensor is then approximated as

$$\bar{\mathbf{B}}_{\text{int}}(\mathbf{t}, \mathbf{X}) = \int_{H(\mathbf{X})} (\mathbf{t}(\mathbf{t}, \mathbf{X}, \mathbf{X}') - \mathbf{t}(\mathbf{t}, \mathbf{X}', \mathbf{X})) dV' . \quad (9)$$

While the correspondence formulation overcomes the limitations of the bond-based and ordinary state-based models, it introduces stability issues from zero-energy modes [34,35], which are oscillatory displacement patterns that produce a vanishing deformation gradient and therefore vanishing internal forces [47]. The bond-associated approach addresses this shortcoming by computing bond-wise deformation gradients that capture local deformation variations.

2.3. Bond-associated quadrature points

The bond-associated quadrature point (BAQP) approach, introduced by Behzadinasab et al. [38,48,49], addresses the zero-energy mode instability by computing deformation gradients at bond-associated quadrature points located at the center of each bond. This is conceptually similar to higher-order quadrature in the finite element method, where additional integration points suppress hourglass modes.

A *bond-associated deformation gradient* $\bar{\mathbf{F}}$ is defined for each bond as

$$\bar{\mathbf{F}}(\mathbf{t}, \mathbf{X}, \mathbf{X}') = \bar{\mathbf{F}}^{\text{avg}}(\mathbf{t}, \mathbf{X}, \mathbf{X}') + (\Delta \mathbf{x} - \bar{\mathbf{F}}^{\text{avg}}(\mathbf{t}, \mathbf{X}, \mathbf{X}') \Delta \mathbf{X}) \otimes \frac{\Delta \mathbf{X}}{|\Delta \mathbf{X}|^2} \quad (10)$$

depending on the average deformation gradient of a bond $\bar{\mathbf{F}}^{\text{avg}}(\mathbf{t}, \mathbf{X}, \mathbf{X}') = \frac{1}{2} (\bar{\mathbf{F}}(\mathbf{t}, \mathbf{X}) + \bar{\mathbf{F}}(\mathbf{t}, \mathbf{X}'))$. The correction term in Eq. (10) ensures that the bond deformation gradient exactly reproduces the actual bond deformation in the bond direction, i.e. $\bar{\mathbf{F}}(\mathbf{t}, \mathbf{X}, \mathbf{X}') \Delta \mathbf{X} = \Delta \mathbf{x}$. This construction eliminates zero-energy modes, since oscillatory displacement patterns that would cancel in the point-wise average are captured by the bond-wise correction [38].

The strain energy density of a point is approximated by averaging the bond-wise strain energy densities over the neighborhood with

$$\bar{\Psi}(\mathbf{t}, \mathbf{X}) = \int_{H(\mathbf{X})} \bar{\omega}_b(\mathbf{X}, \mathbf{X}') \Psi(\bar{\mathbf{F}}(\mathbf{t}, \mathbf{X}, \mathbf{X}')) dV' , \quad (11)$$

where the *averaged bond kernel function* $\bar{\omega}_b : B_0 \times B_0 \rightarrow \mathbb{R}^+$ is defined as

$$\bar{\omega}_b(\mathbf{X}, \mathbf{X}') = \frac{1}{2} \left(\frac{\omega(\Delta \mathbf{X})}{\omega_0(\mathbf{X})} + \frac{\omega(-\Delta \mathbf{X})}{\omega_0(\mathbf{X}')} \right) , \quad (12)$$

with the *kernel integral* $\omega_0 : B_0 \rightarrow \mathbb{R}^+$ defined as

$$\omega_0(\mathbf{X}) = \int_{H(\mathbf{X})} \omega(\Delta \mathbf{X}) dV' . \quad (13)$$

The kernel integral satisfies the normalization condition for all interior points as

$$\omega_0(\mathbf{X}) = 1 \quad \forall \mathbf{X} \in \{ \mathbf{X} \in B_0 \mid H(\mathbf{X}) \subset B_0 \} . \quad (14)$$

This normalization condition is just a statement about internal points, since ω_0 appears in the denominator of the averaged bond kernel function (12) so boundary truncation is correctly accounted for. A commonly used kernel [38] satisfying this condition is the cubic spline kernel

$$\omega(\Delta \mathbf{X}) = \frac{8}{\pi \delta^3} \cdot \begin{cases} 1 - 6 \left(\frac{|\Delta \mathbf{X}|}{\delta} \right)^2 + 6 \left(\frac{|\Delta \mathbf{X}|}{\delta} \right)^3 & \text{if } \frac{|\Delta \mathbf{X}|}{\delta} \leq \frac{1}{2} , \\ 2 \left(1 - \frac{|\Delta \mathbf{X}|}{\delta} \right)^3 & \text{if } \frac{1}{2} < \frac{|\Delta \mathbf{X}|}{\delta} \leq 1 , \\ 0 & \text{if } \frac{|\Delta \mathbf{X}|}{\delta} > 1 . \end{cases} \quad (15)$$

The bond-wise first Piola–Kirchhoff stress tensor is evaluated from the constitutive model as

$$\bar{\mathbf{P}}(\mathbf{t}, \mathbf{X}, \mathbf{X}') = \mathbf{P}(\bar{\mathbf{F}}(\mathbf{t}, \mathbf{X}, \mathbf{X}')) . \quad (16)$$

Introducing the *non-uniform stress tensor* $\mathbf{Z} : [0, \infty) \times B_0 \rightarrow \mathbb{R}^{3 \times 3}$ as

$$\mathbf{Z}(t, \mathbf{X}) = \int_{H(\mathbf{X})} \bar{\omega}_b(\mathbf{X}, \mathbf{X}') \tilde{\mathbf{P}}(t, \mathbf{X}, \mathbf{X}') \left(\mathbf{I} - \frac{\Delta \mathbf{X} \otimes \Delta \mathbf{X}}{|\Delta \mathbf{X}|^2} \right) dV' , \quad (17)$$

the force state can be written compactly as

$$\mathbf{t}(t, \mathbf{X}, \mathbf{X}') = \bar{\omega}_b(\mathbf{X}, \mathbf{X}') \tilde{\mathbf{P}}(t, \mathbf{X}, \mathbf{X}') \frac{\Delta \mathbf{X}}{|\Delta \mathbf{X}|} + \mathbf{Z}(t, \mathbf{X}) \mathbf{B}(\mathbf{X}, \mathbf{X}') . \quad (18)$$

The first term projects the bond stress along the bond direction, while the second term distributes the non-uniform stress correction via the bond shape vector. This formulation is inherently free of zero-energy modes and provides a stable correspondence framework for large deformation problems [38,49].

3. Phase-field peridynamics

Following the brittle fracture approaches of Griffith and Irwin [1,2], a material fails when a critical energy release rate is reached. This assumption provides a criterion for crack growth based on energy considerations, which has to be embedded into an energy minimization setting for the entire structure to determine crack initiation and propagation [14,16,17,50]. Consequently, the total energy of a body is composed of its kinetic energy, the potential energy with the strain energy density of the undamaged material Ψ_0 , and the surface energy contributions from evolving crack boundaries with the critical energy release rate G_c . Here, Ψ_0 denotes the strain energy density Ψ from Section 2, now carrying a subscript 0 to distinguish the undamaged material response from the degraded one in the phase-field setting. With this assumption, the total energy functional reads

$$\mathcal{E}(t) = \int_{B_0 \setminus \Gamma_c(t)} \left(\frac{1}{2} \rho_0 |\dot{\mathbf{U}}|^2 + \Psi_0 \right) dV + \int_{\Gamma_c(t)} G_c dA , \quad (19)$$

where $\Gamma_c(t)$ is the crack surface at time t . An important assumption for thermodynamic consistency is that the crack evolution is irreversible, i.e., $\Gamma_c(t_1) \subset \Gamma_c(t_2)$ for $t_1 < t_2$.

In the **classical phase-field approach** [12–14], the sharp crack topology is regularized via a diffusive crack approximation. To characterize the state of the material, a phase-field parameter s is introduced. For the approximation, the discontinuous set of evolving crack surfaces $\Gamma_c(t)$ is replaced by a continuous surface energy density function, approximating the surface integral via a volumetric integral over the entire body B_0 . This leads to the objective functional for the total energy

$$\mathcal{E}(t) = \int_{B_0} \left(\frac{1}{2} \rho_0 |\dot{\mathbf{U}}|^2 + g(s) \Psi_0 + G_c \gamma_c(s) \right) dV \rightarrow \text{stationary} , \quad (20)$$

where

$$\gamma_c(s) = \frac{1}{c_0} \left(\frac{f_\Gamma(s)}{\ell_c} + \ell_c |\nabla s|^2 \right) \quad (21)$$

is a crack surface density function. The *degradation function* g quantifies the degradation of the material response due to damage. Also, the *crack surface regularization function* f_Γ is used to regularize the crack surface energy contribution based on the phase-field parameter. The regularization term with the *length scale parameter* ℓ_c ensures a diffusive crack approximation over a finite width. The *normalization constant* $c_0 \in \mathbb{R}^+$ ensures that the total crack dissipation corresponds to Griffith's energy release rate per unit crack area and can be calculated analytically for a given choice of f_Γ , cf. [51,52]. For the commonly used AT1 model with $f_\Gamma(s) = s$, the normalization constant is $c_0 = \frac{8}{3}$, while for the AT2 model with $f_\Gamma(s) = s^2$, one obtains $c_0 = 2$ [14,51,52].

In the following, the classical phase-field approach to fracture is adapted to the bond-associated correspondence framework described in Section 2. The length scale parameter ℓ_c is defined as

$$\ell_c := \delta , \quad (22)$$

to maintain consistency with the nonlocal interaction radius in peridynamics. Since the nonlocal averaging over the horizon inherently provides a regularization of the crack approximation, the regularization term $\ell_c |\nabla s|^2$ vanishes in the bond-associated formulation.

3.1. A bond-associated phase-field approximation

To embed the phase-field fracture approach into the bond-associated correspondence framework, the total energy functional is expressed in terms of bond contributions. The *bond phase-field parameter* $s : [0, \infty) \times B_0 \times B_0 \rightarrow [0, 1]$ is introduced as a bond-wise variable, defining the state of damage of the bond between two material points $\mathbf{X}$ and $\mathbf{X}'$ at time t . Following the definition of the point-wise strain energy of the BAQP formulation in Eq. (11), the total energy functional from Eq. (20) can be expressed in terms of bond contributions in a peridynamic setting as

$$\begin{aligned} \mathcal{E}(t) = & \int_{B_0} \frac{1}{2} \rho_0 |\dot{\mathbf{U}}(t, \mathbf{X})|^2 dV \\ & + \int_{B_0} \int_{H(\mathbf{X})} \bar{\omega}_b(\mathbf{X}, \mathbf{X}') \left(g(s(t, \mathbf{X}, \mathbf{X}')) \Psi_0 + \frac{G_c f_\Gamma(s(t, \mathbf{X}, \mathbf{X}'))}{c_0 \delta} \right) dV' dV \rightarrow \text{stationary} . \end{aligned} \quad (23)$$

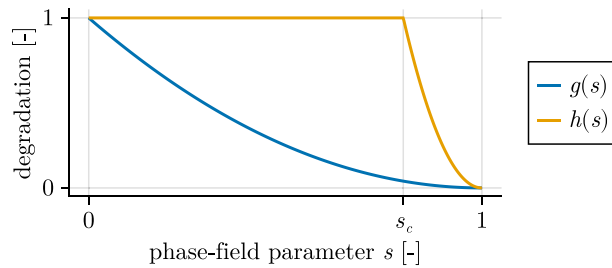


Fig. 2. The bond and kinematic degradation functions g and h over the bond phase-field parameter s .

To keep the notation consistent with the classical peridynamics approach, a fully damaged bond has the value of $s = 1$, while an undamaged bond has the value of $s = 0$. The usage of the BAQP framework and the additional integral over the horizon inherently provides a regularization of the crack approximation over the horizon length scale.

The *total crack dissipation* $\mathcal{E}^F : [0, \infty) \rightarrow \mathbb{R}$ accounts for the total energy dissipated due to crack formation and growth in the body with

$$\mathcal{E}^F(t) = \int_{B_0} \int_{H(X)} \bar{\omega}_b(\mathbf{X}, \mathbf{X}') \frac{G_c f_F(s(t, \mathbf{X}, \mathbf{X}'))}{c_0 \delta} dV' dV, \quad (24)$$

where G_c , c_0 , and δ are constant for each point in the body. A *bond crack dissipation* $E_b^F : [0, \infty) \times B_0 \times B_0 \rightarrow \mathbb{R}$ can be introduced as

$$E_b^F(t, \mathbf{X}, \mathbf{X}') = \frac{G_c f_F(s(t, \mathbf{X}, \mathbf{X}'))}{c_0 \delta}, \quad (25)$$

denoting the energetic contribution of a single bond to the total crack dissipation. With Eq. (25), the total crack dissipation can be expressed as the integral of the bond crack dissipation over the body and its horizon as

$$\mathcal{E}^F(t) = \int_{B_0} \int_{H(X)} \bar{\omega}_b(\mathbf{X}, \mathbf{X}') E_b^F(t, \mathbf{X}, \mathbf{X}') dV' dV. \quad (26)$$

Here, E_b^F is bond-wise local and does not depend on the neighboring bonds. For the remainder of this work, the crack regularization function f_F is chosen as

$$f_F(s) := s(t, \mathbf{X}, \mathbf{X}'), \quad (27)$$

consistent with the AT1 model in the literature [12,51,52]. This leads to a linear increase of the crack surface energy with increasing damage.

The degradation of the material response due to damage is modeled via the *bond degradation function* $g : [0, 1] \rightarrow [0, 1]$. A common choice for the degradation function is the quadratic form

$$g(s) = (1 - s(t, \mathbf{X}, \mathbf{X}'))^2, \quad (28)$$

leading to a smooth degradation of the material response with increasing damage. The bond-wise first Piola–Kirchhoff stress tensor $\tilde{\mathbf{P}}$ from Eq. (16) is modified to account for the degradation due to damage as

$$\tilde{\mathbf{P}}(t, \mathbf{X}, \mathbf{X}') = g(s(t, \mathbf{X}, \mathbf{X}')) \mathbf{P}_0(\tilde{\mathbf{F}}(t, \mathbf{X}, \mathbf{X}')), \quad (29)$$

with the undamaged first Piola–Kirchhoff stress tensor $\mathbf{P}_0$, here denoted with a subscript 0 in contrast to the previous section. This ensures that the force density contribution of a bond vanishes smoothly as the bond becomes fully damaged with $s \rightarrow 1$.

However, in peridynamics, the kinematic approximation via the shape tensor $\mathbf{K}$ and the associated bond shape vectors $\mathbf{B}$ plays a crucial role in ensuring accurate deformation measures that have to be considered when degrading bonds. To maintain numerical stability and the accuracy of the kinematic approximation, a *bond kinematic degradation function* $h : [0, 1] \rightarrow [0, 1]$ is introduced. It is defined as

$$h(s) = \begin{cases} 1 & \text{if } s(t, \mathbf{X}, \mathbf{X}') \leq s_c, \\ \left(\frac{1 - s(t, \mathbf{X}, \mathbf{X}')}{1 - s_c} \right)^2 & \text{if } s(t, \mathbf{X}, \mathbf{X}') > s_c, \end{cases} \quad (30)$$

where $s_c \in [0, 1]$ is a parameter defining a threshold for the degradation of the kinematic approximation, as displayed in Fig. 2. For $s_c = 1$, the kinematic degradation is disabled, while for $s_c = 0$, the kinematic approximation degrades similarly to the energetic contribution with

$$h(s(t, \mathbf{X}, \mathbf{X}')) \Big|_{s_c=0} = g(s(t, \mathbf{X}, \mathbf{X}')). \quad (31)$$

The shape tensor and bond shape vector are modified to account for the kinematic degradation as

$$\mathbf{K}(t, \mathbf{X}) = \int_{\mathcal{H}(\mathbf{X})} \omega(\Delta \mathbf{X}) h(s(t, \mathbf{X}, \mathbf{X}')) \Delta \mathbf{X} \otimes \Delta \mathbf{X} dV', \quad (32)$$

and

$$\mathbf{B}(t, \mathbf{X}, \mathbf{X}') = \omega(\Delta \mathbf{X}) h(s(t, \mathbf{X}, \mathbf{X}')) \mathbf{K}^{-1}(\mathbf{X}) \Delta \mathbf{X}. \quad (33)$$

Note that both the shape tensor and the bond shape vector are now time-dependent due to the time dependence of the phase-field. Without the kinematic degradation, damage under large deformations can lead to unphysical deformation gradients, and fully broken fragments are incorrectly still connected to the intact material via the kinematic approximation.

3.2. Damage evolution and crack criteria

The evolution of the damage state in classical phase-field fracture is derived from a variational principle minimizing the total energy functional with respect to the phase-field parameter s [13,14]. Similarly, here a bond-wise local energy minimization principle is employed to derive the evolution of the bond phase-field parameter s . Considering a static situation and a single bond between two material points $\mathbf{X}$ and $\mathbf{X}'$, the energy contribution of a bond $E_b : [0, \infty) \times B_0 \times B_0 \rightarrow \mathbb{R}$ is defined as

$$E_b(t, \mathbf{X}, \mathbf{X}') = g(s(t, \mathbf{X}, \mathbf{X}')) \Psi_0(\tilde{\mathbf{F}}(t, \mathbf{X}, \mathbf{X}')) + E_b^f(t, \mathbf{X}, \mathbf{X}'). \quad (34)$$

The bond phase-field parameter evolves by minimizing the bond mechanical energy as

$$\frac{\delta E_b}{\delta s} = \frac{dg(s(t, \mathbf{X}, \mathbf{X}'))}{ds} \Psi_0 + \frac{G_c}{c_0 \delta} \frac{df_f(s(t, \mathbf{X}, \mathbf{X}'))}{ds} = 0, \quad (35)$$

leading with Eq. (27) and Eq. (28) to $\frac{df_f(s(t, \mathbf{X}, \mathbf{X}'))}{ds} = 1$ and the relation

$$-2(1 - s(t, \mathbf{X}, \mathbf{X}')) \Psi_0 + \frac{G_c}{c_0 \delta} = 0. \quad (36)$$

Substituting a phase-field value of $s(t, \mathbf{X}, \mathbf{X}') = 0$ yields the *critical crack driving force* Y_c as

$$Y_c := \frac{G_c}{2 c_0 \delta}, \quad (37)$$

defining an elastic strain energy density threshold for damage initiation. Deformations below this threshold do not lead to damage growth, and the bond remains intact with $s = 0$. Only the tensile (positive) part of the undamaged strain energy density, denoted Ψ_0^+ , contributes to damage growth, therefore the bond-wise *crack driving force* $Y : [0, \infty) \times B_0 \times B_0 \rightarrow \mathbb{R}$ is defined as

$$Y(t, \mathbf{X}, \mathbf{X}') = \Psi_0^+(\tilde{\mathbf{F}}(t, \mathbf{X}, \mathbf{X}')). \quad (38)$$

This approach is consistent with well-established phase-field fracture models in the literature [51,52].

In the work of Bilgen et al. [53,54], a physically motivated definition of the crack driving force based on the maximum principal stress σ_1 is proposed. The maximum principal stress is computed as the maximum eigenvalue of the bond-wise Cauchy stress tensor. This leads to the definition of a critical stress σ_c for damage initiation, ensuring that damage only occurs when the maximum principal stress exceeds this critical value. Defining stress as a driving force for damage is particularly appealing for engineering applications, as design criteria often rely on stress-based failure theories. The strain energy density of a bond for a linear elastic material is given as

$$\Psi_0 = \frac{\sigma^2}{2E}, \quad (39)$$

with the scalar stress value σ and Young's modulus E of the bond. The critical stress σ_c is derived from the critical strain energy density in Eq. (37) as

$$\sigma_c := \sqrt{\frac{E G_c}{c_0 \delta}}. \quad (40)$$

Analogously, the critical crack driving force is expressed in terms of the critical stress as

$$Y_c := \frac{\sigma_c^2}{2E}. \quad (41)$$

The crack driving force is then defined based on the maximum principal stress as

$$Y(t, \mathbf{X}, \mathbf{X}') = \frac{\langle \sigma_1(\tilde{\mathbf{F}}(t, \mathbf{X}, \mathbf{X}')) \rangle_+^2}{2E}, \quad (42)$$

with the Macaulay bracket $\langle \bullet \rangle_+ = \max(0, \bullet)$ extracting only tensile contributions.

To ensure the irreversibility of damage, a *history variable* $\mathcal{Y} : [0, \infty) \times B_0 \times B_0 \rightarrow \mathbb{R}$ is introduced, tracking the maximum crack driving force experienced by the bond up to time t as

$$\mathcal{Y}(t, \mathbf{X}, \mathbf{X}') = \max_{t' \leq t} Y(t', \mathbf{X}, \mathbf{X}') . \quad (43)$$

The evolution of the bond phase-field parameter follows from the stationarity condition Eq. (36), with the undamaged strain energy density replaced by the history variable $\mathcal{Y}$. Solving for the bond phase-field parameter and restricting it to the admissible interval yields the closed-form evolution law

$$s(t, \mathbf{X}, \mathbf{X}') = \left\langle 1 - \frac{Y_c}{\mathcal{Y}(t, \mathbf{X}, \mathbf{X}')} \right\rangle_+ . \quad (44)$$

Since the history variable is non-decreasing in time, damage can only grow and remains bounded within the interval $[0, 1]$. The Macaulay bracket reproduces the elastic threshold of Eq. (37), such that bonds with $\mathcal{Y} < Y_c$ remain intact, which is the characteristic feature of the AT1 model and consistent with the choice of f_r in Eq. (27). Above the threshold, the law provides a smooth damage transition governed by the critical crack driving force Y_c and guarantees thermodynamic consistency with non-negative crack dissipation.

To connect the bond-wise defined phase-field parameter s to a point-wise damage variable similar to the classical peridynamics approach, the bond phase-field parameter is averaged over the family of a material point $\mathbf{X}$ as

$$\bar{s}(t, \mathbf{X}) = \int_{\mathcal{H}(\mathbf{X})} \bar{\omega}_b(\mathbf{X}, \mathbf{X}') s(t, \mathbf{X}, \mathbf{X}') dV' . \quad (45)$$

This is identical to the classical definition of a point-wise damage variable in peridynamics, therefore yielding

$$D(t, \mathbf{X}) = \bar{s}(t, \mathbf{X}) . \quad (46)$$

With this definition, the bond-associated phase-field peridynamics framework seamlessly integrates with existing peridynamic damage modeling approaches and the point-wise damage variable can be used for visualization and post-processing purposes.

3.3. Determination of the normalization constant

A critical remaining aspect of the bond-associated phase-field peridynamics framework is the determination of the normalization constant c_0 . The normalization constant must be chosen such that the total crack dissipation corresponds to Griffith's assumption of energy release rate per unit crack area.

In classical phase-field fracture mechanics, the normalization constant is derived through a rigorous mathematical procedure involving Γ -convergence arguments. The approach considers an infinite domain with a one-dimensional damage profile, solves an Euler–Lagrange equation for the optimal damage distribution, and demonstrates convergence to a sharp crack formulation in the limit of vanishing length scale parameter [13,55].

In bond-associated phase-field peridynamics, a planar crack in an infinite domain is considered, and the normalization constant is determined by requiring that the total crack dissipation equals the Griffith energy release. However, the nonlocal nature of the bond-associated peridynamics framework introduces additional geometric considerations, as the crack dissipation depends on the volume of bonds crossing the crack plane weighted by the kernel function. Therefore, the normalization constant depends on the choice of the kernel function ω .

Griffith's assumption states that the energy required to create new crack surface is proportional to the crack area. For a crack growth of area A_Γ , the required total crack dissipation is

$$\mathcal{E}^\Gamma = G_c A_\Gamma . \quad (47)$$

In the following, the normalization constant is derived for general spherical kernel functions, using Eq. (47) as the defining condition for c_0 .

3.3.1. Setting and assumptions

Consider an infinite material domain $\Omega \subset \mathbb{R}^3$ with a planar crack surface Γ_c at position $z = 0$, where the *crack coordinate* $z : B_0 \rightarrow \mathbb{R}$ is defined as the signed perpendicular distance from a material point $\mathbf{X}$ to the crack plane as

$$z(\mathbf{X}) = \mathbf{X} \cdot \mathbf{n}_\Gamma , \quad (48)$$

with the unit crack normal vector $\mathbf{n}_\Gamma$. Points with $z(\mathbf{X}) > 0$ are located above the crack plane, while points with $z(\mathbf{X}) < 0$ are located below. Only points within distance δ of the crack plane have bonds that cross the crack and contribute to the total crack dissipation.

For the derivation, the kernel function is assumed to be spherical, depending only on the bond length $r := |\Delta \mathbf{X}|$, i.e., $\omega(\Delta \mathbf{X}) = \omega_s(r)$ with the scalar kernel function $\omega_s : \mathbb{R}^+ \rightarrow \mathbb{R}^+$. Additionally, the material domain is assumed to be infinite, such that all points involved in the derivation are bulk points with complete spherical neighborhoods. This implies that the kernel integrals $\omega_0(\mathbf{X})$ from Eq. (13) are equal for all material points, i.e., $\omega_0(\mathbf{X}) = \omega_0(\mathbf{X}')$ for all $\mathbf{X}, \mathbf{X}' \in \Omega$. Due to the equality, the simplified notation $\omega_0 := \omega_0(\mathbf{X})$ is used in the following. Note that the normalization condition from Eq. (14) yields $\omega_0 = 1$. However, ω_0 is kept

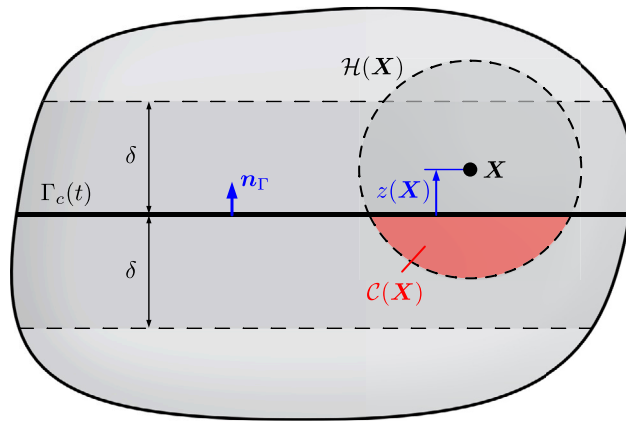


Fig. 3. Spherical cap geometry for the derivation of the normalization constant. A material point X at crack coordinate $z = z(X) > 0$ has its spherical neighborhood $\mathcal{H}(X)$ intersected by the crack plane at $z = 0$. The spherical cap $\mathcal{C}(X)$ contains all neighbors below the crack plane whose bonds cross the crack.

explicitly in the following derivation to maintain generality for arbitrary kernel functions. Under these assumptions, the averaged bond kernel function from Eq. (12) simplifies to

$$\bar{\omega}_b(\mathbf{X}, \mathbf{X}') = \frac{\omega(\Delta \mathbf{X})}{\omega_0}. \quad (49)$$

3.3.2. Crack dissipation energy density of a material point

The point-wise crack dissipation energy density $E_X^\Gamma: [0, \infty) \times B_0 \rightarrow \mathbb{R}$ for a material point X is defined as the weighted integral of the bond crack dissipation E_b^Γ from Eq. (25) over all neighbors as

$$E_X^\Gamma(t, \mathbf{X}) = \int_{\mathcal{H}(X)} \bar{\omega}_b(\mathbf{X}, \mathbf{X}') E_b^\Gamma(t, \mathbf{X}, \mathbf{X}') dV'. \quad (50)$$

The total crack dissipation can be obtained by integrating E_X^Γ over the material domain. For a fully propagated sharp crack, all bonds crossing the crack have $s = 1$, and all bonds not crossing the crack have $s = 0$. Consequently, the crack regularization function evaluates to $f_\Gamma(s) = 1$ for crossing bonds and $f_\Gamma(s) = 0$ for non-crossing bonds.

A bond between X at height $z(X)$ and X' at height $z(X')$ crosses the crack if and only if the endpoints are on opposite sides of the crack plane, i.e., $z(X) \cdot z(X') < 0$. For a point X at crack coordinate $z = z(X) > 0$ (above the crack), all neighbors X' with $z(X') < 0$ (below the crack) form the set of crossing bonds. This set of neighbors constitutes a spherical cap $\mathcal{C}(X)$, defined as

$$\mathcal{C}(X) = \left\{ X' \in \mathcal{H}(X) \mid z(X) \cdot z(X') < 0 \right\}. \quad (51)$$

The point-wise crack dissipation then reduces to an integral over the spherical cap as

$$E_X^\Gamma(\mathbf{X}) = \int_{\mathcal{C}(X)} \frac{\omega_s(r)}{\omega_0} \frac{G_c}{c_0 \delta} dV' = \frac{G_c}{c_0 \delta \omega_0} \int_{\mathcal{C}(X)} \omega_s(r) dV'. \quad (52)$$

Since the crack is planar, for any two material points X_1 and X_2 with equal crack coordinates $z(X_1) = z(X_2)$, the spherical caps $\mathcal{C}(X_1)$ and $\mathcal{C}(X_2)$ are geometrically congruent. Combined with the fact that the spherical kernel function ω_s depends only on the bond length r , the kernel-weighted cap integrals are identical, i.e.,

$$\int_{\mathcal{C}(X_1)} \omega_s(r) dV' = \int_{\mathcal{C}(X_2)} \omega_s(r) dV'. \quad (53)$$

This motivates the definition of the general spherical cap $\mathcal{C}_z(z) := \mathcal{C}(X)$ for any point $X \in \Omega$ at a given crack coordinate $z = z(X)$. Consequently, the point-wise crack dissipation of all points with equal crack coordinate is identical, i.e., $E_X^\Gamma(X_1) = E_X^\Gamma(X_2)$. Therefore, the point-wise crack dissipation can be expressed as a function of the crack coordinate alone, i.e.,

$$E_z^\Gamma(z) = \frac{G_c}{c_0 \delta \omega_0} \int_{\mathcal{C}_z(z)} \omega_s(r) dV'. \quad (54)$$

3.3.3. Volume of the spherical cap

The volume of the spherical cap depends mainly on the crack coordinate $z(X)$ of the material point, as illustrated in Fig. 3. Since the crack is considered planar and the neighborhood a sphere of radius δ centered at height z above the crack plane, the cap volume is given by the standard equation

$$V^{\mathcal{C}}(z) = \frac{\pi}{3} (\delta - |z|)^2 (2\delta + |z|). \quad (55)$$

The absolute value ensures that the symmetry of the cap volume is captured for points below the crack plane with $-\delta < z < 0$, enforcing $V^C(z) = V^C(-z)$. The total volume of the neighborhood sphere is $V^H = \frac{4}{3}\pi\delta^3$. The *geometric volume fraction function* $f_V : [-\delta, \delta] \rightarrow [0, \frac{1}{2}]$ is defined as the ratio of the cap volume to the sphere volume, yielding

$$f_V(z) = \frac{V^C(z)}{V^H} = \frac{(\delta - |z|)^2(2\delta + |z|)}{4\delta^3} . \quad (56)$$

This function satisfies $f_V(0) = \frac{1}{2}$, meaning that a point on the crack plane has exactly half of its neighborhood volume on each side. Additionally, $f_V(\pm\delta) = 0$, indicating that a point at distance δ from the crack has no neighborhood volume on the other side.

Introducing the *normalized crack coordinate* $\xi := |z|/\delta \in [0, 1]$, the volume fraction can be expressed as

$$f_V(\xi) = \frac{V^C(\xi\delta)}{V^H} = \frac{(\delta - \xi\delta)^2(2\delta + \xi\delta)}{4\delta^3} = \frac{(1 - \xi)^2(2 + \xi)}{4} \quad \forall \xi \in [0, 1] . \quad (57)$$

Expanding the numerator leads to the explicit polynomial form

$$f_V(\xi) = \frac{1}{2} - \frac{3\xi}{4} + \frac{\xi^3}{4} \quad \forall \xi \in [0, 1] . \quad (58)$$

Similarly, the normalized form satisfies $f_V(0) = \frac{1}{2}$ and $f_V(1) = 0$. This normalized form will be used in the subsequent derivation of the normalization constant.

3.3.4. Integration over the crack band

The total crack dissipation is obtained by integrating the point-wise crack dissipation energy density over the material domain. Only points forming a band of thickness 2δ centered on the crack plane are expected to contribute. Due to the symmetry of the spherical cap geometry, points at height z above the crack and at height $-z$ below the crack contribute equally to the total crack dissipation. The total dissipated energy therefore consists of equal contributions from both sides of the crack.

For the upper block with $0 < z < \delta$, the crack dissipation is integrated as follows. Consider a thin layer of material at height z with infinitesimal thickness dz and crack area A_I . The volume of this layer is $A_I dz$. Since E_X^I is an energy density (energy per unit volume), the contribution to the total crack dissipation from this layer is

$$d\mathcal{E}_+^I = E_z^I(z) A_I dz . \quad (59)$$

The total contribution from the upper block is obtained by integration as

$$\mathcal{E}_+^I = A_I \int_0^\delta E_z^I(z) dz . \quad (60)$$

Utilizing the normalized crack coordinate ξ , a *kernel-weighted volume fraction function* $f_\omega : [0, 1] \rightarrow [0, \frac{1}{2}]$ can be defined as

$$f_\omega(\xi) = \frac{1}{\omega_0} \int_{C_z(\xi\delta)} \omega_s(r) dV' . \quad (61)$$

This function gives the fraction of the total kernel weight contained in the spherical cap at height $z = \xi\delta$. It satisfies the same boundary conditions as the geometric volume fraction function, i.e., $f_\omega(0) = \frac{1}{2}$ and $f_\omega(1) = 0$. Evaluating the cap integral in spherical coordinates centered at $\mathbf{X}$ with the polar axis along the crack normal $\mathbf{n}_I$, and introducing the *normalized radial coordinate* $\rho := r/\delta \in [0, 1]$, the cap fraction is obtained in closed form as

$$f_\omega(\xi) = \frac{\int_\xi^1 \omega_s(\rho\delta) \rho(\rho - \xi) d\rho}{2 \int_0^1 \omega_s(\rho\delta) \rho^2 d\rho} . \quad (62)$$

Eq. (62) is exact for any spherical kernel function and reduces to the geometric volume fraction function Eq. (58) for a constant kernel.

With this definition, the point-wise crack dissipation from Eq. (52) becomes

$$E_z^I(z) = \frac{G_c}{c_0 \delta} f_\omega(z/\delta) . \quad (63)$$

Inserting this expression into Eq. (60) and changing to the normalized coordinate $\xi = z/\delta$ with $dz = \delta d\xi$, the contribution from the upper block yields

$$\mathcal{E}_+^I = A_I \frac{G_c}{c_0 \delta} \int_0^\delta f_\omega(z/\delta) dz = \frac{G_c A_I}{c_0} \int_0^1 f_\omega(\xi) d\xi . \quad (64)$$

Due to the symmetry of the spherical cap geometry, the contribution from the lower block is equal to the upper block contribution. The total crack dissipation is therefore

$$\mathcal{E}^I = 2 \mathcal{E}_+^I = 2 \frac{G_c A_I}{c_0} \int_0^1 f_\omega(\xi) d\xi . \quad (65)$$

The cap fraction f_ω enters Eq. (65) only through the sharp-crack profile, on which the bond crack regularization $f_r(s)$ equals 1 on the crossing bonds and 0 on all others. A general damage state replaces it by the point-wise *kernel-weighted band average* of the crack regularization function,

$$\hat{f}_r(t, \mathbf{X}) := \int_{H(\mathbf{X})} \bar{\omega}_b(\mathbf{X}, \mathbf{X}') f_r(s(t, \mathbf{X}, \mathbf{X}')) dV' , \quad (66)$$

a time-dependent field over the body. Since the averaged bond kernel Eq. (49) integrates to unity over the family by Eq. (13), the band average is a convex combination of the bond values with $0 \leq \hat{f}_r \leq 1$, and it recovers the cap fraction on the sharp-crack profile, $\hat{f}_r = f_\omega$. For the planar crack considered here the damage state depends on position only through the distance to the crack plane, so the band average reduces to a function $\hat{f}_r(t, \xi)$ of time and the band coordinate $\xi = z/\delta$. Substituting $f_\omega \rightarrow \hat{f}_r$ in Eqs. (63) and (65) generalizes the derivation to any damage profile in the crack band.

3.3.5. Derivation of the normalization constant

Applying the Griffith condition Eq. (47), the total crack dissipation Eq. (65) must equal $G_c A_r$, leading to

$$G_c A_r = 2 \frac{G_c A_r}{c_0} \int_0^1 \hat{f}_r(t, \xi) d\xi . \quad (67)$$

Here, c_0 is the fixed material constant of the energy functional Eq. (20) and the factor 2 stems from the crack-band symmetry $z \in (-\delta, \delta)$. The identity holds at a given instant only if c_0 equals the *apparent normalization*

$$\bar{c}_0(t) = 2 \int_0^1 \hat{f}_r(t, \xi) d\xi . \quad (68)$$

The sharp-crack profile with $\hat{f}_r = f_\omega$ is stationary and defines a fixed value, but the evolution law Eq. (44) cannot realize its jump between crossing and non-crossing bonds and instead damages a band around the crack. This wake damage raises $\hat{f}_r$ above f_ω and grows as the crack opens, so $\bar{c}_0(t)$ is not constant and no fixed c_0 satisfies Eq. (67) throughout the loading unless the wake damage becomes stationary.

The crack opening enters the band through the deformation gradient. For a point at ξ above an open crack of opening Δ_r , only the crossing bonds, i.e. the neighbors in the spherical cap $C_z(\xi\delta)$, carry the relative displacement jump $\Delta_r \mathbf{n}_r$ across the crack faces, while the non-crossing bonds transmit none. The cap's kernel-weighted reach along the crack normal is measured by its *first normal moment*

$$m_\omega(\xi) = \frac{1}{\omega_0} \int_{C_z(\xi\delta)} \omega_s(r) |\Delta \mathbf{X} \cdot \mathbf{n}_r| dV' , \quad (69)$$

the direct counterpart of the cap fraction f_ω Eq. (61), the *zeroth* moment of the same cap. Both integrate the kernel over $C_z(\xi\delta)$ and differ only by the crack-normal lever arm $|\Delta \mathbf{X} \cdot \mathbf{n}_r|$, so that f_ω counts the crossing bonds and sets the dissipation, whereas m_ω weights each by its normal reach and sets the wake stretch. Inserting the crossing-bond jump into the deformation gradient Eq. (4), whose shape tensor Eq. (5) is isotropic for a spherical kernel, the cap contribution aligns with the crack normal and the non-crossing bonds feel the uniaxial stretch

$$\tilde{\mathbf{F}}(\xi) = \mathbf{I} + \varepsilon_r(\xi) \mathbf{n}_r \otimes \mathbf{n}_r . \quad (70)$$

Its amplitude scales with the opening and the first moment, $\varepsilon_r(\xi) \propto \Delta_r m_\omega(\xi)$, through a proportionality factor set by the inverse shape tensor. Since that factor cancels out, we denote the normalized stretch to its crack-plane value as the *normalized wake amplitude*

$$q_\omega(\xi) = \frac{\varepsilon_r(\xi)}{\varepsilon_r(0)} = \frac{m_\omega(\xi)}{m_\omega(0)} = \frac{\int_\xi^1 \omega_s(\rho\delta) \rho (\rho^2 - \xi^2) d\rho}{\int_0^1 \omega_s(\rho\delta) \rho^3 d\rho} , \quad (71)$$

with $q_\omega : [0, 1] \rightarrow [0, 1]$, $q_\omega(0) = 1$, and $q_\omega(1) = 0$.

The kinematic degradation h Eq. (30) freezes this growing wake damage. Once the crossing bonds exceed s_c they lose their weight in the shape tensor Eq. (32), the crack faces decouple, the stretch Eq. (70) is screened, and the history variable freezes the wake at the level reached at that instant. With the *driving level* $\Lambda := \mathcal{Y}/Y_c$, the evolution law Eq. (44) reads $s = \langle 1 - \Lambda^{-1} \rangle_+$, so a bond reaches s_c at the *freeze level* $\Lambda_f = (1 - s_c)^{-1}$. The crack driving force Eq. (42) is the tensile strain energy and hence quadratic in the stretch Eq. (70). Fixing the scaling by the innermost wake bond, which reaches Λ_f at $\xi \rightarrow 0$ with $q_\omega(0) = 1$, the driving level of a wake bond at ξ is

$$\Lambda(\xi) = \Lambda_f q_\omega(\xi)^2 . \quad (72)$$

Substituting Eq. (72) into the evolution law yields the opening-independent *frozen-wake profile*

$$\hat{s}_f(\xi) = \left\langle 1 - \frac{1 - s_c}{q_\omega(\xi)^2} \right\rangle_+ . \quad (73)$$

On this profile the wake bonds at ξ are all driven by the common stretch of Eq. (70) and hence share the single damage $\hat{s}_f(\xi)$, so their contribution to the band average is $f_r(\hat{s}_f(\xi))$. Splitting the family into the crossing bonds, which fill the cap weight $f_\omega(\xi)$ from

Table 1

Normalization constant c_0 for the two kernel functions used in this work, comparing the sharp-crack value of Eq. (76) with the frozen-wake value from Eq. (75) for three critical phase-field values s_c .

c_0 , computed for ... kernel function	Sharp crack	Frozen wake		
		$s_c = 0.90$	$s_c = 0.95$	$s_c = 0.99$
cubic B-spline, Eq. (15)	0.221	0.655	0.778	1.012
linear, Eq. (98)	0.300	0.876	1.030	1.303

Eq. (61) with $f_F = 1$, and the non-crossing wake bonds sharing the remaining weight $1 - f_w(\xi)$, the band average Eq. (66) becomes the stationary *frozen band average*

$$\hat{f}_F^f(\xi) = f_w(\xi) + (1 - f_w(\xi)) f_F(\hat{s}_F(\xi)), \quad (74)$$

independent of the opening and hence of time. The apparent normalization Eq. (68) then loses its time dependence, yielding the material *frozen-wake normalization constant*

$$c_0 = 2 \int_0^1 \hat{f}_F^f(\xi) d\xi. \quad (75)$$

Its integrand involves only the kernel, the crack regularization, the evolution law, and the critical phase-field value s_c , on which c_0 depends as a fixed material parameter. No fitted constant and no reference to the loading enter, and it is evaluated once by quadrature when the material is defined. Two special cases confirm the construction. At $s_c = 0$ the screening is immediate, the frozen-wake profile vanishes, and Eq. (75) recovers the sharp-crack value

$$c_0|_{s_c=0} = 2 \int_0^1 f_w(\xi) d\xi. \quad (76)$$

As $s_c \rightarrow 1$ the kinematic degradation is disabled, the freeze level Eq. (72) diverges, and c_0 tends to 2, so that no finite fracture energy exists. The critical phase-field value s_c must therefore be defined as $s_c < 1$, and the value $s_c = 0.95$ used throughout this work is admissible. Table 1 lists the resulting constants, which exceed the sharp-crack values by a factor of three to four. The detailed derivation of Eqs. (70) and (75), the proof that the wake damage freezes, and the bounds on s_c are subject of ongoing work. The numerical verification is given in Section 5.3.

3.4. Remarks on the regularization length scale and kernel choice

The role of the horizon as the regularization length scale deserves further discussion. In the present formulation, the horizon fully replaces the classical phase-field length scale. With $\ell_c := \delta$, the width of the diffuse damage band is set by the nonlocal interaction radius, and the averaging over the neighborhood takes over the regularizing role of the gradient term $\ell_c |\nabla s|^2$. Nevertheless, the two length scales enter the formulations differently. In the classical setting, ℓ_c controls the width of the optimal one-dimensional damage profile of the point-wise phase field. In the PFPD setting, the damage band emerges from the set of bonds crossing the crack surface together with the frozen wake, and its extent is bounded by the horizon. Since the normalization constant Eq. (75) enforces the Griffith dissipation $G_c A_F$ for every value of δ , the fracture energy is independent of the horizon, which therefore acts, as in the elastic case, solely as the nonlocal interaction length. In the vanishing-horizon limit $\delta \rightarrow 0$, the crack band of thickness 2δ contracts to the sharp crack surface while the dissipated energy per unit crack area remains G_c , so that a Griffith-type sharp-crack description is recovered formally. A rigorous analysis of this limit is beyond the scope of this work. Related convergence results have been established for nonlocal fracture models with continuous damage by Lipton and coworkers [28–30] and for finite-horizon nonlocal gradient functionals by Cueto et al. [26,27].

The choice of the kernel function affects the normalization constant considerably, as shown in Table 1. Once this dependence is accounted for by evaluating c_0 for the employed kernel, the resulting fracture behavior is insensitive to the kernel choice, as the boundary tension test in Section 5.3 confirms with consistent branching patterns for the cubic B-spline and the linear kernel across all horizon ratios. Evolving damage zones around a propagating crack do not require an additional renormalization, since the wake damage accompanying the crack is precisely what the frozen-wake construction of Section 3.3 incorporates into c_0 .

4. Discretization and algorithmic implementation

This section presents the spatial discretization of the peridynamic equations introduced in Section 2, the connection to reproducing kernel methods, and the algorithmic implementation of the PFPD model.

4.1. Spatial discretization

To solve the previously described equations numerically, the continuous body is represented as a discrete set of material points. A discretized peridynamic body $B_0^N = \{X_k \in B_0 \mid k = 1, \dots, N\}$ is defined as a set of N points in the reference domain B_0 . Each point X_k occupies a finite volume $V_k > 0$ in the reference configuration. In a uniform discretization, the points are arranged in a regular grid with a characteristic *point spacing* Δx , with $V_k = \Delta x^3$ in three-dimensional space. The *horizon ratio* m_δ relates the horizon to the point spacing as

$$\delta = (m_\delta + \epsilon_\delta) \Delta x, \quad (77)$$

with the numerical parameter ϵ_δ , usually chosen as $\epsilon_\delta = 0.015$, ensuring that the horizon is strictly larger than the point spacing. The *discrete neighborhood* of a point X_k is defined as

$$\mathcal{H}_k^N := \mathcal{H}^N(X_k) = \{X_n \in B_0^N \mid 0 < |\Delta X_{kn}| \leq \delta\}, \quad (78)$$

with the discrete bond $\Delta X_{kn} = X_n - X_k$ and neighbor volume $V_n > 0$. In the discrete setting, integrals over the neighborhood are approximated by sums over all neighbors as

$$\int_{\mathcal{H}(X_k)} \omega(\Delta X)(\bullet) dV' \approx \sum_{X_n \in \mathcal{H}_k^N} \omega(\Delta X_{kn})(\bullet) V_n. \quad (79)$$

This quadrature rule is applied to all neighborhood integrals, including the shape tensor in Eq. (5), the deformation gradient in Eq. (4), the internal force density in Eq. (9), as well as the strain energy density and force state of the BAQP formulation in Eq. (11) and Eq. (18).

4.2. Reproducing kernel peridynamics notation

The BAQP correspondence formulation can be equivalently expressed in the framework of reproducing kernel (RK) methods [36, 56, 57]. This connection arises from the fact that the shape tensor and bond shape vector of the correspondence formulation are special cases of the RK moment matrix and shape function derivatives, respectively [56].

In the RK framework, the moment matrix is defined as

$$\mathbf{M}_{[n]}(X_k) = \sum_{X_n \in \mathcal{H}_k^N} \omega(\Delta X_{kn}) \mathbf{Q}_{[n]}(\Delta X_{kn}) \otimes \mathbf{Q}_{[n]}(\Delta X_{kn}) V_n, \quad (80)$$

with the monomial basis vector $\mathbf{Q}_{[n]} \in \mathbb{R}^m$ containing m monomial terms up to order n . The RK shape function derivatives are given by

$$\frac{d\phi_n(X_k)}{dX_k} = \omega(\Delta X_{kn}) \mathbf{Q}_{[n]}^\nabla \mathbf{M}_{[n]}^{-1}(X_k) \mathbf{Q}_{[n]}(\Delta X_{kn}) V_n, \quad (81)$$

where $\mathbf{Q}_{[n]}^\nabla$ denotes the *target derivative matrix* that selects the desired derivative order and direction. In three-dimensional space, this matrix is defined as

$$\mathbf{Q}_{[n]}^\nabla := \begin{bmatrix} \frac{\partial \mathbf{Q}_{[n]}(\mathbf{0})}{\partial X_1} & \frac{\partial \mathbf{Q}_{[n]}(\mathbf{0})}{\partial X_2} & \frac{\partial \mathbf{Q}_{[n]}(\mathbf{0})}{\partial X_3} \end{bmatrix}^\top, \quad (82)$$

and contains the partial derivatives of the monomial basis vector evaluated at the zero vector, with each row corresponding to a Cartesian derivative direction.

Choosing the linear monomial basis $\mathbf{Q}_{[1]}(\Delta X) = \Delta X$ (abbreviated as C1), the target derivative matrix reduces to $\mathbf{Q}_{[1]}^\nabla = \mathbf{I}$, and the moment matrix reduces exactly to the shape tensor

$$\mathbf{M}_{[1]}(X_k) = \mathbf{K}(X_k), \quad (83)$$

and the shape function derivatives become

$$\frac{d\phi_n(X_k)}{dX_k} = \mathbf{B}(X_k, X_n) V_n. \quad (84)$$

This reveals that the correspondence formulation is inherently a first-order accurate reproducing kernel approximation. Higher-order accuracy can be achieved by enriching the monomial basis, for instance with the RK1 basis $\mathbf{Q}_{[1]}(\Delta X) = [1, \Delta X_1, \Delta X_2, \Delta X_3]^\top$ that includes a constant term to enforce partition of unity. With the RK1 basis, the target derivative matrix in three-dimensional space becomes

$$\mathbf{Q}_{[1]}^\nabla = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (85)$$

Using an enriched basis improves the approximation quality, especially near boundaries where the neighborhood is truncated [57, 58].

In the RK notation, the deformation gradient from Eq. (7) is expressed as

$$\bar{\mathbf{F}}(t, \mathbf{X}_k) = \mathbf{I} + \sum_{\mathbf{X}_n \in \mathcal{H}_k^N} \Delta \mathbf{U}_{kn} \otimes \frac{d\phi_n(\mathbf{X}_k)}{d\mathbf{X}_k} = \sum_{\mathbf{X}_n \in \mathcal{H}_k^N} \Delta \mathbf{x} \otimes \frac{d\phi_n(\mathbf{X}_k)}{d\mathbf{X}_k}, \quad (86)$$

where $\Delta \mathbf{U}_{kn} = \mathbf{U}(t, \mathbf{X}_n) - \mathbf{U}(t, \mathbf{X}_k)$ is the bond displacement. The algorithms below use this notation, since it naturally accommodates the kinematic degradation required for the phase-field approach and generalizes to higher-order bases.

4.3. Discrete PFPD formulation and algorithms

To write the equations in a discretized form, the *discrete bond phase-field parameter* is introduced as

$$s_{kn} := s(t, \mathbf{X}_k, \mathbf{X}_n), \quad (87)$$

with the discrete material points $\mathbf{X}_k$ and $\mathbf{X}_n \in \mathcal{H}_k^N$ in a discretized body B_0^N . Analogously, the *discrete history variable* and the *discrete crack driving force* are defined as bond-wise quantities for each pair of discrete material points as

$$\mathcal{Y}_{kn} := \mathcal{Y}(t, \mathbf{X}_k, \mathbf{X}_n), \quad Y_{kn} := Y(t, \mathbf{X}_k, \mathbf{X}_n). \quad (88)$$

The *discrete point damage* is the point-wise damage variable obtained from averaging the bond phase-field parameters over the family of $\mathbf{X}_k$, as defined in Eq. (46), yielding

$$D_k := D(t, \mathbf{X}_k). \quad (89)$$

Analogously to the degradation of the shape tensor and bond shape vector in Eqs. (32) and (33), the PFPD framework can be expressed using a reproducing kernel peridynamics (RKPd) notation, cf. [36,56]. The shape tensor and bond shape vector are replaced by the moment matrix and shape function derivatives in the reproducing kernel setting. The moment matrix is degraded by the bond phase-field parameter as

$$\mathbf{M}_{[n]}(t, \mathbf{X}_k) = \sum_{\mathbf{X}_n \in \mathcal{H}_k^N} \omega(\Delta \mathbf{X}_{kn}) h(s_{kn}) \mathbf{Q}_{[n]}(\Delta \mathbf{X}_{kn}) \otimes \mathbf{Q}_{[n]}(\Delta \mathbf{X}_{kn}) V_n, \quad (90)$$

where the kinematic degradation function h uses the discrete bond phase-field parameter s_{kn} and the critical phase-field value s_c , as defined in Eq. (30). The shape function derivatives are then degraded similarly as

$$\frac{d\phi_n(t, \mathbf{X}_k)}{d\mathbf{X}_k} = \omega(\Delta \mathbf{X}_{kn}) h(s_{kn}) \mathbf{Q}_{[n]}^\top \mathbf{M}_{[n]}^{-1}(t, \mathbf{X}_k) \mathbf{Q}_{[n]}(\Delta \mathbf{X}_{kn}) V_n. \quad (91)$$

This kinematic degradation ensures that the inversion of the moment matrix remains stable even for finite deformations and largely damaged bonds, while maintaining the capability for full removal of the energetic contribution of fully damaged bonds with $s_{kn} = 1$. Due to the delayed application of the kinematic degradation with s_c , the provided scheme is numerically much more stable than a direct degradation of the moment matrix with the same degradation function as the energetic contribution.

To highlight the differences between the standard BAQP model with critical stretch damage and the novel phase-field peridynamics (PFPD) model, both approaches are presented as algorithms for the evaluation of the internal force density $\bar{\mathbf{B}}_{\text{int}}$ within a single time step. The time integration is handled by a standard explicit time integration scheme such as Velocity Verlet [24], and is identical for both models. The algorithmic differences are entirely contained in the force evaluation procedure.

In the standard **BAQP model**, fracture is modeled by deleting bonds whose bond strain exceeds a critical value ε_c . A deleted bond carries no force and is excluded from the kinematic approximation. The evaluation of $\bar{\mathbf{B}}_{\text{int}}$ proceeds in four steps, as described in Algorithm 1. In Step 0, the bond strain ε_b is evaluated for each bond and compared against the critical value. Bonds exceeding the threshold are deactivated by setting the failure flag d_{kn} to zero, and the point damage D_k is updated as the volume-weighted fraction of failed bonds. In Step 1, the moment matrix $\mathbf{M}_{[n]}$ and the shape function derivatives are reassembled for all points where the damage state changed, using only the remaining active bonds with $d_{kn} = 1$. Step 2 assembles the nonlocal deformation gradient $\bar{\mathbf{F}}$ for all points. In Step 3, the bond-associated deformation gradient $\bar{\mathbf{F}}$ is computed for each active bond as the average of the nonlocal deformation gradients at both endpoints, corrected to reproduce the actual bond deformation exactly. The first Piola–Kirchhoff stress tensor $\bar{\mathbf{P}}$ is then evaluated from the constitutive law. The force state $\mathbf{t}$ is assembled from two contributions: the direct bond stress projection along the bond direction and the non-uniform stress correction $\mathbf{Z}$ distributed via the shape function derivatives. The internal force density $\bar{\mathbf{B}}_{\text{int}}$ is accumulated by summing the force state contributions over all active bonds.

In the **PFPD model**, bonds are not deleted when the critical crack driving force is exceeded. Instead, all bonds remain part of the kinematic approximation, and the stress response is continuously degraded via the degradation function $g(s)$. The evaluation of $\bar{\mathbf{B}}_{\text{int}}$ follows the same four-step structure, as described in Algorithm 2 but differs in several key aspects. In Step 0, the point damage D_k is computed from the bond phase-field parameters of the previous time step using the kernel-weighted averaging from Eq. (46), replacing the volume-weighted failure fraction of the BAQP model. In Step 1, the moment matrix is updated for all points where any bond phase-field parameter exceeds the critical value s_c . Notably, the binary failure flag d_{kn} of the BAQP model is replaced by the continuous kinematic degradation function $h(s_{kn})$, which ensures a gradual reduction of the kinematic contribution for highly damaged bonds while preserving the stability of the moment matrix. Step 2 is identical to the BAQP model. The main difference lies in Step 3: the bond-associated deformation gradient is computed for *all* bonds, regardless of their damage state. The undamaged first

Piola–Kirchhoff stress tensor $\mathbf{P}_0$ is evaluated first from the constitutive law. Subsequently, the crack driving force Y_{kn} is determined from the tensile strain energy density or the maximum principal stress, as introduced in Eqs. (38) and (42), respectively. The history variable $\mathcal{Y}_{kn}$ is updated by taking the maximum of its previous value and the current crack driving force, and the bond phase-field parameter s_{kn} is evolved accordingly. The degraded stress tensor $\tilde{\mathbf{P}} = g(s_{kn}) \mathbf{P}_0$ is then used to assemble the non-uniform stress correction and the internal force density, following the same procedure as in the BAQP model.

The key differences between both algorithms are summarized as follows. In the BAQP model (Algorithm 1, bonds are abruptly deleted when the critical strain is exceeded, and the moment matrix is reassembled using only the remaining active bonds. This binary bond deletion progressively degrades the kinematic approximation and can lead to numerical instabilities, especially in regions with many deleted bonds. Moreover, the stress response of a bond transitions instantaneously from undamaged to fully inactive, which does not allow for a smooth dissipation of energy during crack propagation. In contrast, the PFPD model (Algorithm 2 retains all bonds and continuously degrades the stress contribution of damaged bonds via the degradation function $g(s)$. The kinematic approximation is only gradually affected through $h(s)$, preserving the stability of the moment matrix even under large-scale fracture. The phase-field evolution via the history variable $\mathcal{Y}$ ensures irreversibility and thermodynamic consistency of the damage process. Another notable difference concerns the input and output quantities of the algorithms: while the BAQP model operates on binary bond failure flags d_{kn} , the PFPD model requires the continuous history variable $\mathcal{Y}_{kn}$ and the bond phase-field parameter s_{kn} to be stored and updated for each bond at every time step.

Algorithm 1: BAQP with critical stretch, evaluation of $\bar{\mathbf{B}}_{\text{int}}$

Input: Positions $\mathbf{X}_k, \mathbf{x}_k$; bond failure d_{kn} from previous time step
Output: Force density $\bar{\mathbf{B}}_{\text{int}}(\mathbf{X}_k)$; updated bond failure d_{kn}

Step 0: Bond failure

```

1  foreach point  $\mathbf{X}_k \in B^N$  do
2    foreach bond  $\Delta \mathbf{X}_{kn}$  with  $\mathbf{X}_n \in \mathcal{H}_k^N$  do
3       $\varepsilon_b(\mathbf{X}_k, \mathbf{X}_n) \leftarrow \frac{|\Delta \mathbf{x}_{kn}| - |\Delta \mathbf{X}_{kn}|}{|\Delta \mathbf{X}_{kn}|}$ 
4      if  $\varepsilon_b(\mathbf{X}_k, \mathbf{X}_n) > \varepsilon_c$  and failure is permitted then
5        Deactivate bond:  $d_{kn} \leftarrow 0$ 

```

Step 1: Moment matrix and shape function derivatives

```

6  foreach point  $\mathbf{X}_k \in B^N$  where damage changed do
7     $\mathbf{M}_{[n]}(\mathbf{X}_k) \leftarrow \sum_{\mathbf{X}_n \in \mathcal{H}_k^N} \omega(\Delta \mathbf{X}_{kn}) d_{kn} \mathbf{Q}_{[n]}(\Delta \mathbf{X}_{kn}) \otimes \mathbf{Q}_{[n]}(\Delta \mathbf{X}_{kn}) V_n$ 
8    foreach bond  $\Delta \mathbf{X}_{kn}$  with  $\mathbf{X}_n \in \mathcal{H}_k^N$  do
9       $\frac{d\phi_n(\mathbf{X}_k)}{d\mathbf{X}_k} \leftarrow \omega(\Delta \mathbf{X}_{kn}) d_{kn} \mathbf{Q}_{[n]}^\top \mathbf{M}_{[n]}^{-1}(\mathbf{X}_k) \mathbf{Q}_{[n]}(\Delta \mathbf{X}_{kn}) V_n$ 

```

Step 2: Nonlocal deformation gradient

```

10 foreach point  $\mathbf{X}_k \in B^N$  do
11    $\bar{\mathbf{F}}(\mathbf{X}_k) \leftarrow \mathbf{I} + \sum_{\mathbf{X}_n \in \mathcal{H}_k^N} \Delta U_{kn} \otimes \frac{d\phi_n(\mathbf{X}_k)}{d\mathbf{X}_k}$ 

```

Step 3: Bond stress and force density

```

12 foreach point  $\mathbf{X}_k \in B^N$  do
13   Initialize  $\mathbf{Z}(\mathbf{X}_k) \leftarrow \mathbf{0}$ 
14   Initialize  $\bar{\mathbf{B}}_{\text{int}}(\mathbf{X}_k) \leftarrow \mathbf{0}$ 
15   foreach active bond  $\Delta \mathbf{X}_{kn}$  with  $\mathbf{X}_n \in \mathcal{H}_k^N$  and  $d_{kn} = 1$  do
16      $\tilde{\mathbf{F}}^{\text{av}}(\mathbf{X}_k, \mathbf{X}_n) \leftarrow \frac{1}{2} (\bar{\mathbf{F}}(\mathbf{X}_k) + \bar{\mathbf{F}}(\mathbf{X}_n))$ 
17      $\tilde{\mathbf{F}}(\mathbf{X}_k, \mathbf{X}_n) \leftarrow \tilde{\mathbf{F}}^{\text{av}}(\mathbf{X}_k, \mathbf{X}_n) + (\Delta \mathbf{x}_{kn} - \tilde{\mathbf{F}}^{\text{av}}(\mathbf{X}_k, \mathbf{X}_n) \Delta \mathbf{X}_{kn}) \otimes \frac{\Delta \mathbf{X}_{kn}}{|\Delta \mathbf{X}_{kn}|^2}$ 
18      $\tilde{\mathbf{P}}(\mathbf{X}_k, \mathbf{X}_n) \leftarrow \mathbf{P}(\tilde{\mathbf{F}}(\mathbf{X}_k, \mathbf{X}_n))$ 
19      $\mathbf{Z}(\mathbf{X}_k) \leftarrow \mathbf{Z}(\mathbf{X}_k) + \bar{\omega}_b(\mathbf{X}_k, \mathbf{X}_n) \tilde{\mathbf{P}}(\mathbf{X}_k, \mathbf{X}_n) \left( \mathbf{I} - \frac{\Delta \mathbf{X}_{kn} \otimes \Delta \mathbf{X}_{kn}}{|\Delta \mathbf{X}_{kn}|^2} \right) V_n$ 
20   foreach active bond  $\Delta \mathbf{X}_{kn}$  with  $\mathbf{X}_n \in \mathcal{H}_k^N$  and  $d_{kn} = 1$  do
21      $\mathbf{t}(\mathbf{X}_k, \mathbf{X}_n) \leftarrow \bar{\omega}_b(\mathbf{X}_k, \mathbf{X}_n) \tilde{\mathbf{P}}(\mathbf{X}_k, \mathbf{X}_n) \frac{\Delta \mathbf{X}_{kn}}{|\Delta \mathbf{X}_{kn}|} + \mathbf{Z}(\mathbf{X}_k) \frac{d\phi_n(\mathbf{X}_k)}{d\mathbf{X}_k} \frac{1}{V_n}$ 
22      $\bar{\mathbf{B}}_{\text{int}}(\mathbf{X}_k) \leftarrow \bar{\mathbf{B}}_{\text{int}}(\mathbf{X}_k) + \mathbf{t}(\mathbf{X}_k, \mathbf{X}_n) V_n$ 
23      $\bar{\mathbf{B}}_{\text{int}}(\mathbf{X}_n) \leftarrow \bar{\mathbf{B}}_{\text{int}}(\mathbf{X}_n) - \mathbf{t}(\mathbf{X}_k, \mathbf{X}_n) V_k$ 

```

Algorithm 2: PFPD with phase-field damage, evaluation of $\bar{\mathbf{B}}_{\text{int}}$ **Input:** Positions $\mathbf{X}_k, \mathbf{x}_k$; history $\mathcal{Y}_{kn}$; phase-field s_{kn} from previous time step**Output:** Force density $\bar{\mathbf{B}}_{\text{int}}(\mathbf{X}_k)$; updated history $\mathcal{Y}_{kn}$ and phase-field s_{kn}

Step 1: Moment matrix and shape function derivatives

```

1 foreach point  $\mathbf{X}_k \in \mathcal{B}^N$  where  $s_{kn} > s_c$  do
2    $\mathbf{M}_{[n]}(\mathbf{X}_k) \leftarrow \sum_{\mathbf{X}_n \in \mathcal{H}_k^N} \omega(\Delta \mathbf{X}_{kn}) h(s_{kn}) \mathbf{Q}_{[n]}(\Delta \mathbf{X}_{kn}) \otimes \mathbf{Q}_{[n]}(\Delta \mathbf{X}_{kn}) V_n$ 
3   foreach bond  $\Delta \mathbf{X}_{kn}$  with  $\mathbf{X}_n \in \mathcal{H}_k^N$  do
4      $\frac{d\phi_n(\mathbf{X}_k)}{d\mathbf{X}_k} \leftarrow \omega(\Delta \mathbf{X}_{kn}) h(s_{kn}) \mathbf{Q}_{[n]}^\nabla \mathbf{M}_{[n]}^{-1}(\mathbf{X}_k) \mathbf{Q}_{[n]}(\Delta \mathbf{X}_{kn}) V_n$ 

```

Step 2: Nonlocal deformation gradient

```

5 foreach point  $\mathbf{X}_k \in \mathcal{B}^N$  do
6    $\bar{\mathbf{F}}(\mathbf{X}_k) \leftarrow \mathbf{I} + \sum_{\mathbf{X}_n \in \mathcal{H}_k^N} \Delta \mathbf{U}_{kn} \otimes \frac{d\phi_n(\mathbf{X}_k)}{d\mathbf{X}_k}$ 

```

Step 3: Bond stress, damage update, and force density

```

7 foreach point  $\mathbf{X}_k \in \mathcal{B}^N$  do
8   Initialize  $\mathbf{Z}(\mathbf{X}_k) \leftarrow \mathbf{0}$ 
9   Initialize  $\bar{\mathbf{B}}_{\text{int}}(\mathbf{X}_k) \leftarrow \mathbf{0}$ 
10  foreach bond  $\Delta \mathbf{X}_{kn}$  with  $\mathbf{X}_n \in \mathcal{H}_k^N$  do
11     $\bar{\mathbf{F}}^{\text{av}}(\mathbf{X}_k, \mathbf{X}_n) \leftarrow \frac{1}{2} (\bar{\mathbf{F}}(\mathbf{X}_k) + \bar{\mathbf{F}}(\mathbf{X}_n))$ 
12     $\bar{\mathbf{F}}(\mathbf{X}_k, \mathbf{X}_n) \leftarrow \bar{\mathbf{F}}^{\text{av}}(\mathbf{X}_k, \mathbf{X}_n) + (\Delta \mathbf{x}_{kn} - \bar{\mathbf{F}}^{\text{av}}(\mathbf{X}_k, \mathbf{X}_n) \Delta \mathbf{X}_{kn}) \otimes \frac{\Delta \mathbf{X}_{kn}}{|\Delta \mathbf{X}_{kn}|^2}$ 
13     $\mathbf{P}_0(\mathbf{X}_k, \mathbf{X}_n) \leftarrow \mathbf{P}(\bar{\mathbf{F}}(\mathbf{X}_k, \mathbf{X}_n))$ 
14     $\mathbf{Y}_{kn} \leftarrow \Psi_0^+(\bar{\mathbf{F}}(\mathbf{X}_k, \mathbf{X}_n))$  or  $\frac{\langle \sigma_1(\bar{\mathbf{F}}(\mathbf{X}_k, \mathbf{X}_n)) \rangle_+^2}{2E}$ 
15     $\mathcal{Y}_{kn} \leftarrow \max(\mathcal{Y}_{kn}, \mathbf{Y}_{kn})$ 
16     $s_{kn} \leftarrow \min(1, \mathcal{Y}_{kn} / (\mathcal{Y}_{kn} + \mathbf{Y}_c))$ 
17     $\bar{\mathbf{P}}(\mathbf{X}_k, \mathbf{X}_n) \leftarrow g(s_{kn}) \mathbf{P}_0(\mathbf{X}_k, \mathbf{X}_n)$ 
18     $\mathbf{Z}(\mathbf{X}_k) \leftarrow \mathbf{Z}(\mathbf{X}_k) + \bar{\omega}_b(\mathbf{X}_k, \mathbf{X}_n) \bar{\mathbf{P}}(\mathbf{X}_k, \mathbf{X}_n) \left( \mathbf{I} - \frac{\Delta \mathbf{X}_{kn} \otimes \Delta \mathbf{X}_{kn}}{|\Delta \mathbf{X}_{kn}|^2} \right) V_n$ 
19  foreach bond  $\Delta \mathbf{X}_{kn}$  with  $\mathbf{X}_n \in \mathcal{H}_k^N$  do
20     $\mathbf{t}(\mathbf{X}_k, \mathbf{X}_n) \leftarrow \bar{\omega}_b(\mathbf{X}_k, \mathbf{X}_n) \bar{\mathbf{P}}(\mathbf{X}_k, \mathbf{X}_n) \frac{\Delta \mathbf{X}_{kn}}{|\Delta \mathbf{X}_{kn}|} + \mathbf{Z}(\mathbf{X}_k) \frac{d\phi_n(\mathbf{X}_k)}{d\mathbf{X}_k} \frac{1}{V_n}$ 
21     $\bar{\mathbf{B}}_{\text{int}}(\mathbf{X}_k) \leftarrow \bar{\mathbf{B}}_{\text{int}}(\mathbf{X}_k) + \mathbf{t}(\mathbf{X}_k, \mathbf{X}_n) V_n$ 
22     $\bar{\mathbf{B}}_{\text{int}}(\mathbf{X}_n) \leftarrow \bar{\mathbf{B}}_{\text{int}}(\mathbf{X}_n) - \mathbf{t}(\mathbf{X}_k, \mathbf{X}_n) V_k$ 

```

5. Numerical examples

The capabilities of the PFPD model are demonstrated through a series of numerical examples. First, mode I and mode II crack propagation in a notched plate are investigated to verify the correct crack path prediction. Then, the boundary tension test (BTT) is used to verify the normalization constant derived in Section 3.3 and to demonstrate the ability of the PFPD model to capture crack branching. Finally, the Kalthoff-Winkler experiment is simulated to assess the performance of the model under impact-induced fracture conditions. The simulations are performed in three dimensions using Velocity Verlet time integration [24]. If not stated otherwise, a critical phase-field value of $s_c = 0.95$, and the Saint Venant–Kirchhoff constitutive model with linear elastic material parameters are employed. The critical phase-field value is not a free parameter of the model. It enters the frozen-wake normalization constant through Eq. (75) and is therefore accounted for in the fracture energy, as listed in Table 1. It is bounded away from 1, where no finite normalization constant exists, and away from small values, where bonds decouple kinematically while still carrying a significant share of their elastic energy. The value $s_c = 0.95$ lies within this admissible range. Prior to each simulation, the normalization constant c_0 is calculated based on the employed kernel function and the chosen critical phase-field value s_c . The open-source Julia package `Peridynamics.jl` [59,60] is used for all simulations. The PFPD formulation presented in this work is currently implemented in a private development branch of the package and will be released in a future version.

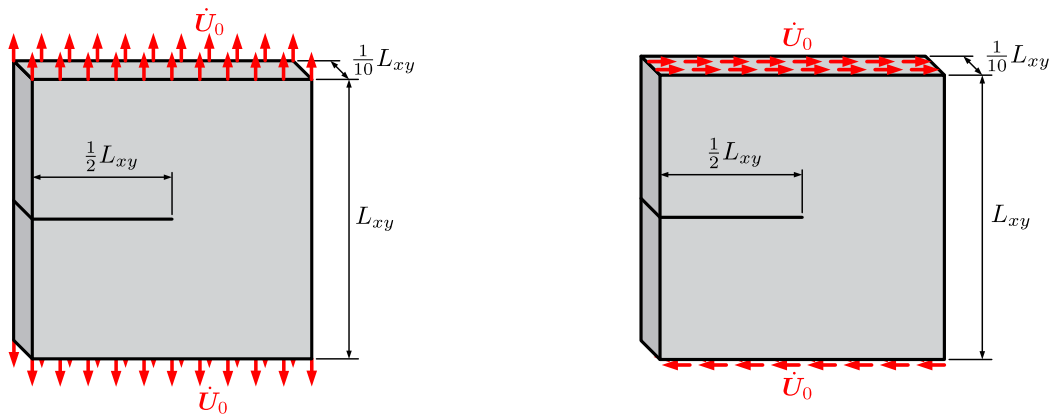


Fig. 4. Setup for the mode I (left) and mode II (right) crack propagation examples.

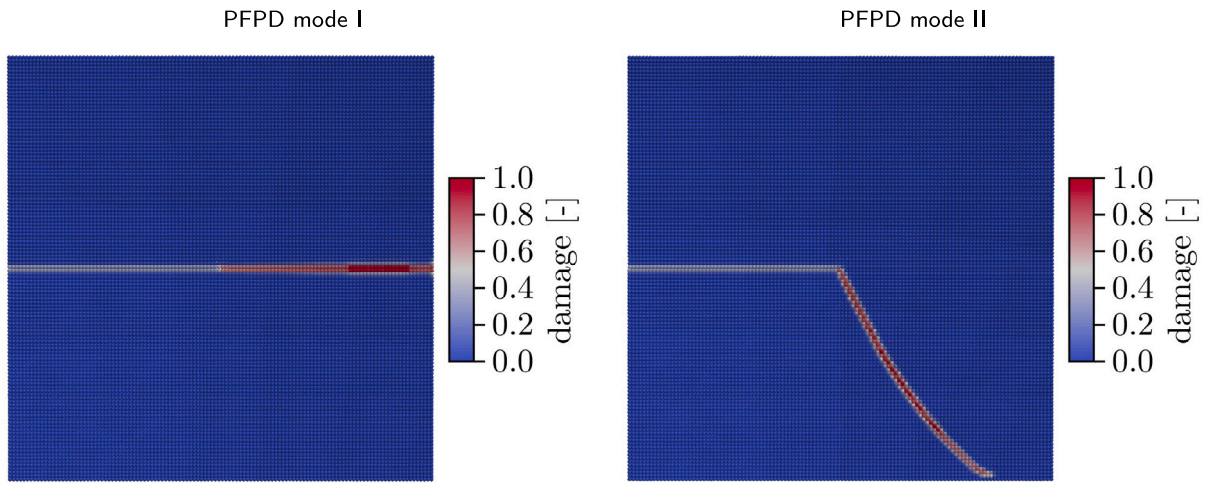


Fig. 5. Mode I and mode II crack propagation in 3D at time $t = 171 \mu\text{s}$ for the PFPD model.

5.1. Mode I and mode II crack propagation

As a first verification of the PFPD model, mode I and mode II crack propagation in a notched plate are investigated. A rectangular plate with dimensions $L_{xy} \times L_{xy} \times \frac{1}{10} L_{xy}$ with $L_{xy} = 100 \text{ mm}$ and a pre-defined central notch of length $\frac{1}{2} L_{xy}$ is considered. The plate is uniformly discretized with point spacing $\Delta x = L_{xy}/120$, yielding a $120 \times 120 \times 12$ point cloud with $N = 172\,800$ material points. The pre-notch is introduced as a *sharp crack* before the simulation. The bond phase-field parameter of all bonds crossing the notch surface is initialized as $s = 1$, while all other bonds remain undamaged with $s = 0$, consistent with the sharp-crack profile considered in Section 3.3.2. Since $g(1) = 0$ and $h(1) = 0$, these bonds carry neither an energetic nor a kinematic contribution, which is equivalent to their complete removal. It is emphasized that this one-time initialization only defines the initial crack topology and must not be confused with a fracture criterion. During the simulation, no bonds are deleted, and damage evolves exclusively through the continuous phase-field evolution law Eq. (44). As points directly adjacent to the pre-notch have roughly half of their bonds crossing the notch surface, they carry a point damage of approximately 0.5, making the initial crack directly visible in all damage plots. One layer of boundary points at the top and bottom surfaces is defined as a no-failure zone. The material parameters are density $\rho_0 = 2500 \text{ kg/m}^3$, Young's modulus $E = 32\,000 \text{ MPa}$, Poisson's ratio $\nu = 0.25$, and critical energy release rate $G_c = 100 \text{ J/m}^2$. A constant time step of $\Delta t = 0.14 \mu\text{s}$ is used for the time integration.

The loading is imposed via a Dirichlet boundary condition $\dot{U}_D$ on the upper and lower boundaries as

$$\dot{U}(t, \mathbf{X}) = \dot{U}_D(t, \mathbf{X}) \quad \forall \mathbf{X} \in B_0 \cap \partial_D B_0, \quad (92)$$

where $\partial_D B_0$ denotes the set of points on the upper and lower boundaries, and the prescribed boundary velocity is constant with $\dot{U}_D(t, \mathbf{X}) = \dot{U}_0$. For mode I, a Dirichlet boundary condition with $\dot{U}_0 = \pm \dot{U}_0 \mathbf{e}_2$ and $\dot{U}_0 = 0.2 \text{ m/s}$ is applied at the upper and lower boundaries, pulling the plate apart symmetrically, as shown in Fig. 4 (left). For mode II, the loading is changed to shear, and the

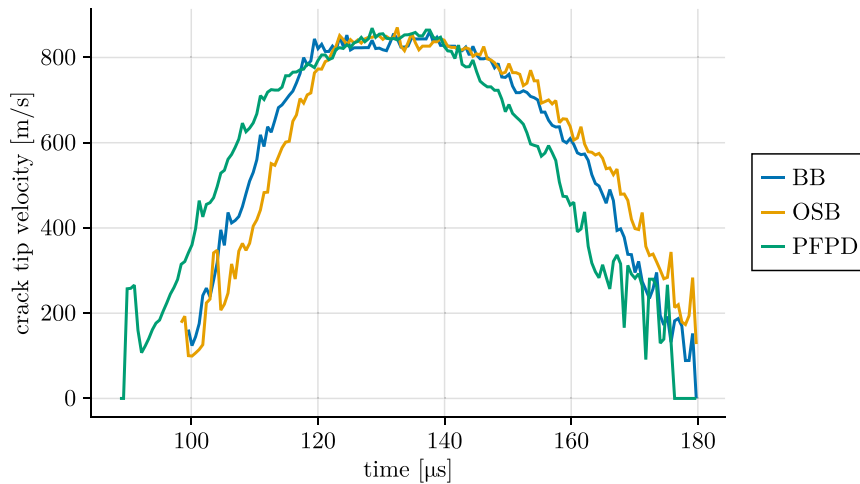


Fig. 6. Crack tip velocities over the propagation time for the mode I test with the BB, OSB, and PFPD models.

upper and lower halves are displaced in opposite horizontal directions with $\dot{U}_0 = \pm \dot{U}_0 e_1$ and $\dot{U}_0 = 0.1$ m/s, as depicted in Fig. 4 (right). All other geometric and material parameters remain identical.

In Fig. 5, the crack patterns obtained with the PFPD model are shown. For mode I, the crack propagates as a straight horizontal line from the initial notch tip, which is the expected behavior for symmetric tensile loading. For mode II, the crack deviates from the initial notch direction and follows a curved path, consistent with the expected behavior for shear-dominated fracture. Both results are in agreement with results from standard bond-based and other correspondence peridynamics formulations, confirming that the PFPD model correctly reproduces the expected crack paths for both loading modes. Notably, in our tests the standard BAQP model with critical stretch damage becomes unstable in this mode II case due to shape tensor degradation. The asymmetric bond deletion under shear loading leads to an ill-conditioned shape tensor and eventually to numerical failure. To the best of our knowledge, this instability of the bond-associated formulation has not been documented in the literature so far. The PFPD model, in contrast, remains stable throughout the simulation, since bonds are not deleted but continuously degraded via the degradation function.

To quantify the comparison with the standard formulations beyond the crack path, the crack tip position is additionally tracked over time for the mode I test and compared to results obtained with the bond-based (BB) and ordinary state-based (OSB) models using the same discretization and material parameters. The resulting crack tip velocities are shown in Fig. 6. All three models produce a very similar overall velocity evolution. After initiation, the crack accelerates to a plateau of approximately 850 m/s, well below the half Rayleigh wave speed of the material, and decelerates again as the crack approaches the opposite plate boundary. The PFPD crack initiates about $10 \mu\text{s}$ earlier than in the BB and OSB models and correspondingly arrests slightly earlier, while the OSB model shows a slightly slower initial acceleration than the BB model. Apart from these small shifts, the velocity curves nearly coincide. Note that the extraction of initiation time and crack tip position is not entirely trivial. In all models, the crack tip is not a single material point but a region of points with growing damage, from which the tip position is determined by thresholding the damage field. Small differences in this tracking, in particular between the continuous phase-field damage of the PFPD model and the binary bond damage of the standard models, can therefore shift the curves slightly, and the minor deviations near initiation and arrest should not be overinterpreted.

5.2. Verification of the energy dissipation with mode I fracture

While the boundary tension test in Section 5.3 verifies the normalization constant c_0 indirectly through the reproduction of consistent crack paths, a direct quantitative verification is obtained by measuring the energy actually dissipated during crack growth. To this end, the mode I test from Section 5.1 is revisited, extracting the dissipated fracture energy from a global energy balance and relating it to the newly created crack area. If the frozen-wake normalization constant Eq. (75) is correct, the dissipated energy per unit crack area must equal the input critical energy release rate G_c , independent of the critical phase-field value s_c that enters the normalization.

The plate geometry, and material parameters are identical to the mode I test in Section 5.1, and only the discretization with a point spacing of $\Delta x = L_{xy}/60$ and the boundary conditions are different to the previous results. Instead of the prescribed Dirichlet velocity, the plate is loaded by a Neumann traction $\bar{\mathbf{B}}_{\text{ext}}(t, \mathbf{X}) = \bar{\mathbf{B}}_{\text{N}}(t, \mathbf{X})$ for $\mathbf{X} \in \mathcal{B}_0 \cap \partial_{\text{N}} \mathcal{B}_0$ on one layer of points at the upper and lower boundaries. A linearly ramped stress is applied to keep the plate close to quasi-static conditions, defined as

$$\bar{\mathbf{B}}_{\text{N}}(t, \mathbf{X}) = \pm \frac{\sigma_{\text{max}}}{\Delta x} \phi(t) e_2, \quad \phi(t) = \begin{cases} t/t_r & \text{if } t \leq t_r, \\ 1 & \text{if } t > t_r, \end{cases} \quad (93)$$

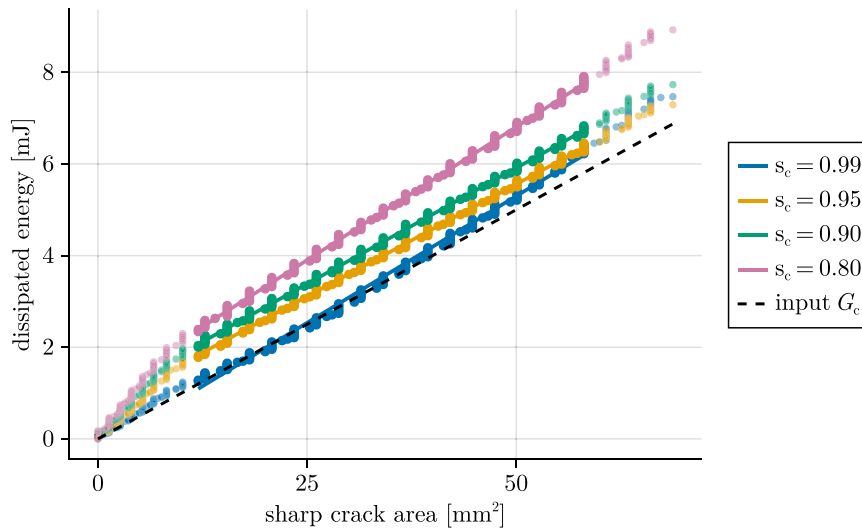


Fig. 7. Dissipated fracture energy $\mathcal{E}^F$ over the sharp crack area $A = a_{\text{tip}} 10^{-1} L_{xy}$ in the mode I test for four critical phase-field values s_c , with linear fits over the propagation window to extract the effective fracture energy G_c^{eff} .

Table 2

Effective fracture energy G_c^{eff} from the slope of the dissipated energy over the sharp crack area for the six simulated critical phase-field values s_c , compared to the input critical energy release rate $G_c = 100 \text{ J/m}^2$.

s_c	0.80	0.85	0.90	0.95	0.97	0.99
G_c^{eff} in J/m^2	116.68	108.06	101.61	98.44	101.80	106.98
deviation from G_c	+16.7%	+8.1%	+1.6%	-1.6%	+1.8%	+7.0%

with maximum applied stress $\sigma_{\text{max}} = 4 \text{ MPa}$ and ramp time $t_r = 400 \mu\text{s}$, equal to the total simulation time. The gradual ramp keeps the kinetic energy small, so that the released elastic energy is converted almost entirely into fracture energy.

The dissipated fracture energy $\mathcal{E}^F$ is obtained from the global energy balance of the plate as

$$\mathcal{E}^F(t) = \mathcal{W}_{\text{ext}}(t) - \mathcal{E}_{\text{kin}}(t) - \mathcal{E}_{\text{el}}(t), \quad (94)$$

where $\mathcal{W}_{\text{ext}}$ denotes the external work performed by the applied boundary forces, $\mathcal{E}_{\text{kin}}$ the kinetic energy, and $\mathcal{E}_{\text{el}}$ the elastic strain energy stored in the degraded body. The external work is accumulated over time from the instantaneous power of the boundary forces. The crack surface created up to time t is approximated as a sharp planar crack of area

$$A(t) = a_{\text{tip}}(t) \frac{1}{10} L_{xy}, \quad (95)$$

where the crack extension a_{tip} is determined by crack-tip tracking of the damage field and the width $10^{-1} L_{xy}$ is the plate width. Following Griffith's assumption Eq. (47), the dissipated energy grows linearly with the crack area, $\mathcal{E}^F = G_c A$, so that the slope of $\mathcal{E}^F$ over the crack area A yields the effective fracture energy G_c^{eff} realized by the model.

Fig. 7 shows the dissipated energy over the sharp crack area for four representative critical phase-field values $s_c \in \{0.99, 0.95, 0.90, 0.80\}$, together with the linear fits over the propagation window. For all values, the dissipated energy grows linearly with the crack area, confirming the Griffith-type scaling $\mathcal{E}^F = G_c A$. Near crack initiation, the curves lie slightly higher for smaller s_c . This offset reflects the build-up of the diffuse process zone rather than the fracture energy itself. With an earlier activation of the kinematic degradation, the damage wake dragged along by the crack is wider and stores more energy relative to the tracked sharp crack tip. This process zone must first develop as the crack starts to grow, which raises the dissipation during the initial crack extension. Once it is fully formed, the slope becomes consistent across all values of s_c , so the effective fracture energy G_c^{eff} is extracted from the linear regime of steady crack propagation and is summarized for all simulated values in Table 2.

Overall, the model reproduces the input critical energy release rate G_c well over an intermediate window of critical phase-field values that contains the default value $s_c = 0.95$, and the agreement degrades only towards the extremes of the tested range. This confirms that the frozen-wake normalization constant delivers the intended Griffith energy dissipation directly from the global energy balance for the relevant range of s_c , with the default choice $s_c = 0.95$ lying well within it.

5.3. Verification of the normalization with the boundary tension test

To verify the correctness of the normalization constant c_0 derived in Section 3.3, the boundary tension test (BTT) is employed. The BTT, originally introduced by Bourdin et al. [55] for phase-field fracture models, is a standard benchmark for dynamic crack

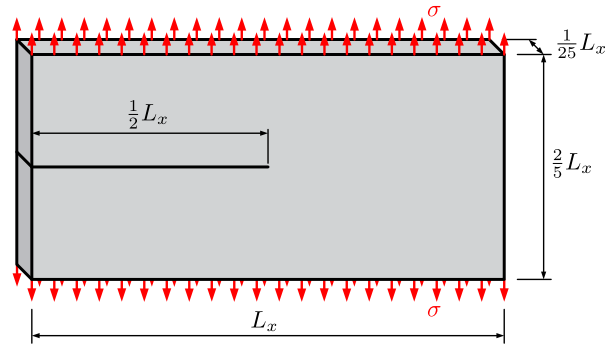


Fig. 8. Setup of the boundary tension test (BTT).

branching. A rectangular plate with dimensions $L_x \times \frac{2}{5} L_x \times \frac{1}{25} L_x$ and $L_x = 100$ mm is considered. The plate is uniformly discretized with $N_x \times N_y \times N_z$ points and point spacing $\Delta x = \frac{2}{5} L_x / N_y$. A pre-defined crack extends from the left side to the center of the plate. It is introduced as a sharp crack analogously to Section 5.1 and is therefore visible in all damage plots as a band with a point damage of approximately 0.5. The material parameters are density $\rho_0 = 2450$ kg/m³, Young's modulus $E = 32\,000$ MPa, Poisson's ratio $\nu = 0.25$, and critical energy release rate $G_c = 3$ J/m². The setup is shown in Fig. 8.

The loading is applied through Neumann boundary conditions $\bar{B}_N(t, \mathbf{X})$ at the upper and lower boundaries with

$$\bar{B}_{\text{ext}}(t, \mathbf{X}) = \bar{B}_N(t, \mathbf{X}) \quad \forall \mathbf{X} \in B_0 \cap \partial_N B_0, \quad (96)$$

where $\partial_N B_0$ denotes one layer of points on the upper and lower boundaries. A constant dynamic stress of $\sigma = 1$ MPa is applied as body force with

$$\bar{B}_N(t, \mathbf{X}) = \frac{\pm \sigma}{\Delta x} \mathbf{e}_2, \quad (97)$$

at the upper and lower boundaries, respectively.

If the normalization constant is correct, the critical crack driving force Y_c from Eq. (37) yields consistent fracture behavior regardless of the choice of kernel function and horizon ratio. To this end, the BTT is simulated with different horizon ratios $m_\delta \in \{3, 4, 5, 6\}$ for two different kernel functions, each with a discretization of $N_y = 100$.

5.3.1. Results with the cubic B-spline kernel

Fig. 9 shows the crack branching patterns obtained with the cubic B-spline kernel from Eq. (15). All four simulations produce very similar crack paths, with the same branching point and comparable branching angles. After branching, the cracks initially propagate at an angle of approximately 30° with respect to the initial crack plane. In the later propagation phase, the angle reduces to approximately 22° . As expected, higher horizon ratios lead to a thicker damage zone, which is a direct consequence of the increased nonlocal interaction radius. The same results are obtained for different discretization values N_y , confirming the consistency of the results and the correctness of the normalization constant for this kernel function.

5.3.2. Results with the linear kernel

To ensure that the normalization constant is also correct for other kernel functions, the simulations are repeated with the linear kernel

$$\omega(\Delta \mathbf{X}) = \frac{3}{\pi \delta^3} \left(1 - \frac{|\Delta \mathbf{X}|}{\delta} \right), \quad (98)$$

which fulfills the normalization condition from Eq. (14) with $\omega_0(\mathbf{X}) = 1$. The results for this kernel are shown in Fig. 10. Again, all simulations produce very similar crack paths, and the results are also comparable to those obtained with the cubic B-spline kernel in Fig. 9. The branching angles of approximately 30° in the initial phase and 22° in the later propagation phase are also obtained with this kernel. The branching angles and branching points are consistent across both kernels and all horizon ratios, with deviations within the measurement accuracy of the angle, confirming that the normalization constant is correct for different kernel functions. The obtained crack patterns are in good agreement with the experimental observations reported in the literature [40,55].

5.3.3. Crack tip velocity

The crack tip velocity over the crack propagation time frame is shown in Fig. 11 for the PFPD model with $N_y = 100$, together with results from the bond-based (BB) model for reference. The crack tip velocity of the PFPD model does not significantly exceed the half Rayleigh wave speed $\frac{1}{2} c_R$, which is the theoretically expected limiting velocity for crack branching in brittle materials. The Rayleigh wave speed is computed from an approximation equation [61] as

$$c_R = c_s \frac{0.862 + 1.14 \nu}{1 + \nu}, \quad (99)$$

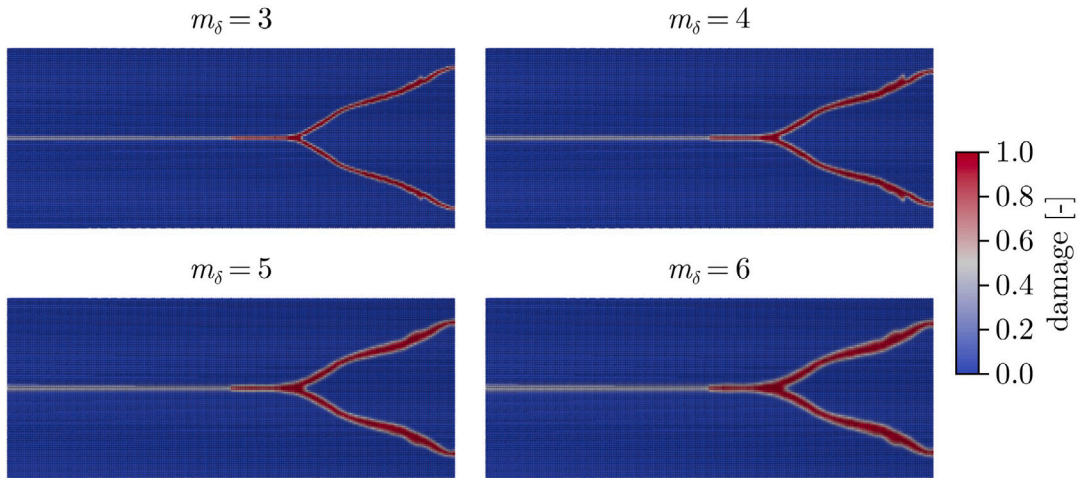


Fig. 9. Crack branching patterns for different horizon ratios in the BTT for the PFPD model with the cubic B-spline kernel from Eq. (15).

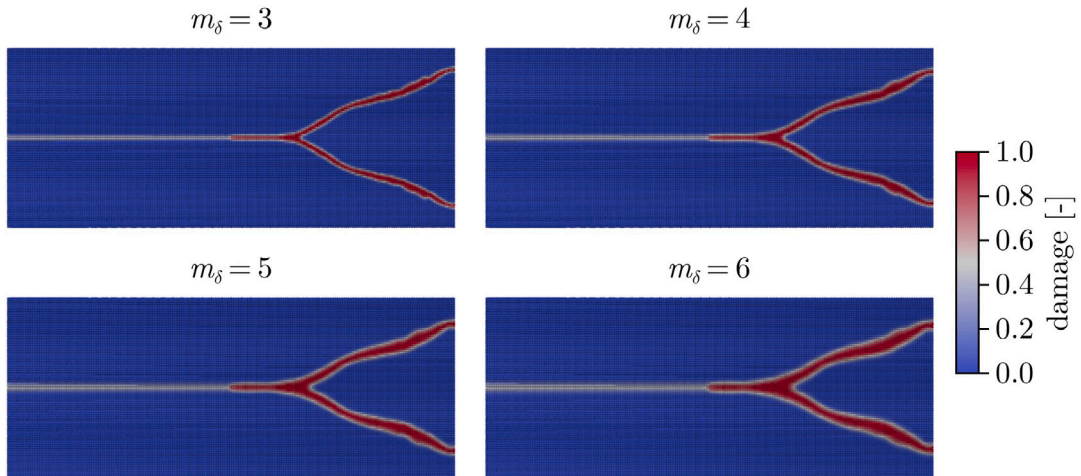


Fig. 10. Crack branching patterns for different horizon ratios in the BTT for the PFPD model with the linear kernel from Eq. (98).

with the shear wave speed $c_s = \sqrt{G/\rho_0}$ and the shear modulus $G = E/(2(1 + \nu))$. The velocity curve of the PFPD model is smoother compared to the BB results, without the strong peaks observed for the standard formulations. This observation is attributed to the improved kinematic approximation provided by the correspondence-based framework.

5.3.4. Discretization convergence

To verify that the results are consistent across different spatial discretizations, the BTT is simulated with different grid resolutions $N_y \in \{60, 80, 100, 120\}$, corresponding to point spacings $\Delta x = \frac{2}{5}L_x/N_y$ of approximately 0.67 mm, 0.50 mm, 0.40 mm, and 0.33 mm, respectively. All other parameters remain identical to the setup described above, with the cubic B-spline kernel from Eq. (15) and a horizon ratio of $m_\delta = 3$.

In Fig. 12, the damage fields for the different grid resolutions are shown. The crack branching pattern is captured for all resolutions, with similar branching angles and branching points across the different discretizations. As expected, finer discretizations produce narrower damage bands while preserving the same crack topology. The crack paths are consistent with the results obtained for different horizon ratios in Fig. 9 and with phase-field fracture simulations in the literature [55], confirming that the PFPD model correctly reproduces the expected fracture behavior under mesh refinement.

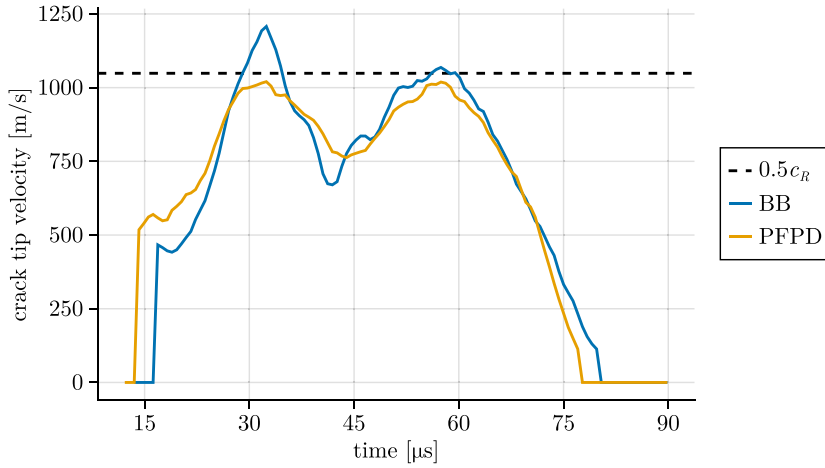


Fig. 11. Crack tip velocity over the propagation time in the BTT for the PFPD model compared to the bond-based (BB) model, normalized by the half Rayleigh wave speed $\frac{1}{2}c_R$.

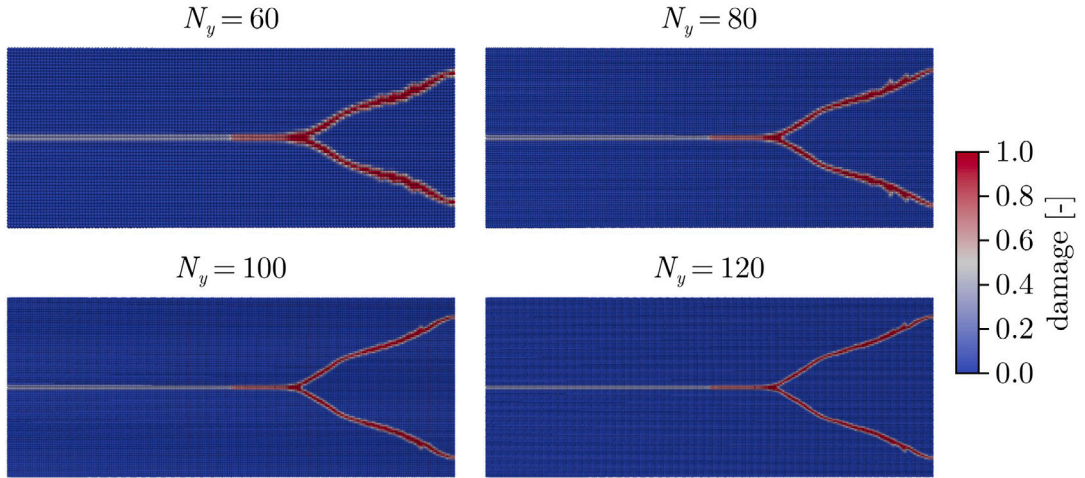


Fig. 12. Crack branching patterns in the BTT for different discretizations $N_y \in \{60, 80, 100, 120\}$ with the PFPD model.

5.4. Kalthoff-Winkler experiment

As a final benchmark, the Kalthoff-Winkler experiment [62] is simulated. This experiment is a well-established test for dynamic fracture under impact loading, where a projectile impacts a pre-notched steel plate, leading to mode II dominated crack initiation and propagation. The plate has a width of 100 mm, a height of 200 mm, and a thickness of 9 mm, featuring two parallel pre-notches with a length of 50 mm and a cylindrical section diameter of 50 mm. The projectile impact is modeled using a Dirichlet boundary condition as in Eq. (92), with $\dot{U}_D(t, X) = -v(t)e_2$ applied to the first five point layers. The velocity function follows a linear ramp to the impact velocity v_0 over a ramp time t_1 , defined as

$$v(t) = \begin{cases} v_0 \frac{t}{t_1} & \text{if } t \leq t_1, \\ v_0 & \text{if } t > t_1, \end{cases} \quad (100)$$

with impact velocity $v_0 = 16.5$ m/s and ramp time $t_1 = 10$ μs. Three layers of boundary points on the opposite side of the impact are defined as no-failure zones. The plate is discretized with $111 \times 222 \times 10$ material points and a point spacing of $\Delta x = 0.9$ mm. The material parameters are density $\rho_0 = 8000$ kg/m³, Young's modulus $E = 190\,000$ MPa, Poisson's ratio $\nu = 0.3$, and critical energy release rate $G_c = 34$ kJ/m². The time integration is performed with a constant time step of $\Delta t = 0.1$ μs. The experimental setup is illustrated in Fig. 13.

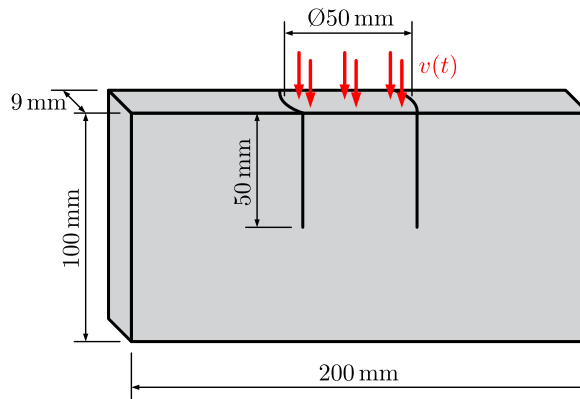


Fig. 13. Setup of the Kalthoff-Winkler experiment.

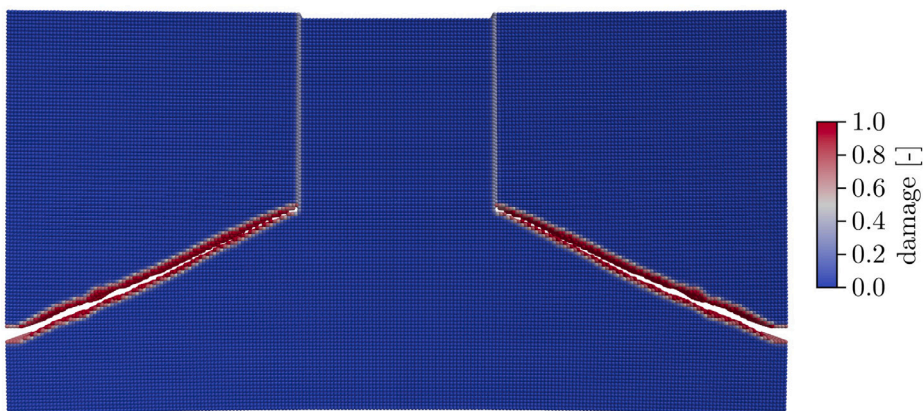


Fig. 14. Results of the Kalthoff-Winkler experiment at $t = 0.2$ ms for the PFPD model.

In Fig. 14, the damage field of the PFPD model at time $t = 0.2$ ms is shown. The pre-notches are introduced as sharp cracks analogously to Section 5.1 and are visible as bands of damage of approximately 0.5. The cracks initiate at both notch tips and propagate at an angle close to 70° to the original notch direction, which is in good agreement with the experimentally observed crack angle [62].

The crack tip trajectory is shown in Fig. 15, compared to results obtained with the ordinary state-based (OSB) peridynamics model using the same discretization and material parameters. Both models initially propagate along the characteristic 70° direction, and their trajectories nearly coincide during the first half of the crack propagation. In the later stage, both trajectories flatten slightly, resulting in overall crack angles of approximately 67° for the PFPD model and 62° for the OSB model. The PFPD trajectory thus remains closer to the experimentally observed 70° angle over the entire crack path. This improved agreement may be attributed to the continuous damage evolution in the PFPD framework, which allows for a more gradual and physically consistent crack propagation compared to the abrupt bond deletion in the standard models.

The crack tip velocity over the propagation time frame is shown in Fig. 16. Both models initiate at nearly the same time of approximately $24 \mu\text{s}$ and yield similar velocity profiles. During the phase of fastest propagation between approximately $35 \mu\text{s}$ and $55 \mu\text{s}$, both models temporarily exceed the half Rayleigh wave speed $\frac{1}{2}c_R$ with peak velocities of about $1600 \text{ m/s} \approx 0.57 c_R$, remaining well below the Rayleigh wave speed throughout the simulation. In the subsequent deceleration phase, the PFPD velocities are slightly lower than the OSB results, and the PFPD crack arrests slightly earlier. The close agreement of initiation time, velocity profile, and arrest between both models confirms that the continuous phase-field degradation reproduces the fracture dynamics of the established bond-deletion approach for this impact-loaded benchmark.

Regarding the computational cost, both simulations were performed with MPI parallelization on 16 ranks on a workstation with an AMD Ryzen 9 5950X processor. The total runtime is 95.93 s for the OSB model and 431.30 s for the PFPD model, a factor of approximately 4.5. This overhead is inherent to the bond-associated correspondence framework, which requires the assembly of a deformation gradient and a stress evaluation for every bond, rather than to the phase-field damage model itself. A bond-based simulation of this setup is not included in the comparison, since the BB model is restricted to a Poisson's ratio of $\nu = 0.25$, which does not match the value $\nu = 0.3$ of the steel plate.

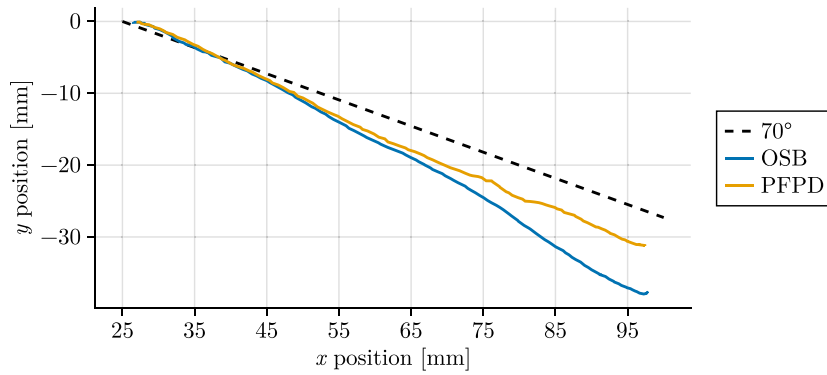


Fig. 15. Crack tip trajectory in the Kalthoff-Winkler experiment for the PFPD model compared to the ordinary state-based (OSB) model, with the characteristic 70° angle from experiments indicated.

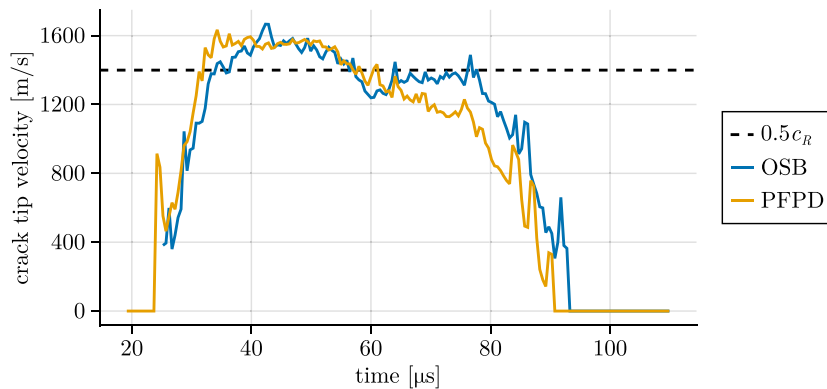


Fig. 16. Crack tip velocity in the Kalthoff-Winkler experiment for the PFPD model compared to the ordinary state-based (OSB) model.

6. Conclusion

A phase-field peridynamics (PFPD) approach has been developed that embeds a phase-field fracture model into the bond-associated correspondence framework. Instead of abruptly removing bonds when a critical deformation is exceeded, the approach continuously degrades their energetic contribution through a bond phase-field parameter, thereby avoiding the numerical instabilities inherent to bond deletion in the correspondence formulation. Additionally, a kinematic degradation function has been introduced to gradually reduce the kinematic contribution of highly damaged bonds, preserving the stability of the shape tensor and the accuracy of the deformation gradient approximation even under large-scale fracture.

An analytical expression for the normalization constant of the damage model has been derived for general spherical kernel functions by requiring that the total crack dissipation equals the Griffith energy release rate per unit crack area. By reducing the problem to a ratio of two one-dimensional integrals over the unit interval, an exact closed-form expression is obtained that can be evaluated analytically for polynomial kernels.

Numerical examples confirm the capabilities of the PFPD model. Mode I and mode II simulations yield the expected crack paths, with the PFPD model remaining stable in the mode II case where the standard BAQP model with bond deletion becomes unstable in our tests due to shape tensor degradation. A direct verification via the global energy balance of the mode I test confirms that the dissipated energy per unit crack area reproduces the input critical energy release rate for a wide range of critical phase-field values. Consistent crack branching patterns in the boundary tension test verify the correctness of the normalization constant across different horizon ratios and kernel functions, with branching angles and branching points in good agreement with experimental observations. Crack tip velocities do not significantly exceed the half Rayleigh wave speed, consistent with the theoretically expected limiting velocity for crack branching. In the Kalthoff-Winkler experiment, crack angles close to the experimentally observed 70° are obtained, with the PFPD trajectory following the experimental angle more closely than the OSB reference model.

Overall, the proposed framework retains the full accuracy of the bond-associated correspondence formulation while providing a thermodynamically consistent description of fracture that avoids the numerical instabilities associated with bond deletion approaches. Since bonds are never removed but only degraded, the kinematic approximation remains well-conditioned throughout the simulation, making the PFPD model a robust and versatile tool for dynamic fracture problems.

CRediT authorship contribution statement

Kai Partmann: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Resources, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Christian Wieners:** Writing – review & editing, Supervision, Project administration, Methodology, Funding acquisition, Formal analysis, Conceptualization. **Michael Ortiz:** Writing – review & editing, Validation, Supervision. **Kerstin Weinberg:** Writing – review & editing, Validation, Supervision, Software, Resources, Project administration, Methodology, Funding acquisition, Conceptualization.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Kai Partmann reports financial support was provided by German Research Foundation. Kerstin Weinberg reports financial support was provided by German Research Foundation. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

Data will be made available on request.

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