



Estimates of the gap between extreme solutions of nonlinear difference equations

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Abstract. We study bounded solutions of nonlinear difference equations on arbitrary index sets within a unified fixed-point framework. Working in the Banach lattice $l^\infty(X, \mathbb{R})$, we consider equations of the form

$$M(u) = \lambda u + D_f(u) + w,$$

where M is an increasing linear operator, D_f is the diagonal operator induced by a scalar super-linear nonlinearity $f \in C^1(\mathbb{R}, \mathbb{R})$, and $w \in l^\infty(X, \mathbb{R})$. Our abstract results apply to a broad class of discrete models, including discrete Laplace-type operators on $\mathbb{R}^n$, Banach limits and difference equations with spatial structure.

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1. Introduction

The study of bounded solutions of nonlinear difference equations has seen substantial development over the last decades, motivated both by intrinsic discrete dynamics and by the discretization of differential and integral equations. Classical approaches often focus on second- and higher-order scalar equations, which frequently arise from the discretization of ordinary and partial differential equations [1, 4, 5, 9, 10], on asymptotic properties of solutions [1, 5, 11, 12], or on boundedness and stability for specific classes of problems [1–3, 5–7], such as Sturm-Liouville type equations, equations with advanced argument, and Volterra-type schemes. Bounded solutions of certain vector-valued difference equations are investigated in [8]. A bridge between differential and difference equations is provided by Mawhin’s work [9, 10] on bounded solutions. For linear problems, he compares the bounded input - bounded output problem for

the continuous equation

$$u'(t) + \lambda u(t) = f(t) \quad (t \in \mathbb{R})$$

with its discrete analogue

$$u_{k+1} = \lambda u_k + w_k \quad (k \in \mathbb{Z}),$$

identifying the exact ranges of λ for which unique solvability in $l^\infty(\mathbb{Z}, \mathbb{R})$ and a maximum (or anti-maximum) principle holds, and explaining the differences in terms of the spectral properties of the underlying operators. For nonlinear equations, he develops a lower-upper solution method and Landesman-Lazer type conditions for bounded solutions of Duffing-type problems in the discrete setting.

The present paper contributes to this framework by considering more general nonlinear equations of the form

$$M(u) = \lambda u + D_f(u) + w$$

on the Banach lattice $l^\infty(X, \mathbb{R})$, where X is an arbitrary index set, M is an increasing linear operator with spectral radius $\rho(M)$, D_f is the nonlinear diagonal operator induced by a scalar super-linear nonlinearity $f \in C^1(\mathbb{R}, \mathbb{R})$, and $w \in l^\infty(X, \mathbb{R})$. Our approach is purely functional analytic and exploits both Banach's contraction principle and Tarski's fixed point theorem in the ordered Banach space. Under a structural lower bound on f' we show that for $0 \leq \lambda \leq \rho(M)$ the equation admits a smallest and a largest bounded solution in $l^\infty(X, \mathbb{R})$, and under natural additional assumptions on M one can derive explicit estimates of the gap between these extremal solutions, with sharp dependence on $\rho(M) - \lambda$ in suitable equivalent norms. For $\lambda > \rho(M)$ the bounded solution is unique. In many cases we can approximate solutions by pointwise or uniformly convergent successive approximations of an equivalent fixed point equation.

The generality of the index set X allows us to encompass a wide class of difference equations, including discrete Laplace-type operators, Banach limits, and difference equations on $\mathbb{Z}^n$ or $\mathbb{R}^n$.

2. Main result

Let $X \neq \emptyset$ be a set of arbitrary cardinality, and let $l^\infty(X, \mathbb{R})$ denote the Banach space of all bounded functions $u : X \rightarrow \mathbb{R}$ endowed with the supremum norm $\|u\|_\infty = \sup_{x \in X} |u(x)|$. We consider $l^\infty(X, \mathbb{R})$ as ordered by the cone K of nonnegative functions, that is $u \leq v : \Leftrightarrow u(x) \leq v(x) \ (x \in X)$. Let $\mathbf{1} \in l^\infty(X, \mathbb{R})$ denote the constant function $\mathbf{1}(x) = 1 \ (x \in X)$.

Let $M : l^\infty(X, \mathbb{R}) \rightarrow l^\infty(X, \mathbb{R})$ be a linear operator, and let M be increasing in the given order, that is $M[K] \subseteq K$. Note that, in particular, M is continuous

with operator norm $\|M\|_\infty = \|M(\mathbf{1})\|_\infty$. If $p \in l^\infty(X, \mathbb{R})$ is from the interior $\text{int}K$ of K (that is $\inf_{x \in X} p(x) > 0$), then

$$\|u\|_p := \sup_{x \in X} \frac{|u(x)|}{p(x)}$$

is an equivalent norm on $l^\infty(X, \mathbb{R})$ and again the corresponding operator norm is $\|M\|_p = \|M(p)\|_p$. Let $\rho(M)$ denote the spectral radius of M .

For continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ we consider the nonlinear diagonal operator $D_f : l^\infty(X, \mathbb{R}) \rightarrow l^\infty(X, \mathbb{R})$, $D_f(u) = f \circ u$.

Theorem 1. *Let $f \in C^1(\mathbb{R}, \mathbb{R})$ be such that*

$$f'(t) \geq b|t|^\beta \quad (t \in \mathbb{R})$$

for some $b > 0$ and $\beta > 0$, let $\lambda \geq 0$ and let $w \in l^\infty(X, \mathbb{R})$. Then, the equation

$$M(u) = \lambda u + D_f(u) + w \quad (1)$$

has a smallest and a largest solution $\underline{u}$ and $\bar{u}$ in $l^\infty(X, \mathbb{R})$. If $\lambda > \rho(M)$ then $\bar{u} = \underline{u}$. If additional $M(p) \leq \rho(M)p$ for some $p \in \text{int}K$, then

$$\forall \lambda \in [0, \rho(M)] : \|\bar{u} - \underline{u}\|_p \leq 2\|p^{-1}\|_\infty \left(\frac{\beta+1}{b} (\rho(M) - \lambda) \right)^{1/\beta}. \quad (2)$$

In particular, we have $\bar{u} = \underline{u}$ if $\lambda = \rho(M)$.

Remark. 1. Admissible functions f in Theorem 1 are for example $t \mapsto bt|t|^\beta$ with $b, \beta > 0$, or $t \mapsto \sinh(t) - t$ (with $b = 1$ and $\beta = 2$).

2. The assumption $M(p) \leq \rho(M)p$ for some $p \in \text{int}K$ implies $\|M\|_p = \rho(M)$. In many applications one has $M(\mathbf{1}) = \text{const} \cdot \mathbf{1}$. In this case we have $\|\cdot\|_1 = \|\cdot\|_\infty$, $\|M\|_\infty = \rho(M)$, and (2) reads

$$\|\bar{u} - \underline{u}\|_\infty \leq 2 \left(\frac{\beta+1}{b} (\|M\|_\infty - \lambda) \right)^{1/\beta} \quad (0 \leq \lambda \leq \|M\|_\infty).$$

3. The bound $\rho(M)$ is sharp for unique solvability of (1) as can be seen by the trivial one-dimensional case ($|X| = 1$): The polynomial equation

$$u = \lambda u + u^3$$

has three solutions in $\mathbb{R}$ if $\lambda \in [0, 1)$. Thus, already

$$u(x) = \lambda u(x) + u(x)^3 \quad (x \in \mathbb{R})$$

has uncountably many solutions in $l^\infty(\mathbb{R}, \mathbb{R})$. The question whether $\bar{u} = \underline{u}$ if $\lambda = \rho(M)$ in the general case seems to be open.

In the proof of Theorem 1 we will utilize the following lemma.

Lemma 1. Let $g \in C^1(\mathbb{R}, \mathbb{R})$ be such that

$$g'(t) \geq b|t|^\beta \quad (t \in \mathbb{R})$$

where $b > 0$ and $\beta > 0$. Then $g : \mathbb{R} \rightarrow \mathbb{R}$ is increasing and bijective, and $g^{-1} : \mathbb{R} \rightarrow \mathbb{R}$ is increasing and Hölder continuous, that is

$$|g^{-1}(t) - g^{-1}(s)| \leq c|t - s|^\alpha \quad (t, s \in \mathbb{R})$$

with

$$c = 2^{\beta/(\beta+1)} \left(\frac{\beta+1}{b} \right)^{1/(\beta+1)} \quad \text{and} \quad \alpha = \frac{1}{1+\beta}.$$

Proof. Since $g'(t) > 0$ ($t \neq 0$) and $g'(t) \geq b > 0$ ($|t| \geq 1$) clearly g is strictly increasing and $g(t) \rightarrow \pm\infty$ ($t \rightarrow \pm\infty$). We will use the known inequalities

$$t^p - s^p \geq (t - s)^p \quad (0 \leq s \leq t, p \geq 1), \quad (3)$$

$$s^p + t^p \geq 2^{1-p}(s + t)^p \quad (s, t \geq 0, p \geq 1). \quad (4)$$

For $0 \leq s \leq t$ we have

$$g(t) - g(s) \geq \int_s^t b\tau^\beta d\tau = \frac{b}{\beta+1}(t^{\beta+1} - s^{\beta+1}) \geq (3) \frac{b}{\beta+1}(t - s)^{\beta+1},$$

and for $s \leq t \leq 0$ we get

$$g(t) - g(s) \geq \frac{b}{\beta+1}((-s)^{\beta+1} - (-t)^{\beta+1}) \geq (3) \frac{b}{\beta+1}(t - s)^{\beta+1}.$$

In case $s \leq 0 \leq t$ we have

$$\begin{aligned} g(t) - g(s) &\geq \int_s^0 b(-\tau)^\beta d\tau + \int_0^t b\tau^\beta d\tau \\ &= \frac{b}{\beta+1}((-s)^{\beta+1} + t^{\beta+1}) \geq (4) \frac{b}{\beta+1}2^{-\beta}(t - s)^{\beta+1}. \end{aligned}$$

Since $2^{-\beta} < 1$ we have

$$\forall t, s \in \mathbb{R} : |g(t) - g(s)| \geq \frac{b}{\beta+1}2^{-\beta}|t - s|^{\beta+1},$$

hence

$$\forall t, s \in \mathbb{R} : |t - s| \leq 2^{\beta/(\beta+1)} \left(\frac{\beta+1}{b} \right)^{1/(\beta+1)} |g(t) - g(s)|^{1/(\beta+1)},$$

from which the assertion follows. $\square$

Proof of Theorem 1. We set $g : \mathbb{R} \rightarrow \mathbb{R}$, $g(t) = f(t) + \lambda t$ and note that $g'(t) \geq f'(t) \geq b|t|^\beta$ ($t \in \mathbb{R}$), so that Lemma 1 applies to g .

For $u \in l^\infty(X, \mathbb{R})$ we have

$$\begin{aligned} M(u) &= \lambda u + D_f(u) + w \Leftrightarrow M(u)(x) - w(x) = g(u(x)) \quad (x \in X). \\ &\Leftrightarrow u(x) = g^{-1}(M(u)(x) - w(x)) \quad (x \in X). \end{aligned}$$

Thus, we consider the increasing mapping $\Phi : l^\infty(X, \mathbb{R}) \rightarrow l^\infty(X, \mathbb{R})$ defined as

$$\Phi(u) = D_{g^{-1}}(M(u) - w).$$

Lemma 1 yields

$$\|\Phi(u) - \Phi(v)\|_\infty \leq c\|M(u) - M(v)\|_\infty^\alpha \leq c\|M\|_\infty^\alpha \|u - v\|_\infty^\alpha. \quad (5)$$

For $r > 0$ let $B(0, r)$ be the closed ball with center 0 and radius r in $l^\infty(X, \mathbb{R})$.

For $u \in B(0, r)$ we get by (5)

$$\|\Phi(u)\|_\infty \leq \|\Phi(u) - \Phi(0)\|_\infty + \|\Phi(0)\|_\infty \leq c\|M\|_\infty^\alpha r^\alpha + \|\Phi(0)\|_\infty.$$

Since $0 < \alpha < 1$ there is a unique fixed point r_0 of

$$r \mapsto c\|M\|_\infty^\alpha r^\alpha + \|\Phi(0)\|_\infty$$

in $(0, \infty)$. Note that

$$c\|M\|_\infty^\alpha r^\alpha + \|\Phi(0)\|_\infty < r \quad (r > r_0). \quad (6)$$

We now have $\Phi[B(0, r_0)] \subseteq B(0, r_0)$. Since each closed ball in $l^\infty(X, \mathbb{R})$ is a complete lattice and since $\Phi : B(0, r_0) \rightarrow B(0, r_0)$ is increasing, Tarski's Fixed Point Theorem [13] yields the existence of a smallest and a largest fixed point $\underline{u}$ and $\bar{u}$ in $B(0, r_0)$ and both solve (1). Moreover there is no solution of (1) outside of $B(0, r_0)$: $\Phi(u) = u$ with $\|u\|_\infty > r_0$ leads by (6) to

$$\|u\|_\infty = \|\Phi(u)\|_\infty \leq c\|M\|_\infty^\alpha \|u\|_\infty^\alpha + \|\Phi(0)\|_\infty < \|u\|_\infty,$$

a contradiction. Next, let $\lambda > \rho(M)$, and note that

$$0 \leq (g^{-1})'(t) \leq \frac{1}{\lambda} \quad (t \in \mathbb{R}),$$

since $g'(t) \geq \lambda$. Choose $\rho(M) < \mu < \lambda$ and set

$$p := (\mu I - M)^{-1}(\mathbf{1}) = \frac{1}{\mu} \sum_{k=0}^{\infty} \frac{1}{\mu^k} M^k(\mathbf{1}) \geq \frac{1}{\mu} \mathbf{1}.$$

Note that $p \in \text{int}K$ and

$$M(p) = \mu p - \mathbf{1} \leq \mu p.$$

Now for all $x \in X$ we have

$$|\Phi(u)(x) - \Phi(v)(x)| \leq \frac{1}{\lambda} |M(u - v)(x)| \leq \frac{1}{\lambda} M(\|u - v\|_p p)(x) \leq \frac{\mu}{\lambda} \|u - v\|_p p(x),$$

hence

$$\|\Phi(u) - \Phi(v)\|_p \leq \frac{\mu}{\lambda} \|u - v\|_p \quad (u, v \in l^\infty(X, \mathbb{R}))$$

with $\mu/\lambda < 1$. This shows $\underline{u} = \bar{u}$ (or, alternatively, unique solvability of $\Phi(u) = u$ via Banach's Fixed Point Theorem).

If $p \in \text{int}K$ is such that $M(p) \leq \rho(M)p$ and $\lambda \leq \rho(M)$ we set $d := \bar{u} - \underline{u}$, and have

$$M(d) - \lambda d \leq M(\|d\|_p p) - \lambda d = \|d\|_p M(p) - \lambda d \leq \|d\|_p \rho(M)p - \lambda d,$$

and

$$M(d) - \lambda d = D_f(\bar{u}) - D_f(\underline{u}),$$

hence

$$\frac{D_f(\bar{u}) - D_f(\underline{u})}{p} \leq \|d\|_p \rho(M) \mathbf{1} - \lambda \frac{d}{p}.$$

Now, Lemma 1 applied to f in place of g , yields

$$\frac{d}{p} \leq \frac{c}{p^{1-\alpha}} \left(\frac{D_f(\bar{u}) - D_f(\underline{u})}{p} \right)^\alpha \leq \frac{c}{p^{1-\alpha}} \left(\|d\|_p \rho(M) \mathbf{1} - \lambda \frac{d}{p} \right)^\alpha.$$

Now, we choose a sequence $(x_j)_{j \in \mathbb{N}}$ in X such that $d(x_j)/p(x_j) \rightarrow \|d\|_p$ ($j \rightarrow \infty$). By passing to a suitable subsequence, we can assume w.l.o.g. that $p(x_j) \rightarrow \gamma$ ($j \rightarrow \infty$) for some $\gamma \geq \frac{1}{\|p^{-1}\|_\infty}$. We obtain

$$\|d\|_p \leq \frac{c}{\gamma^{1-\alpha}} (\|d\|_p \rho(M) - \lambda \|d\|_p)^\alpha = \frac{c}{\gamma^{1-\alpha}} \|d\|_p^\alpha (\rho(M) - \lambda)^\alpha.$$

In case $\lambda = \rho(M)$ we get $\|d\|_p = 0$, which means $\underline{u} = \bar{u}$. If $0 \leq \lambda < \rho(M)$ we get (dividing by $\|d\|_p^\alpha$ if $\|d\|_p > 0$ and trivially if $\|d\|_p = 0$)

$$\|d\|_p^{1-\alpha} \leq \frac{c}{\gamma^{1-\alpha}} (\rho(M) - \lambda)^\alpha,$$

that is

$$\begin{aligned} \|d\|_p &\leq \frac{1}{\gamma} c^{1/(1-\alpha)} (\rho(M) - \lambda)^{\alpha/(1-\alpha)} \leq \|p^{-1}\|_\infty c^{1/(1-\alpha)} (\rho(M) - \lambda)^{\alpha/(1-\alpha)} \\ &= 2 \|p^{-1}\|_\infty \left(\frac{\beta + 1}{b} (\rho(M) - \lambda) \right)^{1/\beta}. \end{aligned}$$

□

From the proof of Theorem 1 we see that all successive approximations for the fixed point equation $\Phi(u) = u$ are convergent to the unique solution of (1) if $\lambda > \rho(M)$. Surprisingly, assuming the existence of a sub-eigenvector $p \in \text{int}K$ with $M(p) \leq \rho(M)p$ the same is true in case $\lambda = \rho(M)$.

Theorem 2. *Let f, M, w and p be as in Theorem 1, let $\Phi : l^\infty(X, \mathbb{R}) \rightarrow l^\infty(X, \mathbb{R})$ be defined as*

$$\Phi(u) = D_{g^{-1}}(M(u) - w),$$

with $g(t) = f(t) + \rho(M)t$ ($t \in \mathbb{R}$). Then for each $u_1 \in l^\infty(X, \mathbb{R})$ the sequence (u_n) of successive approximations $u_{n+1} = \Phi(u_n)$ ($n \in \mathbb{N}$) is convergent in $l^\infty(X, \mathbb{R})$ towards the unique fixed point of Φ as $n \rightarrow \infty$.

Proof. Let u be the unique fixed point of Φ . Let r_0 be as in the proof of Theorem 1 and choose $r_1 \geq \max\{r_0, \|u_1\|_\infty\}$. Set $\underline{v}_1 := -r_1 \mathbf{1}$, $\bar{v}_1 := r_1 \mathbf{1}$, and let $(\underline{v}_n)$, $(\bar{v}_n)$ be the corresponding successive approximations. From the choice

of r_1 and monotonicity of Φ it is clear that $\underline{v}_n \leq u_n \leq \bar{v}_n$ and $\underline{v}_n \leq u \leq \bar{v}_n$ ($n \in \mathbb{N}$). Since

$$\|\Phi(\pm r_1 \mathbf{1})\|_\infty \leq c \|M\|_\infty^\alpha r_1^\alpha + \|\Phi(0)\|_\infty \leq r_1$$

we have $\underline{v}_2 \geq \underline{v}_1$ and $\bar{v}_2 \leq \bar{v}_1$. Hence $(\underline{v}_n)$ and $(\bar{v}_n)$ are increasing and decreasing, respectively. Thus both sequences converge pointwise to limit functions $\underline{v}, \bar{v} \in l^\infty(X, \mathbb{R})$, respectively, with $\underline{v} \leq u \leq \bar{v}$. Thus it is sufficient to show that $\|\bar{v}_n - \underline{v}_n\|_p \rightarrow 0$ ($n \rightarrow \infty$). Set $d_n := \bar{v}_n - \underline{v}_n$ ($n \in \mathbb{N}$) and note that $(\|d_n\|_p)$ is decreasing, hence convergent. For $n \in \mathbb{N}$ we have

$$M(d_n) - \rho(M)d_{n+1} = D_f(\bar{v}_{n+1}) - D_f(\underline{v}_{n+1}),$$

and

$$M(d_n) - \rho(M)d_{n+1} \leq M(\|d_n\|_p p) - \rho(M)d_{n+1} \leq \rho(M)(\|d_n\|_p p - d_{n+1}),$$

hence

$$\frac{D_f(\bar{v}_{n+1}) - D_f(\underline{v}_{n+1})}{p} \leq \rho(M) \left(\|d_n\|_p \mathbf{1} - \frac{d_{n+1}}{p} \right).$$

Application of Lemma 1 yields

$$\frac{d_{n+1}}{p} \leq \frac{c}{p^{1-\alpha}} \left(\frac{D_f(\bar{v}_{n+1}) - D_f(\underline{v}_{n+1})}{p} \right)^\alpha \leq \frac{c}{p^{1-\alpha}} \rho(M)^\alpha \left(\|d_n\|_p \mathbf{1} - \frac{d_{n+1}}{p} \right)^\alpha.$$

For fixed n we can (similar as in the proof of Theorem 1) find a sequence (x_j) in X such that $d_{n+1}(x_j)/p(x_j) \rightarrow \|d_{n+1}\|_p$ and in addition $p(x_j) \rightarrow \gamma_n$ ($j \rightarrow \infty$). We obtain

$$\begin{aligned} \|d_{n+1}\|_p &\leq \frac{c}{\gamma_n^{1-\alpha}} \rho(M)^\alpha (\|d_n\|_p - \|d_{n+1}\|_p)^\alpha \\ &\leq c \|p^{-1}\|_\infty^{1-\alpha} \rho(M)^\alpha (\|d_n\|_p - \|d_{n+1}\|_p)^\alpha \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

□

3. Applications

With regard to difference equations we now consider M of the form

$$M(u) = \sum_{k=1}^m \lambda_k (u \circ q_k) \quad (u \in l^\infty(X, \mathbb{R})),$$

with functions $q_k : X \rightarrow X$ and $\lambda_k \geq 0$ ($k = 1, \dots, m$). Note that here $M(\mathbf{1}) = (\sum_{k=1}^m \lambda_k) \mathbf{1}$, hence $\|M\|_\infty = \rho(M) = \sum_{k=1}^m \lambda_k$. In this case $\bar{u}, \underline{u}$ can be obtained by monotone iteration. In this setting (1) reads

$$\sum_{k=1}^m \lambda_k (u \circ q_k) = \lambda u + D_f(u) + w,$$

and

$$\Phi(u)(x) = g^{-1} \left(\sum_{k=1}^m \lambda_k u(q_k(x)) - w(x) \right) \quad (x \in X, u \in l^\infty(X, \mathbb{R})).$$

As in the proof of Theorem 2 we can start monotone iteration of Φ at $\underline{v}_1 := -r_1 \mathbf{1}$, $\bar{v}_1 := r_1 \mathbf{1}$ for some $r_1 \geq r_0$ and obtain increasing and decreasing sequences $(\underline{v}_n)$ and $(\bar{v}_n)$ with pointwise limit functions $\underline{v}, \bar{v} \in l^\infty(X, \mathbb{R})$, respectively. Since for each fixed $x \in X$ the expression $M(u)(x)$ only depends on a finite subset of X we have $\Phi(\underline{v}) = \underline{v}$ and $\Phi(\bar{v}) = \bar{v}$. From $\bar{u} \leq \bar{v}_1$ we get inductively $\bar{u} \leq \bar{v}_n$ ($n \in \mathbb{N}$) hence $\bar{u} \leq \bar{v}$, and by maximality of $\bar{u}$ we have $\bar{u} = \bar{v}$, and $\underline{u} = \underline{v}$ the same way.

If $q : X \rightarrow X$ is a function with

$$\sum_{k=1}^m \lambda_k u \circ q_k \circ q = \sum_{k=1}^m \lambda_k u \circ q \circ q_k \quad (u \in l^\infty(X, \mathbb{R})), \quad (7)$$

then $\bar{u}$ and $\underline{u}$ inherit the property $w \circ q = w$ if it exists: Clearly $\bar{v}_1 \circ q = \bar{v}_1$ and by induction $\bar{v}_{n+1} \circ q = \bar{v}_{n+1}$ ($n \in \mathbb{N}$). As $n \rightarrow \infty$ we obtain $\bar{u} \circ q = \bar{u}$, and we obtain $\underline{u} \circ q = \underline{u}$ in the same way.

If X is in addition a metric space (X, d) and if $q_k \in C(X, X)$ ($k = 1, \dots, m$) and $w \in C_b(X, \mathbb{R})$ (the space of bounded continuous functions), then $(\underline{v}_n)$ and $(\bar{v}_n)$ are increasing and decreasing sequences in $C_b(X, \mathbb{R})$ and therefore $\underline{u}$ and $\bar{u}$ are lower and upper semi-continuous, respectively. In particular in case $\underline{u} = \bar{u} =: u$ we get that the unique solution $u \in l^\infty(X, \mathbb{R})$ of (1) is continuous.

This proves the following result for difference equations.

Theorem 3. *Let (X, d) be a metric space, $q_k \in C(X, X)$ and $\lambda_k \geq 0$ ($k = 1, \dots, m$), $w \in C_b(X, \mathbb{R})$, and $f \in C^1(\mathbb{R}, \mathbb{R})$ as in Theorem 1. Then for $\lambda \geq \sum_{k=1}^m \lambda_k$ the equation*

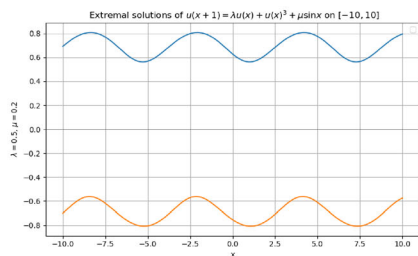
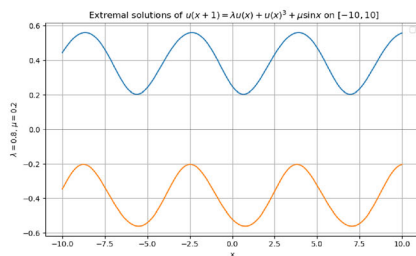
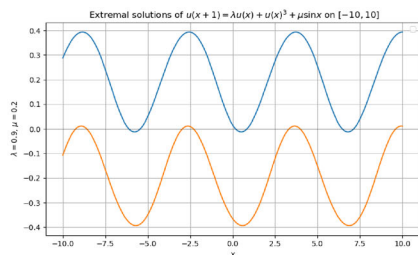
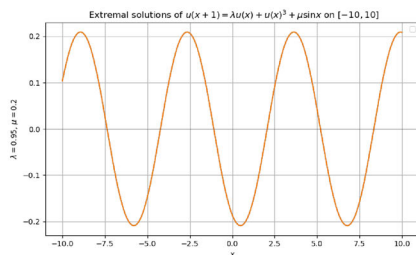
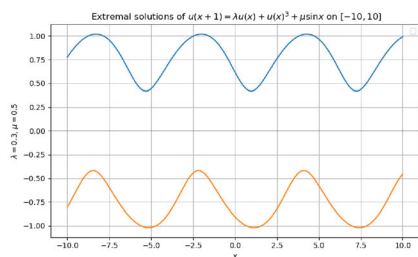
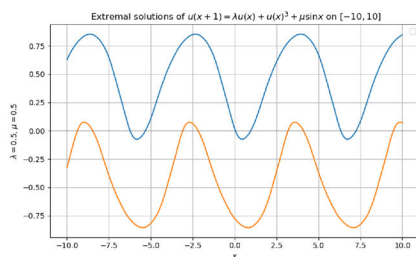
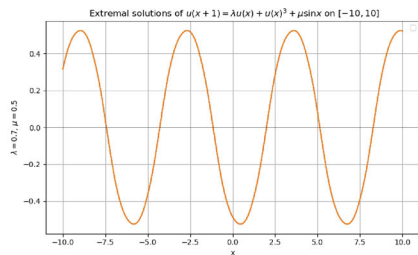
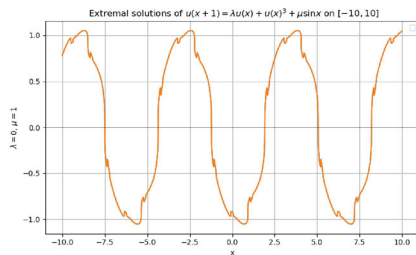
$$\sum_{k=1}^m \lambda_k (u \circ q_k) = \lambda u + D_f(u) + w \quad (8)$$

has a unique solution u in $l^\infty(X, \mathbb{R})$, and this solution is continuous on X . For $0 \leq \lambda < \sum_{k=1}^m \lambda_k$ equation (8) has a lower semi-continuous smallest solution $\underline{u}$ and an upper semi-continuous largest solution $\bar{u}$ in $l^\infty(X, \mathbb{R})$, with

$$\|\bar{u} - \underline{u}\|_\infty \leq 2 \left(\frac{\beta + 1}{b} \left(\left(\sum_{k=1}^m \lambda_k \right) - \lambda \right) \right)^{1/\beta}.$$

Example. In the following examples we always use for simplicity the nonlinearity $f(t) = t^3$. Here $f'(t) = 3t^2$, hence $b = 3$, $\beta = 2$, so

$$\frac{\beta + 1}{b} = 1, \quad \frac{1}{\beta} = \frac{1}{2}.$$

(a) $\lambda = 0.5, \mu = 0.2$ (b) $\lambda = 0.8, \mu = 0.2$ (c) $\lambda = 0.9, \mu = 0.2$ (d) $\lambda = 0.95, \mu = 0.2$ (e) $\lambda = 0.3, \mu = 0.5$ (f) $\lambda = 0.5, \mu = 0.5$ (g) $\lambda = 0.7, \mu = 0.5$ (h) $\lambda = 0, \mu = 1$ FIGURE 1. $\bar{u}, \underline{u}$ for different values of λ and μ in Example 4

1. For $\delta > \frac{2}{3\sqrt{3}}$ and $M : l^\infty(\mathbb{R}, \mathbb{R}) \rightarrow l^\infty(\mathbb{R}, \mathbb{R})$, $M(u) = u$ the equation

$$u(x) = u(x)^3 + \delta \sin(x)$$

is a simple example where the solutions $\underline{u}$ and $\bar{u}$ are discontinuous. To see this just note that for each $x \in \mathbb{R}$ the cubic equation

$$t^3 - t + \delta \sin(x) = 0$$

has one, two or three solutions if $|\delta \sin(x)| >, =$ or $< \frac{2}{3\sqrt{3}}$, respectively. In particular the successive approximations $(\underline{v}_n)$ and $(\bar{v}_n)$ we constructed above are sequences in $C_b(\mathbb{R}, \mathbb{R})$, and thus are pointwise but not uniformly convergent to $\underline{u}$ and $\bar{u}$, respectively. Note that this is in contrast to the assertion of Theorem 2.

2. Let $h_1, \dots, h_m \in \mathbb{R}^n$, and let $\Delta_d : l^\infty(\mathbb{R}^n, \mathbb{R}) \rightarrow l^\infty(\mathbb{R}^n, \mathbb{R})$ denote the discrete Laplace-type operator

$$\Delta_d(u)(x) = \sum_{k=1}^m (u(x + h_k) - u(x)).$$

Note that

$$M : l^\infty(\mathbb{R}^n, \mathbb{R}) \rightarrow l^\infty(\mathbb{R}^n, \mathbb{R}), \quad M(u)(x) := \sum_{k=1}^m u(x + h_k)$$

has spectral radius $\rho(M) = m$ since $M(\mathbf{1}) = m\mathbf{1}$. For each $w \in l^\infty(\mathbb{R}^n, \mathbb{R})$ and $\lambda \geq 0$ the equation

$$\Delta_d(u)(x) = \lambda u(x) + u(x)^3 + w(x) \quad (x \in \mathbb{R}^n) \quad (9)$$

has, by Theorem 3, a unique solution u in $l^\infty(\mathbb{R}^n, \mathbb{R})$ with $u \in C_b(\mathbb{R}^n, \mathbb{R})$ if $w \in C_b(\mathbb{R}^n, \mathbb{R})$. If $\lambda \in [-m, 0)$ and $w \in l^\infty(\mathbb{R}^n, \mathbb{R})$ there is a largest and a smallest solution of (9) with $\|\bar{u} - \underline{u}\|_\infty \leq 2\sqrt{-\lambda}$. Moreover assume in addition that $h_1, \dots, h_m$ is a periodic orbit of a matrix $A \in \mathbb{R}^{n \times n}$, that is

$$h_2 = Ah_1, \dots, h_m = Ah_{m-1}, \quad h_1 = Ah_m.$$

With $q_k(x) = x + h_k$ ($k = 1, \dots, m$) and $q(x) = Ax$ we then have (7) for each $u \in l^\infty(\mathbb{R}^n, \mathbb{R})$. Thus, if $w \in l^\infty(\mathbb{R}^n, \mathbb{R})$ is such that $w(x) = w(Ax)$ ($x \in \mathbb{R}^n$), then $\underline{u}(x) = \underline{u}(Ax)$ and $\bar{u}(x) = \bar{u}(Ax)$ ($x \in \mathbb{R}^n$).

3. Let $m : l^\infty(\mathbb{N}, \mathbb{R}) \rightarrow \mathbb{R}$ be a Banach limit and consider the equation

$$m((u_j(1 + \sin(jk)))_{j \in \mathbb{N}}) = \lambda u_k + u_k^3 + w_k \quad (k \in \mathbb{N}). \quad (10)$$

Using the shift invariance of m it is easy to see that $m((\sin(jk))_{j \in \mathbb{N}}) = 0$ for each $k \in \mathbb{N}$, so the operator

$$M : l^\infty(\mathbb{N}, \mathbb{R}) \rightarrow l^\infty(\mathbb{N}, \mathbb{R}), \quad M(u) := (m((u_j(1 + \sin(jk)))_{j \in \mathbb{N}}))_{k \in \mathbb{N}}$$

has spectral radius $\rho(M) = 1$ since $M(\mathbf{1}) = \mathbf{1}$. Thus, (10) is uniquely solvable in $l^\infty(\mathbb{N}, \mathbb{R})$ for each $w \in l^\infty(\mathbb{N}, \mathbb{R})$ if $\lambda \geq 1$, and in case $\lambda \in [0, 1)$ there is a largest and a smallest solution with $\|\bar{u} - \underline{u}\|_\infty \leq 2\sqrt{1 - \lambda}$.

4. The pictures in (Fig. 1) show the numerical solutions $\underline{u}$ and $\bar{u}$ of

$$u(x+1) = \lambda u(x) + u(x)^3 + \mu \sin(x) \quad (x \in \mathbb{R})$$

for some values of λ and μ , and plotted for $-10 \leq x \leq 10$. Note that here

$$M : l^\infty(\mathbb{R}, \mathbb{R}) \rightarrow l^\infty(\mathbb{R}, \mathbb{R}), \quad M(u)(x) = u(x+1)$$

with $\|M\|_\infty = \rho(M) = 1$, thus again $\|\bar{u} - \underline{u}\|_\infty \leq 2\sqrt{1-\lambda}$. Note that $\underline{u}$ and $\bar{u}$ are 2π -periodic since (7) holds for $q(x) = x + 2\pi$. The iterations of Φ are started at $\underline{v}_1 := -8 \cdot 1$ and $\bar{v}_1 := 8 \cdot 1$. In this example uniqueness seems to occur numerically for λ before the spectral radius $\rho(M) = 1$, the sooner the bigger μ is chosen.

If this is not merely a numerical phenomenon it would be interesting to know which other properties of the equations under consideration enforce uniqueness.

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