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Hypoplasticity for Sand With Explicitly Defined Asymptotic States

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ABSTRACT

A hypoplastic model for sand with defined asymptotic states is proposed. The fourth-order tensor A is defined by modifying the compression relation of the reference hypoplastic model by von Wolffersdorff (HP). This enables the mathematical formulation of a hypoplastic model for sand including the asymptotic state boundary surface (ASBS) using the A -based mathematical framework. The resulting asymptotic hypoplastic model (AHP) for sand can be formulated either using an implicitly defined ASBS (AHPi) or, based on the further proposed double f_s formulation, using an explicitly defined ASBS (AHPe). Especially the AHPi reproduces the overall stress-strain behavior of the reference HP model quite well. However, larger discrepancies occur between AHPe and HP, particularly for dense soils.

To further address cyclic loading effects, the AHP models are coupled with the generalized intergranular strain (GIS) concept. Element simulations confirm consistent responses of the HP+GIS model and the proposed constitutive formulations and demonstrate that the latter provides a basis for the future development of ASBS-based hypoplastic models for sand.

1 | Introduction

The modeling of asymptotic states, that is striven-for states under a prescribed proportional strain path, is a fundamental aspect of advanced constitutive models for soils. The sum of all such states is referred to as the asymptotic state boundary surface (ASBS) [1–4]. In elasto-plastic constitutive models, the ASBS is derived from the yield surface and the hardening rules, and is therefore always explicitly defined. In hypoplastic constitutive models, asymptotic states can also be modeled [4, 5], although they do not necessarily need to be explicitly defined. For clays, hypoplastic formulations with an explicitly defined ASBS have already been developed [6–8], which are mathematically characterized by the existence of a defined fourth-order tensor A . The advantage of such a formulation lies in the independent description of the

asymptotic behavior from the elastic part of the model. For sands, however, such a formulation, and thus such a tensor, has not yet been proposed.

This study introduces two hypoplastic constitutive models for sand which are based on the fourth-order tensor A , allowing the models to be expressed mathematically in the same framework as formulations with an explicitly defined ASBS. The novel models modify the base hypoplastic model by von Wolffersdorff [9] (HP) and are therefore referred to as AHP. They can be expressed using the tensor A in the A -based mathematical framework [6–8]. Depending on whether the asymptotic pyknotropy factor f_d^A and the asymptotic strain rate direction $\mathbf{d}$ are explicitly defined, the ASBS is either defined implicitly or explicitly in the AHP. The models are therefore referred to as AHPi or AHPe.

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To enable the AHP models to approximate cyclic loading as well, they are further coupled with the generalized intergranular strain (GIS) concept [10], resulting in the AHPi+GIS and the AHPe+GIS formulations.

2 | Asymptotic State Boundary Surface (ASBS)

The asymptotic behavior of sand was first mentioned by Goldscheider [11] based on experimental observations in two “rules.” These describe the evolution of the state of a soil specimen (expressed as the effective stress path $\sigma(t)$) resulting from a monotonic, proportional strain path $\varepsilon(t)$. A strain path $\varepsilon(t)$ or a stress path $\sigma(t)$ is proportional if it represents a straight line through the origin in strain space or stress space:

1. The response to a proportional strain path starting from a stress-free initial state $\sigma_0 = \mathbf{0}$ (and only to such a strain path) is a proportional stress path.
2. For a proportional strain path with an initial stress $\sigma_0 \neq \mathbf{0}$, the stress tends asymptotically, in the stress space toward the straight line through the origin that would arise under the same proportional strain path starting from a stress-free initial state.

These “Goldscheider rules” can be generalized by including additional state variables α (e.g., void ratio e or fabric tensor F). Asymptotic states are then defined as all states of the soil that are approached due to a monotonic, proportional strain path.

The concept of an asymptotic state can be seen as a generalization of the critical state concept [12, 13]. For a given monotonic volume decreasing deformation,

$$\text{tr}(\dot{\varepsilon}) = \dot{\varepsilon}_{\text{vol}} < 0 \quad (1)$$

the resulting path in the n -dimensional vector space $\mathbb{R}^n$ of state variables will asymptotically approach, regardless of the initial condition, not a single well-defined state, but rather a set of asymptotic states along an asymptotic path. Under these conditions, the soil tends toward a proportional stress path with a non-vanishing stress rate [5, 14]¹.

$$\dot{\sigma} \neq \mathbf{0} \quad \text{but} \quad \vec{\sigma} = \vec{\sigma} = \text{const} \quad (2)$$

The well-known critical state [12, 13] represents special asymptotic state that is reached under a given proportional and volume-preserving strain path from a non stress-free initial state with

$$\text{tr}(\dot{\varepsilon}) = \dot{\varepsilon}_{\text{vol}} = 0. \quad (3)$$

If the void ratio is included as a state variable, $\dot{e} = 0$ follows directly from Equation 3. In the critical state, it must further hold

$$\dot{\sigma} = \mathbf{0} \quad \text{and} \quad \dot{\alpha} = \mathbf{0} \quad (4)$$

[15, 16]. Although typical and well-known so-called critical state models, such as the modified and original Cam-Clay models [13,

17], explicitly mention the critical state, they implicitly describe asymptotic states for each load direction [6].

It becomes evident from Equation 4 that, for a monotonic volume-preserving strain path, a well-defined state is reached depending on the initial condition. This critical state remains unchanged under a continued monotonic deformation and can therefore be referred to as a stationary state [18, 19], although the terms critical state and stationary state are often experimentally used to distinguish between drained and undrained conditions [18]. However, the critical state should also be understood as a set of states, since there exist infinitely many proportional, volume-preserving strain paths in the full tensorial space, each leading to a distinct critical state. Li and Dafalias [15] demonstrated the uniqueness of these critical states based on thermodynamic considerations.²

It is worth noting that asymptotic states are also referred to as swept-out-memory states or attractors [1, 2].

The ASBS can be visualized in the n -dimensional vector space $\mathbb{R}^n$ of the state variables (see Figure 2, described later). The critical states are part of the ASBS. The asymptotic behavior of soil described above is schematically illustrated for axisymmetric conditions under volume-preserving and volume-reducing monotonic and proportional strain paths in Figure 1 (crit: $\dot{\varepsilon}_{\text{vol}} = 0$; iso: $\dot{\varepsilon}_1 = \dot{\varepsilon}_2 = \dot{\varepsilon}_3 < 0$). Different proportional strain paths can be characterized by an angle ψ_ε , as shown in Figure 1a. Asymptotically, different proportional strain paths lead to different proportional stress paths with constant stress ratios, represented by the angle ψ_σ (Figure 1b). Figure 1c further illustrates that each proportional strain path tends to a distinct normal compression line (NCL) in the ep -diagram. Volume-preserving proportional strain paths thus tend to the critical state line (CSL). Every NCL represents a part of the ASBS.

3 | Asymptotic States in (Hypoplastic) Constitutive Models

In elasto-plastic constitutive models, which can generally be expressed as

$$\dot{\sigma} = C^{\text{el}}(\dot{\varepsilon} - \dot{\varepsilon}^{\text{pl}}) \quad (5)$$

with the elastic stiffness tensor C^{el} and the plastic strain rate $\dot{\varepsilon}^{\text{pl}}$, the ASBS coincides with the state boundary surface (SBS). The SBS in such models is explicitly defined by the yield surface and the hardening law [6].

Hypoplastic constitutive models provide an alternative to the elasto-plastic framework. In these constitutive models, the norm of the strain rate $\|\dot{\varepsilon}\|$ replaces the plastic strain rate $\dot{\varepsilon}^{\text{pl}}$ to represent irreversible deformations. Hypoplastic formulations do not require a decomposition of the total strain rate into elastic and plastic components and do not employ a yield surface or a plastic potential. Instead, hypoplastic models are formulated in terms of a single tensorial equation linking the effective stress rate $\dot{\sigma}$ with the strain rate $\dot{\varepsilon}$. A comprehensive overview of hypoplastic models is provided by Mašin [8].

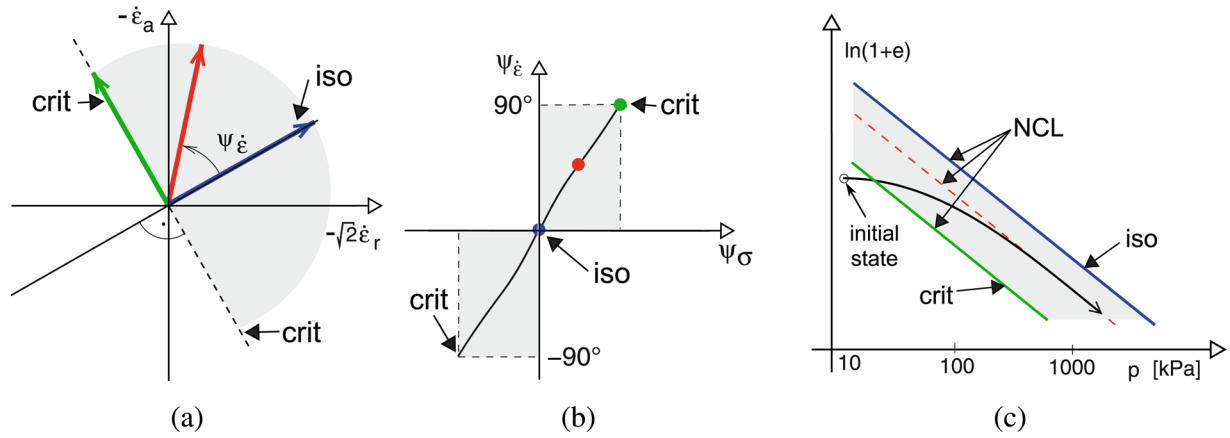


FIGURE 1 | Soil behavior resulting from volume-preserving and volume-reducing proportional strain paths under axisymmetric conditions: (a) different strain paths (crit: $\dot{\epsilon}_{vol} = 0$; iso: $\dot{\epsilon}_1 = \dot{\epsilon}_2 = \dot{\epsilon}_3 < 0$) lead to (b) different proportional stress paths and (c) different normal compression lines (NCL), modified from [4].

Early hypoplastic formulations used only the effective stress σ as a state variable [22, 26–28]. The critical state [12, 13] concept was later incorporated by introducing the void ratio e as an additional state variable [20, 21]. Most modern hypoplastic constitutive models for soils are expressed in the well-known general form

$$\dot{\sigma} = f_s L : \dot{\epsilon} + f_d f_d N \|\dot{\epsilon}\|, \quad (6)$$

where L is a fourth-order tensor, N a second-order tensor, and f_s and f_d are the barotropy and pyknotropy factors, respectively.

The widely used hypoplastic model for sand proposed by von Wolffersdorff [9] (HP) which incorporates the failure criterion of Matsuoka and Nakai [29], can be expressed in the form of Equation 6. The mathematical formulations for L , N , and f_d in the HP model are adopted in the proposed modified HP (AHP) model and are provided in the Appendix³.

However, the two state variables of the HP model (σ , e) are generally insufficient to capture the full complexity of soil behavior. Therefore, several extensions have been developed, such as the intergranular strain concept [30] for modeling cyclic behavior, which has been continuously improved in recent years [10, 31–33]. Other developments include constitutive efforts to account for an anisotropic microstructure (fabric) [34, 35] or for the simulation of the cyclic mobility [36, 37]. These extensions, however, are beyond the scope of this paper.

In hypoplastic constitutive models, asymptotic states can be reproduced, but they are not necessarily explicitly defined [4, 5, 7, 14]. It should be emphasized that demonstrating the existence of an ASBS within a specific model is not equivalent to explicitly incorporating it into the model formulation.

For the HP, for example, the ASBS is not explicitly defined [14]. However, it can be found numerically. A corresponding three-dimensional representation of the ASBS for the HP under axisymmetric conditions in the pqe -diagram is shown in Figure 2. The irregular shape becomes apparent. This shape is not explicitly defined but is a consequence of the model itself.

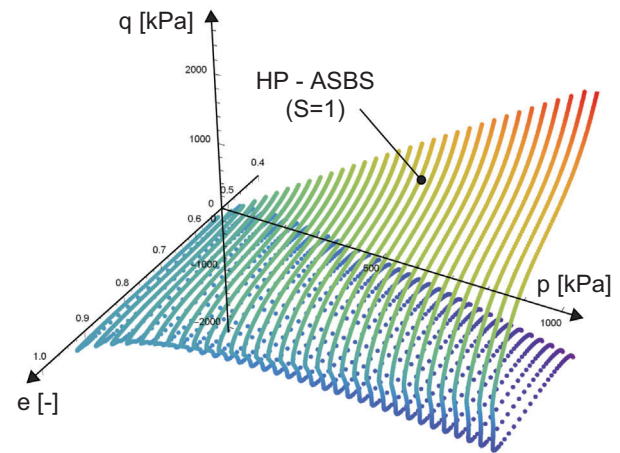


FIGURE 2 | Numerically determined ASBS for the HP model for sand in the 3D epq -diagram under axisymmetric conditions for the parameter set from Table 2 [10].

Following the pioneering works of Mašin [6–8], several hypoplastic models for clay have been developed that include an explicitly defined ASBS. For clay, it is often assumed that the shape of the ASBS for cross-sections of constant void ratio e is invariant, while only the size of the cross-sections changes with e .

As described by Mašin [6–8], a hypoplastic constitutive model incorporating an explicitly defined ASBS can be formulated as

$$\dot{\sigma} = f_s L : \dot{\epsilon} - \frac{f_d}{f_d^A} A : d \|\dot{\epsilon}\|. \quad (7)$$

In Equation 7, the fourth-order tensor L represents the linear stiffness. The fourth-order tensor A is defined through the explicitly defined ASBS. The superscript $\sqcup^A$ denotes a quantity corresponding to an asymptotic state. The asymptotic pyknotropy factor f_d^A represents the value of the pyknotropy factor f_d for the projection of the current state onto the ASBS and the shape of the ASBS can be controlled by its explicit formulation.

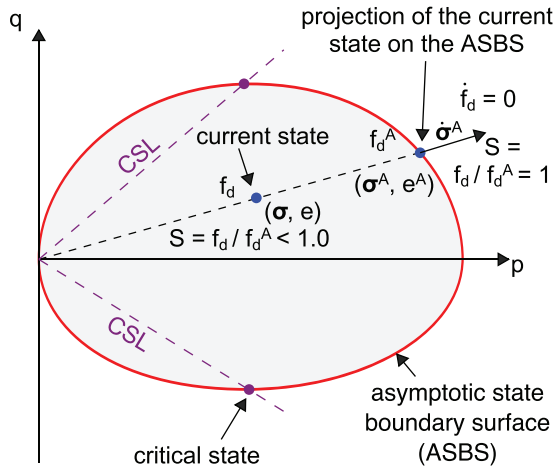


FIGURE 3 | Simplified schematic illustration of a projection of the current state onto the ASBS of hypoplastic constitutive formulations under axisymmetric conditions in the pq -plane [10].

Such a projection is illustrated symbolically for an axisymmetric condition in Figure 3.

The ratio f_d/f_d^A , which was introduced in [10] as the so-called state mobilization $S = f_d/f_d^A$, quantifies the distance of the current state from the ASBS. The soil behavior becomes less irreversible for states inside and further away from the ASBS ($f_d < f_d^A$). For any point on the ASBS, the condition $S = f_d/f_d^A = 1$ holds true.

The second-order tensor $\mathbf{d}$ describes the normalized asymptotic strain rate $\dot{\epsilon}^A$ corresponding to the projected state on the ASBS and is defined as

$$\mathbf{d} = \frac{\dot{\epsilon}^A}{\|\dot{\epsilon}^A\|} = \vec{\epsilon}^A. \quad (8)$$

When an ASBS is explicitly defined, the fourth-order tensor $\mathbf{A}$, the asymptotic pyknosity factor f_d^A , and the asymptotic strain rate $\mathbf{d}$ are known or explicitly defined.

In many hypoplastic models, however, the ASBS is not explicitly defined. Nevertheless, the ASBS can be extracted and visualized from existing formulations even without explicit definition [5, 6, 10, 14]. Using

$$\mathbf{N} = -\frac{\mathbf{A} : \mathbf{d}}{f_s f_d^A} \quad (9)$$

Equation 7 can be reformulated in terms of the known second-order tensor $\mathbf{N}$ as

$$\dot{\sigma} = f_s L : \dot{\epsilon} + f_s \frac{f_d}{f_d^A} f_d^A \mathbf{N} \|\dot{\epsilon}\|. \quad (10)$$

Compared to Equation 7, any hypoplastic model expressed in the form of Equation 6 can be rewritten as Equation 10, provided that the asymptotic pyknosity factor f_d^A is known. Note that due to the inclusion of the state mobilization $S = f_d/f_d^A$, Equation 10 already provides a more insightful formulation than Equation 6.

If the asymptotic pyknosity factor f_d^A is not known, either because it is not explicitly defined or can only be derived as a function of the current state itself [5], it is possible to define a quantity with similar characteristics like $S = 1.0$ for a state located on the ASBS. Such a quantity was introduced by Mugele et al. [10] and defined for the HP model. While the HP can therefore be expressed in the form of Equation 10, the AHP models derived in the following can be directly written according to Equation 7. Thus, the AHP models represent the first hypoplastic models for sand written using the fourth-order tensor $\mathbf{A}$, offering promising new perspectives for the constitutive modeling of sand and also for the future definition of hybrid models for both sand and clays.

4 | Asymptotic Hypoplasticity (AHP)

The proposed AHP formulations are based on a modified compression relation, which allows the concept originally introduced for clay models to derive the fourth-order tensor $\mathbf{A}$ [5–7] to be adapted to a constitutive model for sand.

The HP model is based on the compression relation by Bauer [20]

$$e = e_0 \exp \left[-\left(\frac{3p}{h_s} \right)^n \right], \quad (11)$$

which can be written in rate form as

$$\dot{e}/e = -\left(\frac{3p}{h_s} \right)^{(n-1)} \frac{3n}{h_s} \dot{p}. \quad (12)$$

It turns out that the formulation of the compression relation by Bauer [20] prevents a definition of the tensor $\mathbf{A}$. Therefore, a modified compression relation is proposed, which can be formulated in rate form

$$\dot{e}/(e+1) = -\left(\frac{3p}{h_s^*} \right)^{(n^*-1)} \frac{3n^*}{h_s^*} \dot{p} \quad (13)$$

or as a function

$$e = (e_0 + 1) \exp \left[-\left(\frac{3p}{h_s^*} \right)^{n^*} \right] - 1. \quad (14)$$

The material parameters h_s^* and n^* for the proposed compression relation do not coincide with h_s and n from the relation of Bauer [20] and must be calibrated accordingly. A calibration of h_s^* and n^* can be done analogously to h_s and n , for instance, using an oedometric or an isotropic compression test on an initially loose specimen. Details can be found in [8, 38, 39]. In general, h_s resp. h_s^* influences the slope of the compression curve, whereas n resp. n^* controls its curvature. As a first estimate, one may assume $h_s^* = 10h_s$ and $n^* = n$.

The proposed new compression relation of the AHP (Equation 14), the compression relation of the HP according to Bauer [20], and the well-known compression relation of Butterfield [40]

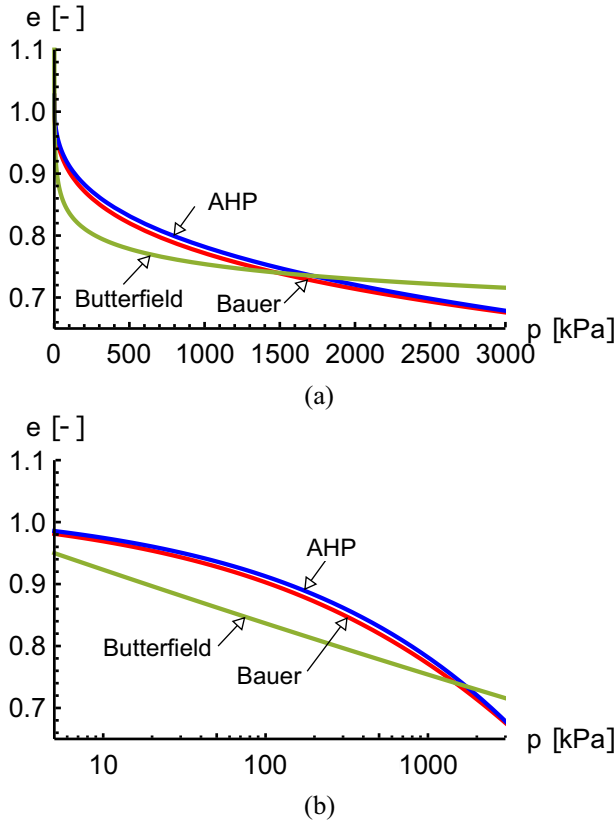


FIGURE 4 | Compression relations a) in the ep -space and b) in the $e \log p$ -space: relation by Bauer [20] for parameters from Table 2 ($e_0 = e_{i0} = 1.027$, $h_s = 111.746$ MPa and $n = 0.346$), the proposed AHP compression relation with parameters from Table 1 ($e_0 = e_{i0} = 1.027$, $h_s^* = 10h_s = 1117.46$ MPa and $n^* = n = 0.346$), and the relation by Butterfield [40] ($N_0 = 0.7$, $\lambda = 0.02$, $p_{\text{ref}} = 1$ kPa).

$$\ln(1 + e) = N_0 - \lambda \ln \left(\frac{p}{p_{\text{ref}}} \right) \quad (15)$$

or

$$e = \exp \left[N_0 - \lambda \ln \left(\frac{p}{p_{\text{ref}}} \right) \right] - 1 \quad (16)$$

are briefly discussed. The latter requires a reference value N_0 , a reference pressure p_{ref} , and the material parameter λ . The Butterfield [40] relation is included in many constitutive models (in particular in constitutive formulations for clays) [7, 17]. Of particular importance for this paper is the fact that Equation 15 was used to develop a tensor-based hypoplastic model for clay [5, 7], which was possible only because of the mathematical structure of the Butterfield [40] relation.

All three compression relations are shown in Figure 4 in the ep - and $e \log p$ -planes. The compression relation by Bauer [20] and the AHP compression relation are further compared in Figure 5 for different parameter sets. In the relevant pressure range, the proposed AHP compression relation closely follows the relation by Bauer [20], whereas the Butterfield relation deviates from it. The aforementioned assumption of $h_s^* = 10h_s$ seems to be

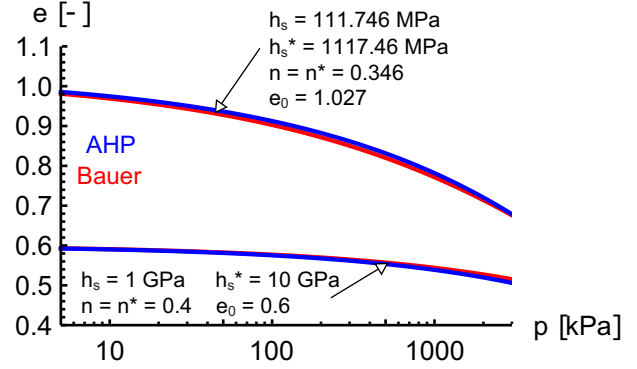


FIGURE 5 | Comparison of the compression relation by Bauer [20] and the proposed compression relation for the AHP for different parameter sets.

appropriate even though the assumption $n^* = n$ might not lead to the best fit for small values of the exponent.

For the theoretical limit $p \rightarrow 0$ applies

$$\begin{aligned} \lim_{p \rightarrow 0} e(p)^{\text{Bauer}} &= e_0, \\ \lim_{p \rightarrow 0} e(p)^{\text{Butterfield}} &= \infty, \\ \lim_{p \rightarrow 0} e(p)^{\text{AHP}} &= e_0. \end{aligned} \quad (17)$$

This limit behavior is more realistic in the Bauer [20] relation and in the proposed AHP relation than in the Butterfield [40] relation. For $p \rightarrow \infty$ applies

$$\begin{aligned} \lim_{p \rightarrow \infty} e(p)^{\text{Bauer}} &= 0, \\ \lim_{p \rightarrow \infty} e(p)^{\text{Butterfield}} &= -1, \\ \lim_{p \rightarrow \infty} e(p)^{\text{AHP}} &= -1. \end{aligned} \quad (18)$$

In this case, the Bauer [20] relation is more realistic than the compression relation by Butterfield [40] and the proposed AHP compression relation. This comparison shows that the proposed compression relation (Equation 14) represents, in a sense, a compromise between the relation by Bauer [20] established for sand and the relation established for clay by Butterfield [40].

For the proposed compression relation of the AHP, the effective pressure corresponding to a void ratio of $e = 0$ is given by

$$p(e = 0) = \frac{h_s^*}{3} [\ln(e_0 + 1)]^{1/n^*}. \quad (19)$$

Similar to the compression relation by Bauer [20] implemented in the HP, the novel proposed compression relation for the AHP defines the so-called limit void ratios within the AHP, that is, the pressure-dependent loosest and densest void ratios ($e_i(p)$ and $e_d(p)$) as well as the void ratio in the critical state ($e_c(p)$). These limit void ratios are obtained from the respective compression relations by using e_{d0} , e_{c0} , and e_{i0} , for e_0 .

For the practical parameter set given in Table 1, $e_d(p)$ becomes zero at a pressure of approximately $p(e_d = 0) = 30.1$ MPa. The

TABLE 1 | Parameter set for the AHP models (AHPi and AHPe) for Zbraslav Sand (ZS)

	φ_c	e_{i0}	e_{c0}	e_{d0}	h_s^*	n^*	α	β
	[°]	[-]	[-]	[-]	[MPa]	[-]	[-]	[-]
ZS	34	1.027	0.893	0.520	1117.46	0.346	0.15	2.2

critical and loosest void ratios become zero at approximately $p(e_c = 0) = 101.7$ MPa and $p(e_i = 0) = 136.5$ MPa, respectively. These pressures, even $p(e_d = 0)$, are well beyond the usual calibration range of hypoplastic constitutive models and within a range where grain breakage is to be expected. Consequently, the zero crossing of the proposed novel compression relation of the AHP occurs far outside the pressure range typically relevant for hypoplastic constitutive models, which is generally limited to a few MPa at most.

Using Equation 14 combined with all remaining constitutive relations of the HP, the approach proposed by Mašin and Herle [5] can be used to derive the novel AHP models. The AHP formulations include two state variables: the effective Cauchy stress σ and the void ratio e . For asymptotic states, $\dot{\sigma} \parallel \sigma$ holds with $\dot{f}_d^A = 0$. In rate-independent constitutive models applies for an asymptotic state

$$\dot{\sigma} = \gamma \vec{\sigma} \parallel \vec{\varepsilon}^A \quad (20)$$

with a scalar multiplier γ . A rate-independent hypoplastic model on the ASBS can be rewritten as

$$\gamma \vec{\sigma} = f_s^A \mathbf{L} : \vec{\varepsilon}^A + f_s^A f_d^A \mathbf{N}. \quad (21)$$

Using the void ratio evolution

$$\dot{e}^A = (1 + e^A) \text{tr}(\vec{\varepsilon}^A) \quad (22)$$

and $\dot{p} = -\text{tr}(\dot{\sigma})/3$, the proposed AHP compression relation (Equation 13) can be written as

$$\text{tr}(\vec{\varepsilon}^A) = \left(\frac{-\text{tr}(\sigma)}{h_s^*} \right)^{(n^*-1)} \frac{n^*}{h_s^*} \text{tr}(\dot{\sigma}). \quad (23)$$

Equation 23 and Equation 20 can be combined to

$$\text{tr}(\vec{\varepsilon}^A) = \left(\frac{-\text{tr}(\sigma)}{h_s^*} \right)^{(n^*-1)} \frac{n^*}{h_s^*} \text{tr}(\gamma \vec{\sigma}), \quad (24)$$

and rewritten:

$$\text{tr}(\vec{\varepsilon}^A) = \frac{(-\text{tr}(\sigma))^{n^*} \text{tr}(\gamma \vec{\sigma})}{-\text{tr}(\sigma)} h_s^{*(-n^*)} n^*. \quad (25)$$

By introducing the scalar quantity

$$\lambda^* = (-\text{tr}(\sigma))^{n^*} h_s^{*(-n^*)} n^* = n^* \left(\frac{3p}{h_s^*} \right)^{n^*} \quad (26)$$

and $\text{tr}(\vec{\mathbf{L}})/\text{tr}(\mathbf{L}) = 1/\|\mathbf{L}\|$ one obtains

$$\text{tr}(\vec{\varepsilon}^A) = -\frac{(-\text{tr}(\sigma))^{n^*} h_s^{*(-n^*)} n^*}{\|\sigma\|} \gamma = -\frac{\lambda^*}{\|\sigma\|} \gamma. \quad (27)$$

The introduced scalar λ^* depends on the mean effective pressure $p = -\text{tr}(\sigma)/3$. This highlights an important difference from hypoplastic models with an explicitly defined ASBS for clays [6, 7]: for clays, λ^* is a constant model parameter (the compression index) and does not depend on the current stress [6, 7, 14, 41].

For the scalar multiplier γ follows

$$\gamma = -\|\sigma\| \frac{\text{tr}(\vec{\varepsilon}^A)}{\lambda^*} \quad (28)$$

and combining Equation 28 with Equation 21 yields to

$$-\frac{\sigma}{\lambda^*} \text{tr}(\vec{\varepsilon}^A) = f_s^A \mathbf{L} : \vec{\varepsilon}^A + f_s^A f_d^A \mathbf{N}. \quad (29)$$

Now, the fourth-order tensor $\mathbf{A}$ can be defined

$$\mathbf{A} = f_s^A \mathbf{L} + \frac{1}{\lambda^*} \sigma \mathbf{1} \quad (30)$$

such that

$$\mathbf{A} : \vec{\varepsilon}^A = f_s^A \mathbf{L} : \vec{\varepsilon}^A + \frac{\sigma}{\lambda^*} \text{tr}(\vec{\varepsilon}^A) \quad (31)$$

holds. Equation 29 can then be written as

$$\mathbf{A} : \vec{\varepsilon}^A + f_s^A f_d^A \mathbf{N} = \mathbf{0}. \quad (32)$$

Note that the second-order tensor $\mathbf{N}$ is treated as a known quantity in the above equations. In contrast, in conventional A-based formulations for clays $\mathbf{N}$ is typically not explicitly defined but instead calculated using Equation 9. However, in the context of the development of A-based formulations for sand, the specific $\mathbf{N}$ -tensor of the HP model is used. To avoid confusion, $\mathbf{N}^{\text{HP}}$ is therefore denoted in the following.

With $\|\vec{\varepsilon}^A\| = 1$ this leads to

$$\begin{aligned} f_d^A &= \|\mathbf{A}^{-1} : (f_s^A \mathbf{N}^{\text{HP}})\|^{-1} \\ &= \left\| \left(\mathbf{L} + \frac{1}{f_s^A \lambda^*} \sigma \mathbf{1} \right) : \mathbf{N}^{\text{HP}} \right\|^{-1} \end{aligned} \quad (33)$$

and⁴

$$\mathbf{d} = \vec{\varepsilon}^A = -\frac{\mathbf{A}^{-1} : \mathbf{N}^{\text{HP}}}{\|\mathbf{A}^{-1} : \mathbf{N}^{\text{HP}}\|}. \quad (34)$$

Equations 33 and 34 require the inverse of the previously introduced tensor $\mathbf{A}$, thereby achieving an ASBS of the AHP that is as similar as possible to that of the HP. However, it remains to be shown in the future that the tensor $\mathbf{A}$ is invertible at all for all stress and density states. An alternative further development will be, however, the definition of f_d^A and $\mathbf{d}$ without using $\mathbf{A}$, which would result in an ASBS of the AHP that is less similar to that of the HP, but whose shape could be freely chosen.

Until now, the barotropy factor f_s , and the asymptotic barotropy factor f_s^A , which were considered symbolically, have not yet been defined, and therefore need to be derived in detail. In hypoplastic models for sand, the barotropy factor is generally derived by enforcing consistency between the rate of mean effective stress $\dot{p}$ and the corresponding change in void ratio $\dot{e}^A$ during monotonic isotropic compression (i.e., on the ASBS) between the chosen compression relation and the constitutive model itself [20, 21, 41]. Such a derived factor should be referred to as asymptotic barotropy factor f_s^A to be more precise. Note that the barotropy factor f_s in A-based formulations is usually derived using the unloading path [6].

Since the AHP uses a modified compression relation, the barotropy factor must be adjusted compared to that of HP. From the combination of isotropic compression in the HP [41, 42], which still holds for the AHP,

$$\dot{p} = \frac{-f_s^A}{3(1+e^A)} X \dot{e}^A \quad (35)$$

with

$$X = 3 + a^2 - a \sqrt{3} \left(\frac{e_{i0} - e_{d0}}{e_{c0} - e_{d0}} \right)^\alpha \quad (36)$$

and Equation 13 for $e = e_i$ and considering the density factor to include a stiffness dependency on the void ratio $f_e = (e_c/e)^\beta$ [20] (asymptotically $f_e^A = (e_c/e^A)^\beta$), the asymptotic barotropy factor in the AHP is obtained as

$$f_s^A = \frac{h_s^*}{n^*} \left(\frac{e_i}{e^A} \right)^\beta \left(\frac{-\text{tr}(\sigma)}{h_s^*} \right)^{1-n^*} X^{-1}. \quad (37)$$

Equation 37 is defined in terms of the asymptotic void ratio e^A . It is not explicitly known, even for the HP model reformulated using the A tensor with Bauer's compression law replaced by that defined in Equation 14. Thus, an additional assumption is needed to define the formulation of the A-based hypoplastic sand model. Two options, denoted as AHPi and AHPe, are described in the subsequent paragraphs. It is worth noting that, as many hypoplastic constitutive models including the reference HP model considered in this paper, the proposed AHP models are undefined at a mean effective pressure of zero. Accordingly, the models are formulated for positive mean effective pressures ($p > 0$).

4.1 | AHP With Implicit ASBS: AHPi

In the first approach, the ASBS is not explicitly defined and is only implicitly contained in the AHP model. Therefore, this formulation is referred to as AHPi. For a given state, it is assumed that the latter is an asymptotic state. Assuming $e^A = e$, the so-called implicit asymptotic barotropy factor f_s^{Ai} can be calculated according to Equation 37⁵:

$$f_s^{Ai} = \frac{h_s^*}{n^*} \left(\frac{e_i}{e} \right)^\beta \left(\frac{-\text{tr}(\sigma)}{h_s^*} \right)^{1-n^*} X^{-1}. \quad (38)$$

With f_s^{Ai} the tensor A^i can be determined using Equation 30 and $\mathbf{d}^i$ can be calculated using Equation 34. The so-called implicit asymptotic pyknotropy factor f_d^{Ai} can be obtained using Equation 33. Finally, the resulting AHPi⁶ reads

$$\dot{\sigma} = f_s^{Ai} \mathbf{L} : \dot{\varepsilon} - \frac{f_d}{f_s^{Ai}} A^i : \mathbf{d}^i \|\dot{\varepsilon}\|. \quad (39)$$

Note that all quantities $\mathbf{L}^i$ are calculated using the implicit asymptotic barotropy factor f_s^{Ai} .

4.2 | AHP With Explicit ASBS: AHPe

In the second approach, the ASBS will be explicitly defined and the model is therefore further called AHPe. Equation 37 is modified by exchanging e^A with e_c and β with $\beta(\sin \varphi_{mob}/\sin \varphi_c)^2$ to the explicit asymptotic barotropy factor

$$f_s^{Ae} = \frac{h_s^*}{n^*} \left(\frac{e_i}{e_c} \right)^{\beta \left(\frac{\sin \varphi_{mob}}{\sin \varphi_c} \right)^2} \left(\frac{-\text{tr}(\sigma)}{h_s^*} \right)^{1-n^*} X^{-1}. \quad (40)$$

Doing so, Equation 40 becomes an explicit function of the effective stress only. Due to the modified exponent $\beta(\sin \varphi_{mob}/\sin \varphi_c)^2$ applies $f_s^{Ae} = f_s^A$ for $e^A = e_i$ and for $e^A = e_c$, which ensures the consistency of the asymptotic critical and isotropic states between the original HP and the novel AHPe. Note that the mobilized friction angle in Equation 40 needs to be determined using the failure criterion after Matsuoka and Nakai [29]⁷, which is considered in the original HP and in the AHP formulations.

Using the explicit asymptotic barotropy factor f_s^{Ae} , the now explicitly defined quantities $\mathbf{d}^e(f_s^{Ae})$ and $f_d^{Ae}(f_s^{Ae})$ can be determined using Equation 33 and Equation 34, which impose the ASBS shape and asymptotic loading direction $\mathbf{d}^e$ of the HP model. Hereafter, all quantities $\mathbf{L}^e$ are computed using the explicit asymptotic barotropy factor f_s^{Ae} .

To ensure predictions as close as possible of AHPe when compared to the original HP model, a so-called double f_s formulation is adopted and first proposed within this paper. Thereby, f_s^{Ai} from Equation 38 is used in the linear part of the hypoplastic equation (including the L-part in A according to Equation 30), whereas the new f_s^{Ae} , governing asymptotic states, is used in the non-linear part of the formulation. The resulting AHPe reads

$$\dot{\sigma} = f_s^{Ai} \mathbf{L} : \dot{\varepsilon} - \frac{f_d}{f_s^{Ae}} A^i : \mathbf{d}^e \|\dot{\varepsilon}\| \quad (41)$$

and is the first hypoplastic model for sand which incorporates a fully explicitly defined ASBS. An advantage of the AHPe formulation compared to AHPi is that either $\mathbf{d}^e$ or f_d^{Ae} can in possible future developments be replaced by different formulations explicitly defining the shape of the ASBS and the asymptotic loading direction, while having explicit formulation of the other component available.

In the form presented in this paper, $\mathbf{d}^e$ and f_d^{Ae} are calculated using expressions taken over from HP model, but they can in the future developments be replaced by other formulations.

TABLE 2 | Parameter set for the HP for Zbraslav Sand (ZS), adopted from Duque et al. [43]

	φ_c	e_{i0}	e_{c0}	e_{d0}	h_s	n	α	β
	[°]	[-]	[-]	[-]	[MPa]	[-]	[-]	[-]
ZS	34	1.027	0.893	0.520	111.746	0.346	0.15	2.2

4.3 | On the AHP Parameter Determination

Both AHP models (AHPi and AHPe) require eight material parameters, where only h_s^* and n^* differ from the original HP parameters due to the modified compression relation. The AHP parameters for ZS in Table 1 were, for simplicity, obtained from the HP parameters in Table 2 by setting $h_s^* = 10h_s$ and $n^* = n$. The HP parameters are taken from Duque et al. [43]. For this parameter choice, the proposed AHP compression relation and the relation by Bauer [20] yield very similar curves in the relevant stress range, as already shown in Figure 4.

In general, the parameter calibration procedures described in the literature for the HP model [8, 39] are likewise suitable for the AHP models.

5 | Comparison of the AHP Models and the HP

The AHP models are based on the HP and the differences between the AHP models and the HP are theoretically motivated. It is therefore to be expected that these models, with appropriate parameter choices, will produce very similar simulation results. This is demonstrated in the following by comparative element simulations, for which the AHPi and the AHPe have been implemented in an Abaqus umat.for subroutine⁸. The simulations were carried out using the freely available program code Incremental Driver [45].

For the AHP (AHPi and AHPe), the ZS parameter set from Table 1 is used and for the HP, the parameter set for ZS given in Table 2 is applied.

Figure 6 shows monotonic isotropic compression tests starting from an isotropic initial stress state of $p_0 = 1$ kPa and an initial void ratio of $e_0 = 1$. In addition to the simulations, the underlying compression relations of the loosest state $e_i(p)$ are plotted. The initial state was chosen to lie closely to $e_i(p)$. It can be seen that the AHP and HP simulations each converge toward $e_i(p)$ and then follow the respective compression relations exactly. The consistency condition for the isotropic compression [42] is therefore fulfilled in all considered models, underscoring the importance of the difference between the barotropy factors in the AHP formulations and in the HP.

Figure 7 considers the oedometric compression of specimens with three different initial void ratios ($e_0 = 0.85$, $e_0 = 0.7$ and $e_0 = 0.55$) with an isotropic initial stress state of $p_0 = 1$ kPa. The compression curves of the HP and AHP formulations are, as expected, very similar. Only the AHPe simulations deviate slightly from the other curves. It should be noted for all simulations that the conformity of the graphs could be improved by adjusting

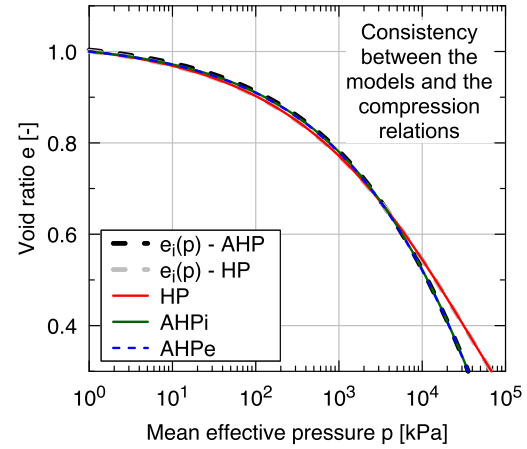


FIGURE 6 | Comparison of the AHPi and the AHPe with the HP and the respective underlying compression relations $e_i(p)$ based on monotonic isotropic compression of an initially loose ZS specimen ($e_0 = 1$ and $p_0 = 1$ kPa): void ratio e as a function of mean effective pressure p .

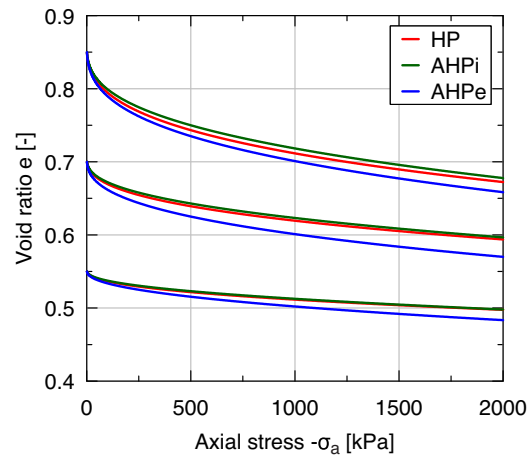
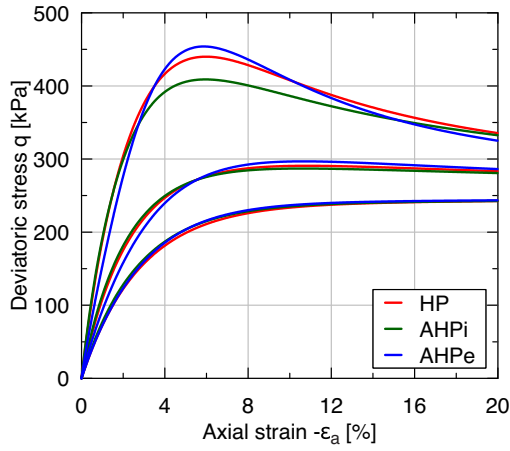


FIGURE 7 | Comparison of the AHPi and the AHPe with the HP based on a monotonic oedometric compression test on ZS: void ratio e as a function of axial stress σ_a .

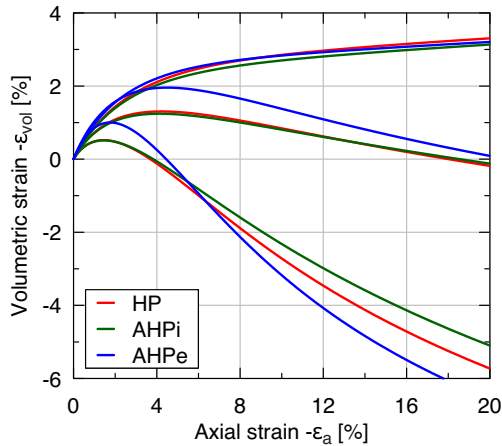
additional parameters (e.g., α and β). However, due to the already close correlation, no such adjustments have been applied within this work.

Figure 8 shows monotonic drained triaxial compression tests with an isotropic initial stress of $p_0 = 100$ kPa and the same initial void ratios. The AHP models yield results comparable to the HP, although the peak strength and dilatancy in the dense sand are slightly underestimated by the AHPi and overestimated using the AHPe compared to the HP. Monotonic undrained triaxial compression tests for the same initial states are shown in Figure 9. Again, nearly identical simulation results are obtained for the AHPi and the HP. However, the AHPe cannot adequately predict the initial density-dependence of the simulations, which can be seen especially in the effective stress path in Figure 13b.

The asymptotic behavior of the three models is investigated by simulating axisymmetric and non-axisymmetric proportional strain paths. Principal stress and principal strain states are



(a)



(b)

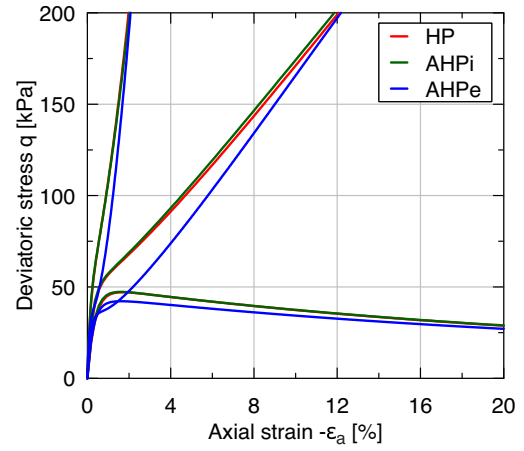
FIGURE 8 | Comparison of the AHPi and the AHPe with the HP based on monotonic drained triaxial compression tests on ZS: (a) $q\varepsilon_a$ -diagram and (b) strain path in the $\varepsilon_{vol}\varepsilon_a$ -diagram.

considered under the assumption of coaxiality, that is, the off-diagonal components of all tensors are zero. Four different proportional strain paths are considered:

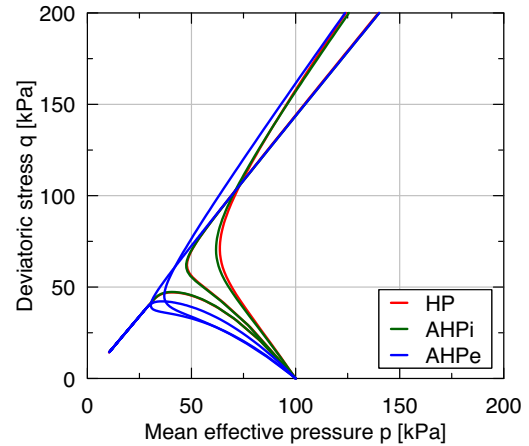
- isotropic compression (iso): $\dot{\varepsilon}_1 = \dot{\varepsilon}_2 = \dot{\varepsilon}_3 < 0$,
- axisymmetric isochoric ($\dot{\varepsilon}_{vol} = 0$) deformation (crit 2D): $\dot{\varepsilon}_1 = -2\dot{\varepsilon}_2 = -2\dot{\varepsilon}_3 < 0$,
- non-axisymmetric isochoric ($\dot{\varepsilon}_{vol} = 0$) deformation (crit 3D): $\dot{\varepsilon}_1 = -3\dot{\varepsilon}_2 = -3\dot{\varepsilon}_3/2 < 0$, and
- non-axisymmetric compressive path (3D): $\dot{\varepsilon}_1 = 3\dot{\varepsilon}_2/2 = 3\dot{\varepsilon}_3 < 0$.

Two different axisymmetric initial stress states with the same void ratio ($e = 0.7$) are considered (ini 1: $\sigma_1 = 2\sigma_2 = 2\sigma_3 = -200$ kPa and ini 2: $\sigma_1 = \sigma_2/2 = \sigma_3/2 = -100$ kPa).

The resulting effective stress paths are shown in the $\sigma_1\sigma_2$ - and $\sigma_1\sigma_3$ -diagrams in Figure 10. Figure 10a and Figure 10b show the results for the HP, Figure 10c and Figure 10d present the results for the AHPi, and Figures 10e and 10f show the results obtained with the AHPe. First, it can be observed that the compressive



(a)



(b)

FIGURE 9 | Comparison of the AHPi and the AHPe with the HP based on monotonic undrained triaxial compression tests on ZS: (a) $q\varepsilon_a$ -diagram and (b) effective stress path in the pq -diagram.

strain paths (iso and 3D) approach and, for practical purposes, reach a specific proportional stress path in all diagrams and for all three models, independently of the initial stress state. The non-axisymmetric compressive strain path (3D) leads to a non-axisymmetric proportional stress path, whereas the stress path induced by the isotropic strain path (iso) is isotropic and therefore also axisymmetric.

The isochoric paths (crit 2D and crit 3D), which are characterized by constant volume, asymptotically approach a critical state, characterized by a vanishing stress rate under continued deformation.⁹ For the axisymmetric isochoric strain path (crit 2D), an axisymmetric stress state is likewise approached asymptotically, whereas the non-axisymmetric isochoric strain path (crit 3D) asymptotically approaches a non-axisymmetric stress state.

The simulations shown in Figure 10 demonstrate that all three constitutive models (HP, AHPi, and AHPe) exhibit convergent asymptotic behavior, indicating the existence of the ASBS. Furthermore, comparison of the simulations for the individual constitutive models shows that the AHPi and AHPe yield very similar asymptotic states to the HP, although their initial response

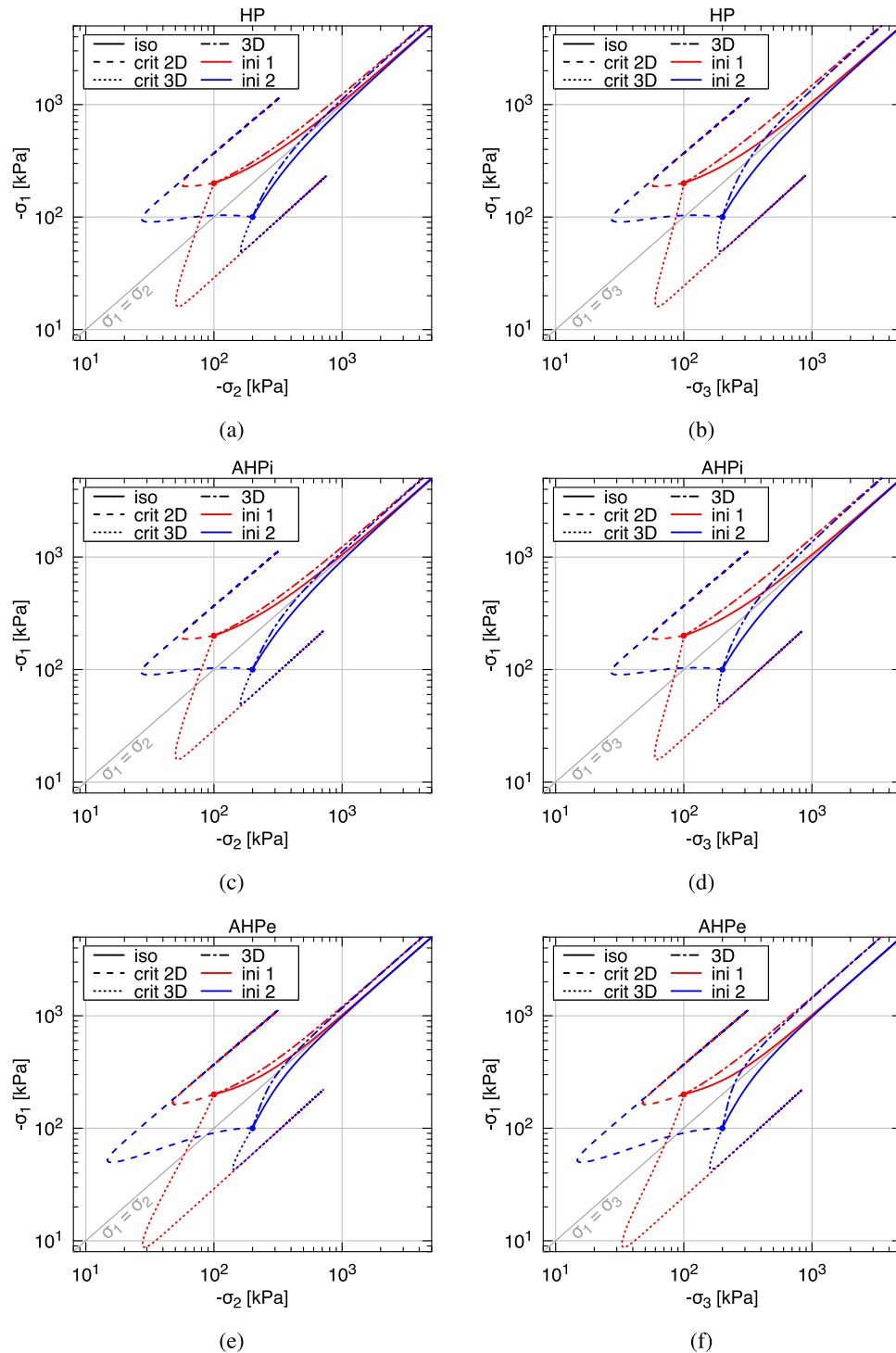


FIGURE 10 | Response due to different proportional strain paths on ZS in the $\sigma_1\sigma_2$ - and $\sigma_1\sigma_3$ -diagrams of (a)+(b) the HP, (c)+(d) the AHPi and (e)+(f) the AHPe for two different initial states.

at the onset of deformation differs slightly from that of the HP. As described previously, these deviations are more pronounced for the AHPe than for the AHPi.

The simulations shown in Figure 10 further demonstrate that the AHPi and AHPe can reproduce the HP also for non-axisymmetric conditions.

So-called response envelopes are polar diagrams in the stress rate space for unit strain rates [2]. Such response envelopes allow a graphical evaluation of the tangential stiffness predicted by a constitutive model of rate type for various strain rate directions and states. Figure 11 illustrates the response envelopes for the three models (HP, AHPi, and AHPe) across six different mean stress states, considering two densities for each. This comparison

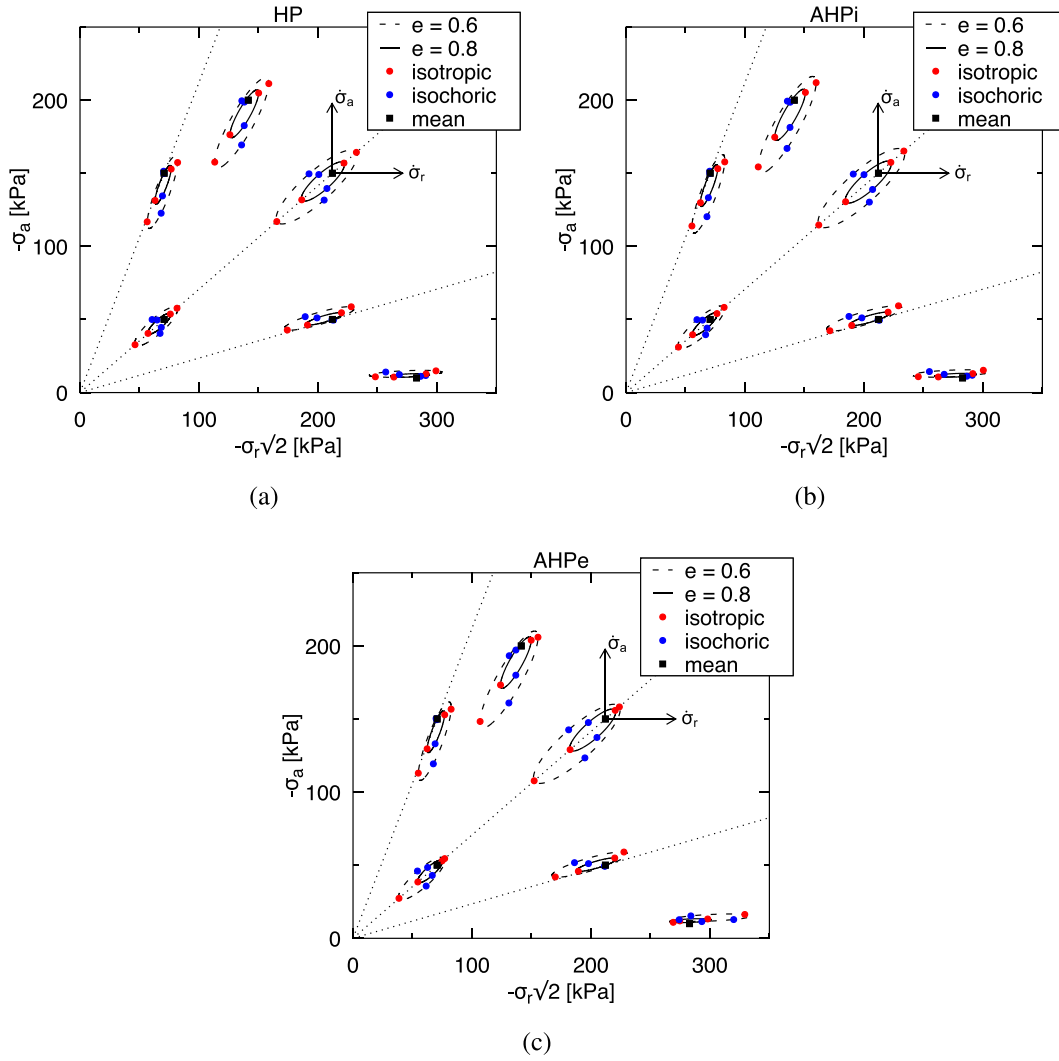


FIGURE 11 | Response envelopes of (a) the HP, (b) the AHPi, and (c) the AHPe for two different void ratios and six different mean stress states.

reaffirms the similarity between the models, although the deviation between the AHPe and HP envelopes is more pronounced than that between AHPi and HP.

Given the overall similarity between the AHPi, the AHPe, and the HP, it is also worth noting that limitations of the HP are in principle to be expected for the AHP formulations as well. These include an underestimation of dilatancy and the possibility of reaching tensile stress states in dense soils, the lack of fabric effects, and an overall insufficiently pronounced nonlinearity [46].

6 | Coupling With the GIS Concept (AHP+GIS)

As in the HP, the AHP formulations are expected to show an overall too soft and insufficiently elastic response upon unloading and reloading, the so-called ratcheting. Under cyclic loading, ratcheting leads to an overestimated accumulation of irreversible strains. To address this issue, Niemunis and Herle [30] introduced for the HP the concept of the intergranular strain (IS) with the state variable $\mathbf{h}$, which renders the model significantly stiffer and more elastic due to a loading direction change. The concept has

been adopted in many other hypoplastic models. However, the original IS concept also shows some deficiencies, primarily the so-called overshooting, an excessive overestimation of shear strength due to unloading and reloading with $\Delta\epsilon \approx R$, and a nearly linear accumulation of plastic strains under cyclic loading. Detailed discussions can be found in [10, 46, 47].

To overcome these deficiencies, Mugele et al. [10] proposed the concept of the generalized intergranular strain (GIS) and coupled it with the HP. This approach likewise uses the state variable $\mathbf{h}$ and, optionally, a state variable of the cyclic preloading Ω to represent nonlinear accumulation effects. The GIS concept is based on the scalar factor

$$k = [\rho^{\chi_R} m_T + (1 - \rho^{\chi_R}) m_R] + \begin{cases} \rho^{\chi_R} (1 - m_T) \vec{h} : \vec{\epsilon} & \text{if } \vec{h} : \vec{\epsilon} > 0 \\ -\rho^{\chi_R} (m_R - m_T) \vec{h} : \vec{\epsilon} & \text{if } \vec{h} : \vec{\epsilon} \leq 0 \end{cases} \quad (42)$$

and the exponent

$$\gamma(\Omega) = \gamma_\chi \chi(\Omega). \quad (43)$$

TABLE 3 | GIS parameters of Zbraslav Sand (ZS) for the coupled AHPi+GIS, AHPe+GIS and the HP+GIS with deactivated cyclic preloading ($\chi_0 = \chi_{\max} = \gamma_\chi = 1$).

	R	m_R	m_T	β_R	χ_R
	[-]	[-]	[-]	[-]	[-]
ZS	10^{-4}	5	2.5	0.1	4.0

Further details are provided in [10]. Representative of coupling a hypoplastic model with an explicitly defined ASBS with the GIS concept, the AHPi and AHPe formulations can be combined with the GIS by a direct modification of Equation 7 respectively Equation 39 (AHPi) or Equation 41 (AHPe):

$$\dot{\sigma} = k f_s L : \dot{\varepsilon} - k \left(\frac{f_d}{f_d^A} \right)^{(k\gamma)} A : \mathbf{d} \|\dot{\varepsilon}\|. \quad (44)$$

As the HP+GIS, the AHPi+GIS and the AHPe+GIS use four state variables: the effective stress σ , the void ratio e , the intergranular strain $\mathbf{h}$, and the cyclic preloading Ω . All constitutive relations of AHP+GIS models are summarized concisely in the Appendix. The influence of Ω can be deactivated, which reduces the number of state variables and, accordingly, the number of parameters. In such cases, the exponent γ would become a constant and would be defined just by the parameter γ_χ . For $\gamma_\chi = 1$, the number of GIS parameters is reduced to the number of parameters in the original IS concept. The cyclic preloading effect is discussed in detail in the appendix as it lies outside the scope of this article, and the version of the models with deactivated influence of Ω and $\gamma_\chi = 1$ is further considered.

As suggested by the previous comparison between the HP and the AHP, the AHPi+GIS and the AHPe+GIS likewise yields simulation results very similar to the HP+GIS. This is illustrated by simulations of drained triaxial tests with an un- and reloading step after $-\varepsilon_a = 10\%$ of $\Delta\varepsilon_a = R$ on medium-dense ZS ($e_0 = 0.7$, $p_0 = 100$ kPa, $\mathbf{h}_0 = -R/\sqrt{3}\delta$) the models with deactivated cyclic preloading (AHP and HP parameters combined with the GIS parameters from Table 3) in Figure 12. For reference, Figure 12 also includes simulations with HP and the AHP formulations ($m_R = m_T = 1$) with an unloading step of $\Delta\varepsilon_a = m_R R$.

It can be seen from the evolution of the deviatoric stress in Figure 12a, that the AHPi+GIS and the AHPe+GIS deliver results that are highly comparable to the results of the HP+GIS. Both prevent ratcheting, which is evident in the simulations with HP and both AHPi and AHPe. Due to the constitutive consideration of the ASBS within the GIS concept via the state mobilization S , no overshooting occurs in either the HP+GIS, the AHPi+GIS, or the AHPe+GIS. It should be noted that overshooting is most pronounced for this unloading increment. The very similar simulation results for all models are also apparent from the strain paths in Figure 12b.

Finally, it is discussed that the extension of the AHPe to the AHPe+GIS can be used to overcome the qualitative shortcoming of the AHPe shown in Figure 9, namely the incorrect modeling of the density dependent mechanical soil behavior at the beginning

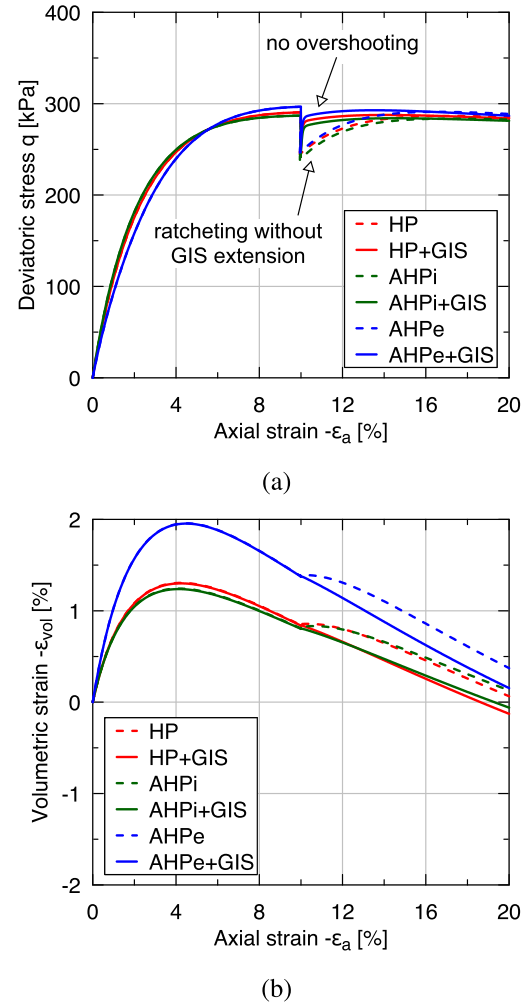


FIGURE 12 | Simulations of drained triaxial tests with an unloading and reloading phase using the AHPi, the AHPi+GIS, the AHPe, the AHPe+GIS with the HP and the HP+GIS for a medium-dense ZS specimen ($p_0 = 100$ kPa, $e_0 = 0.7$, $\mathbf{h}_0 = -R\delta/\sqrt{3}$ and $\Omega_0 = 0$): (a) deviatoric stress q and (b) volumetric strain ε_{vol} as functions of axial strain ε_a .

of the undrained triaxial compression tests. Figure 13 therefore presents the simulations from Figure 9, but now with the coupled models HP+GIS, AHPi+GIS, and AHPe+GIS. The initial conditions correspond to those in Figure 9, supplemented by $\mathbf{h}_0 = \mathbf{0}$. The GIS extension generally increases the elasticity of the models at the beginning of the deformation and reduces the nonlinear component, which results in effective stress paths that are more perpendicular to the p -axis under undrained conditions. This general effect is qualitatively evident in the HP+GIS, AHPi+GIS, and AHPe+GIS formulations. Because the GIS extension reduces the nonlinear component of the models, the GIS can actually compensate for the shortcoming of the AHPe model regarding its lack of density dependence. Quantitatively, however, the AHPe+GIS still shows differences compared to the AHPi+GIS and the HP+GIS.

7 | Summary and Conclusion

In this study, two asymptotic hypoplastic models (AHP) for sand using the A-based mathematical expression are proposed as the

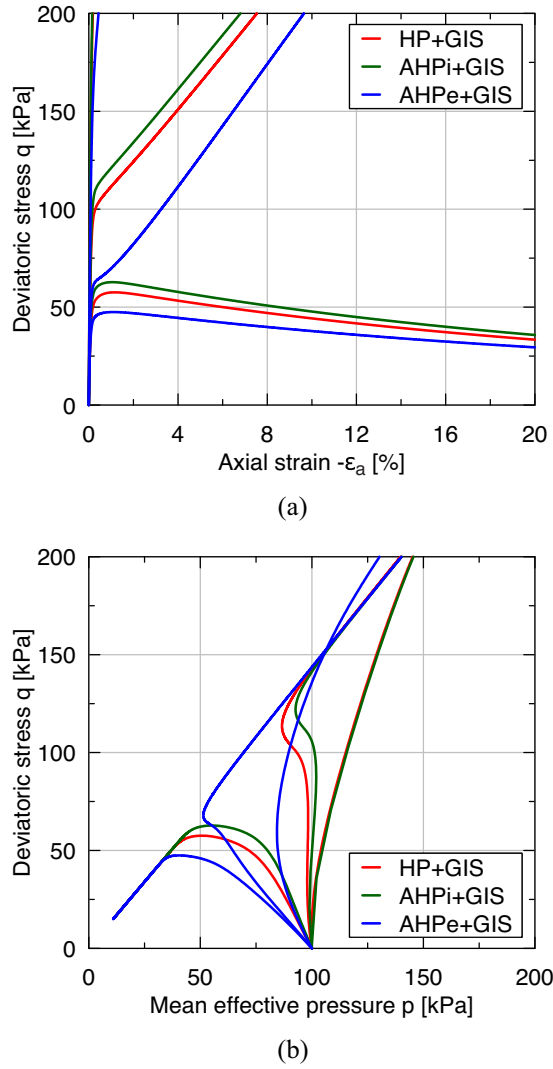


FIGURE 13 | Comparison of the AHPI+GIS and the AHPE+GIS with the HP+GIS based on monotonic undrained triaxial compression tests on ZS: (a) $q\varepsilon_a$ -diagram and (b) effective stress path in the pq -diagram.

first hypoplastic constitutive models for sand in such theoretical framework. The AHPi can indeed be written using a fourth-order tensor $\mathbf{A}$, but the ASBS is still not explicitly defined. The latter is additionally explicitly defined in the AHPE. Both novel models are derived using an adjusted compression relationship and generally closely follow the hypoplastic model for sand proposed by von Wolffersdorff [9] (HP). Especially the AHPi reproduces the overall stress-strain behavior of the reference HP model quite well. However, larger discrepancies occur between AHPE and HP, particularly for dense soils.

The AHP formulations have been further coupled with the concept of the generalized intergranular strain (GIS) [10] to account for cyclic loading without well-known issues like overshooting. The shown AHP+GIS models represent the first hypoplastic constitutive equations for sand that account for the increased stiffness at small strains while being formulated in the tensor $\mathbf{A}$ -based form. This work thus provides a decisive theoretical contribution to the constitutive modeling of sand. It can also serve as a basis for future development of hybrid hypoplastic sand/clay

models, as they both now share the same $\mathbf{A}$ -based structure with the possibility of an explicit definition of asymptotic states.

8 | Notation

The conventional mechanical sign convention is used throughout this paper. Extension strain and tensile stress are positive. Second-order tensors are written in bold letters (e.g., $\boldsymbol{\sigma}$ or $\mathbf{m}$), with $\|\boldsymbol{\sqcup}\|$ denoting the Frobenius norm and $\text{tr}(\boldsymbol{\sqcup})$ representing the sum of diagonal components of the corresponding second-order tensor $\boldsymbol{\sqcup}$. The deviatoric part is denoted with $\boldsymbol{\sqcup}^*$. Normalized tensors are denoted as $\hat{\boldsymbol{\sqcup}} = \boldsymbol{\sqcup}/\|\boldsymbol{\sqcup}\|$. Proportional quantities are denoted

TABLE 4 | List of symbols.

$\boldsymbol{\sigma}$	effective stress tensor (tension positive)
e	void ratio
$\mathbf{h}$	intergranular strain tensor
Ω	cyclic preloading variable
$\boldsymbol{\alpha}$	additional state variables
$\boldsymbol{\varepsilon}$	strain tensor (compression negative)
$\varepsilon_{\text{vol}} = \text{tr}(\boldsymbol{\varepsilon})$	volumetric strain
$\dot{\boldsymbol{\sigma}}$	effective stress rate
$\dot{\boldsymbol{\varepsilon}}$	strain rate
$p = -\text{tr}(\boldsymbol{\sigma})/3$	Roscoe mean effective pressure
$q = \ \boldsymbol{\sigma}^*\ \sqrt{3/2} \stackrel{\text{triax}}{=} \sigma_r - \sigma_a$	Roscoe deviatoric stress
$\dot{\varepsilon}_{\text{vol}} = \text{tr}(\dot{\boldsymbol{\varepsilon}})$	volumetric strain rate
$I_D = (e_{c0} - e)/(e_{c0} - e_{d0})$	relative density
$\boldsymbol{\delta}$	unit tensor of second-order (Kronecker symbol)
$\mathbf{I}$	symmetrizing unit tensor of fourth-order
$\mathbf{C}^{\text{el}}$	elastic stiffness tensor
$\dot{\boldsymbol{\varepsilon}}^{\text{pl}}$	plastic strain rate
$\mathbf{L}, \mathbf{A}$	fourth-order stiffness tensors in hypoplastic models
$\mathbf{N}$	second-order tensor
$\mathbf{d}$	normalized asymptotic strain rate
f_s	barotropy factor
f_d	pyknotropy factor
S	state mobilization
k	scalar factor
$\boldsymbol{\sqcup}^A$	quantity corresponding to an asymptotic state
$\boldsymbol{\sqcup}^i$	implicitly defined quantity
$\boldsymbol{\sqcup}^e$	explicitly defined quantity

by $\sqcup||\sqcup$. Fourth-order tensors are symbolized using capital sans-serif letters (e.g., $\mathbf{A}$). The symbol $\cdot$ denotes multiplication with one dummy index (single contraction). The multiplication with two dummy indices (double contraction) is written using a colon, for example, $\mathbf{a} : \mathbf{b}$. Dyadic multiplication is written as $\mathbf{ab}$. A list of symbols is provided in Table 4.

Author Contributions

Luis Mugele: conceptualization, methodology, software, validation, formal analysis, investigation, writing – original draft, visualization, funding acquisition. **David Mašin:** conceptualization, methodology, validation, resources, writing – review & editing, supervision, project administration, funding acquisition. **Hans Henning Stutz:** conceptualization, methodology, resources, writing – review & editing, supervision, project administration, funding acquisition.

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Conflicts of Interest

The authors declare no conflicts.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Endnotes

¹The concept of asymptotic states is also applicable to a monotonic volume increasing deformation with $\text{tr}(\dot{\epsilon}) = \dot{\epsilon}_{\text{vol}} > 0$. These can not be reached from stress-free state, but are equally important. Asymptotically, the stress-free state $\sigma = \mathbf{0}$ with $\dot{\sigma} = \mathbf{0}$ is approached in this case [4].

²Osinov [19] showed that Equation 4 should in fact be written using the material time derivative $d\sqcup/dt = \dot{\sqcup}$ of the effective stress. In the literature, however, when discussing the critical state or ideal plastic flow (failure) in hypoplasticity, an objective stress rate is often used [20–25]. In a typical scenario with $\|\mathbf{W}\| \ll \|\mathbf{D}\|$ they deviate only slightly and for a spin tensor $\mathbf{W} = \mathbf{0}$ both formulations are equivalent.

³It will be shown later that f_s differs between the HP and AHP formulations.

⁴Alternatively, the asymptotic pyknorotropy factor f_d^A can be derived similarly to [5, 10] as

$$f_d^A = \sqrt{\left[\|\mathbf{B}\|^2 + \left(\frac{\|\mathbf{C}\| \text{tr} \mathbf{B}}{G - \text{tr} \mathbf{C}} \right)^2 + \frac{2(\mathbf{B} : \mathbf{C}) \text{tr} \mathbf{B}}{G - \text{tr} \mathbf{C}} \right]^{-1}}$$

with

$$\begin{aligned} \mathbf{B} &= \mathbf{L}^{-1} : \mathbf{N}^{\text{HP}}, \\ \mathbf{C} &= 1/f_s^A (\mathbf{L}^{-1} : \dot{\sigma}), \\ G &= \frac{n^*}{h_s^*} \text{tr}(\dot{\sigma}) \left(\frac{3p}{h_s^*} \right)^{(n^*-1)} = -\frac{\lambda^*}{\|\dot{\sigma}\|} \end{aligned}$$

which is equivalent to Equation 33.

⁵Note that the missing factor $(1 + e_i)/e_i$ in Equation 37 compared to the barotropy factor of the HP is canceled due to the novel compression relation.

⁶The AHPi model is presented in [44], where it is referred to as the MHP model. The quantity f_d^{Ai} is thereby called f_d^{A*} .

⁷ $\sin^2(\varphi_{\text{mob}}) = \frac{9I_3 + I_1 I_2}{I_3 + I_1 I_2}$ with $I_1 = \text{tr}(\sigma)$, $I_2 = \frac{1}{2}[\sigma : \sigma - (I_1)^2]$, and $I_3 = \det(\sigma)$

⁸The implementation is available from the first author upon reasonable request.

⁹The presented simulations for the crit 2D and crit 3D paths are continued until the stress rate has effectively vanished, corresponding to an axial strain of up to $\epsilon_a = -200\%$.

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Appendix A: On the Cyclic Preloading in the GIS Concept

An improvement of the original IS concept by Niemunis and Herle [30], which is incorporated into the GIS concept by Mugele et al. [10], is the constitutive consideration of cyclic preloading effects (i.e., the number and intensity of previous cycles) for accumulation calculations. The cyclic preloading concept considered in the GIS formulation is based on earlier works [32, 33, 48].

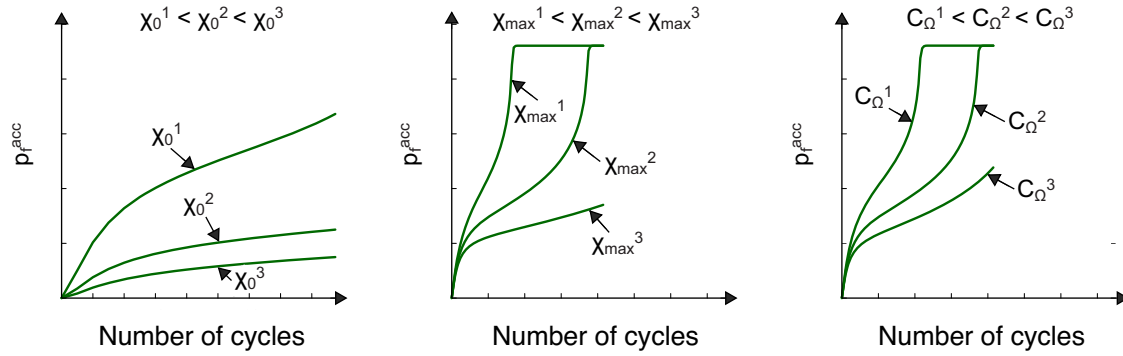


FIGURE A1 | Schematic representation of the calibration of the GIS parameters χ_0 , χ_{max} , and C_Ω based on the accumulation of pore water pressure p_f in a cyclic undrained triaxial test with a specified deviatoric stress amplitude, modified from Duque et al. [33].

TABLE A1 | GIS parameters of Zbraslav Sand (ZS) for the coupled HP+GIS formulation including the effect of the cyclic preloading variable Ω from Mugele et al. [10]

	GIS (related to h)					GIS (related to Ω)				
	R	m_R	m_T	β_R	χ_R	γ_χ	χ_0	χ_{max}	C_Ω	γ_Ω
	[-]	[-]	[-]	[-]	[-]	[-]	[-]	[-]	[-]	[-]
ZS	10^{-4}	5	2.5	0.1	4.0	1.7	0.8	1.5	45	1.0

The following remark regarding the cyclic preloading with the state variable Ω and the corresponding parameter calibration therefore applies analogously to these models from the literature, as well as to the AHPi+GIS and AHPE+GIS. In this appendix, however, the HP+GIS from Mugele et al. [10] is considered, which was also used for comparison in this paper.

Table A1 shows the GIS parameters for ZS from [10]. The corresponding HP parameters can be found in Table 2.

References for the calibration of the five GIS parameters related to Ω can be found in [33]. For calibration, it is recommended to use the accumulation of pore water pressure in a cyclic undrained triaxial test with a constant deviatoric stress amplitude q^{ampl} and an isotropic effective initial stress state. No cyclic preloading should be imposed in the test, and the cyclic deformation should occur immediately after the isotropic compression. Consequently, an initialization of $\Omega_0 = 0$ is justified. The initialization and corresponding parameters of the constitutive equations for all other state variables of the material model under consideration (σ , e , h , ...) should already have been calibrated.

First, the parameter γ_χ must be chosen such that the simulation without taking Ω into account ($\chi_0 = \chi_{max} = 1$) shows a more pronounced accumulation of pore water pressure than the experiment. Based on this, the remaining four GIS parameters can be calibrated.

The influence of χ_0 , χ_{max} , and C_Ω on pore water pressure accumulation as a function of the number of cycles in an undrained cyclic triaxial test with given stress cycles is shown in Figure A1. The parameter χ_0 controls the accumulation rate during the first few cycles. The larger χ_0 is, the slower the buildup of pore water pressure during the first cycles. It should be noted that in basic models that account for the influence of an anisotropic microstructure (fabric), an increased buildup of pore water pressure may also occur during the first cycles as a result of the correspondingly initialized state variables of the fabric. This fact must be considered when calibrating the GIS parameters, and primarily when calibrating χ_0 . The parameter χ_{max} controls the accumulation rate

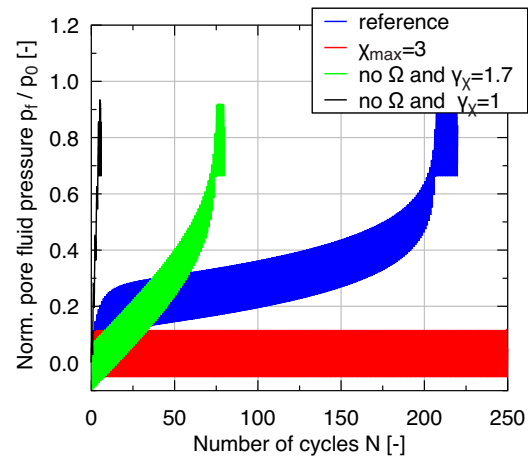


FIGURE A2 | Evolution of the normalized pore fluid pressure p_f / p_0 in undrained cyclic triaxial tests simulated using the HP+GIS and different options concerning the cyclic preloading variable Ω .

after a large number of cycles with small strain amplitudes. The larger χ_{max} is, the smaller the corresponding accumulation effects are. The transition from faster accumulation at the beginning of the test to slower accumulation after a larger number of cycles is controlled by C_Ω . The larger C_Ω is, the faster the accumulation rate decreases.

The parameter γ_Ω has a similar effect. The rapid accumulation of pore water pressure toward the end of the test at a given stress amplitude results from the basic model, provided that the larger strain amplitude resulting from the decreasing mean effective pressure due to barotropy is accurately represented by the latter.

Using the parameters for ZS (Table A1 and Table 2), it is demonstrated that the calibration of the constitutive model parameters for considering Ω must be carried out carefully. For this purpose, a cyclic undrained triaxial test with a prescribed deviatoric stress amplitude is analyzed using the HP+GIS with the GIS parameters from Table A1 and a stress amplitude $q^{ampl} = 50$ kPa. An isotropic stress state $p_0 = 200$ kPa, a void ratio $e_0 = 0.6915$ ($I_{D0} = 0.5$), an intergranular strain $h_0 = 0$, and $\Omega_0 = 0$ are initialized. The resulting normalized pore fluid pressure as a function of the applied cycles is shown in Figure A2.

With the parameters from Table A1, the desired S-shaped accumulation curve of the pore fluid pressure is obtained. However, if the parameter χ_{max} is chosen too large (e.g., $\chi_{max} = 3$), the accumulation is practically deactivated, which is unrealistic. Depending on the base model, the

direction of accumulation can generally even reverse if the GIS parameters are insufficiently calibrated.

The influence of Ω can be practically deactivated, for instance, by setting $\chi_0 = \chi_{\max} = 1$. In this case, the exponent γ is described by the parameter γ_χ . The larger the value of γ_χ , the lower the general accumulation rate. Therefore, two variants with the influence of Ω deactivated are considered in Figure A2. In the first one, $\gamma_\chi = 1.7$ remains unchanged and in the second variant, $\gamma_\chi = 1$ is set. The latter variant reduces the GIS parameters to those of the original IS concept, whereby even in this form of the GIS concept, overshooting is constitutively prevented.

Appendix B: Equations of Constitutive Models AHPi+GIS and AHPe+GIS

B.1 | General Equations

$$\begin{aligned} \mathbf{L} &= \frac{1}{\dot{\boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}}} [(F^{\text{HP}})^2 \mathbf{I} + a^2 \dot{\boldsymbol{\sigma}} \dot{\boldsymbol{\sigma}}] \\ \mathbf{N}^{\text{HP}} &= \frac{F^{\text{HP}}}{\dot{\boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}}} a (\dot{\boldsymbol{\sigma}} + \dot{\boldsymbol{\sigma}}^*) \\ f_d &= \left(\frac{e - e_d}{e_c - e_d} \right)^\alpha \\ \dot{\boldsymbol{\sigma}} &= \frac{\boldsymbol{\sigma}}{\text{tr } \boldsymbol{\sigma}} \quad \dot{\boldsymbol{\sigma}}^* = \dot{\boldsymbol{\sigma}} - \frac{1}{3} \delta \\ a &= \sqrt{\frac{3}{8}} \frac{(3 - \sin \varphi_c)}{\sin \varphi_c} \\ F^{\text{HP}} &= \sqrt{\frac{1}{8} \tan^2 \psi + \frac{2 - \tan^2 \psi}{2 + \sqrt{2} \tan \psi \cos(3\theta)}} - \frac{1}{2\sqrt{2}} \tan \psi \\ \cos(3\theta) &= -\sqrt{6} \frac{\text{tr}(\dot{\boldsymbol{\sigma}}^* \cdot \dot{\boldsymbol{\sigma}}^* \cdot \dot{\boldsymbol{\sigma}}^*)}{[\dot{\boldsymbol{\sigma}}^* : \dot{\boldsymbol{\sigma}}^*]^{3/2}} \\ \tan \psi &= \sqrt{3} \|\dot{\boldsymbol{\sigma}}^*\| \\ \lambda^* &= (-\text{tr}(\boldsymbol{\sigma}))^{n^*} h_s^{*(-n^*)} n^* = n^* \left(\frac{3p}{h_s^*} \right)^{n^*} \\ \dot{\mathbf{h}} &= \begin{cases} (1 - \vec{\mathbf{h}} \vec{\mathbf{h}} \rho^{\beta_R}) : \dot{\boldsymbol{\varepsilon}} & \text{if } \vec{\mathbf{h}} : \dot{\boldsymbol{\varepsilon}} > 0 \\ \dot{\boldsymbol{\varepsilon}} & \text{if } \vec{\mathbf{h}} : \dot{\boldsymbol{\varepsilon}} \leq 0 \end{cases} \\ \vec{\mathbf{h}} &= \begin{cases} \frac{\mathbf{h}}{\|\mathbf{h}\|} & \text{if } \mathbf{h} \neq \mathbf{0} \\ \mathbf{0} & \text{if } \mathbf{h} = \mathbf{0} \end{cases} \quad \rho = \frac{\|\mathbf{h}\|}{R} \\ k &= [\rho^{\chi_R} m_T + (1 - \rho^{\chi_R}) m_R] + \\ &\quad \begin{cases} \rho^{\chi_R} (1 - m_T) \vec{\mathbf{h}} : \vec{\boldsymbol{\varepsilon}} & \text{if } \vec{\mathbf{h}} : \dot{\boldsymbol{\varepsilon}} > 0 \\ -\rho^{\chi_R} (m_R - m_T) \vec{\mathbf{h}} : \vec{\boldsymbol{\varepsilon}} & \text{if } \vec{\mathbf{h}} : \dot{\boldsymbol{\varepsilon}} \leq 0 \end{cases} \\ \gamma &= \gamma_\chi \chi \quad \chi = \chi_0 + \Omega(\chi_{\max} - \chi_0) \end{aligned}$$

$$\dot{\Omega} = C_\Omega (1 - \rho^{\gamma_\Omega} - \Omega) \|\dot{\boldsymbol{\varepsilon}}\|$$

$$\begin{aligned} e_i &= (e_{i0} + 1) \exp \left[- \left(\frac{3p}{h_s^*} \right)^{n^*} \right] - 1 \\ e_d &= (e_{d0} + 1) \exp \left[- \left(\frac{3p}{h_s^*} \right)^{n^*} \right] - 1 \\ e_c &= (e_{c0} + 1) \exp \left[- \left(\frac{3p}{h_s^*} \right)^{n^*} \right] - 1 \end{aligned}$$

$$\dot{\boldsymbol{\varepsilon}} = (1 + e) \text{tr}(\dot{\boldsymbol{\varepsilon}})$$

$$X = \left[3 + a^2 - a \sqrt{3} \left(\frac{e_{i0} - e_{d0}}{e_{c0} - e_{d0}} \right)^\alpha \right]$$

B.2 | AHPi+GIS

$$\begin{aligned} \dot{\boldsymbol{\sigma}} &= k f_s^{Ai} \mathbf{L} : \dot{\boldsymbol{\varepsilon}} - k \left(\frac{f_d}{f_s^{Ai}} \right)^{(k\gamma)} \mathbf{A}^i : \mathbf{d}^i \|\dot{\boldsymbol{\varepsilon}}\| \\ f_s^{Ai} &= \frac{h_s^*}{n^*} \left(\frac{e_i}{e} \right)^\beta \left(\frac{-\text{tr } \boldsymbol{\sigma}}{h_s^*} \right)^{(1-n^*)} X^{-1} \\ \mathbf{A}^i &= f_s^{Ai} \mathbf{L} + \frac{1}{\lambda^*} \boldsymbol{\sigma} \mathbf{1} \\ \mathbf{d}^i &= - \frac{(\mathbf{A}^i)^{-1} : \mathbf{N}^{\text{HP}}}{\|(\mathbf{A}^i)^{-1} : \mathbf{N}^{\text{HP}}\|} \\ f_d^{Ai} &= \|(\mathbf{A}^i)^{-1} : (f_s^{Ai} \mathbf{N}^{\text{HP}})\|^{-1} \end{aligned}$$

B.3 | AHPe+GIS

$$\begin{aligned} \dot{\boldsymbol{\sigma}} &= k f_s^{Ai} \mathbf{L} : \dot{\boldsymbol{\varepsilon}} - k \left(\frac{f_d}{f_s^{Ae}} \right)^{(k\gamma)} \mathbf{A}^i : \mathbf{d}^e \|\dot{\boldsymbol{\varepsilon}}\| \\ f_s^{Ai} &= \frac{h_s^*}{n^*} \left(\frac{e_i}{e} \right)^\beta \left(\frac{-\text{tr } \boldsymbol{\sigma}}{h_s^*} \right)^{(1-n^*)} X^{-1} \\ f_s^{Ae} &= \frac{h_s^*}{n^*} \left(\frac{e_i}{e_c} \right)^\beta \left(\frac{\sin \varphi_{\text{mob}}}{\sin \varphi_c} \right)^2 \left(\frac{-\text{tr } \boldsymbol{\sigma}}{h_s^*} \right)^{1-n^*} X^{-1} \\ \mathbf{A}^i &= f_s^{Ai} \mathbf{L} + \frac{1}{\lambda^*} \boldsymbol{\sigma} \mathbf{1} \\ \mathbf{A}^e &= f_s^{Ae} \mathbf{L} + \frac{1}{\lambda^*} \boldsymbol{\sigma} \mathbf{1} \\ \mathbf{d}^e &= - \frac{(\mathbf{A}^e)^{-1} : \mathbf{N}^{\text{HP}}}{\|(\mathbf{A}^e)^{-1} : \mathbf{N}^{\text{HP}}\|} \\ f_d^{Ae} &= \|(\mathbf{A}^e)^{-1} : (f_s^{Ae} \mathbf{N}^{\text{HP}})\|^{-1} \end{aligned}$$