



Research paper

Interpretation of soliton energy ratios in directional wave fields

Yu-Chen Lee ^{a,b} , Sander Wahls ^c ^a Delft Center for Systems and Control, Mechanical Engineering, Delft University of Technology, Mekelweg 5, Delft, 2628 CD, South Holland, The Netherlands^b Coastal Ocean Monitoring Center, National Cheng Kung University, University Road No. 1, Tainan, 701401, Taiwan^c Karlsruhe Institute of Technology, Institute of Industrial Information Technology, Hertzstraße 16, Karlsruhe, 76187, Baden-Württemberg, Germany

ARTICLE INFO

Keywords:

Soliton
Soliton energy ratio
Nonlinear Fourier transform
Inverse scattering method
Nonlinear Schrödinger equation
Directional spreading

ABSTRACT

Highly nonlinear soliton-dominated sea states in deep water have been explored in previous studies through nonlinear theory and physical experiments. Deep water solitons are known to retain their characteristic properties during evolution within one-dimensional wave fields that are described by the nonlinear Schrödinger (NLS) equation, which is the most important model for their study. However, realistic ocean wave fields are typically multidirectional, a condition that can destabilize solitons and challenge the direct application of NLS-based interpretations. To address this gap, we generate directional JONSWAP wave fields with different directional spreadings that can be directly compared to field measurements to discuss the directional effects on soliton-dominated seas. The soliton energy ratio derived from the nonlinear Fourier spectrum is used as an indicator to assess dominant components of directional sea states. We investigate the relation between the soliton energy ratio obtained from the original time series, where the full directionality is present, to that obtained from the time series with reduced directionality using a directional filtering approach. For small directional spreading without filtering, the soliton energy ratio in the main direction is only slightly lower compared to initial wave fields, as expected. With increasing directional spreading, the soliton energy ratio decreases more significantly in most cases. However, in highly nonlinear soliton-dominated directional sea states, soliton energy is found to be concentrated in the main propagation direction, as the soliton ratio remains large after the directional interference has been removed.

1. Introduction

When an ensemble of solitons dominates the dynamics of a nonlinear dispersive system, the resulting statistical state is referred to as a soliton gas, which has gained popularity in theoretical studies (Zakharov, 1971; El and Kamchatnov, 2005; El et al., 2011; Suret et al., 2024). Solitons were first mathematically described by the Korteweg–de Vries (KdV) equation (Zabusky and and, 1965), which models the evolution of weakly nonlinear and dispersive unidirectional free surface waves in shallow water with a constant depth. For deep water waves, solitons are envelope solitons, which are obtained by the nonlinear Schrödinger (NLS) equation (Zakharov, 1968; Yuen and Lake, 1982). Envelope solitons are regarded as a fundamental mechanism for unidirectional rogue wave formation in deep water (Slunyaev, 2006). The NLS equation is a fundamental model for describing the propagation of weakly nonlinear, narrow-banded wave envelopes in deep water with a constant depth in one dimension. The NLS equation can be analytically solved using the Nonlinear Fourier Transform (NFT), also known as the direct scattering transform in the inverse scattering method, which

is a powerful technique developed for integrable nonlinear systems (Shabat and Zakharov, 1972; Ablowitz and Segur, 1981). Analogously to the linear Fourier transform, the NFT decomposes complex nonlinear waveforms into simpler components. This decomposition is based on the eigenvalues and eigenfunctions of an associated linear scattering problem, which together define the nonlinear Fourier spectrum.

The nonlinear Fourier spectrum comprises two parts: a continuous spectrum, which corresponds to dispersive radiation waves, and a discrete spectrum, which represents the number and characteristics of solitons. In optical communications, the NFT has shown that supposedly linear signals in nonlinear fiber channels can actually contain solitonic components (Wahls et al., 2015, Fig. 8), (Sedov et al., 2018; Turitsyn et al., 2019). To quantify this solitonic content, we adapt a parameter known as the soliton energy ratio, which measures how closely a given sea state resembles a soliton gas (Ablowitz and Segur, 1981; Lee and M. Brühl, 2023). It can be calculated directly from the nonlinear Fourier spectrum as the ratio of soliton energy to total energy, ranging from zero representing a purely radiation wave dominated state to one

* Corresponding author at: Delft Center for Systems and Control, Mechanical Engineering, Delft University of Technology, Mekelweg 5, Delft, 2628 CD, South Holland, The Netherlands.

E-mail address: y.lee.delft@gmail.com (Y.-C. Lee).

<https://doi.org/10.1016/j.apor.2026.105304>

Received 13 August 2025; Received in revised form 9 October 2026; Accepted 11 October 2026

Available online 19 October 2026

0141-1187/© 2026 The Authors. Published by Elsevier Ltd. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

indicating a system fully dominated by solitons. Unfortunately, it can be difficult to interpret the soliton energy ratio in practice when dealing with more complex sea states, particularly those with directionality. The one-dimensional NLS equation assumes unidirectional wave propagation, so directional spreading violates the assumptions of the NLS model, causing the nonlinearity to become defocusing in the lateral direction (Onorato et al., 2009). As a result, solitons become transversely unstable when evolving in multidimensional wave fields. Other factors such as wind forcing and wave breaking can also contribute to unstable solitons (Rapti et al., 2004; Carter and Contreras, 2008; Onorato and Proment, 2012; Brunetti et al., 2014; Alberello and Părau, 2022), but these effects are not studied here.

Highly nonlinear sea states dominated by NLS solitons have been successfully generated in controlled wave flume experiments (Suret et al., 2020; Fache et al., 2024). Under ideal experimental conditions, the collective behavior of soliton ensembles can be characterized statistically. However, the idealized conditions of laboratory experiments cannot be directly compared to the real ocean, primarily due to the presence of directional spreading. The NFT has been used to detect solitons in field measurements of sea states, including those with small directional spreading, such as in Currituck Sound, North Carolina (Osborne et al., 2019), and those with larger directional spreading, such as in the Northeast Pacific Ocean (Lee et al., 2024) and the Northeast Atlantic Ocean (Onorato et al., 2021). It remains uncertain how to properly explain the detected solitons under directional seas, given that the NLS equation assumes unidirectional wave propagation. It is known that solitons usually do not remain stable and reduce their amplitudes during evolution in the presence of strong directional effects. A recent study however has observed that solitons can persist under directional wave fields (Slunyaev, 2024). It is found that under moderate directional sea states in deep water, solitons can persist continuously for a few tens of wave periods. Under conditions with minimal directional spreading, these solitons can maintain an even longer lifetime.

The gap between idealized theory and real oceanic conditions motivates the present study to further investigate the interpretation of soliton energy ratios calculated from the nonlinear spectrum in directional seas. To better align theoretical assumptions with realistic sea conditions, we focus on waves propagating in the primary direction, thereby constraining directional effects to more closely approximate the validity of the NLS regime. By removing energy outside specific directional ranges, we generate modified wave fields that propagate within a narrower range of directions. These modified wave fields with different spreading scenarios are compared with the original sea states to examine how the soliton energy ratio changes between the initial condition and that in the main direction only. The directional spreading settings in our simulations are designed to be directly comparable to those observed in realistic oceanic sea states.

This paper is organized as follows. In Section 2, we review the nonlinear Fourier transform based on the NLS equation. Section 3 describes how we generate wave fields from directional spectra. The extracted time series are processed with the nonlinear Fourier transform to calculate soliton energy ratios. The statistical relationships between the soliton energy ratio and fundamental wave parameters are presented in Section 4. The primary findings of this study, examining how truncated directionality influences soliton energy ratio, are detailed in Section 5. Finally, we present our conclusions in Section 6.

2. Methods

2.1. Nonlinear Schrödinger equation for deep water waves

The nonlinear dynamics of deep water waves can be described by the nonlinear Schrödinger (NLS) equation (Zakharov, 1968; Yuen and Lake, 1982), where deep water waves are represented by the complex wave envelope $A(x, t) = \eta(x, t) \exp(i(k_0 x - \omega_0 t))$. Here, $\eta(x, t)$ is the surface elevation, ω_0 the carrier frequency, k_0 and the wave number.

This expression represents a decomposition of the wave field into a slowly varying envelope and a rapidly oscillating carrier wave. The focusing 1D NLS equation is considered valid for $k_0 h \geq 1.363$, where h is the water depth. When considering one-dimensional spatial deep water wave envelopes propagating in time t , the 1D spatial NLS equation is given by

$$i [A_t + C_g A_x] + \mu A_{xx} + \nu |A|^2 A = 0, \quad (1)$$

where $C_g = \omega_0 / 2k_0$ is the group velocity, $\mu = -\omega_0 / 8k_0^2$ is the coefficient of the second-order dispersion term and $\nu = -\omega_0 k_0^3 / 2$ is the coefficient of the cubic nonlinear term. The subscripts x and t indicate partial derivatives with respect to these variables. The coefficients are given here for the case of infinite water depth. These coefficients can be revised when considering finite water depth (Hasimoto and Ono, 1972; Alber, 1978).

Field measurements of surface waves are typically single-point time series measurements, such as those obtained from wave height gauges and buoys. As a result, a temporal NLS equation describing the spatial evolution is required. The transformation from the spatial NLS equation into the temporal NLS equation can be followed in Refs. Karpman (2016), Wahls et al. (2020). By applying the spatial-temporal transformation, the temporal NLS equation in normalized form can be expressed as

$$iu_X + u_{TT} + 2|u|^2 u = 0. \quad (2)$$

where $u(X, T) = \rho A(X, T)$ is the scaled envelope, $\rho = \sqrt{C_g^2 \nu / 2\mu}$ is a nonlinear parameter that converts the physical wave field into the nonlinear domain Osborne (2010), $X = \mu / C_g^3 x$ is the scaled space and $T = t - x / C_g$ is the scaled time. We note that these transformations lead to the unexpected result that the unit of u is s^{-3} . This is no problem in our case, since their sole purpose is to bring the data into an intermediate computational domain where it can be processed more easily. Nevertheless, in cases where the normalized data is of direct interest, slightly extended transformations that make u dimensionless are available (Chabchoub and Grimshaw, Eq. 13). Formally, the NLS equation describes the evolution of unidirectional deep-water waves over a flat bottom, capturing the effects of both dispersion and nonlinearity. The main assumptions in its derivation beyond unidirectionality are narrow bandwidth and weak nonlinearity.

2.2. Nonlinear Fourier transform for vanishing boundary conditions

The Nonlinear Fourier Transform (NFT) is a useful tool for solving the NLS equation by taking advantage of its integrability (Shabat and Zakharov, 1972; Ablowitz and Segur, 1981). It is based on the Zakharov-Shabat system, a Lax pair formulation linked to the NLS equation (Lax, 1968). The direct spatial or temporal evolution of complex envelopes governed by the NLS equation can be complicated, but the evolution of nonlinear spectral modes found by the NFT is much simpler (Aktosun, 2022). As a result, the NLS equation can be interpreted as governing the evolution of nonlinear spectral modes that propagate effectively in a linear way. Eventually, one can apply the inverse nonlinear Fourier transform (iNFT) to the nonlinear spectrum to obtain the solution after propagation (Turitsyn et al., 2017).

In this study, we adopt vanishing boundary conditions to simplify NFT computations (Shabat and Zakharov, 1972; Turitsyn et al., 2017). The Zakharov-Shabat system underlying this NFT is

$$\Psi_T = Q(\zeta) \Psi, \quad Q = \begin{pmatrix} -i\zeta & u(X_0, T) \\ -u(X_0, T)^* & i\zeta \end{pmatrix}, \quad \Psi = \Psi(T; \zeta) \quad (3)$$

where ζ is the complex spectral parameter, and $*$ represents complex conjugation. Two pairs of linearly independent Jost functions $\Psi =$

$[\phi \ \bar{\phi}]$ and $\Psi = [\psi \ \bar{\psi}]$ are found by solving Eq. (3). The Jost functions are defined by

$$\begin{aligned}\phi(T; \zeta) &= \begin{bmatrix} 0 \\ e^{-i\zeta T} \end{bmatrix} + o(1) \quad \text{as } T \rightarrow -\infty, \\ \psi(T; \zeta) &= \begin{bmatrix} e^{i\zeta T} \\ 0 \end{bmatrix} + o(1) \quad \text{as } T \rightarrow \infty,\end{aligned}\quad (4)$$

The above pairs of functions are related through the scattering coefficients $a(\zeta)$, $b(\zeta)$, $\bar{a}(\zeta)$, $\bar{b}(\zeta)$

$$\begin{aligned}\Phi(T; \zeta) &= a(\zeta)\psi(T; \zeta) + b(\zeta)\bar{\psi}(T; \zeta), \\ \bar{\Phi}(T; \zeta) &= \bar{b}(\zeta)\psi(T; \zeta) + \bar{a}(\zeta)\bar{\psi}(T; \zeta).\end{aligned}\quad (5)$$

The scattering coefficients $a(\zeta)$ and $b(\zeta)$ form the basis for constructing the nonlinear spectrum and are sufficient to describe the signal completely. The nonlinear spectrum consists of two parts: the continuous spectrum and the discrete spectrum. The continuous spectrum is formed by the *reflection coefficient*

$$r(\zeta) = b(\zeta)/a(\zeta), \quad \zeta \in \mathbb{R}. \quad (6)$$

The discrete spectrum contains the discrete eigenvalues $\{\zeta_k = \xi_k + i\eta_k\}$, $k = 1, \dots, N$, where N is the total number of eigenvalues. They are the zeros of the scattering coefficient $a(\zeta)$ with positive imaginary part. The eigenvalues are complemented by the residues (Ablowitz et al., 1974; Chimmalgi et al., 2019)

$$r_k = b(\zeta_k)/a'(\zeta_k), \quad \text{where } a'(\zeta) = \partial a(\zeta)/\partial \zeta \quad (7)$$

The continuous spectrum represents the radiation components in the time series. These dispersive waves are an integral part of the overall solution, but do not maintain a localized shape. The discrete spectrum consists of a set of isolated eigenvalues, each associated with a soliton. Solitons are stable, localized wave structures that not only retain their shape and speed over time in the absence of interactions, but also recover their initial shape after interactions have taken place. Note that a soliton in the NLS equation is always an envelope soliton (Yuen and Lake, 1982; Osborne, 2010; Slunyaev et al., 2013; Chabchoub et al., 2016; Slunyaev, 2018; Xia et al., 2021). For small-amplitude envelopes, the nonlinear Fourier spectrum typically contains only linear components, meaning there are no discrete eigenvalues present in the discrete spectrum (Boffetta and Osborne, 1992; Wahls et al., 2020). Under unidirectional propagation, typical rogue waves can be represented as a nonlinear superposition of an envelope soliton and background waves (Slunyaev, 2006).

For our study, the NFT is computed by the software library “Fast Nonlinear Fourier Transforms” (FNFT) (Wahls et al., 2018). Fast NFT can match the accuracy of conventional NFT by using exponential splitting schemes to achieve faster numerical computation (Wahls and Poor, 2013, 2015; Prins and Wahls, 2018). More information about FNFT can be obtained online at <https://github.com/FastNFT/FNFT>.

2.3. Calculation of soliton energy ratio

As mentioned previously, the nonlinear spectrum comprises a discrete spectrum related to the solitons and a continuous spectrum related to dispersive radiation waves. The total energy of the time series is calculated by integrating the absolute value of the signal squared over time. It equals the sum of the energy in the solitons and the dispersive radiation waves (Ablowitz and Segur, 1981; Lee and M. Brühl, 2023):

$$E_{\text{total}} = \int_{-\infty}^{\infty} |u(t)|^2 dt = E_{\text{sol}} + E_{\text{rad}}, \quad (8)$$

where the soliton energy is calculated based on the imaginary parts of discrete eigenvalues,

$$E_{\text{sol}} = 4 \sum_{k=1}^N \text{Im}(\zeta_k), \quad (9)$$

and the radiation wave energy is calculated based on the reflection coefficient

$$E_{\text{rad}} = 4 \int_{-\infty}^{\infty} \log(1 + |r(\zeta)|^2). \quad (10)$$

The above formulas allow us to define the soliton energy ratio as

$$E_{\text{sol_ratio}} = E_{\text{sol}}/E_{\text{total}} = (E_{\text{total}} - E_{\text{rad}})/E_{\text{total}}. \quad (11)$$

where $E_{\text{sol_ratio}}$ ranges from zero to one. If the soliton energy ratio is high, the solitons are the dominant components in the system. We use the formula that utilizes the radiation energy to compute the energy ratio because it is easier to compute the continuous spectrum numerically.

3. Simulation setup for directional wave fields

The directional wave fields are generated from two-dimensional directional spectra with random phases. The directional spectrum $S(f, \theta)$ is a product of a frequency spectrum $S(f)$ and a directional spreading function $D(\theta)$:

$$S(f, \theta) = S(f)D(\theta), \quad (12)$$

where f is the frequency and θ is the wave direction. We adopt a Gaussian distribution, which is commonly used to model directional seas (Borgman, 1969, 1979; Benoît, 1993; Hashimoto, 1997). The function is expressed as

$$D(\theta) = \frac{1}{\sqrt{2\pi}\sigma_\theta} \exp\left(-\frac{(\theta - \theta_p)^2}{2\sigma_\theta^2}\right) \quad \text{with } \theta \in [0, 2\pi) \quad (13)$$

where θ represents the direction, θ_p is the peak direction set to π , and σ_θ is the amount of spreading (standard deviation). The directional spreading function $D(\theta)$ is defined over the range from 0 to 2π , and its integral is nearly 1 because some energy is truncated in the Gaussian tails when the spreading is large. This function describes how the wave energy is distributed around a central peak direction, with a narrow width indicating more directional focus and a broader width representing a wider spread of energy. There are no opposite directional components propagating backward against the central peak direction. Other forms of directional spreading functions can be found in Massel and Brinkman (1998).

The frequency spectrum is based on the JONSWAP (Joint North Sea Wave Project) spectrum (Hasselmann et al., 1973), which extends the Pierson–Moskowitz spectrum by introducing a peak enhancement parameter γ to account for the enhancement of wave energy around the spectral peak. The JONSWAP spectrum is widely used for representing wind-generated sea states and is defined as

$$S(f) = \alpha g^2 (2\pi)^{-4} f^{-5} \exp\left[-\frac{5}{4} \left(\frac{f_p}{f}\right)^4\right] \gamma \exp\left[-\frac{(f-f_p)^2}{2\sigma^2 f_p^2}\right], \quad (14)$$

where $S(f)$ is the spectral energy density as a function of frequency f , α is the Phillips constant, g is the acceleration due to gravity, f_p is the peak frequency, γ is the peak enhancement factor, and σ is the spectral width parameter. The Phillips constant α governs the overall energy level of the wave spectrum reflecting the influence of wind energy input. The peak enhancement parameter γ adjusts the sharpness of the spectral peak and is used to characterize the developmental stage of sea states. The parameter σ is defined piecewise to reflect the asymmetry of the spectral shape around the peak frequency: $\sigma = \begin{cases} 0.07, & f \leq f_p \\ 0.09, & f > f_p \end{cases}$. We

initially consider frequency spectra with Phillips constant values $\alpha = 0.0025, 0.005, 0.0075, 0.01, 0.02, 0.03$ and peak enhancement parameters $\gamma = 1, 2, 3, 4, 5, 6$. This setup is later used to construct the wave fields. These initial parameters have been chosen such that some extreme cases are included. Recall that the NLS equation assumes narrow bandwidth ($\Delta f/f_0 \ll 1$) and weak nonlinearity ($k_0 H_s \ll 1$). Table 1 presents

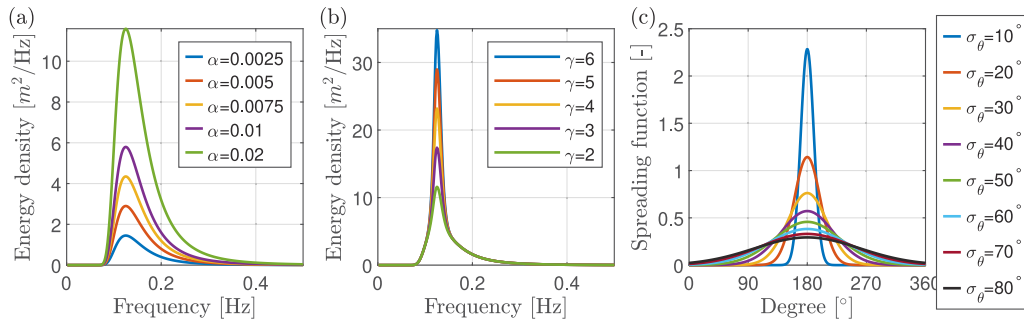


Fig. 1. Representative input components used to construct the two-dimensional wave spectrum. (a) Frequency spectrum corresponding to the Phillips constant α , (b) frequency spectrum associated with the peak enhancement parameter γ , and (c) directional spreading function based on the directional spreading parameter σ_θ .

Table 1

Relative bandwidth of various JONSWAP spectra. The influence of α and T_p on the results is minimal. Δ_f denotes the 3 dB bandwidth.

γ	1	2	3	4	5	6
Δ_f/f_0	0.5762	0.2674	0.2001	0.1740	0.1595	0.1500

Table 2

The nonlinearity parameter $k_0 H_s$ for various JONSWAP spectra. The influence of T_p on the values is minimal.

$\alpha \backslash \gamma$	1	2	3	4	5	6
0.0025	0.0895	0.0999	0.1082	0.1155	0.1220	0.1279
0.0050	0.1265	0.1412	0.1530	0.1633	0.1725	0.1809
0.0075	0.1550	0.1729	0.1874	0.2000	0.2113	0.2216
0.0100	0.1789	0.1997	0.2164	0.2309	0.2439	0.2559
0.0200	0.2530	0.2824	0.3061	0.3266	0.3450	0.3619
0.0300	0.3099	0.3459	0.3749	0.4000	0.4225	0.4432

the relative bandwidths of the considered JONSWAP spectra. From Fig. 8 in Slunyaev et al. (2013), we expect that the narrow bandwidth condition is sufficiently well satisfied for $\Delta f/f_0 \lesssim 0.25$. This is the case in all cases except $\gamma = 1$, where we only have $\Delta f/f_0 = 0.5762$. We will therefore later completely exclude the case $\gamma = 1$.

Table 2 presents the values of the nonlinearity parameter $k_0 H_s$. Experiments and simulations have shown that solitons can exist for $k_0 A_0 \lesssim 0.3$ (Slunyaev et al., 2013), where A_0 here is the soliton amplitude. To relate A_0 to H_s , we use Osborne's investigations of unstable wave packets, which can correspond to solitons on backgrounds. For small-amplitude modulations, the maximum packet amplitude can reach three times the carrier amplitude, $A_{\max} \simeq 3a_0$ (Osborne, 2010, p. 278–281). For random wave trains, the relations $H_s = 4\sigma$ and $a_0 = \sqrt{2}\sigma$ give $a_0 = H_s/(2\sqrt{2})$ (Osborne, 2010, pp. 763–764). At leading order, $H_{\max} \simeq 2A_{\max}$ (Osborne, 2010, p. 301). Therefore, the condition $k_0 A_0 \simeq k_0 A_{\max} \lesssim 0.3$, equivalently $k_0 H_{\max} \lesssim 0.6$, leads to $k_0 H_s \lesssim 0.6\sqrt{2}/3 \approx 0.283$. Accordingly, $k_0 H_s \approx 0.283$ may be regarded as an approximate upper limit for the applicability of the NLS soliton. Most, but not all of the considered JONSWAP spectra fulfill this condition. The cases with $\alpha = 0.02$, $\gamma \geq 4$ and $\alpha = 0.03$ will therefore later be additionally excluded as well.

In Appendix, we furthermore show that the peak frequency in the JONSWAP spectrum can be normalized to one without changing the soliton energy ratio — if time and frequency grids are rescaled. In this paper, we however always consider the same grids to mimic the situation in real-word buoy measurements, where the length of the recording is fixed and in particular independent of the peak period. We will observe later that this lets the estimated soliton energy ratios decrease slightly with the peak period.

To reconstruct a realistic sea surface elevation from a given directional wave spectrum, the surface can be represented as a linear

superposition of oscillations (Tucker et al., 1984; Benoît, 1993; Massel and Brinkman, 1998; Massel, 2017). Specifically, it is represented as a superposition of infinitely many simple cosine waves, each with varying amplitude, frequency, phase, and propagation direction. The sea surface elevation $\eta(x, y, t)$ at a spatial location (x, y) and time t is then given by

$$\eta(x, y, t) = \sum_{i=1}^{N_f} \sum_{j=1}^{N_\theta} \sqrt{2S(f_i, \theta_j) \Delta f \Delta \theta} \cos(k_i(x \cos \theta_j + y \sin \theta_j) - 2\pi f_i t + \phi_{ij}), \quad (15)$$

where N_f and N_θ are the total number of discrete frequency and direction bins, respectively. The term Δf is the frequency increment, and $\Delta \theta$ is the directional increment. The wave number k_i corresponding to frequency f_i is calculated using the deep water dispersion relation. ϕ_{ij} represents a random phase uniformly distributed in the interval $[0, 2\pi]$. This allows us to generate multidirectional wave fields by linearly combining unidirectional wave systems propagating along a discrete set of directions. The directional spreading function determines how wave energy is spread across these directions.

In this study, we consider directional spreadings σ_θ of 10, 20, 30, 40, 50, 60, 70, and 80 degrees, which can be shown from Fig. 1(c). When the spreading is narrow, most of the wave energy is concentrated in a single direction, resulting in long-crested waves. In contrast, wide spreading distributes the energy over many directions, leading to short-crested waves that appear more irregular. We summarize our setup of frequency and directional parameters used to construct the directional spectrum in Table 3. Previous studies have shown that higher values of α and γ can lead to soliton-dominated sea states for deep water waves (Lee and M. Brühl, 2023). We will classify the data by soliton energy ratio to further discuss sea states with different nonlinearity. This will be illustrated in a later section.

Fig. 2 shows typical wave fields generated with different directional spreading angles. The initial setting of the frequency spectrum is considered with $\alpha = 0.01$, $\gamma = 1.0$ and $T_p = 10$ s in this example. A single wave gauge is placed at $x = 0$, $y = 0$ to measure the time series. The time series is recorded for 600 s with a sampling rate of 1.7 Hz. It can be observed that the wave fields change from long-crested seas at 10 degrees (Fig. 2(a)) to short-crested seas at 80 degrees (Fig. 2(b)) as the directional spreading increases.

Using the basic wave parameters of the directional spectrum summarized in Table 3, we generate various wave fields with random phases. Time series are extracted using a virtual wave gauge positioned at the center of each wave field. In total, 2880 time series are included in our dataset. The significant wave height in this dataset ranges from 1 m to 19 m. The distribution of significant wave height for each peak period is illustrated using boxplots in Fig. 3(a). It can be observed that in general larger peak periods are associated with higher significant wave heights in our simulations. We are also interested in the

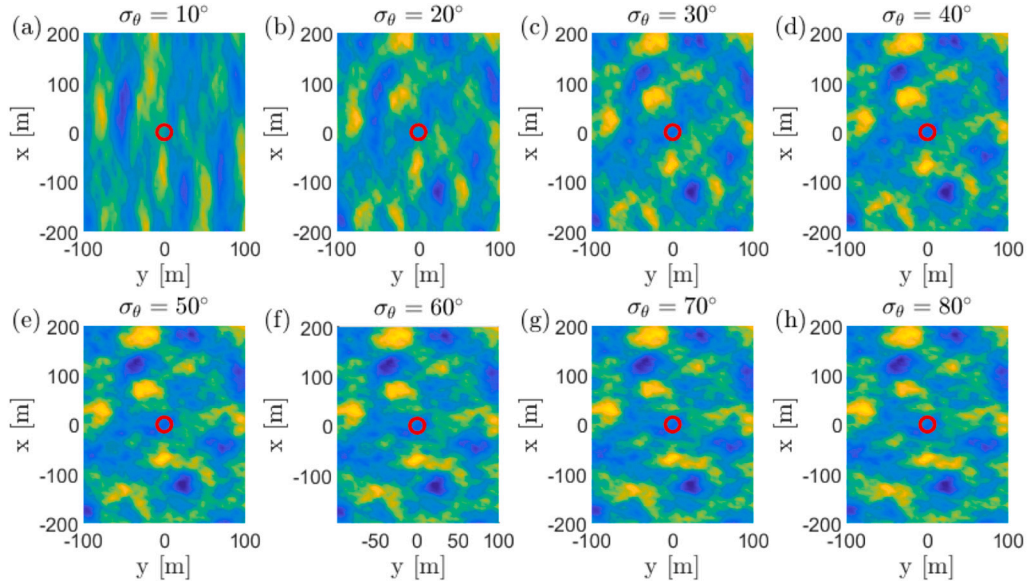


Fig. 2. Directional wave fields with spreading σ_θ varying from (a) 10° to (h) 80° . The wave gauge location at $(x, y) = (0, 0)$ is indicated by a red circle.

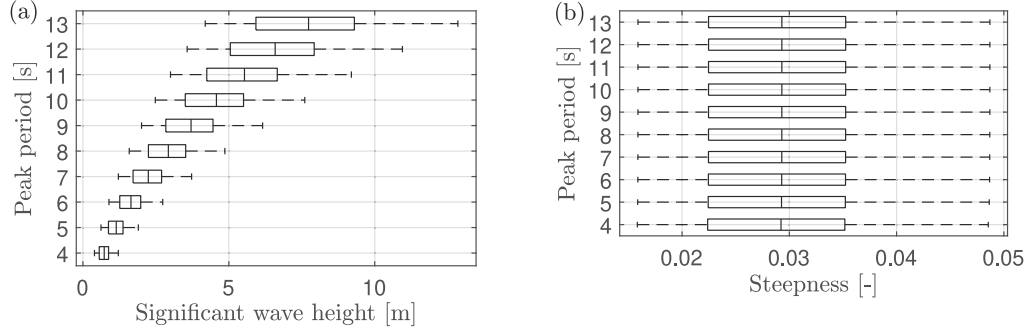


Fig. 3. Basic wave parameters of the dataset. (a) Boxplot of significant wave height versus peak period. (b) Boxplot of wave steepness versus peak period.

Table 3

Parameter setup used to construct the directional wave spectra.

Parameter	Values
Peak period T_p	4, 5, 6, 7, 8, 9, 10, 11, 12, 13
Phillips constant α	0.0025, 0.005, 0.0075, 0.01, 0.02 ^a
Peak enhancement factor γ	2 ^a , 3 ^a , 4 ^a , 5 ^a , 6 ^a
Directional spreading σ_θ	10° , 20° , 30° , 40° , 50° , 60° , 70° , 80°

^a Data corresponding to $(\alpha, \gamma) = (0.02, \gamma \geq 4)$ were excluded from the simulation.

distribution of wave steepness, where the wave steepness is calculated as $\text{Steepness} = 2\pi H_s / g T_p^2$, with H_s representing the significant wave height, calculated as four times the square root of the zeroth moment, and T_p the peak period. The distribution of wave steepness for each peak period is shown in Fig. 3(b). For each peak period, the wave steepness follows the same distribution ranging from 0.016 to 0.049. (We use identical values of α and γ within each group, which results in the same distribution of wave steepness (Annenkov and Shrira, 2014).) This figure provides a basic overview of our dataset.

4. Characteristics of the soliton energy ratio

Before applying the nonlinear Fourier transform based on the NLS equation to all time series, we first provide an exemplary analysis of a single time series to reveal its nonlinear spectrum. The time series analyzed here is extracted from the wave field with parameters $\alpha = 0.03$, $\gamma = 6.0$ and $T_p = 6$ s, as shown by the black line in Fig. 4(a).

The blue line shows the magnitude of the complex envelope, which is calculated from the surface wave elevation using the Hilbert transform (Medina and Hudspeth, 1990). Note that while only the magnitude is shown here, the NFT requires the full complex envelope, including both magnitude and phase information. The envelope time series is used as input for nonlinear Fourier analysis. By the nonlinear Fourier transform based on the NLS equation, we obtain the nonlinear spectrum including continuous spectrum (Fig. 4(b)) and soliton spectrum Fig. 4(c). The continuous spectrum displays the spectral parameter ζ on the x -axis and the absolute value of the reflection coefficient $r(\zeta)$ on the y -axis, representing components of linear dispersive waves. In the soliton spectrum, the real part of the eigenvalue on the x -axis indicates soliton velocity, while the imaginary part on the y -axis represents soliton amplitude. A total of 77 solitons are identified, with most exhibiting small amplitudes. It has been found that a dense ensemble of solitons can give rise to large wave events in the time series through strong nonlinear interactions (Gelash and Agafontsev, 2018). Meanwhile, it is important to note that the solitons of the NLS equation here are envelope solitons (Yuen and Lake, 1982; Osborne, 2010; Slunyaev et al., 2013; Chabchoub et al., 2016; Slunyaev, 2018; Xia et al., 2021). From the nonlinear spectrum, we calculate the energy of the continuous spectrum and the soliton spectrum using Eq. (10) and Eq. (9), respectively. The soliton energy ratio is then obtained using Eq. (11). In this example, the soliton energy ratio is 0.93, indicating a highly soliton-dominated sea state.

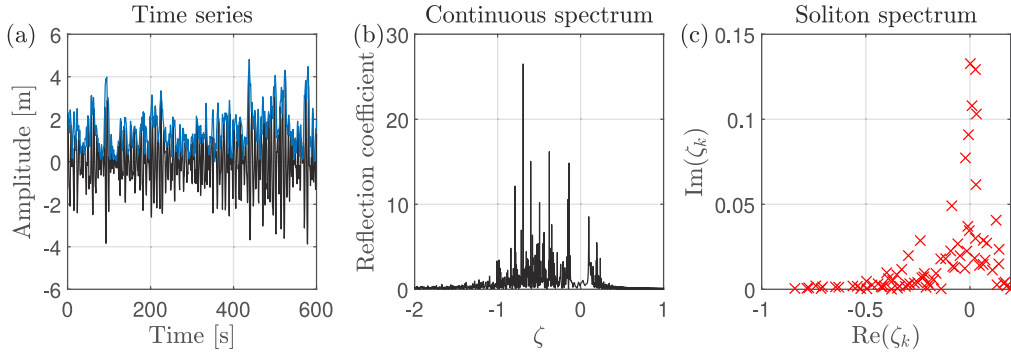


Fig. 4. Example of the nonlinear Fourier transform to time series. (a) Time series (black line) and the magnitude of its complex envelope (blue line). (b) Continuous spectrum. (c) Soliton spectrum.

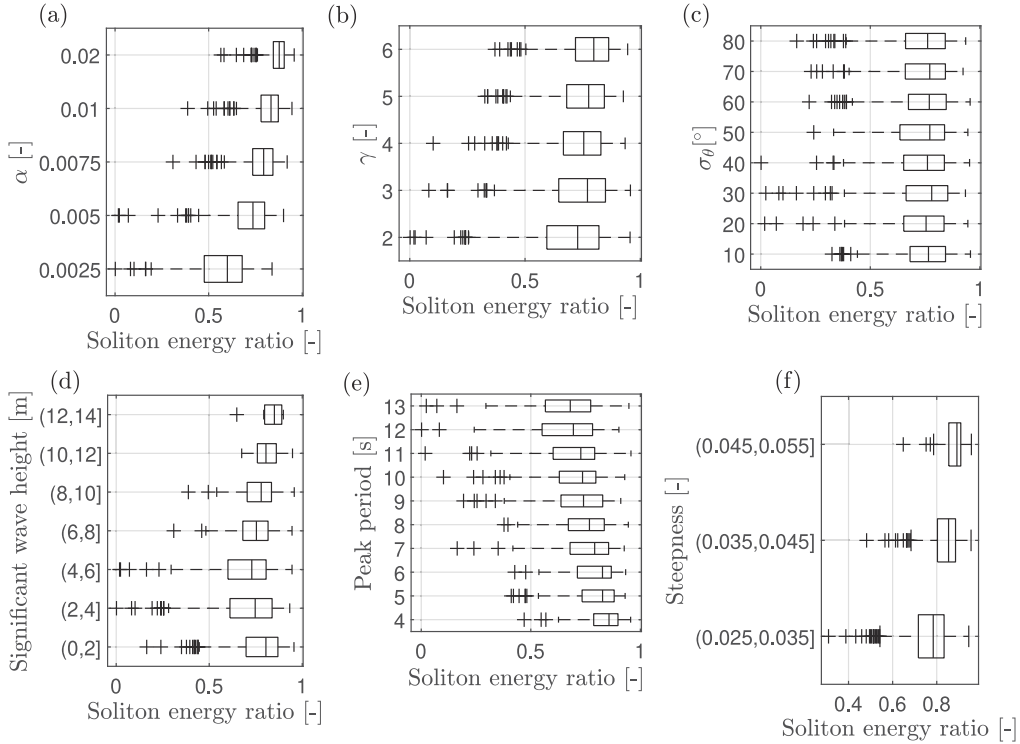


Fig. 5. Boxplots of soliton energy ratio versus key wave parameters: (a) Phillips constant α , (b) peak enhancement factor γ , (c) directional spreading σ_θ , (d) significant wave height, (e) peak period, (f) steepness.

We apply nonlinear Fourier analysis to our dataset and calculate the soliton energy ratios. The results showing the soliton energy ratio across all parameters are presented as a set of boxplots in Fig. 5. The soliton energy ratio is shown on the horizontal axis, while the vertical axes in subplots (a) to (f) correspond to Phillips constant α , peak enhancement parameter γ , directional spreading σ_θ , significant wave height H_s , peak period T_p , and wave steepness, respectively. In Fig. 5(a) and (b), it can be observed that both α and γ increase with higher soliton energy ratios. This indicates that more energy in the system and a sharper spectral peak may be associated with the presence of more solitons. In Fig. 5(c), it can be seen that different directional spreadings do not influence the soliton energy ratio. In reality, directional spreading should destabilize solitons. However, because our measurement relies on a single-point wave gauge within the wave field, it cannot capture the directional characteristics from the recorded time series. We will investigate this further in the next section by truncating the energy in the directional sidebands to assess how spreading affects the soliton energy ratio. In Fig. 5(d), significant wave height does not

exhibit a clear correlation with the soliton energy ratio, suggesting that soliton presence is not directly dependent on wave height alone. Nevertheless, Fig. 5(e) indicates a slight tendency for higher soliton energy ratio to be associated with shorter peak period. This trend is attributed to the use of a fixed recording window of ten minutes. As T_p increases, the average duration of the solitons increases as well, and they are more likely cut off at the boundaries of the recording window. This leads the NFT to overestimate the radiation content in the time series. Consequently, the soliton energy ratio decreases with T_p . More notably, Fig. 5(f) reveals a strong positive correlation between soliton energy ratio and wave steepness, implying that steeper wave conditions are more favorable for soliton existence. Overall, these figures provide a fundamental understanding of the conditions when the soliton energy ratio based on a single buoy measurement is high. However, because of the unaccounted for directionality of the underlying wave field, the detected solitons might not have a clear physical interpretation. To address this issue, we now study how the soliton energy ratio changes when only the main propagation direction is considered. In the absence

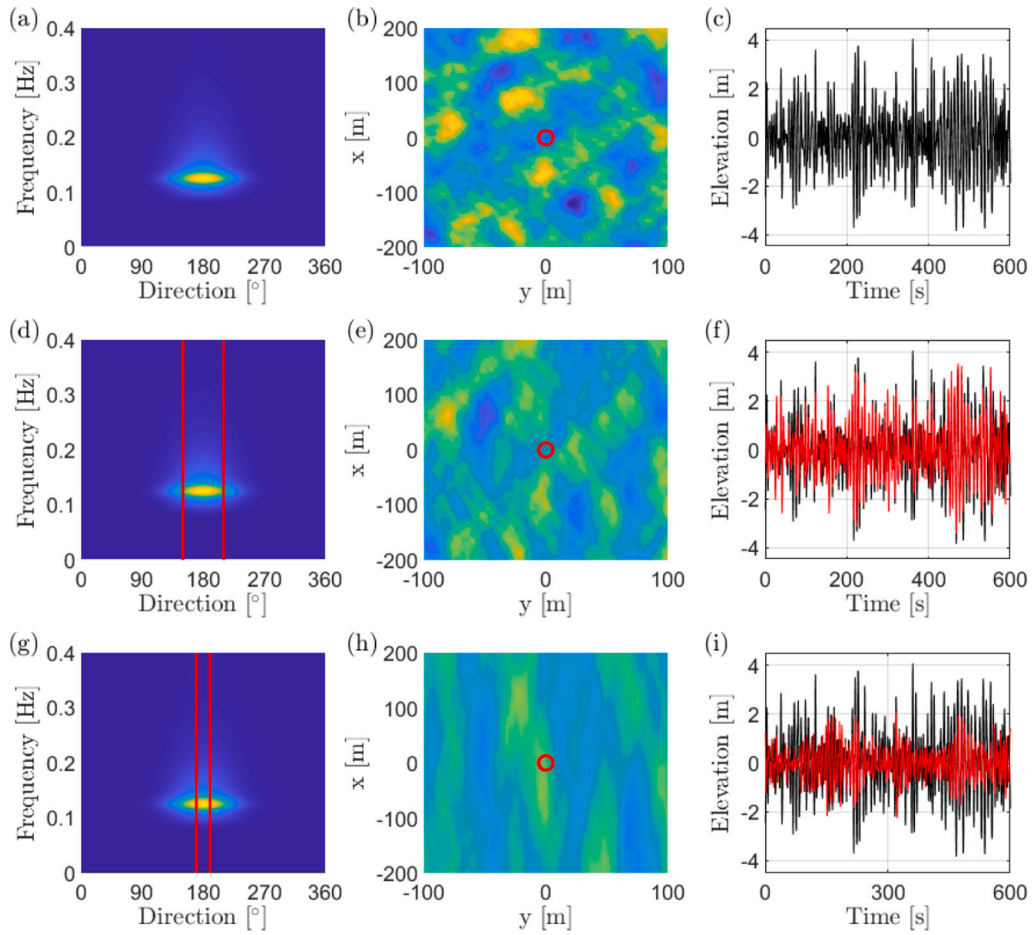


Fig. 6. Directional spectrum truncation and the resulting wave field and time series. The directional spreading is set to $\sigma_\theta = 30^\circ$, and the JONSWAP spectrum is generated with parameters $\alpha = 0.03$, $\gamma = 3$, and $T_p = 8$ s. Retention angles after directional truncation $\Delta\theta = 60^\circ$ (in (d)–(f)) and 20° (in (g)–(i)) are considered. The resulting wave fields are reconstructed from the truncated directional spectra, and time series are extracted using a virtual wave gauge located at the center of the computational domain, as labeled by the red circle.

of directional interference, the detected solitons can remain more stable as the filtered time series evolves nearly unidirectionally.

5. Influence of retention angle after directional truncation

Regarding our objective to investigate when soliton energy ratios calculated in directional seas will remain stable, we address this by removing the energy that is not on principal wave directions to generate modified wave fields with minimal directional interference. Before starting directional truncation on directional wave fields, we emphasize that this study does not account for wave evolution. We consider four retention angles $\Delta\theta$ for directional truncation analysis in the directional domain: 20° , 40° , 60° , and 120° . For each scenario, wave energy outside the specified angular range is removed from the directional spectrum, preserving only the energy within the defined range for subsequent analysis. The retention angles $\Delta\theta$ correspond to varying levels of directional energy removal. Smaller values of $\Delta\theta$ indicate greater truncation of energy in the directional sidebands, while larger values retain a broader range of energy in the principal wave directions. The case $\Delta\theta = 360^\circ$ represents the initial wave field, where no directional energy is removed. While ultimately only small retention angles are of interest, we also consider larger cases to see how the soliton energy ratio depends on this parameter. During the reconstruction stage, following the removal of energy in the sidebands, the modified wave fields are linearly synthesized using the original phase information.

In Fig. 6, we show an example of how to generate modified wave fields with retention angles and the corresponding time series. First, we consider a directional spectrum with spreading set to $\sigma_\theta = 30^\circ$, as shown in Fig. 6(a), which is generated by multiplying a directional spreading function with a JONSWAP spectrum using parameters $\alpha = 0.03$, $\gamma = 3$, and $T_p = 8$ s. The wave field is generated with random phases including a virtual wave gauge located at the center of the wave field (red circle), as shown in Fig. 6(b). The time series is extracted, as shown in Fig. 6(c).

Subsequently, we truncate specific directional ranges to construct wave fields with more concentrated directional characteristics. The retention angles are set as $\Delta\theta = 60^\circ$, and 20° , corresponding to Fig. 6(d) to (f) and (g) to (i), respectively. In Fig. 6(d), the energy outside the retention angle $\Delta\theta = 60^\circ$ (red lines) is filtered out. Using the original phases, we then generate the wave fields corresponding to the truncated spectra, as shown in Fig. 6(e). The time series data are subsequently re-extracted using virtual wave gauges, with the red line in Fig. 6(f). It can be observed that the time series loses some energy after directional truncation, resulting in slightly smaller wave crests and troughs compared to the initial time series (black line). Meanwhile, with smaller retention angles, more energy is filtered out from the spectrum, causing the wave field to become more long-crested, as seen in Fig. 6(h). The wave height in the time series also decreases as more energy is removed, as seen in Fig. 6(i).

After obtaining the time series via directional spectrum truncation, we applied the NFT based on the NLS equation to derive the nonlinear

spectrum and subsequently calculate the soliton energy ratio. The soliton energy ratio of the initial time series is 0.847. For retention angles of $\Delta\theta = 120^\circ, 60^\circ, 40^\circ$, and 20° , the corresponding soliton energy ratios are 0.836, 0.808, 0.796, and 0.726, respectively. Although the soliton energy ratio decreases as wave energy in the directional sidebands is removed, the values remain sufficiently high to be classified as soliton-dominated wave fields. It is worth noting that the directional spreading used in this case, $\sigma_\theta = 30^\circ$, is comparable to the dominant spreading conditions observed in the North Sea (Waseda et al., 2011; Teich et al., 2018). This possibly suggests that if soliton-dominated wave conditions are observed in directional seas, surface waves propagating within a narrow range of the primary direction would be dominated by solitons in the absence of directional interference. The current case is only one example and all simulation results will be presented in the following to support this finding.

To better understand soliton characteristics in the nonlinear spectrum, we classify all wave fields based on the soliton energy ratio into three categories: general cases, soliton-dominated cases, and highly nonlinear soliton-dominated cases. The classification thresholds are defined as follows. Cases with a soliton energy ratio below 60% are categorized as general cases, where the wave field is dominated by radiation waves or exhibits a near-equal contribution from both radiation waves and solitons. Cases with a soliton energy ratio between 60% and 90% are classified as soliton-dominated. Finally, cases with a soliton energy ratio exceeding 90% are considered highly nonlinear soliton-dominated conditions.

Based on the above classification, we first focus on the general cases ($E_{\text{sol, ratio}} < 60\%$), which correspond to sea states dominated by radiation waves or exhibiting a near-equal contribution from radiation waves and solitons. A total of 285 time series are identified in this category. The soliton energy ratios for the initial time series and those obtained after directional truncation under different directional spreading conditions are shown in Fig. 7. In our initial setup, there are eight directional spreading conditions, ranging from 10° to 80° . We only show the cases with spreading angles of $10^\circ, 40^\circ$, and 80° , corresponding to subfigures (a) to (c), respectively, to represent low, moderate, and high directional spreading conditions. Each gray line represents one time series, showing the soliton energy ratio and its variation with the retention angle $\Delta\theta$. We first focus on the case with small directional spreading, $\sigma_\theta = 10^\circ$, shown in Fig. 7(a). It can be seen that a large portion of the soliton energy ratios decrease as the retention angle $\Delta\theta$ becomes smaller. However, in some cases, the soliton energy ratio may increase with a smaller $\Delta\theta$, possibly because the truncated energy in the sidebands is more dominated by radiation waves in those cases. Thus, the removal of radiation wave energy leads to an increased soliton energy ratio. The red line represents the mean soliton energy ratio for all cases in this classification. The mean soliton energy ratios of initial wave fields are around 0.472, which indicates the dataset is nearly equally dominated by radiation waves and solitons. As directional spreading σ_θ increases, the mean soliton energy ratio decreases more significantly when the retention angle $\Delta\theta$ becomes small. For the small directional spreading, $\sigma_\theta = 10^\circ$, shown in Fig. 7(a), the mean soliton energy ratio does not decrease significantly. It only decreases slightly under a small retention angle of $\Delta\theta = 20^\circ$. For moderate directional spreading $\sigma_\theta = 40^\circ$, as shown in Fig. 7(b), the mean soliton energy ratio under $\Delta\theta = 20^\circ$ decreases to 0.127, indicating that only a small portion of solitons still exist, while the remaining energy is mainly dominated by radiation waves. For the largest directional spreading $\sigma_\theta = 80^\circ$, shown in Fig. 7(c), the mean soliton energy ratio under $\Delta\theta = 20^\circ$ drops to 0.055 at $\Delta\theta = 20^\circ$, indicating that the energy in the main direction with narrow directional bands is identified as high radiation waves dominance.

We subsequently examine the results for analysis of soliton dominated cases ($60\% \leq E_{\text{sol, ratio}} < 90\%$), as shown in Fig. 8. There are a total of 1378 time series identified in this class. The mean soliton

energy ratios of initial wave fields are around 0.774. A similar trend of decreasing soliton energy ratios as the retention angle $\Delta\theta$ becomes smaller can be observed. The initial mean soliton energy ratio reduces to 0.702 with a retention angle of $\Delta\theta = 20^\circ$ for small directional spreading $\sigma_\theta = 10^\circ$, to 0.416 for moderate directional spreading $\sigma_\theta = 40^\circ$ and to 0.253 for large directional spreading $\sigma_\theta = 80^\circ$. This indicates that in soliton dominated seas with small spreading, the soliton energy ratio in general decreases less, while for moderate and large spreading, it decreases more.

Finally, we focus on the results of the analysis for highly nonlinear soliton-dominated cases ($E_{\text{sol, ratio}} \geq 90\%$), as shown in Fig. 9. A total of 97 time series have been identified in this classification. The mean soliton energy ratio of the initial wave fields is approximately 0.917. For small directional spreading $\sigma_\theta = 10^\circ$, the mean soliton energy ratio remains nearly unchanged under a retention angle of $\Delta\theta = 20^\circ$. For moderate directional spreading $\sigma_\theta = 40^\circ$, the mean soliton energy ratio decreases slightly to 0.686 under $\Delta\theta = 20^\circ$. For large directional spreading $\sigma_\theta = 80^\circ$, although the mean soliton energy ratio decreases under small retention angles, its value equals to 0.525, which corresponds to being nearly equally dominated by solitons and radiation waves. Overall, the analysis of soliton energy ratios under varying directional spreading and retention angles reveals similar trends across different classifications of sea states.

6. Discussion and conclusions

In this paper, we investigated by how much soliton energy ratios from directional seas reduce when directional interference outside the main direction is filtered out. Here, soliton energy ratio is a parameter used to quantify the nonlinearity of a sea state (Ablowitz and Segur, 1981; Lee and M. Brühl, 2023). We generate directional wave fields under linear superposition of sine waves based on the JONSWAP spectrum with varying directional spreading parameters that are directly comparable to field measurements. The evolution of wave fields is not considered in this study. By removing energy outside specific directional ranges, the modified wave fields retain energy only within a targeted range of directional bands, resulting in reduced directional interference. Time series extracted from virtual wave gauges placed at the center of the wave fields are analyzed using the nonlinear Fourier transform to calculate the soliton energy ratio.

Solitons have been found to persist in small to moderate directional sea states over a number of wave periods (Slunyaev, 2024). Building on this, our results further examine highly nonlinear soliton-dominated wave fields under various directional spreading scenarios to interpret soliton energy ratios in directional wave fields. It is observed that for small directional spreading (e.g., $\sigma_\theta = 10^\circ$), the soliton energy ratio only slightly decreases compared to the initial value under a narrow propagating direction ($\Delta\theta = 20^\circ$) in the main directions. For a broad propagation direction ($\Delta\theta = 120^\circ$), directional filtering may not remove energy from the directional sidebands, causing the soliton energy ratio to remain constant. In contrast, under large directional spreading (e.g., $\sigma_\theta = 80^\circ$), the soliton energy ratio decreases significantly. This indicates that portions of the initially detected solitons cannot retain their energy due to spreading effects. It should be noted that the variation in soliton energy ratio is highly case-dependent and is mainly influenced by the phase structure of the directional wave fields.

Meanwhile, different initial soliton energy ratios can lead to varying degrees of change. For our wave fields characterized by radiation wave-dominated or mixed sea states ($E_{\text{sol, ratio}} < 60\%$), the mean soliton energy ratio decreases slightly under small directional spreading (e.g., $\sigma_\theta = 10^\circ$), but drops significantly under large spreading conditions (e.g., $\sigma_\theta = 80^\circ$). This indicates that solitons can still account for a small portion of the energy in radiation wave-dominated seas when directional spreading is small; however, under large directional spreading, the initial detected solitons may not be representative, as the soliton energy ratio is quite low by directional truncation, indicating

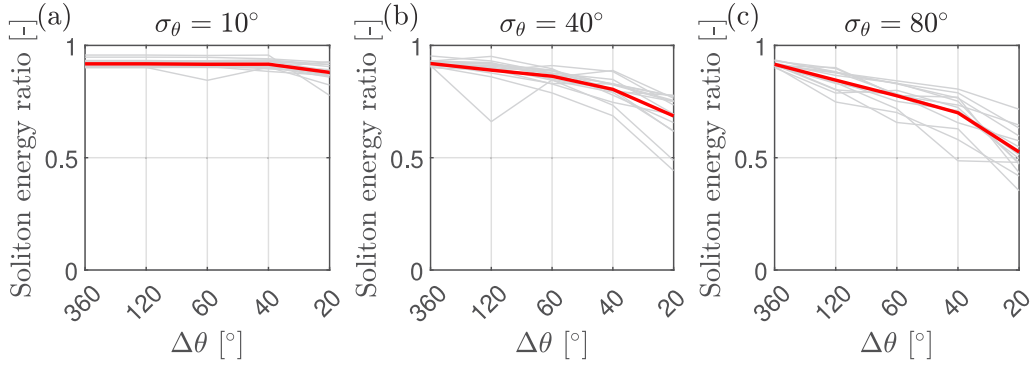


Fig. 7. Effect of directional truncation on the soliton energy ratio for **general cases** ($E_{\text{sol_ratio}} < 60\%$) with directional spreadings from 10° to 80° . The red line is the mean soliton energy ratio for each retention angle.

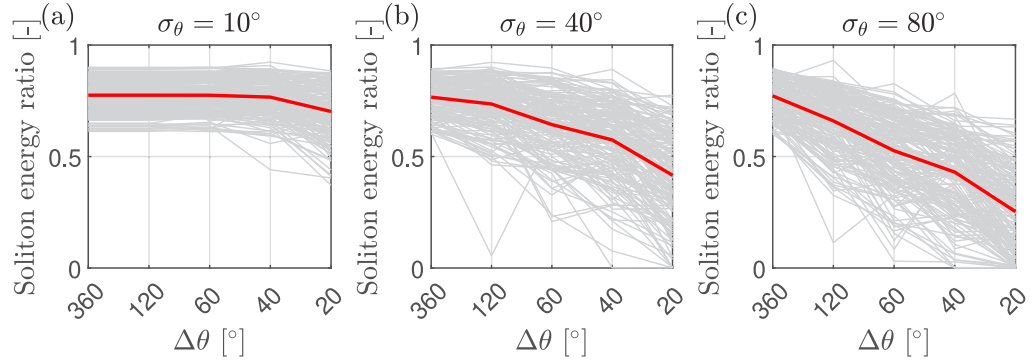


Fig. 8. Effect of directional truncation on the soliton energy ratio for **soliton-dominated cases** ($60\% \leq E_{\text{sol_ratio}} < 90\%$) with directional spreadings from 10° to 80° . The red line is the mean soliton energy ratio for each retention angle.

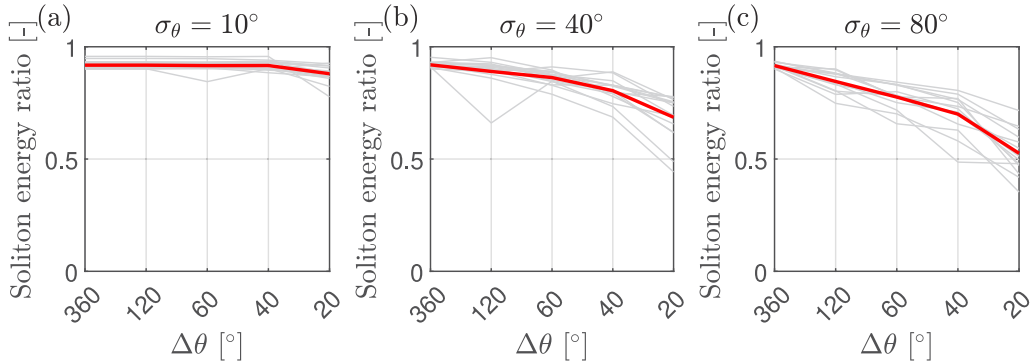


Fig. 9. Effect of directional truncation on the soliton energy ratio for **highly nonlinear soliton-dominated cases** ($E_{\text{sol_ratio}} \geq 90\%$) with directional spreadings from 10° to 80° . The red line is the mean soliton energy ratio for each retention angle.

that almost no solitons are retained in the main directions. For soliton-dominated sea states ($60\% \leq E_{\text{sol_ratio}} < 90\%$), a similar decreasing trend in the mean soliton energy ratio is observed. These seas remain soliton-dominant under small directional spreading but change to radiation wave dominance under larger spreading conditions. Nevertheless, on average significant amounts of soliton energy remain on average even for large spreading. In highly nonlinear soliton-dominated cases ($E_{\text{sol_ratio}} \geq 90\%$), the sea state remains strongly soliton-dominated under small directional spreading; however, with increasing directional spreading, the dominance gradually weakens, although solitons still dominate in the overall energy. This result is particularly noteworthy, as it reveals that soliton energy can remain dominant in the main propagation directions even under conditions of large directional spreading. It should be noted that wave propagation was not considered in this study. Due to deviations from the NLS equation, the solitons detected

by the NFT would undergo changes during their subsequent evolution (Slunyaev, 2021).

Our study has demonstrated that, under highly nonlinear soliton-dominated directional sea states, the waves propagating in the main directions remain dominated by soliton energy at a single measurement point. Even in the absence of directional interference, a soliton may still disintegrate during propagation over a finite distance due to transverse instability (Zakharov and Rubenchik, 1974; Deconinck et al., 2006). Meanwhile, it remains unclear how relevant these highly nonlinear soliton-dominated sea states actually are for the real ocean, as real wave fields may deviate from JONSWAP simulations and some of the considered JONSWAP configurations might occur only rarely in practice. To the best of our knowledge, the soliton energy ratio has been rarely applied in existing studies. In particular, the directional spreading settings used in our simulations can be directly comparable

to field measurement data. The recent observation of soliton gases in deep water around Taiwan provides a relevant case study (Lee and Wahls, 2026). Other measurements such as those conducted in Currituck Sound, North Carolina (Osborne et al., 2019), in the Bay of Biscay, Northeast Atlantic Ocean (Onorato et al., 2021) and in the Northwest Pacific Ocean (Lee et al., 2024) would be valuable for further investigation of the soliton energy ratio in future research. Our study offers a novel approach able to interpret such observations and provides new insight into the physical relevance of solitons in directional seas.

CRedit authorship contribution statement

Yu-Chen Lee: Writing – original draft, Visualization, Validation, Software, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **Sander Wahls:** Writing – review & editing, Validation, Supervision, Software, Resources, Project administration, Methodology, Funding acquisition, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The first author gratefully acknowledges the Ministry of Education, Taiwan, for funding his studies at TU Delft through the 2019 Key Fields Scholarship Program.

Appendix. The peak period does not influence the soliton energy ratio for JONSWAP spectra (if the time and frequency grids are rescaled)

In this appendix, we show that the peak period T_p does not influence the soliton energy ratio if the time and frequency grids are rescaled correctly. We start by noticing that the JONSWAP spectrum satisfies

$$\begin{aligned} S(f; \alpha, \gamma, f_p) &= \alpha g^2 (2\pi)^{-4} f^{-5} \exp \left[-\frac{5}{4} \left(\frac{f_p}{f} \right)^4 \right] \gamma^{\exp \left[-\frac{(f-f_p)^2}{2\sigma^2 f_p^2} \right]} \\ &= \alpha g^2 (2\pi)^{-4} \left(\frac{f}{f_p} \right)^{-5} \left(\frac{f_p}{1} \right)^{-5} \exp \left[-\frac{5}{4} \left(\frac{f_p}{f} \right)^4 \right] \gamma^{\exp \left[-\frac{(f/f_p-1)^2}{2\sigma^2} \right]} \\ &= \frac{\alpha}{f_p^5} g^2 (2\pi)^{-4} \left(\frac{f}{f_p} \right)^{-5} \exp \left[-\frac{5}{4} \left(\frac{1}{f/f_p} \right)^4 \right] \gamma^{\exp \left[-\frac{(f/f_p-1)^2}{2\sigma^2} \right]} \\ &= \frac{1}{f_p^5} S(\tilde{f}; \alpha, \gamma, 1), \quad \text{where} \quad \tilde{f} = \frac{f}{f_p}. \end{aligned}$$

A signal generated from the JONSWAP spectrum $S(f; \alpha, \gamma, f_p)$ thus satisfies

$$\begin{aligned} u(t) &= \sum_{n=1}^N \sqrt{2S(f_n; \alpha, \gamma, f_p)} \Delta f \cos(2\pi f_n t - \phi_n) \\ &= \sum_{n=1}^N \sqrt{2 \frac{1}{f_p^5} S(\tilde{f}_n; \alpha, \gamma, 1)} \Delta f \cos(2\pi f_n t - \phi_n) \\ &= \sum_{n=1}^N \sqrt{2 \frac{1}{f_p^5} S(\tilde{f}_n; \alpha, \gamma, 1)} \Delta \tilde{f} f_p \cos(2\pi \tilde{f}_n f_p t - \phi_n) \\ &= \frac{1}{f_p^2} \sum_{n=1}^N \sqrt{2S(\tilde{f}_n; \alpha, \gamma, 1)} \Delta \tilde{f} \cos(2\pi \tilde{f}_n f_p t - \phi_n). \end{aligned}$$

If we instead sample a signal from the JONSWAP spectrum $S(\tilde{f}; \alpha, \gamma, 1)$ and use the same phases ϕ_n as before, we get

$$\tilde{u}(t) = \sum_{n=1}^N \sqrt{2S(\tilde{f}_n; \alpha, \gamma, 1)} \Delta \tilde{f} \cos(2\pi \tilde{f}_n t - \phi_n).$$

Comparing $u(t)$ and $\tilde{u}(t)$, we see that

$$u(t) = f_p^{-2} \tilde{u}(f_p t).$$

The corresponding signals passed to the NFT are

$$Q(t) = \rho u(t), \quad \tilde{Q}(t) = \tilde{\rho} \tilde{u}(t),$$

where, using the deep-water dispersion relation $\omega_0^2 = g k_0$,

$$\rho = C_g \sqrt{2k_0^2} = \frac{\omega_0}{2k_0} \sqrt{2k_0^2} = \frac{\omega_0^3}{\sqrt{2g}} = \frac{(2\pi)^3}{\sqrt{2g}} f_p^3, \quad \tilde{\rho} = \dots = \frac{(2\pi)^3}{\sqrt{2g}} \tilde{f}_p^3.$$

We see that the signals scaled for the NFT satisfy

$$Q(t) = \frac{(2\pi)^3}{\sqrt{2g}} f_p^3 u(t) = \frac{(2\pi)^3}{\sqrt{2g}} f_p^3 f_p^{-2} \tilde{u}(f_p t) = f_p \tilde{\rho} \tilde{u}(f_p t) = f_p \tilde{Q}(f_p t).$$

This is exactly the relation that appears in the scaling invariance of the NLSE.¹ In this specific case, it can be proven that the change of the nonlinear Fourier transform reduces to scaling operations as well (Yousefi and Kschischang, 2014, Sec. IV.D-4). These scaling operations do not change the soliton energy ratio.

In summary, we found that the soliton energy ratios of $u(t)$ and $\tilde{u}(t)$ are the same. Since this result works for any combination of the random phases, we see that the distribution of the solitons energy ratio resulting from the JONSWAP spectra $S(f; \alpha, \gamma, f_p)$ and $S(\tilde{f}; \alpha, \gamma, 1)$ with random phases must be the same. The peak period $T_p = f_p^{-1}$ therefore cannot influence the distribution of the soliton energy ratios for time series obtained from JONSWAP spectra, if the time and frequency grids are scaled accordingly and α and γ are kept constant.

References

- AbLOWITZ, M.J., Kaup, D.J., Newell, A.C., Segur, H., 1974. The inverse scattering transform—Fourier analysis for nonlinear problems. In: *Studies in Applied Mathematics*, vol. 53, (4), pp. 249–315.
- AbLOWITZ, M.J., Segur, H., 1981. *Solitons and the Inverse Scattering Transform*. SIAM.
- Aktosun, T., 2022. Inverse scattering transform and the theory of solitons. In: *Solitons*. Springer, pp. 47–61.
- Alber, I., 1978. The effects of randomness on the stability of two-dimensional surface wavetrains. *Proc. R. Soc. A* 363 (1715), 525–546.
- Alberello, A., Părău, E.I., 2022. A dissipative nonlinear Schrödinger model for wave propagation in the marginal ice zone. *Phys. Fluids* 34 (6), 061703.
- Annenkov, S., Shkira, V., 2014. Evaluation of skewness and kurtosis of wind waves parameterized by JONSWAP spectra. *J. Phys. Oceanogr.* 44 (6), 1582–1594.
- Benoit, M., 1993. Extensive comparison of directional wave analysis methods from gauge array data. In: *Proceedings of the 2nd International Symposium on Ocean Wave Measurement and Analysis*. ASCE, pp. 740–754.
- Boffetta, G., Osborne, A.R., 1992. Computation of the direct scattering transform for the nonlinear Schrödinger equation. *J. Comput. Phys.* 102 (2), 252–264.
- Borgman, L.E., 1969. Directional spectra models for design use. In: *Offshore Technology Conference*. OTC, OTC-1069.
- Borgman, L.E., 1979. Directional wave spectra from wave sensors. In: *Ocean Wave Climate*. Springer, pp. 269–300.
- Brunetti, M., Marchiando, N., Berti, N., Kasparian, J., 2014. Nonlinear fast growth of water waves under wind forcing. *Phys. Lett. A* 378 (14–15), 1025–1030.
- Carter, J.D., Contreras, C.C., 2008. Stability of plane-wave solutions of a dissipative generalization of the nonlinear Schrödinger equation. *Phys. D: Nonlinear Phenom.* 237 (24), 3292–3296.
- Chabchoub, A., Grimshaw, R.H.J., 2016. The hydrodynamic nonlinear Schrödinger equation: Space and time. *Fluids* 1 (3), 2016.
- Chabchoub, A., Onorato, M., Akhmediev, N., 2016. Hydrodynamic envelope solitons and breathers rogue and shock waves in nonlinear dispersive media. pp. 55–87.
- Chimmalg, S., Prins, P.J., Wahls, S., 2019. Fast nonlinear Fourier transform algorithms using higher order exponential integrators. *IEEE Access* 7, 145161–145176.
- Deconinck, B., Pelinovsky, D.E., Carter, J.D., 2006. Transverse instabilities of deep-water solitary waves. *Proc. R. Soc. A: Math. Phys. Eng. Sci.* 462 (2071), 2039–2061.
- El, G., Kamchatnov, A., 2005. Kinetic equation for a dense soliton gas. *Phys. Rev. Lett.* 95 (20), 204101.

¹ See, e.g., Henderson et al. (1999, Eq. 6). The roles of X and T are reversed in this paper.

- El, G.A., Kamchatnov, A.M., Pavlov, M.V., Zykov, S., 2011. Kinetic equation for a soliton gas and its hydrodynamic reductions. *J. Nonlinear Sci.* 21, 151–191.
- Fache, L., Bonnefoy, F., Ducroz, G., Copie, F., Novkoski, F., Ricard, G., Roberti, G., Falcon, E., Suret, P., El, G., et al., 2024. Interaction of soliton gases in deep-water surface gravity waves. *Phys. Rev. E* 109 (3), 034207.
- Gelash, A., Agafontsev, D., 2018. Strongly interacting soliton gas and formation of rogue waves. *Phys. Rev. E* 98 (4), 042210.
- Hashimoto, N., 1997. Analysis of the directional wave spectrum from field data. *Adv. Coast. Ocean. Eng.* 3, 103–144.
- Hasimoto, H., Ono, H., 1972. Nonlinear modulation of gravity waves. *J. Phys. Soc. Japan* 33 (3), 805–811.
- Hasselmann, K., Barnett, T.P., Bouws, E., Carlson, H., Cartwright, D.E., Enke, K., Ewing, J., Gienapp, A., Hasselmann, D., Kruseman, P., et al., 1973. Measurements of Wind-Wave Growth and Swell Decay During the Joint North Sea Wave Project (JONSWAP). *Ergänzungsheft zur Deutschen Hydrographischen Zeitschrift, Reihe A*.
- Henderson, K.L., Peregrine, D.H., Dold, J.W., 1999. Unsteady water wave modulations: fully nonlinear solutions and comparison with the nonlinear Schrödinger equation. *Wave Motion* 29 (4), 341–361.
- Karpman, V.I., 2016. *Non-Linear Waves in Dispersive Media: International Series of Monographs in Natural Philosophy*, vol. 71, Elsevier.
- Lax, P.D., 1968. Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math.* 21 (5), 467–490.
- Lee, Y.-C., M. Brühl, S. Wahls, 2023. When do JONSWAP spectra lead to soliton gases in deep water conditions?. In: *International Conference on Offshore Mechanics and Arctic Engineering*, Vol. 86878, American Society of Mechanical Engineers, V005T06A101.
- Lee, Y.-C., M. Brühl, D.-J. Doong, Wahls, S., 2024. Nonlinear Fourier classification of 663 rogue waves measured in the philippine sea. *PLoS One* 19 (5), e0301709.
- Lee, Y.-C., Wahls, S., 2026. Field observation of soliton gases in the deep open ocean. *Results Phys.* 108726.
- Massel, S.R., 2017. *Ocean surface waves: their physics and prediction*, vol. 45, World Scientific.
- Massel, S., Brinkman, R., 1998. On the determination of directional wave spectra for practical applications. *Appl. Ocean Res.* 20 (6), 357–374.
- Medina, J.R., Hudspeth, R., 1990. A review of the analyses of ocean wave groups. *Coast. Eng.* 14 (6), 515–542.
- Onorato, M., Cavaleri, L., Randoux, S., Suret, P., Ruiz, M.I., Alfonso, M. De, Benetazzo, A., 2021. Observation of a giant nonlinear wave-packet on the surface of the ocean. *Sci. Rep.* 11, 23606.
- Onorato, M., Proment, D., 2012. Approximate rogue wave solutions of the forced and damped nonlinear Schrödinger equation for water waves. *Phys. Lett. A* 376 (45), 3057–3059.
- Onorato, M., Waseda, T., Toffoli, A., Cavaleri, L., Gramstad, O., Janssen, P., Kinoshita, T., Monbaliu, J., Mori, N., Osborne, A.R., et al., 2009. Statistical properties of directional ocean waves: The role of the modulational instability in the formation of extreme events. *Phys. Rev. Lett.* 102 (11), 114502.
- Osborne, A., 2010. Nonlinear ocean waves and the inverse scattering transform. *Geophys. Ser.* 97.
- Osborne, A.R., Resio, D.T., Costa, A., de León, S., Ponce, Chiriví, E., 2019. Highly nonlinear wind waves in Currituck Sound: dense breather turbulence in random ocean waves. *Ocean. Dyn.* 69, 187–219.
- Prins, P.J., Wahls, S., 2018. Higher order exponential splittings for the fast non-linear Fourier transform of the korteweg-de vries equation. In: *2018 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*. IEEE, pp. 4524–4528.
- Rapti, Z., Kevrekidis, P., Frantzeskakis, D., Malomed, B., 2004. On the modulational instability of the nonlinear Schrödinger equation with dissipation. *Phys. Scr.* 2004, 74, (T113).
- Sedov, E.V., Redyuk, A.A., Fedoruk, M.P., Gelash, A.A., Frumin, L.L., Turitsyn, S.K., 2018. Soliton content in the standard optical OFDM signal. *Opt. Lett.* 43 (24), 5985–5988.
- Shabat, A., Zakharov, V., 1972. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Sov. Phys. JETP* 34 (1), 62.
- Slunyaev, A., 2006. Nonlinear analysis and simulations of measured freak wave time series. *Eur. J. Mech. B Fluids* 25 (5), 621–635.
- Slunyaev, A., 2018. Analysis of the nonlinear spectrum of intense sea wave with the purpose of extreme wave prediction. *Radiophys. Quantum Electron.* 61, 1–21.
- Slunyaev, A., 2021. Persistence of hydrodynamic envelope solitons: Detection and rogue wave occurrence. *Phys. Fluids* 33 (3).
- Slunyaev, A., 2024. Soliton groups and extreme wave occurrence in simulated directional sea waves. *Phys. Fluids* 36 (7), 075606.
- Slunyaev, A., Clauss, G.F., Klein, M., Onorato, M., 2013. Simulations and experiments of short intense envelope solitons of surface water waves. *Phys. Fluids* 25 (6), 065505.
- Suret, P., Randoux, S., Gelash, A., Agafontsev, D., Doyon, B., El, G., 2024. Soliton gas: Theory, numerics, and experiments. *Phys. Rev. E* 109 (6), 061001.
- Suret, P., Tikan, A., Bonnefoy, F., Copie, F., Ducroz, G., Gelash, A., Prabhudesai, G., Michel, G., Cazaubiel, A., Falcon, E., et al., 2020. Nonlinear spectral synthesis of soliton gas in deep-water surface gravity waves. *Phys. Rev. Lett.* 125 (26), 264101.
- Teich, T., Groll, N., Weisse, R., 2018. Long-term statistics of potentially hazardous sea states in the north sea 1958–2014. *Ocean. Dyn.* 68, 1559–1570.
- Tucker, M., Challenor, P.G., Carter, D., 1984. Numerical simulation of a random sea: a common error and its effect upon wave group statistics. *Appl. Ocean Res.* 6 (2), 118–122.
- Turitsyn, S.K., Prilepsky, J.E., Le, S.T., Wahls, S., Frumin, L.L., Kamalian, M., Derevyanko, S.A., 2017. Nonlinear Fourier transform for optical data processing and transmission: advances and perspectives. *Optica* 4 (3), 307–322.
- Turitsyn, S., Sedov, E., Redyuk, A., Fedoruk, M., 2019. Nonlinear spectrum of conventional OFDM and WDM return-to-zero signals in nonlinear channel. *J. Lightwave Technol.* 38 (2), 352–358.
- Wahls, S., Chimmalgil, S., Prins, P.J., 2018. FNFT: A software library for computing nonlinear Fourier transforms. *J. Open Source Softw.* 3 (23), 597.
- Wahls, S., Le, S.T., Prilepsky, J.E., Poor, H.V., Turitsyn, S.K., 2015. Digital backpropagation in the nonlinear Fourier domain. In: *2015 IEEE 16th International Workshop on Signal Processing Advances in Wireless Communications. SPAWC, IEEE*, pp. 445–449.
- Wahls, S., M. Brühl, Y.-M. Fan, Huang, C.-J., 2020. Nonlinear Fourier analysis of free-surface buoy data using the software library FNFT. In: *International Conference on Offshore Mechanics and Arctic Engineering*, Vol. 84386, American Society of Mechanical Engineers, V06BT06A070.
- Wahls, S., Poor, H.V., 2013. Introducing the fast nonlinear Fourier transform. In: *2013 IEEE International Conference on Acoustics Speech and Signal Processing. IEEE*, pp. 5780–5784.
- Wahls, S., Poor, H.V., 2015. Fast numerical nonlinear Fourier transforms. *IEEE Trans. Inform. Theory* 61 (12), 6957–6974.
- Waseda, T., Hallerstig, M., Ozaki, K., Tomita, H., 2011. Enhanced freak wave occurrence with narrow directional spectrum in the north sea. *Geophys. Res. Lett.* 38 (13), L13604.
- Xia, W., Ma, Y., Dong, G., Zhang, J., Ma, X., 2021. Emergence of solitons from irregular waves in deep water. *J. Mar. Sci. Eng.* 9 (12), 1369.
- Yousefi, M.I., Kschischang, F.R., 2014. Information transmission using the nonlinear Fourier transform, part i: Mathematical tools. *IEEE Trans. Inform. Theory* 60 (7), 4312–4328.
- Yuen, H.C., Lake, B.M., 1982. Nonlinear dynamics of deep-water gravity waves. *Adv. Appl. Mech.* 22, 67–229.
- Zabusky, N.J., and, M.D. Kruskal, 1965. Interaction of solitons less in a collisionless plasma and the recurrence of initial states. *Phys. Rev. Lett.* 15 (6), 240.
- Zakharov, V.E., 1968. Stability of periodic waves of finite amplitude on the surface of a deep fluid. *J. Appl. Mech. Tech. Phys.* 9 (2), 190–194.
- Zakharov, V., 1971. Kinetic equation for solitons. *Sov. Phys. JETP* 33 (3), 538–540.
- Zakharov, V.E., Rubenchik, A.M., 1974. Instability of waveguides and solitons in nonlinear media. *Sov. Phys. JETP* 38, 494–500.