Stable self-similar blowup for wave maps and the harmonic map heat flow into perturbed spheres

Zur Erlangung des akademischen Grades eines

DOKTORS DER NATURWISSENSCHAFTEN

von der KIT-Fakultät für Mathematik des
Karlsruher Instituts für Technologie (KIT)
genehmigte

DISSERTATION

von

Alexander Wittenstein

Tag der mündlichen Prüfung: *07. Oktober 2026*

1. Referent Prof. Dr. Tobias Lamm
2. Referentin Prof. Dr. Birgit Schörkhuber

Acknowledgements

I would first like to thank my two supervisors, Tobias Lamm and Birgit Schörkhuber, for introducing me to the topic of geometric analysis, for their guidance and support throughout my studies and for agreeing to serve as referees for this thesis.
I also want to thank my friends and colleagues from the mathematics department. Special thanks go to everyone who contributed to the intense mathematical and non-mathematical discussions during our coffee break chess matches, this includes among others Anna, Björn, Caro, Christopher, David, Friedrich, Gian, Henning, Jonathan, Julia, Kilian, Luca, Lucrezia, Marco, Marvin, Michael, Niklas, Rafael, Richard, Siggi, Siliang, Simeon, Sylvain and Yonas. I am also grateful for the many other events we went to, which made my time as a PhD student much more enjoyable.
I would also like to thank the CRC for its financial support and for the opportunity to participate in its research activities through the project „B5 - Geometric wave equations“.
Last but not least, I would like to thank my family and friends outside the mathematics department for their constant support and encouragement. Thank you for always being there for me and reminding me that there is still a life outside of mathematics.

This work was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173.

Abstract

This thesis is about existence and stability of self-similar blowup solutions to geometric nonlinear heat and wave equations. We study in particular the harmonic map heat flow and the wave maps equation.
Both of these equations can develop singularities in finite time. In this thesis, we study such singularities in the form of self-similar blowup solutions and investigate whether their behaviour is stable under small perturbations of the initial data. In many known results on self-similar blowup for these equations, the target manifold is the perfectly round sphere. The aim of this thesis is to study what happens if the geometry of the sphere is slightly changed. We show that self-similar blowup solutions still exist for small rotationally symmetric perturbations of the sphere and that these solutions are asymptotically nonlinearly stable under small corotational perturbations of the initial data.
The first chapter gives an introduction to geometric partial differential equations, harmonic maps, the harmonic map heat flow, and the wave maps equation. We discuss the basic structures of these equations and explain why singularity formation and stability questions arise naturally in this context.
In the second chapter, we restrict the analysis to corotational maps and derive the corresponding radial equations for their respective radial profiles. We then introduce self-similar solutions for the harmonic map heat flow and for wave maps into the round sphere. After that, we define the class of perturbed spheres considered in this thesis and explain how the perturbation of the geometry of the target manifold changes the corresponding equation.
In the third chapter, we present the strategy used to construct and study self-similar blowup solutions for these perturbed equations. The main idea is to view the perturbed problem as a small perturbation of the known problem for the round sphere. This leads to a perturbative construction of self-similar profiles and to a stability analysis based on the persistence of the relevant spectral properties of the linearized operator.
In the final two chapters, we apply this strategy first to the harmonic map heat flow and then to the wave maps equation. In each case, we prove the existence of self-similar blowup solutions into perturbed spheres and their asymptotic nonlinear stability.
The thesis is mainly based on two works: one work by the author on the harmonic map heat flow [70] and one joint work of the author with R. Donninger and B. Schörkhuber on wave maps [29].

Contents

Abstract

1. Introduction **1**
1.1. Geometric Partial Differential Equations 1
1.2. The Harmonic Map Heat Flow 3
1.3. The Wave Maps Equation 5

2. Self-similar blowup into perturbed spheres **8**
2.1. The corotational ansatz 8
2.2. Self-similar solutions 9
2.3. Perturbed spheres 10

3. Existence and stability of self-similar solutions into perturbed spheres **12**
3.1. Existence of self-similar blowup solutions into S^d_ε 12
3.2. Stability of self-similar blowup solutions into S^d_ε 14

4. Stable blowup for the harmonic map heat flow into perturbed spheres **17**
4.1. Existence of self-similar solutions for the harmonic map heat flow into S^d_ε 23
4.2. Stability analysis 36

5. Stable blowup for supercritical wave maps into perturbed spheres **52**
5.1. Existence of self-similar solutions for the wave maps equation into S^d_ε . . 59
5.2. Stability analysis 73

A. Appendix **90**
A.1. Additional properties of homogeneous Sobolev spaces 90
A.2. Schauder-type estimates for parameter depending nonlinear operators . . 92

Bibliography **97**

1. Introduction

1.1. Geometric Partial Differential Equations

Partial differential equations (PDEs) are equations for an unknown function of several variables that describe relations between these functions and their partial derivatives. These equations provide a natural framework for many phenomena in mathematics and physics. Examples include the distribution of heat in a material, the propagation of sound waves, the vibration of an elastic string or the behaviour of electromagnetic fields.
A fundamental classification in the theory of partial differential equations is the distinction between linear and nonlinear equations. For a homogeneous linear PDE, the so-called *superposition principle* applies, meaning that sums and scalar multiples of solutions are again solutions. This makes it possible to construct more complicated solutions from simpler ones, which is the basis for many classical methods in PDE theory, including for example the Fourier transform or methods involving Green's function.
For nonlinear equations this principle can no longer be applied since the sum of two solutions is usually not necessarily a solution anymore, so that one cannot decompose the problem into simpler independent ones. As a result, many of the standard methods from the linear theory are no longer directly applicable. In particular, questions of local and global well-posedness become more difficult, because one must control both the linear evolution as well as the effect of the nonlinear terms.
In geometric partial differential equations, the nonlinear terms often come from the geometry of the problem itself. They may arise, for example, from the shape of the involved spaces or from geometric conditions placed upon the unknown function. In this way, the nonlinear term already reflects important properties of the underlying geometric problem.
A basic example of this phenomenon is given by *harmonic maps*. To motivate their definition, it is useful to begin with the classical case of *harmonic functions*, i.e. functions that solve the Laplace equation. For a function $U : \mathbb{R}^n \to \mathbb{R}$ its Dirichlet energy is formally given by

$$E[U] = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla U|^2 \, dx,$$

and the associated Euler-Lagrange equation (for a compactly supported variation) to this energy functional is given by

$$\Delta U = 0,$$

which is precisely the Laplace equation. Thus harmonic functions can be variationally characterized as critical points of the Dirichlet energy.

The same concept can be applied to vector-valued functions. For a map $U : \mathbb{R}^n \to \mathbb{R}^m$, the Dirichlet energy is formally defined as

$$E[U] = \frac{1}{2}\int_{\mathbb{R}^n} |DU|^2\, dx = \frac{1}{2}\int_{\mathbb{R}^n} \sum_{\alpha=1}^{n}\sum_{i=1}^{m} |\partial_\alpha U^i|^2\, dx$$

and the corresponding Euler-Lagrange equations are again the Laplace equations but now understood componentwise

$$\Delta U^i = 0, \qquad i = 1, ..., m.$$

This is still a linear non-coupled elliptic system. The reason is that the target space is the flat Euclidean space $\mathbb{R}^m$. There are therefore no restrictions in choosing admissible variations of U. This situation changes if we consider maps $U : M \to N$ between Riemannian manifolds (M, g) and (N, h). The natural analogue of the Dirichlet energy in this case is now

$$E[U] = \frac{1}{2}\int_M |dU|_h^2\, d\,\mathrm{vol}_g, \tag{1.1.1}$$

where dU is the differential of U, $|\cdot|_h$ is the pointwise norm induced by the Riemannian metric h and $d\,\mathrm{vol}_g$ is the Riemannian volume element induced by g. Critical points of this energy are called *harmonic maps* and the harmonic map equation is the associated Euler-Lagrange equation of this energy, which in local coordinates is given by

$$\tau(U) := \Delta_g U^k + g^{\alpha\beta}\Gamma_{ij}^k(U)\partial_\alpha U^i \partial_\beta U^j = 0, \qquad k = 1, ..., \dim N. \tag{1.1.2}$$

Here, $\Delta_g = \frac{1}{\sqrt{|g|}}\partial_\alpha\left(\sqrt{|g|}g^{\alpha\beta}\partial_\beta\right)$ is the Laplace-Beltrami operator and Γ_{ij}^k are the Christoffel-symbols on N. Furthermore, we have used the Einstein summation convention, where repeated indices get summed over. More precisely, the Latin indices range from 1 to $\dim N$, while the Greek indices range from 1 to $\dim M$.
For the precise definitions of these geometric objects we refer to [48] and [47] and for an overview about harmonic maps we refer to [43], [33] and [49].
The harmonic map equation (1.1.2) gives rise to two closely related classes of equations which will be central in what follows. The first one is motivated by the variational character of harmonic maps: Instead of trying to solve the elliptic equation directly, one can deform an initial map in the direction in which the Dirichlet energy decreases most rapidly. This leads to the negative L^2-gradient flow of the Dirichlet energy, which got introduced in the work of Eells and Sampson [32], and it is called the *harmonic map heat flow*. It provides a parabolic approach to the existence problem for harmonic maps: Starting from a given initial map, one studies whether the flow exists for all positive times and whether it converges to a stationary solution which will then be a harmonic map.
A second direction is obtained by changing the geometry of the domain. While harmonic maps are defined for maps from a Riemannian manifold into a Riemannian manifold, one may instead equip the domain with a Lorentzian metric. The corresponding Euler-Lagrange equation is then no longer elliptic, but becomes a hyperbolic system, called the

wave maps equation. In this sense, wave maps are the hyperbolic analogue of harmonic maps and they have the characteristic features of wave equations, such as finite speed of propagation and oscillatory behaviour, but with an additional geometric constraint that the solution takes values in a manifold. This makes wave maps a natural model problem for studying nonlinear hyperbolic phenomena in a geometric setting. They are furthermore closely connected to mathematical physics, where they are often used as simpler model equations for some of the difficulties that also appear in the Einstein field equations of general relativity.
In the following sections, we introduce both equations in more detail and discuss the singularity formation and corresponding stability questions that will be relevant for this thesis.

1.2. The Harmonic Map Heat Flow

Now that we have introduced harmonic maps as critical points of the Dirichlet energy, it is natural to ask how solutions to the harmonic map equation can be constructed. Solving the nonlinear elliptic system from (1.1.2) directly is often difficult. A fundamental idea, introduced by Eells and Sampson in [32], is to replace the elliptic problem by a parabolic one. Instead of directly looking for critical points of the Dirichlet energy, one starts with an initial map and evolves it in the direction in which the energy decreases most rapidly. This process then leads to the *harmonic map heat flow*.
More precisely, let again (M, g) and (N, h) be Riemannian manifolds and take an initial map $U_0 : M \to N$. A map $U : [0, \infty) \times M \to N$ solves the *harmonic map heat flow* with initial data U_0 if it satisfies the following equation

$$\partial_t U = \tau(U), \qquad U(0, \cdot) = U_0.$$

Here, $\tau(U)$ is the tension field from the harmonic map equation (1.1.2), so that in local coordinates the harmonic map heat flow takes the following form

$$\partial_t U^k = \Delta_g U^k + g^{\alpha\beta}\Gamma^k_{ij}(U)\partial_\alpha U^i \partial_\beta U^j, \qquad k = 1, ..., \dim N. \tag{1.2.1}$$

Due to the appearance of the Christoffel-symbols this is a nonlinear parabolic system. If the target manifold would be a flat Euclidean space the nonlinear term would vanish and we would end up with the usual linear heat equation.
Since the first variation formula for the Dirichlet energy is given by the negative of the tension field the equation

$$\partial_t U = \tau(U)$$

is precisely the negative L^2-gradient flow of the Dirichlet energy. We furthermore have the following energy identity along smooth solutions

$$\frac{d}{dt}E[U](t) = -\int_M \langle \partial_t U, \tau(U)\rangle_h \, d\,\mathrm{vol}_g = -\int_M |\partial_t U|^2_h \, d\,\mathrm{vol}_g \leq 0,$$

so that the Dirichlet energy is non-increasing along the flow. This idea was used by Eells and Sampson in [32] to prove that if the target manifold N has non-positive sectional curvature the harmonic map heat flow to any initial datum U_0 exists for all time and converges towards a harmonic map as time goes to infinity.
We note that the curvature assumption plays an important role in this argument. Non-positive sectional curvature leads to strong a priori estimates, which prevent the formation of singularities along the flow. The situation changes when the target manifold does not satisfy such curvature assumptions since the monotonicity of the energy alone is not enough to prevent singularities to form. Although the total energy still decreases along the flow, it may concentrate at smaller and smaller scales. If this happens, the solution can lose regularity in finite time. This phenomenon is usually described as energy concentration or bubbling.
To see this, it is often useful to have a look at the scaling of the equation and how the energy scales accordingly. Indeed, for a map $U : [0, \infty) \times \mathbb{R}^d \to N$ the harmonic map heat flow (1.2.1) is invariant under the parabolic scaling

$$U_\lambda(t, x) = U\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right), \qquad \lambda > 0, \tag{1.2.2}$$

and if we fix the dimension of the domain manifold M to be d the Dirichlet energy scales under this transformation as

$$E[U_\lambda](t) = \lambda^{d-2} E[U]\left(\frac{t}{\lambda^2}\right).$$

Thus, the energy is invariant under the natural scaling precisely in dimension $d = 2$. For this reason, the harmonic map heat flow is called *energy-critical* in this case and *energy-supercritical* for $d \geq 3$.
In the energy-critical case, Struwe proved in [60] the existence of global weak solutions which are smooth except at possibly finitely many singular points. Chang, Ding and Ye later showed in [14] that finite-time singularities can indeed occur, for example for maps into the two-sphere.
In the energy-supercritical case Chen and Struwe proved the existence of global weak solutions which are smooth in a dense subset of M, see [17]. But due to the fact that the energy now becomes smaller at smaller scales $\lambda \to 0$, the energy gives less control over the behaviour of the solution and singularity formation can occur not only via convergence towards a harmonic sphere but also via so-called *self-similar solutions* [61]. That both of these scenarios do actually occur was proven in [35] and [4], see also [18] for a first instance of a blowup result for $d \geq 3$.
Thus, without the curvature assumptions of Eells and Sampson, the harmonic map heat flow is not only a tool for constructing harmonic maps, but also a natural model for studying singularity formation in itself. Once one has found such a blowup solution, it is natural to ask whether this behaviour also occurs for nearby initial data. In other words, one wants to know whether the observed singular behaviour is stable under small perturbations of the initial data. Such stability questions help to distinguish singularities that are dynamically relevant from those that depend on special symmetries or on the special choice of the initial data.

1.3. The Wave Maps Equation

A different direction starting from harmonic maps is obtained by changing the geometry of the domain. Harmonic maps are defined for maps between Riemannian manifolds and arise as critical points of the Dirichlet energy. If the Riemannian metric on the domain is replaced by a Lorentzian metric, one obtains the corresponding hyperbolic analogue of the harmonic map equation.
For the precise mathematical setting we let (M, η) be a Lorentzian manifold[1] and (N, h) a Riemannian manifold. A map $U : M \to N$ is called a *wave map* if it is a critical point of the following action functional

$$\mathcal{L}[U] = \frac{1}{2} \int_M |dU|_h^2 \, d\,\mathrm{vol}_\eta \,.$$

The associated Euler-Lagrange equation is in local coordinates given by

$$\Box_\eta U^k + \eta^{\alpha\beta} \Gamma_{ij}^k(U) \partial_\alpha U^i \partial_\beta U^j = 0, \qquad k = 1, ..., \dim N \tag{1.3.1}$$

and it is called the *wave maps equation*. Here $\Box_\eta$ denotes the wave operator on the Lorentzian domain M and Γ_{ij}^k are again the Christoffel-symbols of the target manifold N. Compared to the harmonic map equation (1.1.2) the Laplace-Beltrami operator is replaced by the Lorentzian wave operator so that the elliptic system turns into a hyperbolic one.
In the important case where the domain is the $(1 + d)$-dimensional Minkowski space $\mathbb{R}^{1+d}$ we use standard coordinates $(t, x_1, ..., x_d)$ and equip the domain with the Lorentzian metric

$$\eta = -dt^2 + dx_1^2 + dx_2^2 + ... + dx_d^2.$$

Then the wave maps equation becomes

$$\partial_t^2 U^k = \Delta_x U^k + \eta^{\alpha\beta} \Gamma_{ij}^k(U) \partial_\alpha U^i \partial_\beta U^j, \qquad k = 1, ..., \dim N. \tag{1.3.2}$$

We remark again that if the target manifold is the Euclidean space, the Christoffel-symbols on the right-hand side vanish and one recovers the free linear wave equation.
This makes the wave maps equation a natural model problem in geometric analysis. On the one hand, it is close enough to the classical wave equation so that one can use tools from the theory of hyperbolic PDEs, such as energy and Strichartz estimates as well as finite speed of propagation and on the other hand the nonlinearity arises naturally from the geometry of the target manifold. For this reason, wave maps have played an important role in the study of nonlinear geometric wave equations and are often viewed as model equations for more complicated systems from mathematical physics and general relativity. In particular, wave maps appear in the physics literature under the name of nonlinear σ-models, see [50]. They are studied as physical field theories in their own right and are also closely related to general relativity, since they share several important geometric and analytic features with the Einstein field equations.

[1]For the relevant background on Lorentzian manifolds, we refer to [2].

Another central feature of the wave maps equation is its conserved energy. For a wave map $U : \mathbb{R}^{1+d} \to N$ its energy is given by

$$E[U](t) = \frac{1}{2} \int_{\mathbb{R}^d} |\partial_t U(t,x)|_h^2 + |\nabla_x U(t,x)|_h^2 \, dx$$

and the time-independence of this functional can formally be seen by differentiating it with respect to time and using the wave maps equation (1.3.2). This conservation law is one of the main tools in the analysis of the wave maps equation and it also shows an important contrast to the harmonic map heat flow: While the energy decreases along the harmonic map heat flow, it is conserved for wave maps. Nevertheless, in both settings singularity formation is closely related to concentration at small scales. As in the parabolic case, the role of the dimension is reflected in the natural scaling of the equation. This is also true for the wave maps equation. We therefore consider

$$U_\lambda(t,x) := U\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right), \quad \lambda > 0 \tag{1.3.3}$$

and under this scaling the energy transforms as

$$E[U_\lambda](t) = \lambda^{d-2} E[U]\left(\frac{t}{\lambda}\right).$$

Thus the energy is also in this case invariant under the natural scaling of the equation precisely in dimension $d = 2$. For this reason, the $(1+2)$-dimensional problem is called energy-critical and in dimensions $d \geq 3$, which will be the main focus of this thesis, it is energy-supercritical. In the latter regime the conserved energy does not control the natural scaling of the equation, which is one of the reasons why finite-time blowup is expected in the energy-supercritical case.

In the energy-subcritical case $d = 1$, on the other hand, the conserved energy gives enough control to rule out the concentration phenomena which are expected to cause singularities. Consequently, smooth finite-energy initial data give rise to global smooth solutions, see for example [42, 38].

The energy-critical case is more subtle, since the conserved energy is exactly invariant under the scaling of the equation. This makes it possible for energy to concentrate at small scales without changing its size. For small energy data, global regularity was proven in the works of Tataru [66, 67] and Tao [63, 64]. The large-energy theory is more delicate and depends strongly on the geometry of the target. Important developments in this direction include for example the work of Shatah and Struwe [57] and Sterbenz and Tataru [59]. On the other hand, for corotational wave maps into the sphere, finite-time blowup was constructed by Krieger, Schlag and Tataru [46] and by Rodnianski and Sterbenz [55].

In the energy-supercritical case the conserved energy is below the scaling of the equation and therefore does not prevent concentration at small scales. This makes finite-time blowup an expected phenomenon in this case. The first construction of a blowup solution is due to Shatah, who considered wave maps into the sphere under symmetry assumptions, see [56]. Under these assumptions, the wave maps equation reduces to a scalar semilinear

wave equation. This reduction has made it possible to construct and study singular solutions in considerable detail, see for example [9, 11, 22, 10].
The existence of finite-time blowup has also been established for certain more general rotationally symmetric target manifolds by Cazenave, Shatah and Tahvildar-Zadeh, see [13]. Their variational argument proves the existence of self-similar solutions, but the approach does not provide enough information about the solutions and the corresponding linearized equations which is needed for a rigorous stability analysis. It is therefore natural to ask whether the stable blowup mechanisms known for the sphere persist when the geometry of the target manifold is changed. This question is one of the main motivations for this thesis. We will study stable blowup for the harmonic map heat flow and the wave maps equation in the energy-supercritical setting, with a particular focus on self-similar solutions. After introducing these solutions in more detail, we will investigate whether the corresponding stable blowup behaviour persists when the geometry of the round sphere is slightly perturbed.

2. Self-similar blowup into perturbed spheres

2.1. The corotational ansatz

Before considering perturbations of the geometry of the target, we first discuss the case where the target manifold is the actual round d-sphere $\mathbb{S}^d$. Thus, for the harmonic map heat flow we consider maps $U : [0,\infty) \times \mathbb{R}^d \to \mathbb{S}^d$, whereas for wave maps we consider maps $U : \mathbb{R}^{1+d} \to \mathbb{S}^d$. In local coordinates the harmonic map heat flow is then given by

$$\partial_t U^k = \Delta_x U^k + \delta^{\alpha\beta}\Gamma^k_{ij}(U)\partial_\alpha U^i \partial_\beta U^j, \qquad k = 1, ..., d \tag{2.1.1}$$

and the wave maps equation by

$$\partial_t^2 U^k = \Delta_x U^k + \eta^{\alpha\beta}\Gamma^k_{ij}(U)\partial_\alpha U^i \partial_\beta U^j, \qquad k = 1, ..., d. \tag{2.1.2}$$

Away from the poles, we can view the sphere as a so-called *warped product manifold*

$$\mathbb{S}^d = (0,\pi) \times_{\sin} \mathbb{S}^{d-1}$$

with metric

$$h = du^2 + \sin^2(u) d\Omega^2,$$

where $d\Omega^2$ is the usual round metric on $\mathbb{S}^{d-1}$. For more details about warped product manifolds we refer the reader to [52, 16, 58]. If we choose spherical coordinates (r,ω) on $\mathbb{R}^d$ and polar coordinates (u,Ω) on $\mathbb{S}^d$, we can write every map U into the sphere as

$$U(t,r,\omega) = (u(t,r,\omega), \Omega(t,r,\omega)).$$

A *corotational map* is now of the form

$$U(t,r,\omega) = (u(t,r),\omega)$$

so that U is fully determined by a scalar function $u(t,r)$, which we call its *radial profile*. Substituting this ansatz into the harmonic map heat flow (2.1.1) and the wave maps equation (2.1.2) reduces both systems to semilinear, d-dimensional, radial heat/wave equations. For the harmonic map heat flow one obtains

$$\partial_t u = \Delta_{\mathrm{rad},d} u - \frac{d-1}{r^2}\sin(u)\cos(u) \tag{2.1.3}$$

and for wave maps

$$\partial_t^2 u = \Delta_{\mathrm{rad},d} u - \frac{d-1}{r^2}\sin(u)\cos(u), \tag{2.1.4}$$

where $\Delta_{\mathrm{rad},d} = \partial_r^2 + \frac{d-1}{r}\partial_r$ is the radial Laplacian in d dimensions.

2.2. Self-similar solutions

We are now going to introduce the blowup solutions which we are going to consider namely so-called *self-similar solutions*. These are special solutions which are adapted to the natural scaling of the corresponding equation and therefore provide natural candidates for finite-time blowup.
Combining the parabolic scaling (1.2.2) with the time translation invariance of the equation leads to the following *self-similar* ansatz for (2.1.3)

$$u(t,r) = f\left(\frac{r}{\sqrt{T-t}}\right), \quad T > 0.$$

Here T denotes the blowup time and f is called the *self-similar profile*. In the *similarity variable*

$$\rho = \frac{r}{\sqrt{T-t}}$$

the corotational harmonic map heat flow (2.1.3) reduces to the ordinary differential equation

$$0 = f''(\rho) + \left(\frac{d-1}{\rho} - \frac{\rho}{2}\right) f'(\rho) - \frac{d-1}{\rho^2}\sin(f(\rho))\,\cos(f(\rho)).$$

For the harmonic map heat flow into the round sphere, self-similar profiles have been studied in several works. Fan proved in [35] that (2.1.3) admits infinitely many self-similar solutions in dimensions $d \in \{3,4,5,6\}$. That these are indeed the only dimensions in which self-similar blowup can occur was proven by Bizoń and Wasserman in [12]. In view of the stability questions discussed above, one is particularly interested in profiles which persist under small perturbations of the initial data. In dimension $d = 3$, Biernat and Donninger constructed such a profile in [6], whose asymptotic nonlinear stability was then proven by Biernat, Donninger and Schörkhuber in [8]. Recently, the corresponding result was extended to dimensions $4 \leq d \leq 6$ in [1]. These results show that, for initial data sufficiently close to the self-similar solution, the flow still blows up in finite time and converges back to the same profile after passing to similarity variables. Thus the profile is not only an example of finite-time blowup, but describes a stable blowup mechanism for the harmonic map heat flow.
For the wave maps equation the natural hyperbolic scaling is given by (1.3.3). If we also combine this with the time translation invariance we end up with the following self-similar ansatz for (2.1.4)

$$u(t,r) = f\left(\frac{r}{T-t}\right), \quad T > 0.$$

Here again T denotes the blowup time and f the self-similar profile. Substituting this ansatz into the corotational wave maps equation (2.1.4) gives

$$0 = (1-\rho^2)f''(\rho) + \left(\frac{d-1}{\rho} - 2\rho\right) f'(\rho) - \frac{d-1}{\rho^2}\sin(f(\rho))\,\cos(f(\rho)),$$

where the similarity variable is now

$$\rho = \frac{r}{T-t}.$$

For wave maps into the round sphere, an important self-similar solution is explicitly given by

$$u_0^T(t,r) = f_0\left(\frac{r}{T-t}\right) \quad \text{for} \quad f_0(\rho) = 2\arctan\left(\frac{\rho}{\sqrt{d-2}}\right), \quad T>0. \tag{2.2.1}$$

This solution is usually referred to as the *ground state self-similar solution* and it was first found by Turok and Spergel in [68] in three space dimensions and Bizoń and Biernat [10] later on found the corresponding profile in higher dimensions $d \geq 4$. Later works then established (nonlinear) stability results for this profile. In particular, it is known that this profile describes a stable self-similar blowup mechanism for corotational wave maps into the round sphere in all energy-supercritical dimensions, see [28, 22, 20, 19, 30, 31, 39]. More recently, mode stability under perturbations without symmetry assumptions has been established first in dimension $d = 3$ and then in all dimensions $d \geq 4$, see [69, 26]. Building on these results, the full nonlinear asymptotic stability of the profile under small perturbations without symmetry assumptions has now been proved for every $d \geq 3$, see [25].

2.3. Perturbed spheres

The results discussed so far concern the round sphere as the target manifold. This is a natural first case to study since the sphere has a high degree of symmetry and this then leads to an explicit reduced equation under the corotational ansatz. It is therefore natural to ask whether the blowup mechanisms described above are tied to this particular symmetry of the target manifold, or whether they persist when the geometry of the target is slightly perturbed. This leads to the class of target manifolds which we denote as *perturbed spheres*.

Definition 2.3.1 (Perturbed spheres)**.** *Let $\alpha \in C^\infty(\mathbb{R})$ be a non-trivial, real-valued, even $2\pi-$periodic function with $\alpha(0) = \alpha(\pi) = 0$. For $|\varepsilon| \leq \varepsilon_0 := (2\|\alpha\|_\infty)^{-1}$ we define the warped product manifold (S^d_ε, h) by*

$$S^d_\varepsilon := (0,\pi) \times_{w_\varepsilon} \mathbb{S}^{d-1} \tag{2.3.1}$$

equipped with the warping function $w_\varepsilon : \mathbb{R} \to \mathbb{R}$,

$$w_\varepsilon(u) := \sin(u)(1 + \varepsilon\,\alpha(u)). \tag{2.3.2}$$

In coordinates $(u,\Omega) \in (0,\pi)\times\mathbb{S}^{d-1}$ the metric on S^d_ε is then given by

$$h = du^2 + w_\varepsilon(u)^2 d\Omega^2$$

with $d\Omega^2$ again denoting the standard round metric on $\mathbb{S}^{d-1} \hookrightarrow \mathbb{R}^d$.

The assumptions on α imply that w_ε is a smooth, odd, 2π-periodic function with

$$w_\varepsilon'(0) = 1, \qquad w_\varepsilon'(\pi) = -1,$$

and which satisfies by the choice of ε_0

$$w_\varepsilon(u) > 0 \quad \text{for} \quad u \in (0, \pi).$$

From these properties now follows that the north and south pole can also be added to S_ε^d so that we obtain a smooth, compact, d-dimensional Riemannian manifold, see [3].
We remark that in the unperturbed case $\varepsilon = 0$, one has

$$w_0(u) = \sin(u)$$

and the corresponding metric is just the round metric on $\mathbb{S}^d$. For small $|\varepsilon| > 0$, the warping function w_ε is a small perturbation of the warping function from the d-sphere, which is why we refer to S_ε^d as a *perturbed sphere*.
If we now consider corotational maps $U : [0, \infty) \times \mathbb{R}^d \to S_\varepsilon^d$ for the harmonic map heat flow and $U : \mathbb{R}^{1+d} \to S_\varepsilon^d$ for the wave maps equation the corresponding equations transform into (see [62] or [13] for a derivation)

$$\partial_t u = \partial_r^2 u + \frac{d-1}{r}\partial_r u - \frac{d-1}{r^2} w_\varepsilon(u)\, w_\varepsilon'(u) \tag{2.3.3}$$

and

$$\partial_t^2 u = \partial_r^2 u + \frac{d-1}{r}\partial_r u - \frac{d-1}{r^2} w_\varepsilon(u)\, w_\varepsilon'(u). \tag{2.3.4}$$

Comparing these equations with the corresponding equations for maps into the round sphere, we see that the perturbation only appears in the nonlinearity. Moreover, expanding this term in ε shows that the zeroth-order term agrees with the one from the round sphere. We will use this fact in the next chapter to construct self-similar blowup solutions for the perturbed equations and then prove their asymptotic nonlinear stability.

3. Existence and stability of self-similar solutions into perturbed spheres

We saw in the previous chapter that the corotational equations (2.3.3) and (2.3.4) into perturbed spheres differ from the corresponding equations into the round sphere only via the nonlinear term. Since this perturbation is additionally small for small values of ε, it is natural to try to construct self-similar blowup solutions for perturbed spheres as perturbations of the already known self-similar profiles for the round sphere.
The argument is based on the fact that the linearized operator around the original blowup profile is invertible (in suitable function spaces) due to the already known asymptotic nonlinear stability. The remaining operator can then be dealt with via parameter depending Schauder-type estimates.
For the stability analysis we then linearize around the just constructed blowup solution and show that the spectral structure from the linearized operator for the sphere persists under small parameters ε. In the following, we give a common overview of the method for both the harmonic map heat flow and the wave maps equation. The detailed arguments are then presented in the last two chapters, where the method is adapted to each equation separately.

3.1. Existence of self-similar blowup solutions into S^d_ε

We have already seen that for corotational maps into perturbed spheres, both the harmonic map heat flow and the wave maps equation can be written as a semilinear radial d-dimensional heat/wave equation for the radial profile u, see (2.3.3) and (2.3.4). Due to the fact that regular corotational maps vanish at the origin, it is convenient to consider the new dependent variable

$$\widetilde{v}(t,r) := r^{-1}u(t,r).$$

This transforms both equations into semilinear radial equations in now $n := d+2$ dimensions

$$\partial_t \widetilde{v}(t,r) = \partial_r^2 \widetilde{v}(t,r) + \frac{d+1}{r}\partial_r \widetilde{v}(t,r) + \frac{d-1}{r^3}\left(r\,\widetilde{v}(t,r) - w_\varepsilon(r\,\widetilde{v}(t,r))\,w_\varepsilon'(r\,\widetilde{v}(t,r))\right)$$

and

$$\partial_t^2 \widetilde{v}(t,r) = \partial_r^2 \widetilde{v}(t,r) + \frac{d+1}{r}\partial_r \widetilde{v}(t,r) + \frac{d-1}{r^3}\left(r\,\widetilde{v}(t,r) - w_\varepsilon(r\,\widetilde{v}(t,r))\,w_\varepsilon'(r\,\widetilde{v}(t,r))\right).$$

Even though the radial Laplacian is now in $d+2$ dimensions the nonlinearity is now generated via a smooth function. Setting $v(t,x) := \widetilde{v}(t,|x|)$ we can reformulate these two equations as a semilinear heat/wave equation on $\mathbb{R}^n$ respectively

$$\partial_t v(t,x) = \Delta_x v(t,x) + \frac{d-1}{|x|^3}\left(|x|\, v(t,x) - w_\varepsilon(|x|\, v(t,x))\, w_\varepsilon'(|x|\, v(t,x))\right)$$

and

$$\partial_t^2 v(t,x) = \Delta_x v(t,x) + \frac{d-1}{|x|^3}\left(|x|\, v(t,x) - w_\varepsilon(|x|\, v(t,x))\, w_\varepsilon'(|x|\, v(t,x))\right).$$

Introducing now similarity coordinates

$$\tau = \log\left(\frac{T}{T-t}\right) \quad \text{and} \quad y = \frac{x}{\sqrt{T-t}}$$

for the harmonic map heat flow and

$$\tau = \log\left(\frac{T}{T-t}\right) \quad \text{and} \quad \xi = \frac{x}{T-t}$$

for the wave maps equation we obtain an evolution equation (after doing a first order reformulation in the wave maps case) of the form

$$\partial_\tau \psi(\tau) = L\psi(\tau) + N_\varepsilon(\psi(\tau)), \tag{3.1.1}$$

where L generates the free heat/wave operator in similarity coordinates and N_ε is the nonlinear part with the parameter ε appearing through the warping function w_ε.
A self-similar profile is now a static solution to this equation. We therefore want to solve

$$0 = L\psi_\varepsilon + N_\varepsilon(\psi_\varepsilon). \tag{3.1.2}$$

For $\varepsilon = 0$, the target manifold is the round sphere and we know in both cases the existence of a stable self-similar profile ψ_0 to this equation. Due to the fact that the geometry of our perturbed spheres is somehow close to the one from the perfectly round sphere, one expects that the self-similar profile for sufficiently small ε can be constructed as a small perturbation of the original blowup profile ψ_0 for the sphere.
We therefore make the ansatz

$$\psi_\varepsilon = \psi_0 + \phi_\varepsilon$$

for a small unknown perturbation ϕ_ε. Substituting this ansatz into the stationary equation (3.1.2) gives a perturbation equation for ϕ_ε which can be written as

$$-L_0\phi_\varepsilon = V_\varepsilon(\psi_0)\phi_\varepsilon + \widetilde{N}_\varepsilon(\phi_\varepsilon) + \mathcal{R}_\varepsilon(\psi_0).$$

Here, L_0 is the linearization around ψ_0, $V_\varepsilon(\psi_0)\phi_\varepsilon$ is the part of the potential which remains after subtracting the linearized operator coming from the sphere, $\widetilde{N}_\varepsilon(\phi_\varepsilon)$ behaves at least

quadratically in its argument and $\mathcal{R}_\varepsilon(\psi_0)$ is the remainder term which arises by inserting the spherical profile into the perturbation equation.
To solve this equation, we use the spatially global framework developed by Glogić in his work [39]. More precisely, we introduce similarity coordinates on the whole space and work in intersection homogeneous Sobolev spaces. The combination of a lower and a higher Sobolev regularity provides both the decay properties needed for the linear evolution and the regularity needed to control the nonlinear terms. For the harmonic map heat flow we take a similar approach as in [1] from Angerer, Kistner and Schörkhuber.
In these function spaces, the spectral analysis of the linearized operator around the spherical profile shows that 0 belongs to the resolvent set of L_0, so that it can be inverted in these function spaces independently of ε. We therefore obtain a fixed-point equation for the perturbation ϕ_ε which can be solved via an application of Banach's fixed-point theorem. The necessary estimates for the operators on the right-hand side follow from parameter depending Schauder-type estimates, see Proposition A.2.1. From this construction we additionally obtain the Lipschitz dependence of ψ_ε with respect to the parameter ε with respect to the relevant norms, which will be crucial for the stability analysis later on.
To further obtain additional regularity and decaying properties of the blowup solution one returns to the ordinary differential equation which is solved by the self-similar profile. Standard ODE arguments then show that the constructed profile is smooth and has additional decaying behaviour at infinity, which will be required later on. Transforming back to the original variables then gives a self-similar finite-time blowup solution for the corresponding harmonic map heat flow and the wave maps equation into perturbed spheres.

3.2. Stability of self-similar blowup solutions into S^d_ε

We now turn to the stability problem. Now that we have constructed the perturbed self-similar profile, the next question is whether the corresponding blowup behaviour persists under small perturbations of the initial data.
We start with the evolution equation in similarity variables from (3.1.1) and make the ansatz

$$\psi(\tau) = \psi_\varepsilon + \phi_\varepsilon(\tau).$$

Plugging this into (3.1.1) and linearizing around ψ_ε leads to an evolution equation for the perturbation

$$\partial_\tau \phi(\tau) = L_\varepsilon \phi(\tau) + \widehat{N}_\varepsilon(\phi(\tau)). \tag{3.2.1}$$

Here $L_\varepsilon = L + L'_\varepsilon$ denotes the linearized operator around the perturbed profile ψ_ε and $\widehat{N}_\varepsilon$ the nonlinear remainder, which is at least quadratic in its argument.
One first shows that the potential L'_ε is bounded. Hence, L_ε generates a strongly continuous semigroup $(S_\varepsilon(\tau))_{\tau\geq 0}$ and the evolution equation (3.2.1) can be written as an integral equation

$$\phi_\varepsilon(\tau) = S_\varepsilon(\tau)u + \int_0^\tau S_\varepsilon(\tau-\tau')\widehat{N}_\varepsilon(\phi_\varepsilon(\tau'))\,d\tau'. \tag{3.2.2}$$

The stability of ψ_ε would follow directly if solutions to arbitrary small initial data u exist globally and decay exponentially as $\tau \to \infty$. The obstruction is that L_ε has an unstable eigenvalue $\lambda = 1$. This eigenvalue is expected, as it comes from the time translation invariance with respect to the blowup time T of the original equation and is therefore not a genuine instability of the blowup mechanism.
The main task is therefore to show that this is indeed the only unstable spectral point of L_ε. For the round sphere, the corresponding result is known: apart from the simple eigenvalue $\lambda = 1$ the spectrum of the linearized operator L_0 is contained in a left half-plane. Since the profile ψ_ε depends Lipschitz continuously on the parameter ε, the potential in L_ε is a small perturbation from the one from L_0.
More precisely, one shows via a Neumann-series argument that the spectrum of L_ε is contained in a stable left half-plane together with a fixed compact region around $\lambda = 1$. The remaining compact region is then treated by using the (Lipschitz-)continuous dependence of the corresponding Riesz projections with respect to ε.
We therefore obtain the required decay of the linearized evolution on the stable subspace. More precisely, if P_ε denotes the spectral projection onto the unstable eigenspace corresponding to $\lambda = 1$ we obtain the existence of an $\omega > 0$ such that for every sufficiently small ε we have a uniform estimate of the form

$$\|S_\varepsilon(\tau)(I - P_\varepsilon)u\| \lesssim e^{-\omega\tau} \|(I - P_\varepsilon)u\| .$$

On the other hand, the evolution grows like e^τ on the unstable subspace. To remove this unstable direction we add a suitable correction term to the initial data. The resulting Cauchy problem can then be solved by a fixed-point argument in a space of exponentially decaying functions. By varying the blowup time T one can find an exponentially decaying solution to the Cauchy problem for which the correction term vanishes, this is then an exponentially decaying solution to the original Cauchy problem.
Transforming back to the original variables then shows that the solution blows up at time T and converges to the perturbed self-similar profile in similarity variables.

General Notations

For two real numbers $A, B \in \mathbb{R}$ we write $A \lesssim B$ if there exists an absolute constant $C > 0$ such that $A \leq CB$ holds. For a constant $\omega \in \mathbb{R}$ we denote by $\mathbb{H}_\omega := \{z \in \mathbb{C} : \mathrm{Re}\,(z) > \omega\}$ the open right half-plane of ω.
Furthermore, if $\mathcal{H}$ is a Hilbert space and $(L, \mathcal{D}(L))$ a closed linear operator on $\mathcal{H}$ we denote by $R_L(\lambda) := (\lambda - L)^{-1}$ the resolvent operator for λ lying in the resolvent set $\rho(L)$. For $n \in \mathbb{N}$ we denote by $C_r^\infty(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n) : f \text{ is radial}\}$ the set of smooth radial functions and by $C_{c,r}^\infty(\mathbb{R}^n)$ the ones with additional compact support. By

$$C_e^\infty[0,\infty) := \{f \in C^\infty([0,\infty)) : f^{(2j+1)}(0) = 0 \text{ for } j \in \mathbb{N}_0\}$$

we define the set of smooth and even functions and note that there is a one-to-one correspondence between $C_r^\infty(\mathbb{R}^n)$ and $C_e^\infty[0,\infty)$. For this we will use the following convention: If $\psi \in C_r^\infty(\mathbb{R}^n)$ is a radial function, we denote its associated radial profile via $\widetilde{\psi} \in C_e^\infty[0,\infty)$, i.e. $\psi(x) = \widetilde{\psi}(|x|)$. Furthermore

$$\mathcal{S}(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n) : \forall \alpha, \beta \in \mathbb{N}_0^n : \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)| < \infty\}$$

denotes the set of Schwartz functions and $\mathcal{S}_r(\mathbb{R}^n)$ denotes the subspace of radially symmetric Schwartz functions.
Next we define the Fourier transform $\mathcal{F}u$ of $u \in C_c^\infty(\mathbb{R}^n)$ via

$$\mathcal{F}u(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} u(x) e^{-i\xi \cdot x}\, dx.$$

For $u, v \in C_c^\infty(\mathbb{R}^n)$ and $s \geq 0$ we define the inner product

$$\langle u, v \rangle_{\dot{H}^s(\mathbb{R}^n)} := \langle |\cdot|^s \mathcal{F}u, |\cdot|^s \mathcal{F}v \rangle_{L^2(\mathbb{R}^n)},$$

and the induced norm $\|u\|_{\dot{H}^s(\mathbb{R}^n)}^2 := \langle u, u \rangle_{\dot{H}^s(\mathbb{R}^n)}$. If $k \in \mathbb{N}_0$ is a nonnegative integer

$$\|u\|_{\dot{H}^k(\mathbb{R}^n)} \simeq \sum_{|\beta|=k} \left\|\partial^\beta u\right\|_{L^2(\mathbb{R}^n)}$$

for all $u \in C_c^\infty(\mathbb{R}^n)$, see [65] Chapter A. Finally, for $s \geq 0$ the homogeneous radial Sobolev space $\dot{H}_r^s(\mathbb{R}^n)$ denotes the space which is obtained by completion of radial test functions $C_{c,r}^\infty(\mathbb{R}^n)$ with respect to the above defined norm.

4. Stable blowup for the harmonic map heat flow into perturbed spheres

In this chapter we are now going to prove the existence and stability of self-similar blowup solutions to the harmonic map heat flow into perturbed spheres. The results of this chapter are contained in a work from the author, see [70].
We start with corotational maps $U : [0,\infty) \times \mathbb{R}^d \to S^d_\varepsilon,\ U(t,r,\omega) = (u(t,r),\omega)$, where the radial profile u solves the following semilinear radial d-dimensional heat equation

$$\partial_t u = \partial_r^2 u + \frac{d-1}{r}\partial_r u - \frac{d-1}{r^2} w_\varepsilon(u)\, w'_\varepsilon(u). \tag{4.0.1}$$

As mentioned above, for $3 \le d \le 6$ and $\varepsilon = 0$, there exists an asymptotically nonlinear stable finite-time blowup solution $\widetilde{f}_0$ to this equation, see [1]. More precisely

$$u_0^T(t,r) = \widetilde{f}_0\left(\frac{r}{\sqrt{T-t}}\right), \quad T > 0 \tag{4.0.2}$$

defines a self-similar solution of (4.0.1).
The first main result of this chapter proves the existence of a self-similar finite-time blowup solution $\widetilde{f}_\varepsilon$ to the harmonic map heat flow into S^d_ε for sufficiently small $|\varepsilon| > 0$.

Theorem 4.0.1. *Let $3 \le d \le 6$. Then there exists an $\varepsilon^* > 0$ such that for every $|\varepsilon| \le \varepsilon^*$ and every $T > 0$ there exists a self-similar solution to equation (4.0.1) whose gradient blows up in $r = 0$ as $t \to T^-$. More precisely, there exists a $u_\varepsilon^T \in C^\infty([0,T) \times [0,\infty)) \cap L^\infty([0,T) \times [0,\infty))$ of the form*

$$u_\varepsilon^T(t,r) = \widetilde{f}_\varepsilon\left(\frac{r}{\sqrt{T-t}}\right) \quad \text{with} \quad \lim_{t \to T^-} |\partial_r u_\varepsilon^T(t,0)| = \infty.$$

Additionally, the profile $\widetilde{f}_\varepsilon$ can be written as

$$\widetilde{f}_\varepsilon(\rho) = \widetilde{f}_0(\rho) + \rho\, \widetilde{\phi}_\varepsilon(\rho)$$

with $\widetilde{f}_0$ being the self-similar solution for $\varepsilon = 0$ defined in Eq. (4.0.2) and a perturbation $\widetilde{\phi}_\varepsilon$ which depends Lipschitz continuously on the parameter ε in the sense that

$$\left\| \widetilde{\phi}_\varepsilon - \widetilde{\phi}_\kappa \right\|_{W^{2,\infty}([0,\infty))} \lesssim |\varepsilon - \kappa|$$

holds for all $|\varepsilon|, |\kappa| \le \varepsilon^*$. Furthermore, $\widetilde{f}_\varepsilon$ is odd with $0 < \widetilde{f}_\varepsilon(\rho) < \pi$ for all $\rho > 0$ and $\lim_{\rho\to\infty} \widetilde{f}_\varepsilon(\rho)$ exists. Additionally, for every $k \in \mathbb{N}$ there are constants $C_k > 0$ such that

$$|\widetilde{f}_\varepsilon^{(k)}(\rho)| \le C_k \langle\rho\rangle^{-2-k} \tag{4.0.3}$$

holds for all $\rho \in [0,\infty)$.

Since the constructed blowup solution differs only via an ε-small perturbation from the original blowup solution we are able to obtain rigorous stability results for these solutions under small corotational perturbations. We set, without loss of generality, the blowup time to $T = 1$ and formulate the result in so-called *normal coordinates* on S^d_ε, as in [58], by setting

$$U^j := u\,\Omega^j, \quad \text{for} \quad j = 1, \dots, d. \tag{4.0.4}$$

In these coordinates, S^d_ε together with its north pole can be identified with the ball $B_\pi(0) \subset \mathbb{R}^d$. Thus the harmonic map heat flow can be written as the following initial value problem on $[0, \infty) \times \mathbb{R}^d$

$$\begin{cases} \partial_t U = \tau(U), \\ U(0, \cdot) = U_0 \end{cases} \tag{4.0.5}$$

for initial data $U_0 : \mathbb{R}^d \to \mathbb{R}^d$.
A corotational map $U : [0, \infty) \times \mathbb{R}^d \to S^d_\varepsilon$ in these coordinates is of the form

$$U(t, x) = u(t, |x|)\frac{x}{|x|}, \tag{4.0.6}$$

where the radial profile u is again a solution to (4.0.1). The stability result can now be formulated in the following way:

Theorem 4.0.2. *Let $3 \le d \le 6$ and take $\varepsilon^* > 0$ as in Theorem 4.0.1. We then define for every $|\varepsilon| \le \varepsilon^*$*

$$U^T_\varepsilon(t, x) := \widetilde{f}_\varepsilon\left(\frac{|x|}{\sqrt{T - t}}\right)\frac{x}{|x|}$$

and consider corotational initial data of the form

$$U_0 = U^1_\varepsilon(0, \cdot) + \nu_0,$$

where $\nu_0 : \mathbb{R}^d \to \mathbb{R}^d$ is defined as $\nu_0(x) = x v_0(|x|)$ for a radial Schwartz function $v_0(|\cdot|) \in \mathcal{S}(\mathbb{R}^d)$. Let $(s, k) \in \mathbb{R} \times \mathbb{N}$ satisfy

$$\frac{d}{2} < s \le \frac{d}{2} + \frac{1}{2d + 2}, \quad k = d + 3. \tag{4.0.7}$$

Then there exists a strictly positive $\overline{\varepsilon} \le \varepsilon^*$ such that for every $|\varepsilon| \le \overline{\varepsilon}$ there are constants $\delta > 0$ and $M_0 > 1$ such that for every $\nu_0(x) = x v_0(|\cdot|)$ as above with

$$\|\nu_0\|_{\dot{H}^s \cap \dot{H}^k(\mathbb{R}^d, \mathbb{R}^d)} < \frac{\delta}{M_0},$$

there exists a $T = T_\varepsilon \in [1 - \delta, 1 + \delta]$ and a unique corotational map $U \in C^\infty([0, T) \times \mathbb{R}^d, \mathbb{R}^d)$ that satisfies (4.0.5) for all $(t, x) \in [0, T) \times \mathbb{R}^d$. The gradient of U blows up at the origin as $t \to T^-$ and we have the decomposition

$$U(t, x) = U^T_\varepsilon(t, x) + \nu\left(t, \frac{x}{\sqrt{T - t}}\right),$$

for a function $\nu : [0,T) \times \mathbb{R}^d \to \mathbb{R}^d$ *which satisfies*

$$\|\nu(t,\cdot)\|_{\dot{H}^r(\mathbb{R}^d,\mathbb{R}^d)} \to 0 \tag{4.0.8}$$

as $t \to T^-$ *for all* $r \in [s,k]$*. In addition,*

$$U(t,(\sqrt{T-t})x) \to \widetilde{f}_\varepsilon(|x|)\frac{x}{|x|}$$

pointwise and uniformly on compact subsets of $\mathbb{R}^d$ *as* $t \to T^-$*.*

Remark 4.0.3. *Although the self-similar profile satisfies* $0 < \widetilde{f}_\varepsilon(\rho) < \pi$ *for all* $\rho > 0$*, the stability result does not assert that the radial profile function coming from the perturbed solution remains in the same interval. If the radial profile crosses* π *the corresponding solution crosses the south pole of* S^d_ε*, which cannot be covered by a single uniform coordinate chart. Nevertheless, the corresponding solution can be smoothly continued via switching from the polar representation* (u,ω) *for* $0 \le u \le \pi$ *to* $(2\pi - u, -\omega)$ *for* $\pi \le u \le 2\pi$*, i.e. continuing the solution on the opposite meridian respectively.*

4.0.1. Related results

In this section we will give a more detailed overview of self-similar blowup for the energy-supercritical harmonic map heat flow. For a more general discussion we refer the reader to for example [49].

As we have already mentioned, the first self-similar blowup constructions are due to Coron and Ghidaglia [18] and Fan [35] in dimensions $3 \le d \le 6$. It was later shown by Bizoń and Wasserman [12] that these are the only dimensions in which self-similar shrinking solutions can exist. We remark that if we allow for k-equivariant maps into spheres of higher dimension than the domain, Gastel showed in [36] that self-similar solutions exist for all supercritical dimensions $d \ge 3$.

The first stability result was obtained by Biernat and Donninger in [6] and together with Schörkhuber in [8] they constructed a spectrally stable self-similar profile in $d = 3$ dimensions and proved its asymptotic nonlinear stability. This result was now recently proven by [1] in dimensions 4 to 6 as well.

In dimensions $d \ge 4$ Glogić, Kistner and Schörkhuber proved the existence and stability of an explicit self-similar blowup profile into a compact, rotationally symmetric target manifold. This result shows that stable self-similar blowup persists also in geometric settings outside of the round sphere. In our work, instead of relying on an explicit profile for a specifically chosen target manifold, we prove that stable self-similar blowup persists under small geometrical perturbations of the sphere. Thus, our result does not only provide another example of stable blowup, but also shows the stability of the blowup mechanism itself.

4.0.2. Outline

We start with the d-dimensional semilinear heat equation (4.0.1) for the radial profile u. Setting $\widetilde{v}(t,r) := r^{-1}u(t,r)$ the equation transforms into a $(d+2)-$dimensional radial

semilinear heat equation with a smooth nonlinearity

$$\left(\partial_t - \partial_r^2 - \frac{d+1}{r}\partial_r\right)\widetilde{v}(t,r) - \frac{d-1}{r^3}\left(r\,\widetilde{v}(t,r) - w_\varepsilon(r\,\widetilde{v}(t,r))\,w_\varepsilon'(r\,\widetilde{v}(t,r))\right) = 0. \quad (4.0.9)$$

For $n := d+2$ and $v(t,x) := \widetilde{v}(t,|x|)$ we can reformulate (4.0.9) as a heat equation on $\mathbb{R}^n$

$$(\partial_t - \Delta_x)v(t,x) = \frac{n-3}{|x|^3}\left(|x|v(t,x) - w_\varepsilon(|x|v(t,x))w_\varepsilon'(|x|v(t,x))\right), \quad x \in \mathbb{R}^n. \quad (4.0.10)$$

We then pass to similarity coordinates

$$\tau = \log\left(\frac{T}{T-t}\right), \quad y = \frac{x}{\sqrt{T-t}}$$

for $T > 0$ and $(t,x) \in [0,T) \times \mathbb{R}^n$ and in these coordinates (4.0.10) turns into an evolution equation of the form

$$\partial_\tau \psi(\tau) = L\psi(\tau) + N_\varepsilon(\psi(\tau)), \quad (4.0.11)$$

where L generates the free heat evolution in similarity coordinates and N_ε is the nonlinear part with the parameter ε appearing through the warping function w_ε. A detailed derivation of this equation is given in Section 4.1 below. We study these operators in intersection homogeneous Sobolev spaces of radial functions

$$X_s^k = \dot{H}_r^s(\mathbb{R}^n) \cap \dot{H}_r^k(\mathbb{R}^n)$$

for suitable exponents $\frac{n}{2} - 1 < s < \frac{n}{2} < k$, $k \in \mathbb{N}$.

Existence of self-similar blowup profiles

Since self-similar solutions become static in similarity coordinates we are looking for a ψ_ε solving

$$L\psi_\varepsilon + N_\varepsilon(\psi_\varepsilon) = 0. \quad (4.0.12)$$

Due to the fact that the geometry of the perturbed spheres is somehow close to the geometry of the perfectly round d-sphere it is natural to look for self-similar solutions which are also somehow close to the original self-similar solution for the sphere. We therefore make the ansatz $\psi_\varepsilon = \psi_0 + \phi_\varepsilon$ where ψ_0 is a static solution to (4.0.11) for $\varepsilon = 0$ and obtain a perturbation equation for ϕ_ε which can be written as

$$-L_0\phi_\varepsilon = V_\varepsilon(\psi_0)\phi_\varepsilon + \widetilde{N}_\varepsilon(\phi_\varepsilon) + \mathcal{R}_\varepsilon(\psi_0). \quad (4.0.13)$$

The key idea here is that we do not want to invert the entire linear part of this equation but only $L_0 := L + V_0(\psi_0)$ which is precisely the linearization around ψ_0 in the case where ε is equal to 0. The warping function w_ε from (2.3.2) is chosen in such a way so that the remaining potential $V_\varepsilon(\psi_0)$ can be made arbitrarily small (for sufficiently small ε).

The same holds true for the remainder term $\mathcal{R}_\varepsilon(\psi_0)$ and the (now quadratically small) nonlinearity $\widetilde{N}_\varepsilon$. A precise definition of all these operators can be found in Section 4.1.
The linearized operator L_0 is known to be self-adjoint in an exponentially weighted L^2-space and that its spectrum consists only of isolated eigenvalues with finite multiplicity. The same result holds true if we restrict the operator onto X_s^k in which space we can then consider the right-hand side of (4.0.13), see [1]. To prove the necessary estimates for these operators we use the parameter-dependent Schauder-type estimates from Proposition A.2.1, where we will assume

$$\frac{n}{2} - 1 < s \leq \frac{n}{2} - 1 + \frac{1}{2n-2}, \quad k = n + 1. \tag{4.0.14}$$

With that we can show that the operators on the right-hand side of (4.0.13) define maps $X_s^k \to X_s^k$ which are Lipschitz-continuous with respect to the parameter ε, see Lemma 4.1.3.
Due to the invertibility of L_0 in X_s^k we rewrite (4.0.13) as a fixed-point equation and apply Banach's fixed-point theorem for sufficiently small $|\varepsilon|$.
The smoothness and decay properties of the resulting solution ψ_ε to Eq. (4.0.12) follow from an ODE analysis similar to [6]. Finally, transforming back to the original variables completes the proof of Theorem 4.0.1.

Stability of the self-similar blowup solution

In Section 4.2 we show the asymptotic nonlinear stability of the blowup solution from Theorem 4.0.1. We start with the semilinear heat equation (4.0.10) on $\mathbb{R}^n$ with perturbed initial data around the self-similar blowup solution that blows up at time $T = 1$. That is we are considering

$$\begin{cases} \partial_t v - \Delta_x v = \frac{n-3}{|x|^3} \left(|x| v - w_\varepsilon(|x|v) w_\varepsilon'(|x|v) \right) \\ v(0,x) = v_\varepsilon^1(0,x) + \varphi_0(x). \end{cases}$$

We then again introduce similarity coordinates to obtain

$$\begin{cases} \partial_\tau \psi(\tau) = L\psi(\tau) + N_\varepsilon(\psi(\tau)), \\ \psi(0) = \psi_\varepsilon^T + \varphi_0^T \end{cases}$$

and note that the only trace of the time parameter T now lies in the initial condition. If we then make the (now time-dependent) ansatz $\psi_\varepsilon(\tau) = \psi_\varepsilon + \phi_\varepsilon(\tau)$, linearize around ψ_ε, we end up with the central evolution equation of this chapter

$$\begin{cases} \partial_\tau \phi_\varepsilon(\tau) = L_\varepsilon \phi_\varepsilon(\tau) + \widehat{N}_\varepsilon(\phi_\varepsilon(\tau)), \\ \phi_\varepsilon(0) = U_{\varepsilon,T}(\varphi_0). \end{cases} \tag{4.0.15}$$

Here L_ε denotes the linearization around the self-similar profile ψ_ε. More precisely, it is given by the heat operator in similarity coordinates together with a potential term depending on ψ_ε. The remaining nonlinearity $\widehat{N}_\varepsilon$ is quadratically in its argument and

$U_{\varepsilon,T}$ is the initial data operator and contains the dependence on the blowup time T. For the precise definition of the above defined operators we refer to Section 4.2.
The stability problem is therefore reduced to the analysis of the Cauchy problem (4.0.15). If one could construct global solutions which decay exponentially as τ goes to ∞ for all sufficiently small initial data, then the asymptotic stability of the profile ψ_ε would follow. However, the linearized operator L_ε has a simple unstable eigenvalue at $\lambda = 1$, see Lemma 4.2.5. Thus one cannot expect exponential decay for arbitrary small initial data without first removing this unstable direction.
Therefore, the main task is to show that this is the only unstable eigenvalue of the linearized operator. For $\varepsilon = 0$ the corresponding spectral properties are known: apart from the simple eigenvalue $\lambda = 1$, the spectrum of the linearized operator L_0 is contained in a left half-plane, see Proposition 4.1.1. We prove that this spectral structure remains the same for L_ε for ε sufficiently small. To do this, we first use a Neumann-series argument to show that the spectrum of L_ε is contained (uniformly for sufficiently small ε) in a left half-plane together with a fixed compact region around the unstable eigenvalue $\lambda = 1$.
For the remaining compact region we use the fact that the Riesz-projections also depend (Lipschitz-)continuous on the parameter ε so that there cannot exist any other spectral points in that region due to dimensional reasons, see for example [44], p.34, Lemma 4.10.
The artificial instability (generated by the time translation invariance with respect to the blowup time $T = 1$) can be dealt with via first suppressing the exponential growth induced by the symmetry eigenvalue $\lambda = 1$ by a correction with values in $\operatorname{ran} P_\varepsilon$ and then one accounts for this using the $T-$dependence of the initial condition to determine the suitable blowup time T_ε.
Combining the above arguments, we can show the following stability result:

Theorem 4.0.4. *Let $5 \leq n \leq 8$ and take $\varepsilon^* > 0$ as in Theorem 4.0.1. For $|\varepsilon| \leq \varepsilon^*$ we define*

$$v_\varepsilon^T(t,x) := \frac{1}{\sqrt{T-t}}\psi_\varepsilon\left(\frac{x}{\sqrt{T-t}}\right)$$

and take $(s,k) \in \mathbb{R} \times \mathbb{N}$ satisfying

$$\frac{n}{2} - 1 < s \leq \frac{n}{2} - 1 + \frac{1}{2n-2}, \quad k = n+1. \tag{4.0.16}$$

Then there exist $\omega > 0$ and $0 < \overline{\varepsilon} \leq \varepsilon^*$ such that for every $|\varepsilon| \leq \overline{\varepsilon}$ there are $\delta > 0$ and $M > 1$ such that the following holds: For every real-valued $\varphi_0 \in \mathcal{S}_r(\mathbb{R}^n)$ satisfying

$$\|\varphi_0\|_{\dot{H}^s \cap \dot{H}^k(\mathbb{R}^n)} < \frac{\delta}{M},$$

there exists a $T = T_\varepsilon \in [1-\delta, 1+\delta]$ and a unique radial solution $v \in C^\infty([0,T) \times \mathbb{R}^n)$ to Eq. (4.0.10) with

$$v(0,\cdot) = v_\varepsilon^1(0,\cdot) + \varphi_0.$$

Moreover, v blows up at $(T,0)$ and can be decomposed as

$$v(t,x) = v_\varepsilon^T(t,x) + \frac{1}{\sqrt{T-t}}\varphi\left(\log\left(\frac{T}{T-t}\right), \frac{x}{\sqrt{T-t}}\right)$$

for all $(t,x) \in [0,T)\times\mathbb{R}^n$, where $\varphi \in C^\infty([0,\infty)\times\mathbb{R}^n)$ is radially symmetric and satisfies

$$\|\varphi(-\log(T-t)+\log T,\cdot)\|_{\dot{H}^r(\mathbb{R}^n)} \lesssim \delta(T-t)^\omega$$

for all $r \in [s,k]$.

To then finally obtain Theorem 4.0.2 we use the equivalence of norms of corotational maps and their radial profiles, see Proposition A.5 [40] .

4.1. Existence of self-similar solutions for the harmonic map heat flow into S^d_ε

From here on out we always assume $3 \le d \le 6$ and that $\alpha \in C^\infty(\mathbb{R})$ satisfies the assumptions from Definition 2.3.1. We then consider for $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \le \varepsilon_0$ the family S^d_ε of warped product manifolds with corresponding warping function w_ε defined as in (2.3.1) and (2.3.2), respectively. Our aim is now to construct for sufficiently small ε a self-similar solution to the following $n := d+2-$dimensional semilinear heat equation

$$\partial_t v - \Delta_x v = \frac{n-3}{|x|^3}\left(|x|v - w_\varepsilon(|x|v)w_\varepsilon'(|x|v)\right). \tag{4.1.1}$$

For this we first of all introduce similarity coordinates

$$\tau = \log\left(\frac{T}{T-t}\right) \quad \text{and} \quad y = \frac{x}{\sqrt{T-t}}$$

so that t and x can be written as

$$t = T - Te^{-\tau} \quad \text{and} \quad x = \sqrt{T}e^{-\frac{\tau}{2}}y$$

and observe that the differential operators transform in the following way

$$\partial_t = \frac{e^\tau}{T}\left(\partial_\tau + \frac{1}{2}y\cdot\nabla_y\right) \quad \text{and} \quad \Delta_x = \frac{e^\tau}{T}\Delta_y.$$

We now set

$$\psi(\tau,y) := \sqrt{T}e^{-\frac{\tau}{2}}v(T-Te^{-\tau}, \sqrt{T}e^{-\frac{\tau}{2}}y)\left[= \sqrt{T-t}\,v(t,x)\right]$$

so that (4.1.1) becomes

$$(\partial_\tau - \Delta_y + \Lambda)\,\psi(\tau,y) = \frac{n-3}{|y|^3}\left(|y|\psi(\tau,y) - w_\varepsilon(|y|\psi(\tau,y))\,w_\varepsilon'(|y|\psi(\tau,y))\right), \tag{4.1.2}$$

where Λ is defined as $\Lambda f(y) = \frac{1}{2}\left(y \cdot \nabla_y f(y) + f(y)\right)$. We now use the notation $\psi(\tau)(y) := \psi(\tau, y)$ and rewrite (4.1.2) as an evolution equation

$$\partial_\tau \psi(\tau) = \widetilde{L}\,\psi(\tau) + N_\varepsilon(\psi(\tau)), \tag{4.1.3}$$

where the linear operator is given by $\widetilde{L} := \Delta - \Lambda$ and the nonlinearity is defined as

$$N_\varepsilon(\psi(\tau))(y) = \frac{n-3}{|y|^3}\eta_\varepsilon(|y|\psi(\tau, y)) \quad \text{for} \quad \eta_\varepsilon(z) := z - w_\varepsilon(z)\, w_\varepsilon'(z). \tag{4.1.4}$$

Since self-similar solutions become static in similarity coordinates we are looking for a ψ_ε solving

$$0 = \widetilde{L}\,\psi_\varepsilon + N_\varepsilon(\psi_\varepsilon). \tag{4.1.5}$$

The idea is now to linearize about the self-similar blowup solution for the sphere by making the ansatz $\psi_\varepsilon = \psi_0 + \phi_\varepsilon$, where ψ_0 is given by $\psi_0(y) = |y|^{-1}\widetilde{f}_0(|y|)$ and $\widetilde{f}_0$ being the self-similar profile from (4.0.2).
Since we have $0 = \widetilde{L}\,\psi_0 + N_0(\psi_0)$ plugging this ansatz into (4.1.5) leads to the following perturbation equation for ϕ_ε

$$0 = \widetilde{L}_0\,\phi_\varepsilon + V_\varepsilon(\psi_0)\phi_\varepsilon + \widetilde{N}_\varepsilon(\phi_\varepsilon) + \mathcal{R}_\varepsilon(\psi_0), \tag{4.1.6}$$

where we have defined $\widetilde{L}_0 := \widetilde{L} + V_0(\psi_0)$ for

$$V_0(\psi_0) = \frac{n-3}{|y|^2}\eta_0'(|y|\psi_0)$$

and the other operators are given by

$$\begin{aligned}
V_\varepsilon(\psi_0) &= \frac{n-3}{|y|^2}\left(\eta_\varepsilon'(|y|\psi_0) - \eta_0'(|y|\psi_0)\right), \\
\mathcal{R}_\varepsilon(\psi_0) &= \frac{n-3}{|y|^3}\left(\eta_\varepsilon(|y|\psi_0) - \eta_0(|y|\psi_0)\right), \\
\widetilde{N}_\varepsilon(u) &= \frac{n-3}{|y|^3}\left(\eta_\varepsilon(|y|(\psi_0+u)) - \eta_\varepsilon(|y|\psi_0) - \eta_\varepsilon'(|y|\psi_0)|y|u\right).
\end{aligned}$$

In the following we introduce the function spaces in which the blowup solution is constructed and in which the stability analysis is carried out later on.

4.1.1. Function spaces

For notational purposes we rename for $0 \le s < k$ the inner product of intersection homogeneous Sobolev spaces on $C^\infty_{c,r}(\mathbb{R}^n)$ via

$$\langle u, v\rangle_{X^k_s} := \langle u, v\rangle_{X^k_s(\mathbb{R}^n)} := \langle u, v\rangle_{\dot{H}^s(\mathbb{R}^n)} + \langle u, v\rangle_{\dot{H}^k(\mathbb{R}^n)}$$

and $X_s^k := X_s^k(\mathbb{R}^n)$ is then defined as the completion of radial test functions $C_{c,r}^\infty(\mathbb{R}^n)$ with respect to the induced norm $\|\cdot\|_{X_s^k}$. Additionally, we define

$$\mathcal{H} := \left\{ f : \mathbb{R}^n \to \mathbb{C} \text{ measurable and radial with } \int_{\mathbb{R}^n} |f(x)|^2 e^{-\frac{|x|^2}{4}}\, dx < \infty \right\}$$

as the exponentially weighted L^2-space of radial functions.

For the rest of this chapter we will always assume the following condition onto the exponents s and k

$$\frac{n}{2} - 1 < s \leq \frac{n}{2} - 1 + \frac{1}{2n-2}, \quad k = n+1. \tag{4.1.7}$$

We will now state the main properties of the free part $\widetilde{L}$ and its linearization $\widetilde{L}_0$ appearing in (4.1.6). A proof of these results can be found in [1] Section 3.

Proposition 4.1.1. *The operators $\widetilde{L}, \widetilde{L}_0 : C_{c,r}^\infty(\mathbb{R}^n) \subset \mathcal{H} \to \mathcal{H}$ are closable. Their closures $(L, \mathcal{D}(L))$ and $(L_0, \mathcal{D}(L_0))$ satisfy $\mathcal{D}(L) = \mathcal{D}(L_0)$, they are both self-adjoint, have compact resolvents and generate strongly continuous semigroups $(S(\tau))_{\tau\geq 0}$ and $(S_0(\tau))_{\tau\geq 0}$ on $\mathcal{H}$. The semigroup $(S(\tau))_{\tau\geq 0}$ admits the explicit representation*

$$[S(\tau)f](x) = e^{-\frac{\tau}{2}} (H_{\alpha(\tau)} * f)(e^{-\frac{\tau}{2}} x), \quad x \in \mathbb{R}^n, \tag{4.1.8}$$

where $H_{\alpha(\tau)}(x) = e^{-\frac{|x|^2}{4\alpha(\tau)}} (4\pi\alpha(\tau))^{-\frac{n}{2}}$ and $\alpha(\tau) := 1 - e^{-\tau}$.
Moreover, the restrictions $L^X := L\big|_{X_s^k}$ and $L_0^X := L_0\big|_{X_s^k}$ with domains $\mathcal{D}(L_0^X) = \mathcal{D}(L^X) = \{f \in \mathcal{D}(L) \cap X_s^k : Lf \in X_s^k\}$ now generate strongly continuous semigroups $(S^X(\tau))_{\tau\geq 0}$ and $(S_0^X(\tau))_{\tau\geq 0}$ on X_s^k and they are given by the restrictions of $(S(\tau))_{\tau\geq 0}$ and $(S_0(\tau))_{\tau\geq 0}$ to X_s^k.
In addition, the following properties hold

1. *[compact perturbation] S_0^X is a compact perturbation of S^X in the sense that the right-hand side of*

$$S_0^X(\tau) - S^X(\tau) = \int_0^\tau S_0^X(\tau - \tau') V_0(\psi_0) S^X(\tau')\, d\tau'$$

   is compact for every $\tau \geq 0$.

2. *[spectral properties] There exists an $\widetilde{\omega} > 0$ with*

$$\sigma(L_0^X) \cap \{\lambda \in \mathbb{C} : \ \mathrm{Re}\,(\lambda) \geq -\widetilde{\omega}\} = \{1\}$$

   and $\lambda = 1$ is a simple eigenvalue whose eigenspace is spanned by $g_0 := \Lambda\psi_0$.

3. [decay on stable subspace] If we define the Riesz projection

$$P_0 := \frac{1}{2\pi i}\int_{\partial B_{1/2}(1)} R_{L_0^X}(\lambda)\,d\lambda$$

then we have that for every $0 < \omega_0 < \widetilde{\omega}$ there holds

$$\left\|S_0^X(\tau)(I-P_0)f\right\|_{X_s^k} \lesssim e^{-\omega_0\tau}\left\|(I-P_0)f\right\|_{X_s^k} \tag{4.1.9}$$

for all $\tau \geq 0$ and $f \in X_s^k$.

4.1.2. Construction of the blowup solution

To now prove the existence of a solution to (4.1.6) we first observe that by Proposition 4.1.1 the operator $\widetilde{L}_0$ is closable in $\mathcal{H}$ and that the restriction of its closure onto X_s^k, namely L_0^X, is invertible.
We can therefore reformulate (4.1.6) as a fixed-point equation

$$\phi_\varepsilon = \left(-L_0^X\right)^{-1}\left(V_\varepsilon(\psi_0)\phi_\varepsilon + \widetilde{N}_\varepsilon(\phi_\varepsilon) + \mathcal{R}_\varepsilon(\psi_0)\right). \tag{4.1.10}$$

We will prove the necessary estimates for the involved operators via parameter depending Schauder-type estimates, see Proposition A.2.1. For this we first need an auxiliary Lemma for η_ε from (4.1.4).

Lemma 4.1.2. *Let $a, b, c \in \mathbb{R}$ and $|\varepsilon| \leq 1$. Then*

$$\eta_\varepsilon(a) = a^3\int_0^1\int_0^1\int_0^1 x^2 y\,\eta_\varepsilon'''(axyz)\,dz\,dy\,dx, \tag{4.1.11}$$

$$\eta_\varepsilon'(a) = a^2\int_0^1\int_0^1 x\,\eta_\varepsilon'''(axy)\,dy\,dx, \tag{4.1.12}$$

and

$$\begin{aligned}&\eta_\varepsilon(a+c) - \eta_\varepsilon(a+b) - \eta_\varepsilon'(a)(c-b)\\ &= (c-b)\int_0^1\int_0^1\int_0^1 (b+x(c-b))(a+y(b+x(c-b)))\eta_\varepsilon'''(z(a+y(b+x(c-b))))\,dz\,dy\,dx.\end{aligned} \tag{4.1.13}$$

Moreover, for every $\ell \in \mathbb{N}_0$ there exists a constant C_ℓ such that

$$|\eta_\varepsilon^{(\ell+3)}(x) - \eta_\kappa^{(\ell+3)}(y)| \leq C_\ell\left(|\varepsilon-\kappa| + |x-y|\right) \tag{4.1.14}$$

holds for every $x, y \in \mathbb{R}$ and $|\varepsilon|, |\kappa| \leq 1$.

Proof. The first three equalities follow from the fact that $\eta_\varepsilon(0) = \eta_\varepsilon'(0) = \eta_\varepsilon''(0) = 0$ together with the fundamental theorem of calculus. We only prove the first equality, since the other two follow from similar ideas.

$$\begin{aligned}\eta_\varepsilon(a) = \int_0^a \eta_\varepsilon'(x)\,dx &= \int_0^a \int_0^x \eta_\varepsilon''(y)\,dy\,dx = \int_0^a \int_0^x \int_0^y \eta_\varepsilon'''(z)\,dz\,dy\,dx \\ &= a^3 \int_0^1 \int_0^1 \int_0^1 x^2 y\, \eta_\varepsilon'''(axyz)\,dz\,dy\,dx.\end{aligned}$$

For (4.1.14) one observes that $\eta_\varepsilon^{(\ell+3)}$ can be written for every $\ell \in \mathbb{N}_0$ as

$$\eta_\varepsilon^{(\ell+3)} = \eta_0^{(\ell+3)} + \varepsilon\, F_1^{(\ell+3)} + \varepsilon^2 F_2^{(\ell+3)}$$

where η_0''', F_1''' and F_2''' are smooth 2π-periodic functions.

□

We can now prove the required norm estimates for the operators via the Schauder-type estimates from Proposition A.2.1.

Lemma 4.1.3. *For any $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \leq 1$ we have $V_\varepsilon(\psi_0) : X_s^k \to X_s^k$ as well as $\mathcal{R}_\varepsilon(\psi_0) \in X_s^k$ with estimates*

$$\|V_\varepsilon(\psi_0)u - V_\kappa(\psi_0)u\|_{X_s^k} \lesssim |\varepsilon - \kappa|\, \|u\|_{X_s^k}, \tag{4.1.15}$$

$$\|\mathcal{R}_\varepsilon(\psi_0) - \mathcal{R}_\kappa(\psi_0)\|_{X_s^k} \lesssim |\varepsilon - \kappa| \tag{4.1.16}$$

for all $|\varepsilon|, |\kappa| \leq 1$ and all $u \in X_s^k$. For the nonlinearity $\widetilde{N}_\varepsilon : X_s^k \to X_s^k$ we obtain the following local Lipschitz bound

$$\|\widetilde{N}_\varepsilon(u) - \widetilde{N}_\kappa(v)\|_{X_s^k} \lesssim \left(\|u\|_{X_s^k} + \|v\|_{X_s^k}\right) \|u - v\|_{X_s^k} + \left(\|u\|_{X_s^k}^2 + \|v\|_{X_s^k}^2\right) |\varepsilon - \kappa| \tag{4.1.17}$$

for all $|\varepsilon|, |\kappa| \leq 1$ and all $u, v \in \mathcal{B}_\delta := \{u \in X_s^k : \|u\|_{X_s^k} \leq \delta\}$ for a fixed $0 < \delta \leq 1$.

Proof. We first prove the estimate for the potential $V_\varepsilon(\psi_0)$. Using (4.1.12) we obtain

$$\begin{aligned}&((V_\varepsilon(\psi_0) - V_\kappa(\psi_0))u)(\xi) = \\ &\qquad (n-3)\psi_0(\xi)^2 u(\xi) \int_0^1 \int_0^1 x \left(\eta_\varepsilon'''(|\xi|\psi_0(\xi)xy) - \eta_\kappa'''(|\xi|\psi_0(\xi)xy)\right) dy\,dx\end{aligned}$$

for every $\xi \in \mathbb{R}^n$. Due to the fact that $F_\varepsilon(x) = \eta_\varepsilon'''(x)$ fulfills the assumptions of the Schauder-type estimates from Proposition A.2.1 and the fact that we have the embedding $X_{s-1}^k \hookrightarrow X_s^k$ we obtain (4.1.15).
For the remainder term $\mathcal{R}_\varepsilon(\psi_0)$ we write with the help of (4.1.11)

$$\begin{aligned}&\frac{n-3}{|\xi|^3}\left(\eta_\varepsilon(|\xi|\psi_0(\xi)) - \eta_\kappa(|\xi|\psi_0(\xi))\right) \\ &\quad = (n-3)\psi_0(\xi)^3 \int_0^1 \int_0^1 \int_0^1 x^2 y \left(\eta_\varepsilon'''(|\xi|\psi_0(\xi)xyz) - \eta_\kappa'''(|\xi|\psi_0(\xi)xyz)\right) dz\,dy\,dx\end{aligned}$$

and conclude the proof by again applying the Schauder estimate from Proposition A.2.1 to the function $F_\varepsilon(x) = \eta_\varepsilon'''(x)$.
For the nonlinearity we first prove

$$\|\widetilde{N}_\varepsilon(u) - \widetilde{N}_\varepsilon(v)\|_{X_s^k} \lesssim \Big(\|u\|_{X_s^k} + \|v\|_{X_s^k}\Big)\|u-v\|_{X_s^k} \tag{4.1.18}$$

for all $|\varepsilon| \leq 1$ and all $u, v \in \mathcal{B}_\delta$ for $0 < \delta \leq 1$. We can first of all write with (4.1.13)

$$\begin{aligned}
&\widetilde{N}_\varepsilon(u)(\xi) - \widetilde{N}_\varepsilon(v)(\xi) = \\
&\frac{n-3}{|\xi|^3}\Big(\eta_\varepsilon\big(|\xi|(\psi_0(\xi)+u(\xi))\big) - \eta_\varepsilon\big(|\xi|(\psi_0(\xi)+v(\xi))\big) - \eta_\varepsilon'(|\xi|\psi_0(\xi))|\xi|(u(\xi)-v(\xi))\Big) \\
&= (n-3)\int_0^1\int_0^1\int_0^1 (u(\xi)-v(\xi))\big(v(\xi)+x(u(\xi)-v(\xi))\big) \\
&\cdot\big(\psi_0(\xi)+y\big(v(\xi)+x(u(\xi)-v(\xi))\big)\big)\cdot\eta_\varepsilon'''\left(z|\xi|\big(\psi_0(\xi)+y(v+x(u-v))\big)\right)dz\,dy\,dx.
\end{aligned}$$

From the Schauder-type estimate from Proposition A.2.1 we then obtain (4.1.18).
To now prove

$$\|\widetilde{N}_\varepsilon(u) - \widetilde{N}_\kappa(u)\|_{X_s^k} \lesssim \|u\|_{X_s^k}^2 |\varepsilon - \kappa| \tag{4.1.19}$$

for all $|\varepsilon|, |\kappa| \leq 1$ and all $u \in \mathcal{B}_\delta$ with $0 < \delta \leq 1$ we write with the help of (4.1.13)

$$\begin{aligned}
\widetilde{N}_\varepsilon(u)(\xi) - \widetilde{N}_\kappa(u)(\xi) = (n-3)\,u(\xi)^2\int_0^1\int_0^1\int_0^1 x\,(\psi_0(\xi)+yxu(\xi)) \\
\cdot\left(\eta_\varepsilon'''\left(z|\xi|\big(\psi_0(\xi)+yxu(\xi)\big)\right) - \eta_\kappa'''\left(z|\xi|\big(\psi_0(\xi)+yxu(\xi)\big)\right)\right)dz\,dy\,dx.
\end{aligned}$$

Also (4.1.19) now follows from an application of Proposition A.2.1 and together with (4.1.18) this implies (4.1.17). As a consequence, and using that $\widetilde{N}_\varepsilon(0) = 0$, we conclude that $\widetilde{N}_\varepsilon$ defines a mapping from X_s^k into itself.

□

With this Lemma in hand we are now able to obtain a fixed point for (4.1.10).

Proposition 4.1.4. *There exist $0 < \delta^* \leq 1$ and $0 < \varepsilon^* \leq 1$, where ε^* is allowed to depend on δ^*, such that for every $|\varepsilon| \leq \varepsilon^*$ the map*

$$K_\varepsilon : \mathcal{B}_{\delta^*} \to \mathcal{B}_{\delta^*},\ K_\varepsilon(\phi) = \big(-L_0^X\big)^{-1}\Big(V_\varepsilon(\psi_0)\phi + \mathcal{R}_\varepsilon(\psi_0) + \widetilde{N}_\varepsilon(\phi)\Big)$$

is a well-defined contraction. Additionally, δ^* and ε^* can be chosen in such a way so that

$$\|K_\varepsilon(\phi) - K_\varepsilon(\psi)\|_{X_s^k} \leq \frac{1}{2}\|\phi-\psi\|_{X_s^k} \tag{4.1.20}$$

holds for every $|\varepsilon| \leq \varepsilon^*$ and all $\phi, \psi \in \mathcal{B}_{\delta^*}$.

Proof. For now we take arbitrary $0 < \delta^* \leq 1$ and $0 < \varepsilon^* \leq 1$. We first show that $K_\varepsilon : \mathcal{B}_{\delta^*} \to \mathcal{B}_{\delta^*}$ is well-defined for every $|\varepsilon| \leq \varepsilon^*$. For this we first observe that due to the invertibility of L_0^X in X_s^k there exists a constant $C > 0$ (independent from ε^* and δ^*) with

$$\|K_\varepsilon(\phi)\|_{X_s^k} \leq C \left(\|V_\varepsilon(\psi_0)\phi\|_{X_s^k} + \|\mathcal{R}_\varepsilon(\psi_0)\|_{X_s^k} + \|\widetilde{N}_\varepsilon(\phi)\|_{X_s^k} \right)$$

for all $|\varepsilon| \leq \varepsilon^*$ and $\phi \in \mathcal{B}_{\delta^*}$. From Lemma 4.1.3 above we then obtain additional constants C_1, C_2 and C_3 with

$$\|V_\varepsilon(\psi_0)\phi\|_{X_s^k} + \|\mathcal{R}_\varepsilon(\psi_0)\|_{X_s^k} + \|\widetilde{N}_\varepsilon(\phi)\|_{X_s^k} \leq C_1\, \varepsilon^* \delta^* + C_2\, \varepsilon^* + C_3\, (\delta^*)^2 .$$

We now first choose $\delta^* < (C_3 C)^{-1}$ and then ε^* so small so that we have $C\left(C_1\, \varepsilon^* + C_2\, (\delta^*)^{-1}\, \varepsilon^* + C_3\, \delta^*\right) \leq 1$, which shows that $K_\varepsilon : \mathcal{B}_{\delta^*} \to \mathcal{B}_{\delta^*}$ is well-defined. To now show the contraction property we first obtain another constant C_4 from Lemma 4.1.3 such that there holds

$$\begin{aligned} \|K_\varepsilon(\phi) - K_\varepsilon(\psi)\|_{X_s^k} &\leq C \left(\|V_\varepsilon(\psi_0)(\phi - \psi)\|_{X_s^k} + \|\widetilde{N}_\varepsilon(\phi) - \widetilde{N}_\varepsilon(\psi)\|_{X_s^k} \right) \\ &\leq C\, (C_1\, \varepsilon^* + C_4\, \delta^*)\, \|\phi - \psi\|_{X_s^k} \end{aligned}$$

for every $\phi, \psi \in \mathcal{B}_{\delta^*}$ and $|\varepsilon| \leq \varepsilon^*$. If we now choose ε^* and δ^* even smaller so that $C\, (C_1\, \varepsilon^* + C_4\, \delta^*) \leq 1/2$ holds the claim follows.

□

The next result shows that the above obtained fixed point solves the perturbation equation (4.1.6) pointwise and that it depends Lipschitz continuously on the parameter ε, which will be crucial for the stability analysis later on.

Proposition 4.1.5. *Take δ^* and ε^* as in Proposition 4.1.4. Then there exists for every $|\varepsilon| \leq \varepsilon^*$ a unique $\phi_\varepsilon \in \mathcal{B}_{\delta^*} \cap \mathcal{D}(L_0^X)$ which satisfies* (4.1.6) *pointwise. We furthermore have*

$$\|\phi_\varepsilon - \phi_\kappa\|_{X_s^k} \lesssim |\varepsilon - \kappa| \tag{4.1.21}$$

for every $|\varepsilon|, |\kappa| \leq \varepsilon^*$.

Proof. From Proposition 4.1.4 we immediately obtain for every $|\varepsilon| \leq \varepsilon^*$ a unique $\phi_\varepsilon \in \mathcal{B}_{\delta^*} \cap \mathcal{D}(L_0^X)$ satisfying $\phi_\varepsilon = K_\varepsilon(\phi_\varepsilon)$. That ϕ_ε now also satisfies (4.1.6) pointwise follows from the fact that L_0^X acts as a differential operator on ϕ_ε due to the embedding $X_s^k(\mathbb{R}^n) \hookrightarrow C^2(\mathbb{R}^n)$. To show the Lipschitz continuity we take $|\varepsilon|, |\kappa| \leq \varepsilon^*$ and obtain with (4.1.20)

$$\|\phi_\varepsilon - \phi_\kappa\|_{X_s^k} = \|K_\varepsilon(\phi_\varepsilon) - K_\kappa(\phi_\kappa)\|_{X_s^k} \leq \frac{1}{2} \|\phi_\varepsilon - \phi_\kappa\|_{X_s^k} + \|K_\varepsilon(\phi_\kappa) - K_\kappa(\phi_\kappa)\|_{X_s^k}$$

and therefore $\|\phi_\varepsilon - \phi_\kappa\|_{X_s^k} \leq 2\, \|K_\varepsilon(\phi_\kappa) - K_\kappa(\phi_\kappa)\|_{X_s^k}$. (4.1.21) now follows from the following estimate which holds for all $\phi \in \mathcal{B}_{\delta^*}$ due to Lemma 4.1.3

$$\begin{aligned} &\|K_\varepsilon(\phi) - K_\kappa(\phi)\|_{X_s^k} \\ &\lesssim \|(V_\varepsilon(\psi_0) - V_\kappa(\psi_0))\, \phi\|_{X_s^k} + \|\mathcal{R}_\varepsilon(\psi_0) - \mathcal{R}_\kappa(\psi_0)\|_{X_s^k} + \|\widetilde{N}_\varepsilon(\phi) - \widetilde{N}_\kappa(\phi)\|_{X_s^k} \lesssim |\varepsilon - \kappa|. \end{aligned}$$

□

From this we obtain the following result about the self-similar profile.

Proposition 4.1.6. *Let $\delta^* > 0$ and $\varepsilon^* > 0$ as in Proposition 4.1.5 and let $\phi_\varepsilon \in \mathcal{B}_{\delta^*}$ be the solution to* (4.1.6) *for every $|\varepsilon| \leq \varepsilon^*$. Then*

$$\psi_\varepsilon := \psi_0 + \phi_\varepsilon \in X_s^k(\mathbb{R}^n) \hookrightarrow C_r^2(\mathbb{R}^n)$$

is a smooth classical solution to (4.1.5)*. Moreover, it is even and we have*

$$\|\psi_\varepsilon - \psi_\kappa\|_{X_s^k} \lesssim |\varepsilon - \kappa|$$

for all $|\varepsilon|, |\kappa| \leq \varepsilon^*$. Furthermore, for every multi-index $\alpha \in \mathbb{N}_0^n$,

$$|\partial^\alpha \psi_\varepsilon(y)| \lesssim_\alpha \langle y\rangle^{-1-|\alpha|} \quad \text{and} \quad |\partial^\alpha (\Lambda\psi_\varepsilon)(y)| \lesssim_\alpha \langle y\rangle^{-3-|\alpha|} \tag{4.1.22}$$

for every $y \in \mathbb{R}^n$ and $|\varepsilon| \leq \varepsilon^*$. The implicit constants can hereby be chosen uniformly with respect to ε. In addition,

$$|\partial^\alpha(\psi_\varepsilon - \psi_\kappa)(y)| \lesssim_\alpha |\varepsilon - \kappa|\langle y\rangle^{-1-|\alpha|} \quad \text{and} \quad |\partial^\alpha (\Lambda\psi_\varepsilon - \Lambda\psi_\kappa)(y)| \lesssim_\alpha |\varepsilon - \kappa|\langle y\rangle^{-3-|\alpha|} \tag{4.1.23}$$

for all $y \in \mathbb{R}^n$ and every $|\varepsilon|, |\kappa| \leq \varepsilon^*$.

Proof. The fact that ψ_ε is a classical solution to (4.1.5) follows from its construction and the embedding $X_s^k(\mathbb{R}^n) \hookrightarrow C_r^2(\mathbb{R}^n)$. The Lipschitz continuity with respect to the X_s^k-norm follows from the above Proposition.
To simplify (and slight abuse of) the notation, we write for the remainder of the proof $\psi_\varepsilon(\rho) := \widetilde{\psi}_\varepsilon(\rho)$ for the radial profile of ψ_ε. This radial profile then satisfies the following second order ODE

$$0 = \psi_\varepsilon''(\rho) + \left(\frac{n-1}{\rho} - \frac{\rho}{2}\right)\psi_\varepsilon'(\rho) - \frac{1}{2}\psi_\varepsilon(\rho) + \frac{n-3}{\rho^3}\eta_\varepsilon(\rho\,\psi_\varepsilon(\rho)), \quad \text{for} \quad \rho > 0. \tag{4.1.24}$$

Since $\eta_\varepsilon(0) = \eta_\varepsilon'(0) = \eta_\varepsilon''(0) = 0$ we can write

$$\eta_\varepsilon(z) = z^3 a_\varepsilon(z) \qquad \text{with} \qquad a_\varepsilon(z) = \frac{1}{2}\int_0^1 (1-t)^2 \eta_\varepsilon'''(tz)\,dt.$$

a_ε then depends smoothly on ε so that we have for every $M > 0$ and $j \in \mathbb{N}_0$

$$\sup_{|\varepsilon|\leq\varepsilon^*,\,|z|\leq M} |a_\varepsilon^{(j)}(z)| < \infty \qquad \text{as well as} \qquad \sup_{|z|\leq M} |a_\varepsilon^{(j)}(z) - a_\kappa^{(j)}(z)| \lesssim_{M,j} |\varepsilon - \kappa| \tag{4.1.25}$$

for every $|\varepsilon|, |\kappa| \leq \varepsilon^*$. We now write the profile equation (4.1.24) as

$$\psi_\varepsilon'' + \frac{n-1}{\rho}\psi_\varepsilon' = F_\varepsilon \qquad \text{with} \quad F_\varepsilon(\rho) = \frac{\rho}{2}\psi_\varepsilon'(\rho) + \frac{1}{2}\psi_\varepsilon(\rho) - (n-3)\psi_\varepsilon(\rho)^3 a_\varepsilon(\rho\,\psi_\varepsilon(\rho)) \tag{4.1.26}$$

and multiplication with ρ^{n-1} gives

$$\big(\rho^{n-1}\psi_\varepsilon'(\rho)\big)' = \rho^{n-1}F_\varepsilon(\rho).$$

Since ψ_ε' is bounded by the C^2-embedding of X_s^k we have $\lim_{\rho\to 0}\rho^{n-1}\psi_\varepsilon'(\rho) = 0$ and integrating from 0 to ρ therefore gives

$$\psi_\varepsilon'(\rho) = \rho^{1-n}\int_0^\rho s^{n-1}F_\varepsilon(s)\,ds = \rho\int_0^1 t^{n-1}F_\varepsilon(t\rho)\,dt =: \rho\,\mathcal{T}F_\varepsilon(\rho).$$

Substituting this into (4.1.26) gives

$$\psi_\varepsilon'' = \mathcal{K}F_\varepsilon, \qquad \text{where} \qquad \mathcal{K}f := f - (n-1)\mathcal{T}f. \tag{4.1.27}$$

From this now follows the smoothness and evenness of ψ_ε via a bootstrapping argument. Differentiation under the integral gives for every $j \in \mathbb{N}_0$

$$(\mathcal{T}f)^{(j)}(\rho) = \int_0^1 t^{n-1+j}f^{(j)}(t\rho)\,dt.$$

Thus, for every $R > 0$ and $j \in \mathbb{N}_0$, we obtain

$$\|\mathcal{T}f\|_{C^j([0,R])} \le \frac{1}{n+j}\,\|f\|_{C^j([0,R])} \quad \text{and therefore} \quad \|\mathcal{K}f\|_{C^j([0,R])} \lesssim_j \|f\|_{C^j([0,R])}.$$

We are now going to prove local bounds for the profile function. That is we prove inductively for every $m \in \mathbb{N}_0$ and $R > 0$

$$\sup_{|\varepsilon|\le\varepsilon^*}\|\psi_\varepsilon\|_{C^m([0,R])} < \infty \qquad \text{and} \qquad \|\psi_\varepsilon - \psi_\kappa\|_{C^m([0,R])} \lesssim_{m,R} |\varepsilon-\kappa| \tag{4.1.28}$$

for all $|\varepsilon|, |\kappa| \le \varepsilon^*$. From $X_s^k(\mathbb{R}^n) \hookrightarrow C_r^2(\mathbb{R}^n)$ the hypothesis immediately follows for $m \le 2$. We now take an $m \ge 2$ and assume that the estimates from (4.1.28) hold. Then, from the definition of F_ε and (4.1.25) we obtain

$$\sup_{|\varepsilon|\le\varepsilon^*}\|F_\varepsilon\|_{C^{m-1}([0,R])} < \infty \qquad \text{as well as} \qquad \|F_\varepsilon - F_\kappa\|_{C^{m-1}([0,R])} \lesssim_{m,R} |\varepsilon-\kappa|$$

Using (4.1.27) and the boundedness of $\mathcal{K}$, we obtain

$$\sup_{|\varepsilon|\le\varepsilon^*}\|\psi_\varepsilon''\|_{C^{m-1}([0,R])} < \infty \qquad \text{and} \qquad \|\psi_\varepsilon'' - \psi_\kappa''\|_{C^{m-1}([0,R])} \lesssim_{m,R} |\varepsilon-\kappa|,$$

so that the local bounds from (4.1.28) follow via induction.
To now prove the decay estimates at infinity, we first define

$$p_\varepsilon(\rho) := \rho\,\psi_\varepsilon(\rho) \quad \text{and} \quad h_\varepsilon(\rho) := p_\varepsilon'(\rho) = \psi_\varepsilon(\rho) + \rho\,\psi_\varepsilon'(\rho),$$

so that h_ε is the radial profile of $2\Lambda\psi_\varepsilon$. We also define

$$q_\varepsilon(\rho) := w_\varepsilon(\rho)w_\varepsilon'(\rho)$$

and since $\eta_\varepsilon(z) = z - q_\varepsilon(z)$ we obtain with (4.1.24)

$$h_\varepsilon' + A\, h_\varepsilon = b_\varepsilon \quad \text{with} \quad A(\rho) := \frac{n-3}{\rho} - \frac{\rho}{2} \quad \text{and} \quad b_\varepsilon(\rho) := \frac{n-3}{\rho^2} q_\varepsilon(p_\varepsilon(\rho)). \tag{4.1.29}$$

If we introduce the following integrating factor

$$\mu(\rho) = \rho^{n-3} e^{-\frac{\rho^2}{4}}$$

it satisfies $\mu'(\rho) = A(\rho)\mu(\rho)$ and therefore

$$(\mu h_\varepsilon(\rho))'(\rho) = (n-3)\mu(\rho)\rho^{-2} q_\varepsilon(p_\varepsilon(\rho)).$$

Since ψ_ε and ψ_ε' are bounded by the C^2-embedding, we know that h_ε grows at most linearly so that we obtain $\lim\limits_{\rho\to\infty} \mu(\rho) h_\varepsilon(\rho) = 0$. Integrating from ρ to ∞ now gives

$$h_\varepsilon(\rho) = -\frac{n-3}{\mu(\rho)} \int_\rho^\infty \mu(s) s^{-2} q_\varepsilon(p_\varepsilon(s))\, ds$$

and since w_ε is smooth and periodic, one has for every $j \in \mathbb{N}_0$

$$\sup_{|\varepsilon| \le \varepsilon^*} \|q_\varepsilon^{(j)}\|_{L^\infty(\mathbb{R})} < \infty.$$

Thus, for $\rho \ge 1$, we obtain

$$|h_\varepsilon(\rho)| \lesssim \rho^{3-n} e^{\frac{\rho^2}{4}} \int_\rho^\infty s^{n-5} e^{-\frac{s^2}{4}}\, ds \lesssim \rho^{-3}. \tag{4.1.30}$$

Since $p_\varepsilon' = h_\varepsilon$ it follows for $\rho_2 \ge \rho_1 \ge 1$

$$|p_\varepsilon(\rho_2) - p_\varepsilon(\rho_1)| \le \int_{\rho_1}^\infty |h_\varepsilon(s)|\, ds \lesssim \rho_1^{-2},$$

so that $\lim\limits_{\rho\to\infty} p_\varepsilon(\rho)$ exists. Since $p_\varepsilon(1) = \psi_\varepsilon(1)$ is uniformly bounded with respect to ε we obtain with the estimate from above

$$|p_\varepsilon(\rho)| \lesssim 1 \quad \text{and therefore} \quad |\psi_\varepsilon(\rho)| \lesssim \rho^{-1}. \tag{4.1.31}$$

We now prove

$$|h_\varepsilon^{(m)}(\rho)| \lesssim_m \rho^{-m-3} \qquad \text{for all} \quad \rho \ge 1$$

via induction on $m \in \mathbb{N}_0$. And since the case $m = 0$ has already been proven in (4.1.30), we can assume that $|h_\varepsilon^{(j)}(\rho)| \lesssim_j \rho^{-j-3}$ for $0 \le j \le m-1$. Since $p_\varepsilon' = h_\varepsilon$ we have

$$|p_\varepsilon^{(j)}(\rho)| = |h_\varepsilon^{(j-1)}(\rho)| \lesssim_j \rho^{-j-2} \qquad \text{for} \quad 1 \le j \le m.$$

From the chain rule and the uniform boundedness of q_ε and its derivatives we obtain

$$\left|\frac{d^j}{d\rho^j} q_\varepsilon(p_\varepsilon(\rho))\right| \lesssim_j \rho^{-j-2} \qquad \text{for} \quad 1 \le j \le m \quad \text{and therefore} \quad |b_\varepsilon^{(m)}(\rho)| \lesssim \rho^{-m-2}.$$

If we now differentiate (4.1.29) m times we get

$$h_\varepsilon^{(m+1)} + A h_\varepsilon^{(m)} = b_\varepsilon^{(m)} - \sum_{j=1}^{m} \binom{m}{j} A^{(j)} h_\varepsilon^{(m-j)} =: R_{\varepsilon,m}. \tag{4.1.32}$$

Now $A'(\rho) = -\frac{n-3}{\rho^2} - \frac{1}{2}$ is bounded for $\rho \ge 1$, whereas for $j \ge 2$ we have $A^{(j)}(\rho) = \mathcal{O}(\rho^{-j-1})$. The induction hypothesis therefore gives

$$|R_{\varepsilon,m}(\rho)| \lesssim_m \rho^{-m-2}.$$

Due to the formula of the integrating factor we can rewrite (4.1.32) as

$$\left(\mu h_\varepsilon^{(m)}\right)' = \mu R_{\varepsilon,m}$$

and from (4.1.29) and its successive derivatives we obtain inductively that all of the derivatives of h_ε have at most polynomial growth, so that we get $\mu(\rho) h_\varepsilon^{(m)}(\rho) \to 0$ as $\rho \to \infty$. From this it follows that

$$h_\varepsilon^{(m)}(\rho) = -\frac{1}{\mu(\rho)} \int_\rho^\infty \mu(s) R_{\varepsilon,m}(s)\, ds$$

and consequently

$$|h_\varepsilon^{(m)}(\rho)| \lesssim \rho^{3-n} e^{\frac{\rho^2}{4}} \int_\rho^\infty s^{n-m-5} e^{-\frac{s^2}{4}}\, ds \lesssim \rho^{-m-3},$$

which closes the induction. Now since

$$h_\varepsilon^{(m)}(\rho) = \rho \psi_\varepsilon^{(m+1)}(\rho) + (m+1)\psi_\varepsilon^{(m)}(\rho),$$

we obtain

$$\psi_\varepsilon^{(m+1)}(\rho) = \frac{h_\varepsilon^{(m)}(\rho)}{\rho} - \frac{m+1}{\rho} \psi_\varepsilon^{(m)}(\rho).$$

And since we have shown that $|\psi_\varepsilon(\rho)| \lesssim \rho^{-1}$ holds, an induction then yields

$$|\psi_\varepsilon^{(m)}(\rho)| \lesssim \rho^{-m-1} \qquad \text{for every} \quad m \in \mathbb{N}_0 \quad \text{and} \quad \rho \ge 1.$$

Now we want to prove the parameter-dependent decay estimates from (4.1.23). For this we define

$$P_{\varepsilon,\kappa}(\rho) := p_\varepsilon(\rho) - p_\kappa(\rho) \quad \text{and} \quad H_{\varepsilon,\kappa}(\rho) := h_\varepsilon(\rho) - h_\kappa(\rho),$$

so that we again have $P'_{\varepsilon,\kappa} = H_{\varepsilon,\kappa}$. Subtracting (4.1.29) for h_ε and h_κ yields

$$H'_{\varepsilon,\kappa} + AH_{\varepsilon,\kappa} = \frac{n-3}{\rho^2} Q_{\varepsilon,\kappa} \qquad \text{with} \quad Q_{\varepsilon,\kappa}(\rho) := q_\varepsilon(p_\varepsilon(\rho)) - q_\kappa(p_\kappa(\rho)).$$

Since q_ε and its derivatives depend Lipschitz continuously on ε, one has

$$|q_\varepsilon(x) - q_\kappa(y)| \lesssim |\varepsilon - \kappa| + |x - y|$$

for all $x, y \in \mathbb{R}$ and therefore $|Q_{\varepsilon,\kappa}(\rho)| \lesssim |\varepsilon - \kappa| + |P_{\varepsilon,\kappa}(\rho)|$. From the formula of the integrating factor we again obtain

$$H_{\varepsilon,\kappa}(\rho) = -\frac{n-3}{\mu(\rho)} \int_\rho^\infty \mu(s) s^{-2} Q_{\varepsilon,\kappa}(s)\, ds.$$

For now take $R > 1$, to be fixed later, and define

$$M_{\varepsilon,\kappa}(R) := \sup_{\rho \geq R} |P_{\varepsilon,\kappa}(\rho)|.$$

Due to (4.1.31) we obtain $M_{\varepsilon,\kappa}(R) < \infty$ uniformly in ε and κ. Since we have for $\rho \geq R$

$$|Q_{\varepsilon,\kappa}(\rho)| \lesssim |\varepsilon - \kappa| + |P_{\varepsilon,\kappa}(\rho)| \leq |\varepsilon - \kappa| + M_{\varepsilon,\kappa}(R),$$

we similarly obtain the following estimate as we did for h_ε for $\rho \geq R$

$$|H_{\varepsilon,\kappa}(\rho)| \lesssim \rho^{3-n} e^{\frac{\rho^2}{4}} \int_\rho^\infty s^{n-5} e^{-\frac{s^2}{4}} |Q_{\varepsilon,\kappa}(s)|\, ds \lesssim \left(|\varepsilon - \kappa| + M_{\varepsilon,\kappa}(R)\right) \rho^{-3}. \tag{4.1.33}$$

Moreover, due to $P'_{\varepsilon,\kappa} = H_{\varepsilon,\kappa}$

$$P_{\varepsilon,\kappa}(\rho) = P_{\varepsilon,\kappa}(R) + \int_R^\rho H_{\varepsilon,\kappa}(s)\, ds$$

and from the local bounds (4.1.28) we obtain

$$|P_{\varepsilon,\kappa}(R)| = R|\psi_\varepsilon(R) - \psi_\kappa(R)| \leq C_R|\varepsilon - \kappa|.$$

It follows

$$M_{\varepsilon,\kappa}(R) \leq |P_{\varepsilon,\kappa}(R)| + \int_R^\infty |H_{\varepsilon,\kappa}(s)|\, ds \lesssim C_R|\varepsilon - \kappa| + R^{-2}\left(|\varepsilon - \kappa| + M_{\varepsilon,\kappa}(R)\right).$$

We now choose $R > 1$ sufficiently large (independently of ε and κ) such that the coefficient on the right-hand side is strictly smaller than 1. Absorbing this term then gives $M_{\varepsilon,\kappa}(R) \lesssim |\varepsilon - \kappa|$, so that we obtain from (4.1.33)

$$|H_{\varepsilon,\kappa}(\rho)| \lesssim |\varepsilon - \kappa| \rho^{-3} \qquad \text{for} \quad \rho \geq R.$$

The estimates for the derivatives of $H_{\varepsilon,\kappa}$ and $P_{\varepsilon,\kappa}$ now follow analogously via induction as for h_ε and p_ε. Overall, we obtain (4.1.23), which finishes the proof.

□

We are now able to prove our first main theorem.

Proof of Theorem 4.0.1. We define $\widetilde{f}_\varepsilon(\rho) = \rho\,\widetilde{\psi}_\varepsilon(\rho)$, where $\widetilde{\psi}_\varepsilon$ is the radial profile from the above constructed ψ_ε solving (4.1.6). From this we immediately obtain that $\widetilde{f}_\varepsilon$ is a smooth and odd function with $\widetilde{f}_\varepsilon(0) = 0$. Furthermore, the limit $L := \lim_{\rho\to\infty} \widetilde{f}_\varepsilon(\rho)$ exists and there exists for every $k \in \mathbb{N}$ a constant $C_k > 0$ with

$$|\widetilde{f}_\varepsilon^{(k)}(\rho)| \leq C_k \langle\rho\rangle^{-2-k} \qquad \text{for all} \quad \rho > 0.$$

We now show $0 < \widetilde{f}_\varepsilon(\rho) < \pi$ for every $\rho > 0$. For this we first observe that $\widetilde{f}_\varepsilon$ solves the following profile equation

$$0 = f''(\rho) + \left(\frac{d-1}{\rho} - \frac{\rho}{2}\right) f'(\rho) - \frac{d-1}{\rho^2} w_\varepsilon(f(\rho))\, w_\varepsilon'(f(\rho)). \tag{4.1.34}$$

We define

$$\mathcal{E}(\rho) := \frac{1}{2}\rho^2 \widetilde{f}_\varepsilon'(\rho)^2 - \frac{d-1}{2} w_\varepsilon(\widetilde{f}_\varepsilon(\rho))^2$$

and differentiation with respect to ρ gives

$$\mathcal{E}'(\rho) = \rho \widetilde{f}_\varepsilon'(\rho)^2 + \rho^2 \widetilde{f}_\varepsilon'(\rho)\widetilde{f}_\varepsilon''(\rho) - (d-1) w_\varepsilon(\widetilde{f}_\varepsilon(\rho)) w_\varepsilon'(\widetilde{f}_\varepsilon(\rho))\widetilde{f}_\varepsilon'(\rho).$$

If we now multiply the profile equation (4.1.34) with $\rho^2 \widetilde{f}_\varepsilon'(\rho)$ and plug this into the above equation we obtain

$$\mathcal{E}'(\rho) = \rho\left(\frac{\rho^2}{2} - (d-2)\right) \widetilde{f}_\varepsilon'(\rho)^2.$$

If we now set $\rho_* := \sqrt{2(d-2)}$ we get $\mathcal{E}'(\rho) \leq 0$ for $0 \leq \rho \leq \rho_*$ and $\mathcal{E}'(\rho) \geq 0$ for $\rho \geq \rho_*$. Since $\mathcal{E}(0) = 0$ we therefore have $\mathcal{E}(\rho) \leq 0$ for all $0 \leq \rho \leq \rho_*$ and due to $\widetilde{f}_\varepsilon'(\rho) = \mathcal{O}(\rho^{-3})$ we get

$$\lim_{\rho\to\infty} \mathcal{E}(\rho) = -\frac{d-1}{2} w_\varepsilon(L)^2 \leq 0.$$

Since $\mathcal{E}$ is nondecreasing on $[\rho_*, \infty)$ and has a nonpositive limit, it follows $\mathcal{E}(\rho) \leq 0$ for $\rho \geq \rho_*$ and therefore

$$\mathcal{E}(\rho) \leq 0 \quad \text{for all} \quad \rho \geq 0. \tag{4.1.35}$$

We now define

$$\rho_0 := \inf\{\rho > 0 : \ \widetilde{f}_\varepsilon(\rho) \notin (0, \pi)\} > 0.$$

Due to the continuity of $\widetilde{f}_\varepsilon$ we must have $\widetilde{f}_\varepsilon(\rho_0) \in \{0, \pi\}$ and therefore

$$\mathcal{E}(\rho_0) = \frac{1}{2}\rho_0^2 \widetilde{f}_\varepsilon'(\rho_0)^2 \geq 0.$$

But due to (4.1.35) we must have $\mathcal{E}(\rho_0) = 0$ and therefore $\widetilde{f}'_\varepsilon(\rho_0) = 0$. Thus, at ρ_0, we have either

$$\widetilde{f}_\varepsilon(\rho_0) = 0, \quad \widetilde{f}'_\varepsilon(\rho_0) = 0 \quad \text{or} \quad \widetilde{f}_\varepsilon(\rho_0) = \pi, \quad \widetilde{f}'_\varepsilon(\rho_0) = 0.$$

But since the profile equation (4.1.34) is a regular second-order ODE at $\rho_0 > 0$ the profile $\widetilde{f}_\varepsilon$ would have to equal $k\pi$ with $k \in \{0, 1\}$ on the entire existence interval by uniqueness, which would be a contradiction to $\widetilde{f}'_\varepsilon(0) \neq 0$. If we now set

$$u_\varepsilon^T(t, r) = \widetilde{f}_\varepsilon\left(\frac{r}{\sqrt{T-t}}\right)$$

we get for the gradient

$$\partial_r u_\varepsilon^T(t, r)\big|_{r=0} = (T-t)^{-1/2}\, \widetilde{f}'_\varepsilon(0) = (T-t)^{-1/2}\left(\widetilde{\psi}_0(0) + \widetilde{\phi}_\varepsilon(0)\right),$$

so that the Theorem follows if we initially choose $\delta^* > 0$ so small so that we have $|\widetilde{\phi}_\varepsilon(0)| < \widetilde{\psi}_0(0)$, which can be done due to the embedding $X_s^k(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$.

□

4.2. Stability analysis

In this section we proceed to investigate the stability of the blowup solution $u_\varepsilon^T \in C^\infty([0, T) \times [0, \infty))$ constructed in Theorem 4.0.1. For this, we consider $\widetilde{v}(t, r) = r^{-1}u(t, r)$ and study the following Cauchy problem on $[0, \infty) \times \mathbb{R}^n$

$$\begin{cases} \partial_t v - \Delta_x v = \frac{n-3}{|x|^3}\left(|x|v - w_\varepsilon(|x|v)w'_\varepsilon(|x|v)\right) \\ v(0, x) = v_\varepsilon^1(0, x) + \varphi_0(x). \end{cases} \tag{4.2.1}$$

Here, v_ε^1 denotes the corresponding blowup solution v_ε^T with blowup time $T = 1$, i.e. v_ε^T is given by

$$v_\varepsilon^T(t, x) = \frac{1}{\sqrt{T-t}}\psi_\varepsilon\left(\frac{x}{\sqrt{T-t}}\right), \quad \psi_\varepsilon(y) = |y|^{-1}\widetilde{f}_\varepsilon(|y|)$$

and φ_0 is a radial real-valued Schwartz function with for now small X_s^k-norm.
Again, introducing similarity coordinates

$$\tau = \log\left(\frac{T}{T-t}\right) \quad \text{and} \quad y = \frac{x}{\sqrt{T-t}}$$

(4.2.1) transforms into (4.1.3) and the blowup solution v_ε^T corresponds to the static solution ψ_ε of Eq. (4.1.3). The Cauchy problem (4.2.1) can therefore be reformulated as

$$\begin{cases} \partial_\tau \psi(\tau) = \widetilde{L}\psi(\tau) + N_\varepsilon(\psi(\tau)), \\ \psi(0) = \psi_\varepsilon^T + \varphi_0^T, \end{cases} \tag{4.2.2}$$

where we use the convention that a superscript T denotes the rescaling $f^T := \sqrt{T} f(\sqrt{T}\cdot)$. $\widetilde{L}$ still denotes the heat operator in similarity coordinates and the nonlinearity N_ε is the one from (4.1.4).
To analyze the dynamics near ψ_ε we decompose the solution as a perturbation of the static profile by writing $\psi(\tau) = \psi_\varepsilon + \phi_\varepsilon(\tau)$ and substitute this ansatz into the equation. This yields an evolution equation for the perturbation ϕ_ε

$$\partial_\tau \phi_\varepsilon(\tau) = \widetilde{L}\phi_\varepsilon(\tau) + N_\varepsilon(\psi_\varepsilon + \phi_\varepsilon(\tau)) - N_\varepsilon(\psi_\varepsilon).$$

Expanding the nonlinearity around ψ_ε we obtain the central evolution equation of this section

$$\begin{cases} \partial_\tau \phi_\varepsilon(\tau) = \widetilde{L}_\varepsilon \phi_\varepsilon(\tau) + \widehat{N}_\varepsilon(\phi_\varepsilon(\tau)), \\ \phi_\varepsilon(0) = \varphi_0^T + \psi_\varepsilon^T - \psi_\varepsilon. \end{cases} \tag{4.2.3}$$

Here, the linearized operator $\widetilde{L}_\varepsilon := \widetilde{L} + L'_\varepsilon$ consists of the heat operator $\widetilde{L}$ in similarity coordinates from (4.1.3) and a potential term L'_ε given by

$$L'_\varepsilon u = V_\varepsilon(\psi_\varepsilon)\, u \quad \text{for} \quad V_\varepsilon(\psi_\varepsilon)(y) = \frac{n-3}{|y|^2} \eta'_\varepsilon(|y|\psi_\varepsilon(y)).$$

The nonlinear remainder $\widehat{N}_\varepsilon$ consists of the higher-order terms and is given by

$$\widehat{N}_\varepsilon(u)(y) = \frac{n-3}{|y|^3} \left(\eta_\varepsilon(|y|(\psi_\varepsilon(y) + u(y))) - \eta_\varepsilon(|y|\psi_\varepsilon(y)) - \eta'_\varepsilon(|y|\psi_\varepsilon(y))|y|u(y)\right). \tag{4.2.4}$$

We note that the dependence on the blowup time T appears only in the initial data. Furthermore, in the unperturbed case $\varepsilon = 0$, the operator $\widetilde{L}_\varepsilon$ coincides with the one from Proposition 4.1.1.

4.2.1. The linearized operator for small parameters

In this section we study the linear operator $\widetilde{L}_\varepsilon$ and its spectral properties. We begin by showing that $\widetilde{L}_\varepsilon$ is closable in $\mathcal{H}$ and that its closure $(L_\varepsilon, \mathcal{D}(L_\varepsilon))$ generates a strongly continuous semigroup $(S_\varepsilon(\tau))_{\tau\geq 0}$ on $\mathcal{H}$. If we restrict this operator onto X_s^k it likewise generates a strongly continuous semigroup on this space, which is obtained from the restriction of $(S_\varepsilon(\tau))_{\tau\geq 0}$ onto X_s^k.
We then turn to the spectral analysis of the restricted operator L_ε^X. We first show that $\lambda = 1$ is a simple eigenvalue, see Lemma 4.2.5, corresponding to the expected instability associated to the time translation symmetry. We then show in Proposition 4.2.6 that this is in fact the only unstable spectral point of L_ε^X. A key ingredient in this analysis is the Lipschitz-dependency of L_ε^X with respect to the parameter ε proved in Proposition 4.2.4, which relies on the Lipschitz continuity of the self-similar solution ψ_ε.

Proposition 4.2.1. *For all $|\varepsilon| \leq \varepsilon^*$ the operator $(\widetilde{L}_\varepsilon, \mathcal{D}(\widetilde{L}_\varepsilon))$ is closable in $\mathcal{H}$ and its closure $L_\varepsilon : \mathcal{D}(L_\varepsilon) \subset \mathcal{H} \to \mathcal{H}$ satisfies $\mathcal{D}(L_\varepsilon) = \mathcal{D}(L)$, is self-adjoint, has compact resolvent and generates a strongly continuous semigroup $(S_\varepsilon(\tau))_{\tau\geq 0}$ of bounded operators on $\mathcal{H}$.*

Proof. We only have to show that $L'_\varepsilon : \mathcal{H} \to \mathcal{H}$ is bounded. The bounded perturbation theorem, see [34] Theorem III.1.3, then implies that $\widetilde{L}_\varepsilon$ is closable in $\mathcal{H}$, that its closure has domain $\mathcal{D}(L_\varepsilon) = \mathcal{D}(L)$ and that it generates a strongly continuous semigroup on $\mathcal{H}$. Moreover, the compactness of the resolvent is preserved under bounded perturbations as well, see [34] Proposition III.1.12, and the self-adjointness then follows from Kato-Rellich's Theorem.
To show the boundedness of L'_ε in $\mathcal{H}$ we write the potential in the following way using (4.1.12)

$$V_\varepsilon(\psi_\varepsilon)(\xi) = (n-3)\psi_\varepsilon(\xi)^2 \int_0^1 x \int_0^1 \eta_\varepsilon'''(|\xi|\psi_\varepsilon(\xi)xy)dy\,dx.$$

Therefore, $V_\varepsilon(\psi_\varepsilon)$ behaves in leading order like ψ_ε^2 and therefore satisfies

$$|\partial^\alpha V_\varepsilon(\psi_\varepsilon)(y)| \lesssim_\alpha \langle y\rangle^{-2-|\alpha|}$$

for every $\alpha \in \mathbb{N}_0^n$ and $y \in \mathbb{R}^n$. With Lemma A.1.4 we infer that $V_\varepsilon(\psi_\varepsilon)$ belongs to $X_s^k(\mathbb{R}^n)$ and is therefore bounded, which then shows the boundedness of L'_ε in $\mathcal{H}$.

□

We now use the embedding $X_s^k(\mathbb{R}^n) \hookrightarrow \mathcal{H}$ to show that these operators also generate strongly continuous semigroups in $X_s^k(\mathbb{R}^n)$, which will be the semigroups in $\mathcal{H}$ restricted onto $X_s^k(\mathbb{R}^n)$.

Proposition 4.2.2. *For all $|\varepsilon| \le \varepsilon^*$ the restrictions $L^X := L\big|_{X_s^k(\mathbb{R}^n)}$ and $L_\varepsilon^X := L_\varepsilon\big|_{X_s^k(\mathbb{R}^n)}$ with domains $\mathcal{D}(L_\varepsilon^X) = \mathcal{D}(L^X) = \{f \in \mathcal{D}(L) \cap X_s^k(\mathbb{R}^n) : Lf \in X_s^k(\mathbb{R}^n)\}$ generate strongly continuous semigroups $(S^X(\tau))_{\tau\ge0}$ and $(S_\varepsilon^X(\tau))_{\tau\ge0}$ on $X_s^k(\mathbb{R}^n)$ respectively and they are given by the restrictions of $(S(\tau))_{\tau\ge0}$ and $(S_\varepsilon(\tau))_{\tau\ge0}$ onto $X_s^k(\mathbb{R}^n)$ respectively.*

Proof. From [1] we infer that $(S(\tau))_{\tau\ge0}$ is also a strongly continuous semigroup on $X_s^k(\mathbb{R}^n)$. Since this space is invariant under $(S(\tau))_{\tau\ge0}$ we know from [34], II.2.3 that $L^X = L\big|_{X_s^k(\mathbb{R}^n)}$ with domain $\mathcal{D}(L^X) = \{f \in \mathcal{D}(L) \cap X_s^k(\mathbb{R}^n) : Lf \in X_s^k(\mathbb{R}^n)\}$ generates a strongly continuous semigroup $(S^X(\tau))_{\tau\ge0}$ which is given by the restriction of $(S(\tau))_{\tau\ge0}$ onto $X_s^k(\mathbb{R}^n)$.
Due to the Algebra property of $X_s^k(\mathbb{R}^n)$ we have

$$\|L'_\varepsilon f\|_{X_s^k} \lesssim \|V_\varepsilon(\psi_\varepsilon)\|_{X_s^k}\,\|f\|_{X_s^k}$$

for all $f \in X_s^k(\mathbb{R}^n)$ and therefore we have by the bounded perturbation theorem that $L_\varepsilon^X = L_\varepsilon\big|_{X_s^k(\mathbb{R}^n)}$ has domain $\mathcal{D}(L_\varepsilon^X) = \mathcal{D}(L^X)$ and also generates a strongly continuous semigroup $(S_\varepsilon^X(\tau))_{\tau\ge0}$, which is given by the restriction of $(S_\varepsilon(\tau))_{\tau\ge0}$ onto $X_s^k(\mathbb{R}^n)$.

□

Proposition 4.2.3. *For every $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \le \varepsilon^*$ the operator $L'_\varepsilon : X_s^k \to X_s^k$ depends Lipschitz continuously on the parameter ε in the sense that*

$$\|L'_\varepsilon u - L'_\kappa u\|_{X_s^k} \lesssim |\varepsilon - \kappa|\,\|u\|_{X_s^k}$$

holds for all $u \in X_s^k$ and all $|\varepsilon|, |\kappa| \le \varepsilon^*$.

Proof. Again we write

$$V_\varepsilon(\psi_\varepsilon)(\xi) = (n-3)\psi_\varepsilon(\xi)^2 \int_0^1 x \int_0^1 \eta_\varepsilon'''(|\xi|\psi_\varepsilon(\xi)xy)dy\,dx$$

so that the Lipschitz continuity follows from the Schauder estimate from Proposition A.2.1 and the Lipschitz continuity of ψ_ε.

□

From this we obtain the Lipschitz dependency of the whole linear operator.

Proposition 4.2.4. *For every $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \leq \varepsilon^*$ the operator L_ε^X satisfies*

$$\left\| L_\varepsilon^X u - L_\kappa^X u \right\|_{X_s^k} \lesssim |\varepsilon - \kappa| \, \|u\|_{X_s^k}$$

for all $u \in \mathcal{D}(L^X) = \mathcal{D}(L_\varepsilon^X) = \mathcal{D}(L_\kappa^X)$ and all $|\varepsilon|, |\kappa| \leq \varepsilon^*$.

As in the unperturbed case $\varepsilon = 0$ the operator L_ε^X has an unstable mode at $\lambda = 1$ generated by the time translation symmetry. More precisely, we have the following result.

Lemma 4.2.5. *For every $|\varepsilon| \leq \varepsilon^*$ we have*

$$L_\varepsilon^X \, g_\varepsilon = g_\varepsilon,$$

where $g_\varepsilon := \Lambda\psi_\varepsilon \in C_r^\infty(\mathbb{R}^n) \cap \mathcal{D}(L_\varepsilon^X)$. Additionally, the eigenfunction g_ε depends Lipschitz continuously on the parameter ε in the sense that

$$\|g_\varepsilon - g_\kappa\|_{X_s^k} \lesssim |\varepsilon - \kappa|$$

holds for all $|\varepsilon|, |\kappa| \leq \varepsilon^*$.

Proof. From the definition we immediately get

$$\widetilde{L}_\varepsilon g_\varepsilon = \Delta g_\varepsilon - \Lambda g_\varepsilon + \frac{n-3}{|y|^2}\eta_\varepsilon'(|y|\psi_\varepsilon(y))g_\varepsilon = g_\varepsilon.$$

Due to the decay of $\Lambda\psi_\varepsilon$, see (4.1.22), we get

$$|\partial^\alpha g_\varepsilon(y)| \lesssim \langle y \rangle^{-3-|\alpha|}$$

for every $\alpha \in \mathbb{N}_0^n$ and $y \in \mathbb{R}^n$. We therefore have $g_\varepsilon \in X_s^k$ by Lemma A.1.4. That g_ε actually belongs to the domain of L_ε can be proven via a cutoff argument. We therefore obtain $g_\varepsilon \in \mathcal{D}(L_\varepsilon^X)$ with $L_\varepsilon^X \, g_\varepsilon = g_\varepsilon$.
The Lipschitz continuity follows from (4.1.23) together with Lemma A.1.4.

□

We now use perturbative methods to exclude any other unstable spectral points of L_ε^X.

Proposition 4.2.6. *For any fixed $0 < \omega_0 < \widetilde{\omega}$, where $\widetilde{\omega}$ is the constant from Proposition 4.1.1, there exists an $\varepsilon^{**} > 0$ such that for every $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \leq \varepsilon^{**}$*

$$\sigma(L_\varepsilon^X) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < -\omega_0\} \cup \{1\}.$$

Moreover, $\lambda = 1$ is a simple eigenvalue of L_ε^X with eigenspace spanned by g_ε. Furthermore, the associated Riesz projection

$$P_\varepsilon := \frac{1}{2\pi i} \int_{\partial B_{1/2}(1)} R_{L_\varepsilon^X}(\lambda)\, d\lambda \tag{4.2.5}$$

satisfies $\operatorname{ran}(P_\varepsilon) = \ker(I - L_\varepsilon^X) = \langle g_\varepsilon \rangle$.

Proof. We first show that

$$\sigma(L_\varepsilon^X) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq -\omega_0\} \subset B_{1/2}(1),$$

so that any spectral point in the right half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq -\omega_0\}$ must lie in a compact region around the isolated eigenvalue $\lambda = 1$. Moreover, there exists a constant $C_{\omega_0} > 0$ such that the resolvent satisfies

$$\left\| R_{L_\varepsilon^X}(\lambda) \right\| \leq C_{\omega_0} \tag{4.2.6}$$

for all $\lambda \in \overline{\mathbb{H}_{-\omega_0}} \setminus B_{1/2}(1)$ and all sufficiently small $|\varepsilon|$.
To prove this we take $\lambda \in \overline{\mathbb{H}_{-\omega_0}} \setminus B_{1/2}(1)$ and $|\varepsilon| \leq \varepsilon^*$. Since λ then belongs to the resolvent set of L_0^X (due to Proposition 4.1.1) we obtain the following identity

$$\lambda - L_\varepsilon^X = \left(I - (L_\varepsilon^X - L_0^X) R_{L_0^X}(\lambda) \right) (\lambda - L_0^X). \tag{4.2.7}$$

Consequently, invertibility of $\lambda - L_\varepsilon^X$ is equivalent to the invertibility of $I - (L_\varepsilon^X - L_0^X) R_{L_0^X}(\lambda)$, which can be proven via a Neumann-series argument. For this, we decompose the resolvent into its stable and unstable components via the spectral projection P_0 from Proposition 4.1.1

$$R_{L_0^X}(\lambda) f = R_{L_0^X}(\lambda)(I - P_0) f + R_{L_0^X}(\lambda) P_0 f. \tag{4.2.8}$$

On the stable subspace, $\operatorname{ran}(I - P_0)$, the semigroup $\left(S_0^X(\tau)\right)_{\tau \geq 0}$ decays exponentially, see (4.1.9), so that we obtain for every $0 < \omega < \widetilde{\omega}$

$$\left\| S_0^X(\tau)\,(I - P_0)\, u \right\|_{X_s^k} \lesssim e^{-\omega\,\tau} \left\| (I - P_0)\, u \right\|_{X_s^k}.$$

Since the restriction of $\left(S_0^X(\tau)\right)_{\tau \geq 0}$ to $\operatorname{ran}(I - P_0)$ coincides with the semigroup generated by the restriction of L_0^X to the stable subspace, we can apply [34], p.55, Theorem 1.10. This yields for every $0 < \omega < \widetilde{\omega}$ the existence of a constant $M_\omega \geq 1$ such that

$$\left\| R_{L_0^X}(\lambda)(I - P_0) \right\| \leq \frac{M_\omega}{\operatorname{Re}\lambda + \omega} \tag{4.2.9}$$

for all $\lambda \in \mathbb{H}_{-\omega}$. By fixing $\omega = \frac{\omega_0+\widetilde{\omega}}{2}$ in (4.2.9) it follows that there exists a constant $C_{\omega_0} > 0$ such that

$$\left\| R_{L_0^X}(\lambda)(I - P_0) \right\| \leq C_{\omega_0}$$

for all $\lambda \in \overline{\mathbb{H}_{-\omega_0}}$. For the second term in Eq. (4.2.8) we exploit the fact that $\lambda = 1$ is a simple eigenvalue with eigenspace spanned by g_0. Therefore, there exists a unique $h_0 \in X_s^k$ with

$$R_{L_0^X}(\lambda)P_0 f = (\lambda - 1)^{-1} \langle h_0, f \rangle_{X_s^k} \, g_0.$$

for every $f \in X_s^k$. Combining (4.2.8) with the previous bounds yields a constant $C_{\omega_0} > 0$ with

$$\left\| R_{L_0^X}(\lambda) \right\| \leq C_{\omega_0}$$

for every $\lambda \in \overline{\mathbb{H}_{-\omega_0}} \setminus B_{1/2}(1)$. From the Lipschitz continuity of L_ε^X, see Proposition 4.2.4, we obtain the bound

$$\left\| (L_\varepsilon^X - L_0^X) R_{L_0^X}(\lambda) \right\| \lesssim |\varepsilon|$$

for every $\lambda \in \overline{\mathbb{H}_{-\omega_0}} \setminus B_{1/2}(1)$ and every $|\varepsilon| \leq \varepsilon^*$. If we now choose $\varepsilon^{**} \leq \varepsilon^*$ sufficiently small so that

$$\left\| (L_\varepsilon^X - L_0^X) R_{L_0^X}(\lambda) \right\| < 1$$

the identity (4.2.7) implies $\lambda \in \rho(L_\varepsilon^X)$. Moreover, we obtain the resolvent representation

$$R_{L_\varepsilon^X}(\lambda) = R_{L_0^X}(\lambda) \left(I - (L_\varepsilon^X - L_0^X) R_{L_0^X}(\lambda) \right)^{-1}$$

from which (4.2.6) follows.
To exclude any other unstable spectral points in the compact region $\overline{B_{1/2}(1)}$ we first notice that we have previously shown that $\partial B_{1/2}(1)$ belongs to the resolvent set of L_ε^X. In particular, the Riesz projection P_ε from (4.2.5) is well-defined. Moreover, combining the resolvent identity with the Lipschitz continuity of L_ε^X and the uniform resolvent bounds on $\partial B_{1/2}(1)$ we obtain for every $\lambda \in \partial B_{1/2}(1)$

$$\left\| R_{L_\varepsilon^X}(\lambda) - R_{L_\kappa^X}(\lambda) \right\| \leq \left\| R_{L_\varepsilon^X}(\lambda) \right\| \left\| L_\varepsilon^X - L_\kappa^X \right\| \left\| R_{L_\kappa^X}(\lambda) \right\| \lesssim |\varepsilon - \kappa|.$$

It follows that the resolvent $R_{L_\varepsilon^X}$ depends Lipschitz continuously on the parameter ε and hence, the same holds for the Riesz projection P_ε. By [44], p.34, Lemma 4.10 we can conclude for every $|\varepsilon| \leq \varepsilon^{**}$,

$$\dim \operatorname{ran}(P_\varepsilon) = \dim \operatorname{ran}(P_0) = 1.$$

Since we already have $\langle g_\varepsilon \rangle \subset \ker(I - L_\varepsilon^X) \subset \operatorname{ran}(P_\varepsilon)$, it follows that all of these spaces coincide. Consequently, the only spectral point of L_ε^X in $B_{1/2}(1)$ is the simple eigenvalue $\lambda = 1$.

□

With these results we can now also prove the exponential decay/growth of the semigroup on the stable/unstable subspace, respectively.

Proposition 4.2.7. *Fix an arbitrary $0 < \omega_0 < \widetilde{\omega}$, where $\widetilde{\omega}$ is the constant from Proposition 4.1.1. Furthermore, let $|\varepsilon| \leq \varepsilon^{**}$, where ε^{**} is the constant from Proposition 4.2.6. Then the projection P_ε commutes with the semigroup S_ε^X. In particular, we have*

$$P_\varepsilon \, S_\varepsilon^X(\tau) = S_\varepsilon^X(\tau) P_\varepsilon = e^\tau \, P_\varepsilon \tag{4.2.10}$$

for all $\tau \geq 0$. For the stable subspace we have

$$\left\| S_\varepsilon^X(\tau)(I - P_\varepsilon)u \right\|_{X_s^k} \lesssim e^{-\omega_0 \tau} \left\| (I - P_\varepsilon)u \right\|_{X_s^k} \tag{4.2.11}$$

as well as

$$\left\| [S_\varepsilon^X(\tau)(I - P_\varepsilon) - S_\kappa^X(\tau)(I - P_\kappa)]u \right\|_{X_s^k} \lesssim e^{-\omega_0 \tau} |\varepsilon - \kappa| \, \|u\|_{X_s^k} \tag{4.2.12}$$

for all $u \in X_s^k, \tau \geq 0$ and $|\varepsilon|, |\kappa| \leq \varepsilon^{**}$.

Proof. Because the semigroup $S_\varepsilon^X(\tau)$ commutes with its generator L_ε^X it also commutes with the associated resolvent $R_{L_\varepsilon^X}$. Consequently, it commutes with the spectral projection P_ε as well. Eq. (4.2.10) then follows from uniqueness of solutions to $\partial_\tau u(\tau) = L_\varepsilon^X u(\tau)$.
The decay on the stable subspace (4.2.11) follows from [54] Theorem A.1 once there exists a constant $M_{\omega_0} > 0$ with

$$\left\| R_{L_\varepsilon^X}(\lambda) \, (I - P_\varepsilon) \right\| \leq M_{\omega_0} \tag{4.2.13}$$

for all $\lambda \in \overline{\mathbb{H}_{-\omega_0}}$ and $|\varepsilon| \leq \varepsilon^{**}$ for a sufficiently small $\varepsilon^{**} > 0$.
From Proposition 4.2.6 we already know that there exists a constant M_{ω_0} such that

$$\left\| R_{L_\varepsilon^X}(\lambda) \right\| \leq M_{\omega_0}$$

holds for every $\lambda \in \overline{\mathbb{H}_{-\omega_0}} \setminus B_{1/2}(1)$ and $|\varepsilon| \leq \varepsilon^{**}$. Since $R_{L_\varepsilon^X}(\lambda) \, (I - P_\varepsilon)$ is analytic in $\mathbb{H}_{-\omega_0}$ and coincides with the resolvent of L_ε^X restricted to the range of $I - P_\varepsilon$, the estimate (4.2.13) extends to all $\lambda \in \overline{B_{1/2}(1)}$ as well.
To prove the Lipschitz estimate of the semigroup on the stable subspace (4.2.12) let $u \in \mathcal{D}(L^X) = \mathcal{D}(L_\varepsilon^X) = \mathcal{D}(L_\kappa^X)$, choose $\omega_0 < \omega_1 < \widetilde{\omega}$, fix $\tau > 0$ and define

$$F(\sigma) := S_\varepsilon^X(\tau - \sigma) S_\kappa^X(\sigma) u \quad \text{for} \quad 0 \leq \sigma \leq \tau.$$

We obtain

$$\begin{aligned} F'(\sigma) &= -S_\varepsilon^X(\tau - \sigma) L_\varepsilon^X S_\kappa^X(\sigma) u + S_\varepsilon^X(\tau - \sigma) L_\kappa^X S_\kappa^X(\sigma) u \\ &= -S_\varepsilon^X(\tau - \sigma)(L_\varepsilon^X - L_\kappa^X) S_\kappa^X(\sigma) u \end{aligned}$$

and integrating over $[0, \tau]$ gives

$$S_\varepsilon^X(\tau) u - S_\kappa^X(\tau) u = \int_0^\tau S_\varepsilon^X(\tau - \sigma)(L_\varepsilon^X - L_\kappa^X) S_\kappa^X(\sigma) u \, d\sigma. \tag{4.2.14}$$

We now apply this to $(I - P_\kappa)u$ instead of just u and obtain

$$S_\kappa^X(\tau)(I-P_\kappa)u = S_\varepsilon^X(\tau)(I-P_\kappa)u - \int_0^\tau S_\varepsilon^X(\tau-\sigma)(L_\varepsilon^X - L_\kappa^X)S_\kappa^X(\sigma)(I-P_\kappa)u\,d\sigma.$$

If we now apply P_ε to this, we obtain with (4.2.10)

$$P_\varepsilon S_\kappa^X(\tau)(I-P_\kappa)u = e^\tau P_\varepsilon(I-P_\kappa)u - \int_0^\tau e^{\tau-\sigma}P_\varepsilon(L_\varepsilon^X - L_\kappa^X)S_\kappa^X(\sigma)(I-P_\kappa)u\,d\sigma.$$

Multiplying by $e^{-\tau}$ we get

$$e^{-\tau}P_\varepsilon S_\kappa^X(\tau)(I-P_\kappa)u = P_\varepsilon(I-P_\kappa)u - \int_0^\tau e^{-\sigma}P_\varepsilon(L_\varepsilon^X - L_\kappa^X)S_\kappa^X(\sigma)(I-P_\kappa)u\,d\sigma$$

and from (4.2.11) we get that the left-hand side converges towards 0 as $\tau \to \infty$ and that the integral on the right-hand side converges. We therefore have

$$P_\varepsilon(I-P_\kappa)u = \int_0^\infty e^{-\sigma}P_\varepsilon(L_\varepsilon^X - L_\kappa^X)S_\kappa^X(\sigma)(I-P_\kappa)u\,d\sigma. \tag{4.2.15}$$

We now obtain with (4.2.14)

$$\begin{aligned} S_\varepsilon^X(\tau)(I-P_\varepsilon)u - S_\kappa^X(\tau)(I-P_\kappa)u &= S_\varepsilon^X(\tau)(P_\kappa - P_\varepsilon)u + [S_\varepsilon^X(\tau) - S_\kappa^X(\tau)](I-P_\kappa)u \\ &= S_\varepsilon^X(\tau)(P_\kappa - P_\varepsilon)u + \int_0^\tau S_\varepsilon^X(\tau-\sigma)(L_\varepsilon^X - L_\kappa^X)S_\kappa^X(\sigma)(I-P_\kappa)u\,d\sigma. \end{aligned}$$

If we now use $S_\varepsilon^X(\tau) = S_\varepsilon^X(\tau)(I-P_\varepsilon) + e^\tau P_\varepsilon$ and (4.2.15) we obtain the following representation formula

$$\begin{aligned} &S_\varepsilon^X(\tau)(I-P_\varepsilon)u - S_\kappa^X(\tau)(I-P_\kappa)u \\ &\quad= S_\varepsilon^X(\tau)(I-P_\varepsilon)(P_\kappa - P_\varepsilon)u + \int_0^\tau S_\varepsilon^X(\tau-\sigma)(I-P_\varepsilon)(L_\varepsilon^X - L_\kappa^X)S_\kappa^X(\sigma)(I-P_\kappa)u\,d\sigma \\ &\qquad + e^\tau\left[P_\varepsilon(P_\kappa - P_\varepsilon)u + \int_0^\tau e^{-\sigma}P_\varepsilon(L_\varepsilon^X - L_\kappa^X)S_\kappa^X(\sigma)(I-P_\kappa)u\,d\sigma\right] \\ &\quad= S_\varepsilon^X(\tau)(I-P_\varepsilon)(P_\kappa - P_\varepsilon)u + \int_0^\tau S_\varepsilon^X(\tau-\sigma)(I-P_\varepsilon)(L_\varepsilon^X - L_\kappa^X)S_\kappa^X(\sigma)(I-P_\kappa)u\,d\sigma \\ &\qquad + e^\tau\left[P_\varepsilon(P_\kappa - I)u + \int_0^\tau e^{-\sigma}P_\varepsilon(L_\varepsilon^X - L_\kappa^X)S_\kappa^X(\sigma)(I-P_\kappa)u\,d\sigma\right] \\ &\quad= S_\varepsilon^X(\tau)(I-P_\varepsilon)(P_\kappa - P_\varepsilon)u + \int_0^\tau S_\varepsilon^X(\tau-\sigma)(I-P_\varepsilon)(L_\varepsilon^X - L_\kappa^X)S_\kappa^X(\sigma)(I-P_\kappa)u\,d\sigma \\ &\qquad - e^\tau\int_\tau^\infty e^{-\sigma}P_\varepsilon(L_\varepsilon^X - L_\kappa^X)S_\kappa^X(\sigma)(I-P_\kappa)u\,d\sigma. \end{aligned}$$

For the first term we obtain with (4.2.11) and the Lipschitz continuity of P_ε for every $u \in \mathcal{D}(L^X)$

$$\left\|S_\varepsilon^X(\tau)(I-P_\varepsilon)(P_\kappa - P_\varepsilon)u\right\|_{X_s^k} \lesssim e^{-\omega_0\tau}|\varepsilon - \kappa|\,\|u\|_{X_s^k}.$$

For the second term we use the Lipschitz continuity of L_ε^X and get

$$\begin{aligned}&\left\|\int_0^\tau S_\varepsilon^X(\tau-\sigma)(I-P_\varepsilon)(L_\varepsilon^X-L_\kappa^X)S_\kappa^X(\sigma)(I-P_\kappa)u\,d\sigma\right\|_{X_s^k}\\ \lesssim |\varepsilon-\kappa|\int_0^\tau e^{-\omega_1(\tau-\sigma)}e^{-\omega_1\sigma}\,d\sigma\,\|u\|_{X_s^k} &= |\varepsilon-\kappa|\tau e^{-\omega_1\tau}\,\|u\|_{X_s^k}\lesssim |\varepsilon-\kappa|e^{-\omega_0\tau}\,\|u\|_{X_s^k}.\end{aligned}$$

For the last term we obtain analogously

$$\begin{aligned}e^\tau\left\|\int_\tau^\infty e^{-\sigma}P_\varepsilon(L_\varepsilon^X-L_\kappa^X)S_\kappa^X(\sigma)(I-P_\kappa)u\,d\sigma\right\|_{X_s^k} &\lesssim |\varepsilon-\kappa|e^\tau\int_\tau^\infty e^{-(1+\omega_0)\sigma}\,d\sigma\,\|u\|_{X_s^k}\\ &\lesssim |\varepsilon-\kappa|e^{-\omega_0\tau}\,\|u\|_{X_s^k},\end{aligned}$$

so that the claim now follows from the density of $\mathcal{D}(L^X)$ in X_s^k.

□

4.2.2. The full nonlinear Cauchy problem for the perturbation

We now turn to the full nonlinear Cauchy problem (4.2.3) but for now with arbitrary small initial data $u \in X_s^k$ that is

$$\begin{cases}\partial_\tau\phi_\varepsilon(\tau) &= \widetilde{L}_\varepsilon(\phi_\varepsilon(\tau))+\widehat{N}_\varepsilon(\phi_\varepsilon(\tau)),\\ \phi_\varepsilon(0) &= u,\end{cases}\tag{4.2.16}$$

By Duhamel's formula, this Cauchy problem can be written as the following integral equation

$$\phi_\varepsilon(\tau)=S_\varepsilon^X(\tau)u+\int_0^\tau S_\varepsilon^X(\tau-\tau')\widehat{N}_\varepsilon(\phi_\varepsilon(\tau'))\,d\tau'.\tag{4.2.17}$$

We fix $\omega := \widetilde{\omega}/2$, where $\widetilde{\omega}$ is the constant from Proposition 4.1.1, and set $\overline{\varepsilon} := \varepsilon^{**}(\omega)$, with $\varepsilon^{**}(\omega)$ as in Proposition 4.2.6. We now first prove a local Lipschitz estimate for the nonlinearity $\widehat{N}_\varepsilon$ defined in (4.2.4).

Lemma 4.2.8. *For every $|\varepsilon|\le\overline{\varepsilon}$ we have $\widehat{N}_\varepsilon : X_s^k \to X_s^k$ as well as*

$$\left\|\widehat{N}_\varepsilon(u)-\widehat{N}_\kappa(v)\right\|_{X_s^k}\lesssim\left(\|u\|_{X_s^k}+\|v\|_{X_s^k}\right)\|u-v\|_{X_s^k}+\left(\|u\|_{X_s^k}^2+\|v\|_{X_s^k}^2\right)|\varepsilon-\kappa|$$

for all $|\varepsilon|,|\kappa|\le\overline{\varepsilon}$ and all $u,v\in\mathcal{B}_\delta\subset X_s^k$ for $0<\delta\le 1$ fixed.

Proof. The proof is analogous to the proof of Lemma 4.1.17 using the Schauder estimate from Proposition A.2.1.

□

The stability of ψ_ε would now follow if we could show global existence and exponential decay of functions satisfying (4.2.17) for arbitrary small initial data $u \in X_s^k$. Unfortunately, the unstable mode at $\lambda = 1$ of the linearized operator L_ε^X prevents us from concluding this immediately. To overcome this we follow the standard approach by subtracting a correction term

$$C(\phi, \varepsilon, u) := P_\varepsilon \left(u + \int_0^\infty e^{-\tau'} \, \widehat{N}_\varepsilon(\phi(\tau')) \, d\tau' \right)$$

to the initial data, which then stabilizes the evolution. More concretely, we introduce the Banach space

$$\mathcal{X} := \{\phi \in C([0,\infty), X_s^k) : \|\phi\|_{\mathcal{X}} := \sup_{\tau>0} e^{\omega\tau} \, \|\phi(\tau)\|_{X_s^k} < \infty\}$$

and want to find a $\phi \in \mathcal{X}$ solving the following integral equation

$$\phi(\tau) = S_\varepsilon^X(\tau) \, [u - C(\phi, \varepsilon, u)] + \int_0^\tau S_\varepsilon^X(\tau - \tau') \widehat{N}_\varepsilon(\phi(\tau')) \, d\tau'. \tag{4.2.18}$$

Since this is a fixed-point equation we define

$$K(\phi, \varepsilon, u)(\tau) := S_\varepsilon^X(\tau) \, [u - C(\phi, \varepsilon, u)] + \int_0^\tau S_\varepsilon^X(\tau - \tau') \widehat{N}_\varepsilon(\phi(\tau')) \, d\tau'$$

and want to show that $K(\cdot, \varepsilon, u)$ is a well-defined contraction on $\mathcal{X}_\delta := \{\phi \in \mathcal{X} : \ \|\phi\|_{\mathcal{X}} \leq \delta\}$ for sufficiently small $\delta > 0$ and fixed (ε, u).

Proposition 4.2.9. *There exist constants $0 < \delta_0 < 1$ and $C_0 > 1$ such that for all $0 < \delta \leq \delta_0$ and $C \geq C_0$ there exists for every $|\varepsilon| \leq \overline{\varepsilon}$ and every $u \in X_s^k$ with $\|u\|_{X_s^k} \leq \frac{\delta}{C}$ a unique function $\phi_\varepsilon(u) \in \mathcal{X}_\delta$ such that (4.2.18) holds for all $\tau \geq 0$.*
Additionally, the map $(u, \varepsilon) \mapsto \phi_\varepsilon(u)$ is Lipschitz continuous in the sense that the following estimate holds for all $u, v \in X_s^k$ with $\|u\|_{X_s^k}, \|v\|_{X_s^k} \leq \frac{\delta}{C}$ and all $|\varepsilon|, |\kappa| \leq \overline{\varepsilon}$

$$\|\phi_\varepsilon(u) - \phi_\kappa(v)\|_{\mathcal{X}} \lesssim \|u - v\|_{X_s^k} + |\varepsilon - \kappa|.$$

Proof. First, we show that the map $K_{(u,\varepsilon)}(\phi) := K(\phi, \varepsilon, u)$ is a well-defined contraction on $\mathcal{X}_\delta$ for fixed $|\varepsilon| \leq \overline{\varepsilon}$ all sufficiently large $C > 1$, sufficiently small $\delta > 0$ and all $u \in X_s^k$ with $\|u\|_{X_s^k} \leq \frac{\delta}{C}$. To see this we take $\phi \in \mathcal{X}_\delta$ and $\tau \geq 0$ and write $K_{(u,\varepsilon)}$ in the following way

$$\begin{aligned} &K_{(u,\varepsilon)}(\phi)(\tau) \\ &= S_\varepsilon^X(\tau)(I - P_\varepsilon)u - \int_\tau^\infty e^{\tau - \tau'} \, P_\varepsilon \, \widehat{N}_\varepsilon(\phi(\tau')) \, d\tau' + \int_0^\tau S_\varepsilon^X(\tau - \tau')(I - P_\varepsilon)\widehat{N}_\varepsilon(\phi(\tau')) \, d\tau'. \end{aligned} \tag{4.2.19}$$

From this we obtain with Proposition 4.2.7 and Lemma 4.2.8

$$\left\| K_{(u,\varepsilon)}(\phi)(\tau) \right\|_{X_s^k} \lesssim \frac{\delta}{C} e^{-\omega\,\tau} + \delta^2 e^{-2\omega\,\tau} + \delta^2 e^{-\omega\,\tau}(e^{-\omega\,\tau} + 1) \lesssim \left(\frac{1}{C} + \delta \right) \delta e^{-\omega\,\tau}$$

so that we have

$$\left\|K_{(u,\varepsilon)}(\phi)(\tau)\right\|_{X_s^k} \leq \delta e^{-\omega\,\tau}$$

if we choose $C \geq C_0$ and $0 < \delta \leq \delta_0$ with C_0 sufficiently large and $\delta_0 > 0$ sufficiently small.
Since the continuity of the mapping $\tau \mapsto K_{(u,\varepsilon)}(\phi)(\tau)$ follows from the dominated convergence we conclude that $K_{(u,\varepsilon)} : \mathcal{X}_\delta \to \mathcal{X}_\delta$ is well-defined.
To show that $K_{(u,\varepsilon)}$ is a contraction on $\mathcal{X}_\delta$ (for potentially even smaller $\delta > 0$) we take $\phi, \psi \in \mathcal{X}_\delta$ and calculate for every $\tau \geq 0$ by using again the representation of $K_{(u,\varepsilon)}$ from Eq. (4.2.19)

$$\begin{aligned}
&\left\|K_{(u,\varepsilon)}(\phi)(\tau) - K_{(u,\varepsilon)}(\psi)(\tau)\right\|_{X_s^k} \\
&\lesssim \int_\tau^\infty e^{\tau-\tau'} \|\widehat{N}_\varepsilon(\phi(\tau')) - \widehat{N}_\varepsilon(\psi(\tau'))\|_{X_s^k}\, d\tau' + \int_0^\tau e^{-\omega\,(\tau-\tau')} \|\widehat{N}_\varepsilon(\phi(\tau')) - \widehat{N}_\varepsilon(\psi(\tau'))\|_{X_s^k}\, d\tau' \\
&\lesssim \left(\delta e^{-2\omega\,\tau} + \delta e^{-\omega\,\tau}(e^{-\omega\,\tau}+1)\right) \|\phi - \psi\|_{\mathcal{X}}\,.
\end{aligned}$$

If we now choose $\delta_0 > 0$ sufficiently small we get

$$\left\|K_{(u,\varepsilon)}(\phi) - K_{(u,\varepsilon)}(\psi)\right\|_{\mathcal{X}} \leq \frac{1}{2}\,\|\phi - \psi\|_{\mathcal{X}}\,,$$

for all $0 < \delta \leq \delta_0$, $C \geq C_0$ and all $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \leq \overline{\varepsilon}$ so that the existence of a unique $\phi_\varepsilon(u) \in \mathcal{X}_\delta$ follows from the contraction mapping principle.
It remains to prove that the map $(u,\varepsilon) \mapsto \phi_\varepsilon(u) \in \mathcal{X}_\delta$ is Lipschitz continuous. For this we take $(u,\varepsilon),(v,\kappa) \in \mathcal{B}_{\frac{\delta}{C}} \times [-\overline{\varepsilon},\overline{\varepsilon}]$ and obtain by the previous considerations functions $\phi_\varepsilon(u), \phi_\kappa(v) \in \mathcal{X}_\delta$ solving

$$\phi_\varepsilon(u)(\tau) = K(\phi_\varepsilon(u),\varepsilon,u)(\tau) \quad \text{and} \quad \phi_\kappa(v)(\tau) = K(\phi_\kappa(v),\kappa,v)(\tau) \quad \forall\,\tau \geq 0.$$

We now show that

$$\|K(\phi_\varepsilon(u),\varepsilon,u) - K(\phi_\kappa(v),\kappa,v)\|_{\mathcal{X}} \lesssim \|u-v\|_{X_s^k} + |\varepsilon-\kappa|.$$

For this we take $\tau \geq 0$ and estimate the terms in (4.2.19) separately. For the first term we simply get from (4.2.12)

$$\left\|S_\varepsilon^X(\tau)(I-P_\varepsilon)u - S_\kappa^X(\tau)(I-P_\kappa)v\right\| \lesssim \frac{\delta}{C} e^{-\omega\,\tau}|\varepsilon-\kappa| + e^{-\omega\,\tau}\,\|u-v\|_{X_s^k}\,.$$

For the second term we apply Lemma 4.2.8 and the Lipschitz continuity of P_ε to get

$$\begin{aligned}
&\int_\tau^\infty e^{\tau-\tau'} \|P_\varepsilon\,\widehat{N}_\varepsilon(\phi_\varepsilon(u)(\tau')) - P_\kappa\,\widehat{N}_\kappa(\phi_\kappa(v)(\tau'))\|_{X_s^k}\, d\tau' \\
&\lesssim |\varepsilon-\kappa| \int_\tau^\infty e^{\tau-\tau'} \|\widehat{N}_\kappa(\phi_\kappa(v)(\tau'))\|_{X_s^k}\, d\tau' + \int_\tau^\infty e^{\tau-\tau'} \|\widehat{N}_\varepsilon(\phi_\varepsilon(u)(\tau')) - \widehat{N}_\kappa(\phi_\kappa(v)(\tau'))\|_{X_s^k}\, d\tau' \\
&\lesssim |\varepsilon-\kappa|\,\delta^2\, e^{-2\omega\,\tau} + \delta\, e^{-2\omega\,\tau}\,\|\phi_\varepsilon(u) - \phi_\kappa(v)\|_{\mathcal{X}}
\end{aligned}$$

and for the last term we similarly get

$$\begin{aligned}
&\int_0^\tau \left\| S_\varepsilon^X(\tau-\tau')(I-P_\varepsilon)\widehat{N}_\varepsilon(\phi_\varepsilon(u)(\tau')) - S_\kappa^X(\tau-\tau')(I-P_\kappa)\widehat{N}_\kappa(\phi_\kappa(v)(\tau'))\right\|_{X_s^k} d\tau' \\
&\lesssim |\varepsilon-\kappa| \int_0^\tau e^{-\omega\,(\tau-\tau')} \|\widehat{N}_\kappa(\phi_\kappa(v)(\tau'))\|_{X_s^k}\, d\tau' \\
&\quad + \int_0^\tau e^{-\omega\,(\tau-\tau')} \|\widehat{N}_\varepsilon(\phi_\varepsilon(u)(\tau')) - \widehat{N}_\kappa(\phi_\kappa(v)(\tau'))\|_{X_s^k}\, d\tau' \\
&\lesssim |\varepsilon-\kappa|\,\delta^2\, e^{-\omega\,\tau} + \delta\, e^{-\omega\,\tau}\, \|\phi_\varepsilon(u)-\phi_\kappa(v)\|_{\mathcal{X}}\,.
\end{aligned}$$

With that we obtain

$$\begin{aligned}
\|\phi_\varepsilon(u)-\phi_\kappa(v)\|_{\mathcal{X}} &= \|K(\phi_\varepsilon(u),\varepsilon,u) - K(\phi_\kappa(v),\kappa,v)\|_{\mathcal{X}} \\
&\lesssim |\varepsilon-\kappa| + \|u-v\|_{X_s^k} + \delta\,\|\phi_\varepsilon(u)-\phi_\kappa(v)\|_{\mathcal{X}}\,.
\end{aligned}$$

For small enough $\delta > 0$ we get $\|\phi_\varepsilon(u)-\phi_\kappa(v)\|_{\mathcal{X}} \lesssim |\varepsilon-\kappa| + \|u-v\|_{X_s^k}$ as desired. □

To now obtain a solution to the original integral equation (4.2.17) we adjust the blowup time T so that the corresponding correction term vanishes for this particular choice of T. We therefore return to the Cauchy problem (4.2.3) but now with the specific initial data and define

$$U_\varepsilon(v,T) := v^T + \psi_\varepsilon^T - \psi_\varepsilon = \sqrt{T}v(\sqrt{T}\cdot) + \sqrt{T}\psi_\varepsilon(\sqrt{T}\cdot) - \psi_\varepsilon.$$

Lemma 4.2.10. *Take $0 < \delta \leq \frac{1}{2}$. For every $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \leq \bar{\varepsilon}$ and every fixed $v \in X_s^k$ the map*

$$U_\varepsilon(v,\cdot) : [1-\delta, 1+\delta] \to X_s^k,\ T \mapsto U_\varepsilon(v,T)$$

is continuous and for every $T \in [1-\delta,1+\delta]$ the initial data operator can be written as

$$U_\varepsilon(v,T) = v^T + (T-1)g_\varepsilon + R_\varepsilon(T), \tag{4.2.20}$$

where R_ε is a remainder term satisfying

$$\|R_\varepsilon(T)\|_{X_s^k} \leq M_\varepsilon |T-1|^2$$

for a constant $M_\varepsilon > 0$.

Proof. The continuity of $U_\varepsilon(v,\cdot)$ follows along the lines as for example Lemma 3.11 in [1]. Eq. (4.2.20) follows from a Taylor expansion applied to the map $[1-\delta,1+\delta] \to X_s^k,\ T \mapsto \psi_\varepsilon^T$ using the following fact

$$\partial_T \sqrt{T}\psi_\varepsilon(\sqrt{T}\cdot)\Big|_{T=1} = \Lambda\psi_\varepsilon = g_\varepsilon.$$

The remainder term $R_\varepsilon(T)$ then satisfies

$$\|R_\varepsilon(T)\|_{X_s^k} \lesssim (T-1)^2 \sum_{j=0}^{2} \left\|\Lambda^j \psi_\varepsilon\right\|_{X_s^k}$$

and we remark that due to the decay of g_ε we have that also Λg_ε belongs to X_s^k so that the right-hand side of the above inequality is finite.

□

Now, we are in the position to prove the central result of this section.

Theorem 4.2.11. *For any $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \leq \bar{\varepsilon}$, there are constants $0 < \delta_\varepsilon < 1$ and $C_\varepsilon > 1$ such that for all $0 < \delta \leq \delta_\varepsilon$ and all $C \geq C_\varepsilon$ the following statement holds: For every $v \in X_s^k$ with $\|v\|_{X_s^k} \leq \frac{\delta}{C^2}$ there exists a $T_\varepsilon = T_\varepsilon(v) \in [1 - \frac{\delta}{C}, 1 + \frac{\delta}{C}]$ and a unique $\phi_\varepsilon \in C([0,\infty); X_s^k)$ satisfying*

$$\phi_\varepsilon(\tau) = S_\varepsilon^X(\tau) U_\varepsilon(v, T_\varepsilon) + \int_0^\tau S_\varepsilon^X(\tau - \tau') \widehat{N}_\varepsilon(\phi_\varepsilon(\tau'))\, d\tau' \quad \text{for all} \quad \tau \geq 0. \tag{4.2.21}$$

Furthermore,

$$\|\phi_\varepsilon(\tau)\|_{X_s^k} \leq \delta e^{-\omega\tau}, \quad \forall \tau \geq 0.$$

Proof. Let $0 < \delta \leq \delta_0$ and $C \geq C_0 \geq 1$ with δ_0 and C_0 as in Proposition 4.2.9. Let $|\varepsilon| \leq \bar{\varepsilon}$ and $v \in X_s^k$ with $\|v\|_{X_s^k} \leq \frac{\delta}{C^2}$. Then we obtain from Lemma 4.2.10

$$\begin{aligned}\|U_\varepsilon(v,T)\|_{X_s^k} &\lesssim \left\|v^T\right\|_{X_s^k} + |T-1| \, \|g_\varepsilon\|_{X_s^k} + \|R_\varepsilon(T)\|_{X_s^k} \\ &\lesssim \frac{\delta}{C^2} + \frac{\delta}{C} A_\varepsilon + \frac{\delta^2}{C^2} M_\varepsilon\end{aligned}$$

for every $T \in [1 - \frac{\delta}{C}, 1 + \frac{\delta}{C}]$ where $A_\varepsilon, M_\varepsilon > 0$ are some constants depending on ε. If we now choose δ sufficiently small and C sufficiently large we obtain for every $T \in [1 - \frac{\delta}{C}, 1 + \frac{\delta}{C}]$ from Proposition 4.2.9 the existence of a unique $\phi_\varepsilon = \phi_\varepsilon(v,T) \in \mathcal{X}_\delta$ which solves

$$\phi_\varepsilon(\tau) = S_\varepsilon^X(\tau) \left[U_\varepsilon(v,T) - C(\phi_\varepsilon, \varepsilon, U_\varepsilon(v,T))\right] + \int_0^\tau S_\varepsilon^X(\tau - \tau') \widehat{N}_\varepsilon(\phi_\varepsilon(\tau'))\, d\tau'.$$

Since C takes values in $\operatorname{ran} P_\varepsilon = \langle g_\varepsilon \rangle$ it is enough to show, given v, the existence of a T such that

$$\langle C(\phi_\varepsilon(v,T), \varepsilon, U_\varepsilon(v,T)), g_\varepsilon \rangle_{X_s^k} = 0. \tag{4.2.22}$$

Due to Lemma 4.2.10 and the definition of C this equation reads as

$$0 = \langle P_\varepsilon v^T, g_\varepsilon \rangle_{X_s^k} + (T-1) \|g_\varepsilon\|_{X_s^k}^2 + \langle P_\varepsilon R_\varepsilon(T), g_\varepsilon \rangle_{X_s^k} + \langle P_\varepsilon \int_0^\infty e^{-\tau'} \widehat{N}_\varepsilon(\phi_\varepsilon(\tau'))\, d\tau', g_\varepsilon \rangle_{X_s^k},$$

which can be written as a fixed-point equation for $T \in [1-\frac{\delta}{C}, 1+\frac{\delta}{C}]$,

$$T = 1 - \langle P_\varepsilon v^T, \widehat{g}_\varepsilon \rangle_{X_s^k} - \langle P_\varepsilon R_\varepsilon(T), \widehat{g}_\varepsilon \rangle_{X_s^k} - \langle P_\varepsilon \int_0^\infty e^{-\tau'} \widehat{N}_\varepsilon(\phi_\varepsilon(\tau'))\, d\tau', \widehat{g}_\varepsilon \rangle_{X_s^k}, \quad (4.2.23)$$

where we have set $\widehat{g}_\varepsilon = g_\varepsilon / \|g_\varepsilon\|_{X_s^k}^2$. Now we obtain from the assumptions on v, the fact that ϕ_ε belongs to $\mathcal{X}_\delta$ and Lemma 4.2.10 as well as Lemma 4.2.8 the following estimate

$$\begin{aligned} &|\langle P_\varepsilon v^T, \widehat{g}_\varepsilon \rangle_{X_s^k}| + |\langle P_\varepsilon R_\varepsilon(T), \widehat{g}_\varepsilon \rangle_{X_s^k}| + |\langle P_\varepsilon \int_0^\infty e^{-\tau'} \widehat{N}_\varepsilon(\phi_\varepsilon(\tau'))\, d\tau', \widehat{g}_\varepsilon \rangle_{X_s^k}| \\ \lesssim &\frac{\delta}{C^2} A_\varepsilon + \frac{\delta^2}{C^2} M_\varepsilon + \delta^2 N_\varepsilon \end{aligned}$$

for again some constants $A_\varepsilon, M_\varepsilon$ and $N_\varepsilon > 0$. If we now choose $C \geq C_\varepsilon$ and $0 < \delta < \delta_\varepsilon$ with $C_\varepsilon > 1$ sufficiently large and $\delta_\varepsilon < 1$ sufficiently small we get that the right-hand side of (4.2.23) is a continuous mapping from $[1-\frac{\delta}{C}, 1+\frac{\delta}{C}]$ into itself so that we obtain by the fixed-point theorem of Brouwer a $T_\varepsilon \in \left[1-\frac{\delta}{C}, 1+\frac{\delta}{C}\right]$ such that equation (4.2.22) is fulfilled. We therefore conclude that the corresponding solution $\phi_\varepsilon(v, T_\varepsilon)$ satisfies (4.2.21). The claimed uniqueness follows along the lines of the proof of Theorem 5.4 in [41].

□

Now we will show the regularity of the just constructed solution.

Proposition 4.2.12. *Let $v \in \mathcal{S}_r(\mathbb{R}^n)$ satisfy the assumptions of Theorem 4.2.11. The from Theorem 4.2.11 guaranteed solution ϕ_ε of Eq.* (4.2.21) *is smooth and satisfies* (4.2.16) *in the classical sense.*

Proof. Due to the regularity of v we first of all obtain $U_\varepsilon(v, T_\varepsilon) \in \mathcal{D}(L_\varepsilon^X)$ and by the local Lipschitz continuity of $\widehat{N}_\varepsilon$ we obtain from standard semigroup theory that ϕ_ε is a strong solution to (4.2.21), i.e. it is a strong solution to (4.2.16).
We therefore have $\phi_\varepsilon \in C([0,\infty), \mathcal{D}(L_\varepsilon^X)) \cap C^1([0,\infty), X_s^k(\mathbb{R}^n))$ with

$$\partial_\tau \phi_\varepsilon(\tau) = L_\varepsilon \phi_\varepsilon(\tau) + \widehat{N}_\varepsilon(\phi_\varepsilon(\tau)). \quad (4.2.24)$$

Due to the embedding $X_s^k(\mathbb{R}^n) \hookrightarrow C^2(\mathbb{R}^n)$ (4.2.24) holds pointwise. If we now use the decomposition $L_\varepsilon = L + L'_\varepsilon$ and the boundedness of L'_ε we get that ϕ_ε solves

$$\phi_\varepsilon(\tau) = S(\tau) U_\varepsilon(v, T_\varepsilon) + \int_0^\tau S(\tau - \tau') \left(L'_\varepsilon \phi_\varepsilon(\tau') + \widehat{N}_\varepsilon(\phi_\varepsilon(\tau')) \right) d\tau'.$$

Using the smoothing properties of the free semigroup, see Appendix A.4 from [41], we obtain $\phi_\varepsilon(\tau) \in C^\infty(\mathbb{R}^n)$ for every $\tau \geq 0$ and using a generalized version of Schwartz Lemma we are allowed to interchange ∂_τ and L_ε to obtain $\phi_\varepsilon \in C^\infty([0,\infty) \times \mathbb{R}^n)$.

□

Now we are finally able to prove our main stability results, Theorem 4.0.4 and Theorem 4.0.2.

Proof of Theorem 4.0.4. Under the assumption stated in Theorem 4.0.4 choose $\omega = \widetilde{\omega}/2$ and $0 < \overline{\varepsilon}$, depending on ω, as at the beginning of Section 4.2.2. For $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \leq \overline{\varepsilon}$ let $\delta = \delta_\varepsilon$ and $C = C_\varepsilon$ denote the constants from Theorem 4.2.11 and let $\varphi_0 \in \mathcal{S}_r(\mathbb{R}^n)$ satisfy $\|\varphi_0\|_{X_s^k} < \frac{\delta}{C^2}$.
Then, by Theorem 4.2.11 and Proposition 4.2.12 there is a $T = T_\varepsilon \in [1 - \frac{\delta}{C}, 1 + \frac{\delta}{C}]$ and a unique radial function $\varphi_\varepsilon \in C^\infty([0,\infty) \times \mathbb{R}^n)$ solving the initial value problem (4.2.3). Moreover,

$$\|\varphi_\varepsilon(\tau, \cdot)\|_{X_s^k} \leq \delta e^{-\omega\tau},$$

for all $\tau \geq 0$. We set

$$v(t,x) := v_\varepsilon^T(t,x) + \frac{1}{\sqrt{T-t}} \varphi_\varepsilon \left(\log\left(\frac{T}{T-t}\right), \frac{x}{\sqrt{T-t}} \right).$$

Then $v \in C^\infty([0,T) \times \mathbb{R}^n)$ by construction and it satisfies Eq. (4.1.1) with

$$v(0,\cdot) = v_\varepsilon^1(0,\cdot) + \varphi_0.$$

Moreover for $r \in [s,k]$,

$$\|\varphi_\varepsilon(-\log(T-t) + \log T, \cdot)\|_{\dot{H}^r(\mathbb{R}^n)} \lesssim \|\phi_\varepsilon(\tau)\|_{X_s^k} \lesssim \delta (T-t)^\omega$$

by definition and Theorem 4.2.11.

□

Proof of Theorem 4.0.2. By the assumptions of Theorem 4.0.2 the initial data are of the form

$$U_0(x) = U_\varepsilon^1(0,x) + x v_0(|x|)$$

Consequently, $v_0 \in C_e^\infty[0,\infty)$ so that $\varphi_0(y) := v_0(|y|)$ for $y \in \mathbb{R}^{d+2}$ is a radially symmetric, real-valued Schwartz function $\varphi_0 \in \mathcal{S}(\mathbb{R}^{d+2})$. By Proposition A.5 and Remark A.6 of [40] there exists a constant $C > 0$ such that

$$\|\varphi_0\|_{X_s^k(\mathbb{R}^{d+2})} \leq C \|\nu_0\|_{X_s^k(\mathbb{R}^d, \mathbb{R}^d)}.$$

If the Sobolev exponents (s,k) satisfy condition (4.0.7), then (4.0.16) holds for $n := d+2$. Let $\omega, \overline{\varepsilon}, \delta, M > 0$ be the constants from Theorem 4.0.4. By setting $M_0 := CM$ and requiring

$$\|\nu_0\|_{X_s^k(\mathbb{R}^d, \mathbb{R}^d)} \leq \frac{\delta}{M_0},$$

we find that φ_0 satisfies the assumptions of Theorem 4.0.4. Hence, there is a $T \in [1-\delta, 1+\delta]$ and a unique radial solution $v \in C^\infty([0,T) \times \mathbb{R}^{d+2})$ to (4.0.10). If we set $v(t,\cdot) = \tilde{v}(t,|\cdot|)$ then $\tilde{v}$ solves Eq. (4.0.9) for $t \in [0,T)$ and can be written as

$$\tilde{v}(t,|x|) = \frac{1}{|x|} \widetilde{f}_\varepsilon \left(\frac{|x|}{\sqrt{T-t}} \right) + \frac{1}{\sqrt{T-t}} \tilde{\varphi} \left(\log\left(\frac{T}{T-t}\right), \frac{|x|}{\sqrt{T-t}} \right)$$

for $\tilde{\varphi}(t,\cdot) \in C_e^\infty[0,\infty)$ satisfying

$$\|\tilde{\varphi}(-\log(T-t)+\log T, |\cdot|)\|_{\dot{H}^r(\mathbb{R}^{d+2})} \lesssim \delta(T-t)^\omega \tag{4.2.25}$$

for all $r \in [s,k]$. We define for $x \in \mathbb{R}^d$, $U(t,x) := x\tilde{v}(t,|x|) \in C^\infty([0,T)\times\mathbb{R}^d,\mathbb{R}^d)$ and find that U can be written as

$$U(t,x) = U_\varepsilon^T(t,x) + \nu\left(t, \frac{x}{\sqrt{T-t}}\right),$$

where ν is a corotational function defined via $\nu(t,x) = x\,\tilde{\varphi}(-\log(T-t)+\log T, |x|)$. The inequality from (4.2.25) now implies (4.0.8) by applying Proposition A.5 and Remark A.6 from [40] and the pointwise as well as the local uniform convergence follow immediately from Sobolev embedding.

□

5. Stable blowup for supercritical wave maps into perturbed spheres

Now that we have proven the existence and stability of self-similar blowup into perturbed spheres for the harmonic map heat flow, we will prove the analogous results for wave maps. The results of this chapter are contained in a work from Donninger, Schörkhuber and the author, see [29].

It is well-known that in the energy-supercritical case and for large classes of warped product target manifolds the wave maps equation (1.3.2) admits finite-time blowup via self-similar solutions [58, 13, 24]. However, besides their existence little is known about the role of these solutions in the generic time evolution. Most results are available for $N = \mathbb{S}^d$, where existence [56], [68], [10] and stability of self-similar blowup have been proven in the past years in various settings [22, 28, 20, 15, 19, 7, 39, 30, 31, 27], see also Section 5.0.1.

We start the analysis with a corotational wave map $U : \mathbb{R}^{1+d} \to S^d_\varepsilon$, $U(t, r, \omega) = (u(t, r), \omega)$, where the radial profile u solves the following semilinear d-dimensional radial wave equation

$$\partial_t^2 u = \partial_r^2 u + \frac{d-1}{r}\partial_r u - \frac{d-1}{r^2} w_\varepsilon(u)\, w'_\varepsilon(u). \tag{5.0.1}$$

If we now again introduce normal coordinates $U = (U^1, ..., U^d)$ with $U^j = u\Omega^j$, S^d_ε can be identified with the ball $B^d_\pi(0) \subset \mathbb{R}^d$ so that we can consider the following initial value problem

$$\begin{aligned} \Box\, U^a + \Gamma^a_{bc}(U)\, \partial^\mu U^b\, \partial_\mu U^c = 0, \quad a \in \{1, \dots, d\} \\ U(0, \cdot) = U_0, \quad \partial_t U(0, \cdot) = U_1 \end{aligned} \tag{5.0.2}$$

on $I \times \mathbb{R}^d$, where $I \subset \mathbb{R}$ is an interval containing zero and data $U_0, U_1 : \mathbb{R}^d \to \mathbb{R}^d$.

A corotational map $U(t, \cdot) : \mathbb{R}^d \to S^d_\varepsilon$ with this identification is now of the form

$$U(t, x) = u(t, |x|)\frac{x}{|x|}. \tag{5.0.3}$$

For $\varepsilon = 0$ Eq. (5.0.1) admits self-similar solutions in all space dimensions $d \geq 3$. The so-called *ground state self-similar solution* is known in closed form, see [68], [10], and given by

$$u_0^T(t, r) = f_0\left(\frac{r}{T-t}\right) \quad \text{for} \quad f_0(\rho) = 2\arctan\left(\frac{\rho}{\sqrt{d-2}}\right), \quad T > 0. \tag{5.0.4}$$

In the past years, the stability of u_0^T has been studied intensively. In this chapter, we address the problem of stable self-similar blowup for wave maps into S^d_ε for $\varepsilon \neq 0$.

For the remainder of the chapter we fix a function α satisfying the conditions from Definition 2.3.1 so that we obtain a corresponding family of warped product manifolds $(S^d_\varepsilon)_{|\varepsilon|\leq\varepsilon_0}$ for $\varepsilon_0 = (2\max_{u\in[0,\pi]}|\alpha(u)|)^{-1}$. Our first Theorem states that for any space dimension $d \geq 3$, if we further restrict to $|\varepsilon| \leq \varepsilon^*$ for $\varepsilon^* \leq \varepsilon_0$ sufficiently small, there exist globally smooth (in space) self-similar blowup profiles for Eq. (5.0.1) for these target manifolds.

Theorem 5.0.1. *Let $d \geq 3$. There exists an $\varepsilon^* > 0$ such that for every $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \leq \varepsilon^*$, Eq. (5.0.1) admits a self-similar solution $u^T_\varepsilon \in C^\infty([0,T)\times[0,\infty)) \cap L^\infty([0,T)\times[0,\infty))$ of the form*

$$u^T_\varepsilon(t,r) = f_\varepsilon\left(\frac{r}{T-t}\right), \quad T > 0,$$

such that its gradient blows up in $r = 0$, i.e.,

$$\lim_{t\to T^-} |\partial_r u^T_\varepsilon(t,0)| = +\infty.$$

The profile f_ε can be written as

$$f_\varepsilon = f_0 + \phi_\varepsilon$$

with f_0 defined in Eq. (5.0.4) and a perturbation $\phi_\varepsilon \in L^\infty([0,\infty)) \cap C^\infty([0,\infty))$. The function ϕ_ε depends Lipschitz continuously on the parameter ε in the sense that

$$\left\|(\cdot)^{-1}(\phi_\varepsilon - \phi_\kappa)\right\|_{W^{2,\infty}([0,\infty))} \lesssim |\varepsilon - \kappa| \tag{5.0.5}$$

for all $|\varepsilon|, |\kappa| \leq \varepsilon^*$ and $\phi_0 = 0$. Furthermore, f_ε satisfies $f_\varepsilon(0) = 0$ as well as $0 < f_\varepsilon(\rho) < \pi$ for all $0 < \rho \leq 1$ and the limit $\lim_{\rho\to\infty} f_\varepsilon(\rho)$ exists. Finally, for every $k \in \mathbb{N}$ there are constants $C_k > 0$ such that

$$|f^{(k)}_\varepsilon(\rho)| \leq C_k \langle\rho\rangle^{-1-k}$$

for all $\rho \in [0,\infty)$.

Since this result shows the existence of self-similar blowup solutions, which can be written as small perturbations of the original blowup solutions for the sphere, we are able to prove rigorous results on the stability of these solutions under small corotational perturbations. We first phrase the result in normal coordinates and again consider without loss of generality small perturbations around the blowup solution with blowup time $T = 1$.

Theorem 5.0.2. *Let $d \geq 3$ and choose $\varepsilon^* > 0$ as in Theorem 5.0.1. For $\varepsilon \in \mathbb{R}$, $|\varepsilon| \leq \varepsilon^*$ define*

$$U^T_\varepsilon(t,x) := f_\varepsilon\left(\frac{|x|}{T-t}\right)\frac{x}{|x|}.$$

Consider corotational initial data of the form

$$U_0 = U^1_\varepsilon(0,\cdot) + \nu_0, \quad U_1 = \partial_t U^1_\varepsilon(0,\cdot) + \nu_1,$$

where for $i \in \{0,1\}$, $\nu_i : \mathbb{R}^d \to \mathbb{R}^d$, $\nu_i(x) = x v_i(|x|)$ *and* $v_i : [0,\infty) \to \mathbb{R}$ *are such that* $v_i(|\cdot|) \in \mathcal{S}(\mathbb{R}^d)$. *Let* $(s,k) \in \mathbb{R} \times \mathbb{N}$ *satisfy*

$$\frac{d}{2} < s \leq \frac{d}{2} + \frac{1}{2d+2}, \quad k > d+2. \tag{5.0.6}$$

Then there exists an $\overline{\varepsilon} \in \mathbb{R}$, $0 < \overline{\varepsilon} \leq \varepsilon^*$ *such that for every* $\varepsilon \in \mathbb{R}$ *with* $|\varepsilon| \leq \overline{\varepsilon}$ *the following holds: There are* $\delta, M_0 > 0$ *such that for* (ν_0, ν_1) *as above satisfying*

$$\|(\nu_0,\nu_1)\|_{\dot{H}^s \cap \dot{H}^k(\mathbb{R}^d,\mathbb{R}^d) \times \dot{H}^{s-1} \cap \dot{H}^{k-1}(\mathbb{R}^d,\mathbb{R}^d)} < \frac{\delta}{M_0}, \tag{5.0.7}$$

there exists a $T = T_\varepsilon \in [1-\delta, 1+\delta]$ *and a unique corotational map* $U \in C^\infty([0,T) \times \mathbb{R}^d, \mathbb{R}^d)$ *that satisfies Eq.* (5.0.2) *for all* $(t,x) \in [0,T) \times \mathbb{R}^d$. *The gradient of* U *blows up at the origin as* $t \to T^-$ *and we have the decomposition*

$$U(t,x) = U_\varepsilon^T(t,x) + \nu\left(t, \frac{x}{T-t}\right),$$

for a function $\nu : [0,T) \times \mathbb{R}^d \to \mathbb{R}^d$ *which satisfies*

$$\|\nu(t,\cdot)\|_{\dot{H}^r(\mathbb{R}^d,\mathbb{R}^d)} + \|((T-t)\partial_t + \Lambda)\nu(t,\cdot)\|_{\dot{H}^{r-1}(\mathbb{R}^d,\mathbb{R}^d)} \to 0 \tag{5.0.8}$$

as $t \to T^-$ *for all* $r \in [s,k]$ *where* Λ *is defined as the operator* $\Lambda = x \cdot \nabla$. *In addition,*

$$U(t,(T-t)x) \to f_\varepsilon(|x|)\frac{x}{|x|}$$

pointwise and uniformly on compact subsets of $\mathbb{R}^d$ *as* $t \to T^-$.

Theorem 5.0.1 and Theorem 5.0.2 guarantee for the considered class of target manifolds the existence of *stable* blowup via corotational self-similar solutions in all space dimensions $d \geq 3$. In low space dimensions, this is the first result of this kind for targets other than the sphere, see Section 5.0.1. The assumptions on the initial data and the underlying topology arise naturally by reformulating the problem as a classical stability problem using similarity coordinates, see Section 5.0.2.

5.0.1. Blowup in energy supercritical wave maps - Known results

As the literature on wave maps is vast, we restrict the discussion to works which are relevant in the context of our main results. In particular, we only discuss the energy supercritical case $d \geq 3$.

Concerning the existence of self-similar blowup for wave maps into the sphere, the first result was obtained by Shatah [56] for maps from $\mathbb{R}^{1+3}$ into $\mathbb{S}^3$. Later, Turok and Spergel [68] found in the same setting an explicit example of a self-similar solution (which is expected to be the one from Shatah). Bizoń and Biernat [10] provided the analogous closed form expression in all space dimensions $d \geq 4$, which is given in Eq. (5.0.4). For $3 \leq d \leq 6$, f_0 can indeed be considered as the "ground state" of a family of self-similar

blowup profiles whose existence has been proven by Bizoń [9] and Biernat, Bizoń and Maliborski [5].
Numerical investigations by Bizoń, Chmaj and Tabor [11], see also [10], indicate that blowup via f_0 is generic (at least within the class of corotational initial data). The first rigorous proof of the stability of f_0 under small corotational perturbations (in $d = 3$ and restricted to a backward light cone) has been established in the series of works by the first author [22], jointly with Aichelburg and the second author [28] and with Costin and Xia [20], respectively. Generalizations to higher dimensions were obtained in [15], [19], where the latter work by Costin, the first author and Glogić provides an important simplification of the ODE analysis underlying the spectral problem. Recent works by the first author and Wallauch [30, 31] establish stability in backward light cones at the optimal regularity level in $d = 3, 4$.
In another line of research the stability is studied in so-called hyperboloidal similarity coordinates, which allow to follow the evolution past the blowup time, at least outside the singularity. This has been initiated by the first two authors together with Biernat in [7] and generalized by the first author and Ostermann in [27]. As already mentioned above, the stability analysis on a strip $[0, T) \times \mathbb{R}^d$ has been implemented recently by Glogić [39] in all space dimensions $d \geq 3$.
Finally, non-self-similar blowup occurs at least in dimensions $d \geq 7$, as has been demonstrated by Ghoul, Ibrahim and Nguyen [37].

The existence of finite-time blowup for wave maps from $\mathbb{R}^{1+d}$ into general warped product manifolds $N^m = (0, a) \times_w \mathbb{S}^{m-1}$, $m \in \mathbb{N}$, $m \geq 2$, $a > 0$, has been addressed by Shatah and Tahvildar-Zadeh [58] and Cazenave, Shatah and Tahvildar-Zadeh [13]: In odd space dimensions $d \geq 3$, by using variational methods and ODE tools, the authors construct $k-$equivariant self-similar profiles ($k = 1$ corresponds to the corotational case) locally in a backward light cone which lead to finite time blowup via finite speed of propagation. The admissible values of k and m depend on the properties of the corresponding warping function w. By lifting these solutions to one space dimension higher, finite time blowup is established in even space dimensions $d \geq 4$. The stability properties of the solutions constructed in [58] and [13] are unknown, except for $N = \mathbb{S}^d$. Finally, for $d \geq 8$, the first author and Glogić [24] provided examples of negatively curved targets which admit an explicit self-similar blowup solution and proved its stability in backward light cones in the case $d = 9$.

5.0.2. Reformulation of the problem and outline of the proof

It is common to write corotational maps as $U(t, x) = xv(t, |x|)$ since v then satisfies a radial nonlinear wave equation with a smooth nonlinearity, see e.g. [58].
To realize this, we change variables and define $\widetilde{v}(t, r) := r^{-1}u(t, r)$ which transforms Eq. (5.0.1) into a $(d + 2)$-dimensional radial semilinear wave equation with a now smooth nonlinearity

$$\left(\partial_t^2 - \partial_r^2 - \frac{d+1}{r}\partial_r\right)\widetilde{v}(t, r) - \frac{d-1}{r^3}\left(r\,\widetilde{v}(t, r) - w_\varepsilon(r\,\widetilde{v}(t, r))\,w_\varepsilon'(r\,\widetilde{v}(t, r))\right) = 0. \quad (5.0.9)$$

In the investigation of Eq. (5.0.9) it is convenient to define $n := d+2$ and $v(t,x) := \widetilde{v}(t,|x|)$ for $x \in \mathbb{R}^n$ and then formulate the equation as a wave equation on $\mathbb{R}^n$,

$$(\partial_t^2 - \Delta_x)v(t,x) = \frac{n-3}{|x|^3}\left(|x|v(t,x) - w_\varepsilon(|x|v(t,x))w_\varepsilon'(|x|v(t,x))\right), \quad x \in \mathbb{R}^n. \quad (5.0.10)$$

We then introduce similarity coordinates

$$\tau = \log\left(\frac{T}{T-t}\right), \quad \xi = \frac{x}{T-t}$$

for $T > 0$ and $(t,x) \in [0,T) \times \mathbb{R}^n$ and rewrite Eq. (5.0.10) as a first order system of the form

$$\partial_\tau \Psi(\tau) = \mathbf{L}\Psi(\tau) + \mathbf{N}_\varepsilon(\Psi(\tau)), \quad (5.0.11)$$

where $\mathbf{L}$ generates the free wave evolution and $\mathbf{N}_\varepsilon$ denotes the nonlinearity which depends on ε via the warping function w_ε, see Section 5.1 for the details. We study the problem in intersection Sobolev spaces of radial functions

$$\mathcal{H}_r^{s,k} = (\dot{H}_r^s(\mathbb{R}^n) \cap \dot{H}_r^k(\mathbb{R}^n)) \times (\dot{H}_r^{s-1}(\mathbb{R}^n) \cap \dot{H}_r^{k-1}(\mathbb{R}^n))$$

for suitable exponents $\frac{n}{2} - 1 < s < \frac{n}{2} < k$, $k \in \mathbb{N}$.

Existence of self-similar blowup profiles

For $\varepsilon = 0$, the blowup solution (5.0.4) transforms into a static solution $\mathbf{\Psi}_0$ of (5.0.11), i.e.,

$$\mathbf{L}\mathbf{\Psi}_0 + \mathbf{N_0}(\mathbf{\Psi}_0) = 0. \quad (5.0.12)$$

In order to find for $\varepsilon \neq 0$ a solution to

$$\mathbf{L}\mathbf{\Psi}_\varepsilon + \mathbf{N}_\varepsilon(\mathbf{\Psi}_\varepsilon) = 0 \quad (5.0.13)$$

we insert the ansatz $\mathbf{\Psi}_\varepsilon = \mathbf{\Psi}_0 + \mathbf{\Phi}_\varepsilon$ into (5.0.13) and write the resulting equation for the perturbation $\mathbf{\Phi}_\varepsilon$ as

$$-\mathbf{L}_0\mathbf{\Phi}_\varepsilon = \mathbf{V}_\varepsilon(\mathbf{\Psi}_0)\mathbf{\Phi}_\varepsilon + \widetilde{\mathbf{N}}_\varepsilon(\mathbf{\Phi}_\varepsilon) + \mathcal{R}_\varepsilon(\mathbf{\Psi}_0) \quad (5.0.14)$$

where $\mathbf{L}_0 := \mathbf{L} + \mathbf{V}_0(\mathbf{\Psi}_0)$ denotes the linearization around $\mathbf{\Psi}_0$ for $\varepsilon = 0$, $\mathbf{V}_\varepsilon(\mathbf{\Psi}_0)$ is a potential term depending on ε via the warping function, $\mathcal{R}_\varepsilon(\mathbf{\Psi}_0)$ denotes the remainder and $\widetilde{\mathbf{N}}_\varepsilon$ the nonlinearity, which is quadratically small in its argument. The spectral analysis of [39] implies that $\mathbf{L}_0$, defined as an unbounded operator on a suitable domain $\mathcal{D}(\mathbf{L}_0) \subset \mathcal{H}_r^{s,k} \to \mathcal{H}_r^{s,k}$, is invertible.
In order to control the right-hand side of (5.0.14) we use the parameter-dependent version of Schauder-type estimates originally stated in [39]. For this we assume

$$\frac{n}{2} - 1 < s \leq \frac{n}{2} - 1 + \frac{1}{2n-2}, \quad k > n, \quad (5.0.15)$$

see Appendix A.2. Using this, we show that the first two terms on the right-hand side of Eq. (5.0.14) define maps $\mathcal{H}_r^{s,k} \to \mathcal{H}_r^{s,k+1}$ which are Lipschitz continuous with respect to the parameter ε and with respect to the argument. Moreover, the remainder is in $\mathcal{H}_r^{s,k+1}$ and depends Lipschitz continuously on ε, see Lemma 5.1.2 - 5.1.4.
With these prerequisites, we solve Eq. (5.0.14) for suitably small $|\varepsilon|$ by applying Banach's fixed-point theorem. The smoothness of the constructed $\mathbf{\Phi}_\varepsilon$ can be shown inductively using the mapping properties of the involved operators together with Sobolev embedding. The claimed decay properties of the resulting smooth solution $\mathbf{\Psi}_\varepsilon = (\psi_{\varepsilon,1}, \psi_{\varepsilon,2})$ to Eq. (5.0.13) do not follow from this perturbative approach (although some decay can be obtained from Strauss-type inequalities). Instead, we use ODE analysis similar to the classical work of Kavian and Weissler [45] to show that $\psi_{\varepsilon,j} = \mathcal{O}(|\xi|^{-j})$ at infinity. This property will be crucial in the analysis. Finally, translating the result back to the original coordinates and variables yields Theorem 5.0.1.

Stability of self-similar blowup

We now consider the time-dependent problem (5.0.11) and linearize around $\mathbf{\Psi}_\varepsilon$ by making the ansatz $\Psi(\tau) = \mathbf{\Psi}_\varepsilon + \Phi_\varepsilon(\tau)$ to obtain

$$\begin{cases} \partial_\tau \Phi_\varepsilon(\tau) = \mathbf{L}\Phi_\varepsilon(\tau) + \mathbf{L}'_\varepsilon \Phi_\varepsilon(\tau) + \widehat{\mathbf{N}}_\varepsilon(\Phi_\varepsilon(\tau)), \\ \Phi_\varepsilon(0) = \mathbf{U}_{\varepsilon,T}, \end{cases} \tag{5.0.16}$$

where $\mathbf{L}$ is again the free wave operator, $\mathbf{L}'_\varepsilon \mathbf{u} = (0, V_\varepsilon(\psi_{\varepsilon,1})u_1)$ for $\mathbf{u} = (u_1, u_2) \in \mathcal{H}_r^{s,k}$ includes the potential term, $\widehat{\mathbf{N}}_\varepsilon$ denotes the nonlinear remainder and $\mathbf{U}_{\varepsilon,T}$ suitably chosen initial data depending on T. It is known that under our assumptions on the exponents (s,k), $\mathbf{L}$ generates a strongly continuous one-parameter semigroup $(\mathbf{S}(\tau))_{\tau\geq 0}$ on $\mathcal{H}_r^{s,k}$, which decays exponentially to zero as $\tau \to \infty$, see [39], [23]. The decay properties of $\psi_{\varepsilon,1}$ at infinity allow us to show compactness of the perturbation $\mathbf{L}'_\varepsilon$. This implies in particular that the full linear operator $\mathbf{L}_\varepsilon := \mathbf{L} + \mathbf{L}'_\varepsilon$ generates a semigroup $(\mathbf{S}_\varepsilon(\tau))_{\tau\geq 0}$. In order to obtain suitable growth bounds for the linearized evolution, we investigate the spectrum of the generator. From time-translation invariance we expect $\lambda = 1$ to be an eigenvalue. We make this rigorous in Lemma 5.2.3 and show that

$$\mathbf{g}_\varepsilon := \begin{pmatrix} \psi_{\varepsilon,1} + \Lambda\psi_{\varepsilon,1} \\ 2\psi_{\varepsilon,1} + 3\Lambda\psi_{\varepsilon,1} + \Lambda^2\psi_{\varepsilon,1} \end{pmatrix}$$

is the unique eigenfunction corresponding to $\lambda = 1$. The decay of $\psi_{\varepsilon,1}$ is crucial in order to show that $\mathbf{g}_\varepsilon \in \mathcal{D}(\mathbf{L}_\varepsilon)$. For $\varepsilon = 0$, the spectral properties of $\mathbf{L}_0$ are well-known. More precisely, we know from [39] that there exists an $\tilde{\omega} > 0$ such that

$$\sigma(\mathbf{L}_0) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < -\tilde{\omega}\} \cup \{1\}.$$

By a perturbation argument, using the fact that $1 \in \sigma(\mathbf{L}_\varepsilon)$, we infer that

$$\sigma(\mathbf{L}_\varepsilon) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < -\omega_0\} \cup \{1\}$$

for some $0 < \omega_0 < \tilde{\omega}$ and sufficiently small $\varepsilon \in \mathbb{R}$. The spectral structure in combination with resolvent bounds proven uniformly in the parameter imply

$$\|\mathbf{S}_\varepsilon(\tau)(\mathbf{I} - \mathbf{P}_\varepsilon)\mathbf{u}\|_{\mathcal{H}^{s,k}} \lesssim e^{-\omega_0 \tau} \|(\mathbf{I} - \mathbf{P}_\varepsilon)\mathbf{u}\|_{\mathcal{H}^{s,k}},$$

and $\mathbf{P}_\varepsilon \mathbf{S}_\varepsilon(\tau) = e^\tau \mathbf{P}_\varepsilon$ for $\mathbf{P}_\varepsilon$ denoting the spectral projection onto the eigenspace spanned by $\mathbf{g}_\varepsilon$, see [54], Theorem A.1. For the nonlinearity $\widehat{\mathbf{N}}_\varepsilon$, we establish Lipschitz bounds by imposing again the above assumptions (5.0.15) on the Sobolev exponents and apply the estimates of Appendix A.2. With these results at hand we construct in Section 5.2.2 global, exponentially decaying strong $\mathcal{H}^{s,k}_r$-solutions of Eq. (5.0.16). For this, we use the standard approach as described in the previous chapter and then upgrade the constructed strong solutions to classical ones in Proposition 5.2.10.
Finally, by defining

$$v_\varepsilon^T(t,x) := \frac{1}{T-t}\psi_\varepsilon\left(\frac{x}{T-t}\right), \quad \psi_\varepsilon(\xi) := |\xi|^{-1} f_\varepsilon(|\xi|) \tag{5.0.17}$$

for $x \in \mathbb{R}^n$ and $t \in [0,T)$ we get the following result.

Theorem 5.0.3. *Let $n \geq 5$ and choose $\varepsilon^* > 0$ as in Theorem 5.0.1. For $\varepsilon \in \mathbb{R}$, $|\varepsilon| \leq \varepsilon^*$, let v_ε^T be defined as in Eq.* (5.0.17) *and let $(s,k) \in \mathbb{R} \times \mathbb{N}$ satisfy*

$$\frac{n}{2} - 1 < s \leq \frac{n}{2} - 1 + \frac{1}{2n-2}, \quad k > n. \tag{5.0.18}$$

Then there exist $\omega > 0$ and $0 < \bar{\varepsilon} \leq \varepsilon^*$ such that for every $\varepsilon \in \mathbb{R}$, $|\varepsilon| \leq \bar{\varepsilon}$ there are $\delta > 0$ and $M > 1$ such that the following holds: For any pair of radial, real-valued functions $\varphi_0, \varphi_1 \in \mathcal{S}(\mathbb{R}^n)$ satisfying

$$\|(\varphi_0, \varphi_1)\|_{\dot{H}^s \cap \dot{H}^k(\mathbb{R}^n) \times \dot{H}^{s-1} \cap \dot{H}^{k-1}(\mathbb{R}^n)} < \frac{\delta}{M}, \tag{5.0.19}$$

there exists $T = T_\varepsilon \in [1-\delta, 1+\delta]$ and a unique radial solution $v \in C^\infty([0,T) \times \mathbb{R}^n)$ to Eq. (5.0.10) *with*

$$v(0,\cdot) = v_\varepsilon^1(0,\cdot) + \varphi_0, \quad \partial_t v(0,\cdot) = \partial_t v_\varepsilon^1(0,\cdot) + \varphi_1.$$

Moreover, v blows up at $(T,0)$ and can be decomposed as

$$v(t,x) = v_\varepsilon^T(t,x) + \frac{1}{T-t}\varphi\left(\log\left(\frac{T}{T-t}\right), \frac{x}{T-t}\right)$$

for all $(t,x) \in [0,T) \times \mathbb{R}^n$, where $\varphi \in C^\infty([0,\infty) \times \mathbb{R}^n)$ is radially symmetric and satisfies

$$\|\varphi(-\log(T-t) + \log T, \cdot)\|_{\dot{H}^r(\mathbb{R}^n)} \lesssim \delta(T-t)^\omega \tag{5.0.20}$$

for all $r \in [s,k]$. Furthermore,

$$\|(\partial_0 + \Lambda + 1)\varphi(-\log(T-t) + \log T, \cdot)\|_{\dot{H}^{r-1}(\mathbb{R}^n)} \lesssim \delta(T-t)^\omega. \tag{5.0.21}$$

Finally, we rephrase the results of Theorem 5.0.3 in terms of normal coordinates using the equivalence of norms of corotational maps and their radial profiles, see [40], to obtain Theorem 5.0.2.

5.1. Existence of self-similar solutions for the wave maps equation into S^d_ε

For $d \geq 3$ and a fixed function α as in Definition 2.3.1 let S^d_ε, for $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \leq \varepsilon_0$, denote the corresponding family of perturbed spheres. For the warping function w_ε as defined in Eq. (2.3.2) we consider for $x \in \mathbb{R}^n$, $n := d+2$ the equation

$$\partial_t^2 v(t,x) - \Delta_x v(t,x) = \frac{n-3}{|x|^3}\left(|x|v(t,x) - w_\varepsilon(|x|v(t,x))w'_\varepsilon(|x|v(t,x))\right). \tag{5.1.1}$$

5.1.1. Reformulation of the problem in similarity coordinates

Restricting to $t \in [0,T)$ for $T > 0$ and $x \in \mathbb{R}^n$ we rewrite Eq. (5.1.1) in similarity coordinates

$$\tau := \log\left(\frac{T}{T-t}\right) \qquad \xi := \frac{x}{T-t}.$$

Note that this transformation maps the strip $S_T := [0,T) \times \mathbb{R}^n$ into the upper half-space $H_+ := [0,\infty) \times \mathbb{R}^n$. Additionally, we define a new rescaled dependent variable via

$$\psi(\tau,\xi) := Te^{-\tau}\, v(T - Te^{-\tau}, Te^{-\tau}\xi).$$

Consequently, the evolution of v inside S_T corresponds to the evolution of ψ inside H_+. Furthermore, the differential operators with respect to t and x are given in the new variables by

$$\partial_t = \frac{e^\tau}{T}(\partial_\tau + \Lambda), \quad \text{and} \quad \partial_{x_i} = \frac{e^\tau}{T}\partial_{\xi_i},$$

for $i \in \{1,\dots,n\}$, where the operator Λ is again defined via $\Lambda f(\xi) := \xi \cdot \nabla f(\xi)$. Eq. (5.1.1) then transforms into

$$\begin{aligned}(\partial_\tau^2 + 3\partial_\tau + 2\partial_\tau\Lambda - \Delta_\xi + \Lambda^2 + 3\Lambda + 2)\psi(\tau,\xi) \\ = \frac{n-3}{|\xi|^3}\left(|\xi|\psi(\tau,\xi) - w_\varepsilon(|\xi|\psi(\tau,\xi))\, w'_\varepsilon(|\xi|\psi(\tau,\xi))\right).\end{aligned} \tag{5.1.2}$$

By defining

$$\psi_1(\tau,\xi) := \psi(\tau,\xi), \quad \psi_2(\tau,\xi) := (\partial_\tau + \Lambda + 1)\psi(\tau,\xi)$$

and $\Psi(\tau) := (\psi_1(\tau,\cdot), \psi_2(\tau,\cdot))$ we get an evolution equation for Ψ of the form

$$\partial_\tau \Psi(\tau) = \widetilde{\mathbf{L}}\Psi(\tau) + \mathbf{N}_\varepsilon(\Psi(\tau)), \tag{5.1.3}$$

where

$$\widetilde{\mathbf{L}} := \begin{pmatrix} -\Lambda - 1 & 1 \\ \Delta & -\Lambda - 2 \end{pmatrix} \tag{5.1.4}$$

is the wave operator in similarity coordinates and the nonlinearity is given by

$$\mathbf{N}_\varepsilon(\mathbf{u}) = \begin{pmatrix} 0 \\ N_\varepsilon(u_1) \end{pmatrix} \quad N_\varepsilon(u_1)(\xi) := \frac{n-3}{|\xi|^3}\eta_\varepsilon(|\xi|u_1(\xi)), \tag{5.1.5}$$

for $\mathbf{u} = (u_1, u_2)$ where

$$\eta_\varepsilon(y) := y - w_\varepsilon(y)w_\varepsilon'(y). \tag{5.1.6}$$

5.1.2. Time-independent solutions

For $\varepsilon = 0$ a static, smooth solution to Eq. (5.1.3) is given by

$$\mathbf{\Psi}_0 := \begin{pmatrix} \psi_{0,1} \\ \psi_{0,2} \end{pmatrix}, \text{ with } \psi_{0,1}(\xi) = |\xi|^{-1} f_0(|\xi|) \text{ and } \psi_{0,2} = \psi_{0,1} + \Lambda\psi_{0,1}, \tag{5.1.7}$$

where the profile f_0 is defined in Eq. (5.0.4). More precisely,

$$\mathbf{0} = \widetilde{\mathbf{L}}\mathbf{\Psi}_0 + \mathbf{N}_0(\mathbf{\Psi}_0). \tag{5.1.8}$$

In order to prove the existence of a static solution for small $\varepsilon \neq 0$, i.e., a smooth function $\mathbf{\Psi}_\varepsilon$ which satisfies

$$\mathbf{0} = \widetilde{\mathbf{L}}\mathbf{\Psi}_\varepsilon + \mathbf{N}_\varepsilon(\mathbf{\Psi}_\varepsilon), \tag{5.1.9}$$

we make the following perturbative ansatz

$$\mathbf{\Psi}_\varepsilon = \mathbf{\Psi}_0 + \mathbf{\Phi}_\varepsilon.$$

By plugging this into Eq. (5.1.9) we obtain an equation for the perturbation $\mathbf{\Phi}_\varepsilon$ which we write in the following form

$$\begin{aligned} \mathbf{0} &= \widetilde{\mathbf{L}}\mathbf{\Phi}_\varepsilon + \mathbf{N}_\varepsilon(\mathbf{\Psi}_0 + \mathbf{\Phi}_\varepsilon) - \mathbf{N}_0(\mathbf{\Psi}_0) \\ &= \widetilde{\mathbf{L}}_0\mathbf{\Phi}_\varepsilon + \mathbf{V}_\varepsilon(\mathbf{\Psi}_0)\mathbf{\Phi}_\varepsilon + \mathcal{R}_\varepsilon(\mathbf{\Psi}_0) + \widetilde{\mathbf{N}}_\varepsilon(\mathbf{\Phi}_\varepsilon), \end{aligned} \tag{5.1.10}$$

where $\widetilde{\mathbf{L}}_0 := \widetilde{\mathbf{L}} + \mathbf{V}_0(\mathbf{\Psi}_0)$,

$$\begin{aligned} \mathbf{V}_0(\mathbf{\Psi}_0)\mathbf{u} &:= \begin{pmatrix} 0 \\ V_0(\psi_{0,1})\, u_1 \end{pmatrix} \quad \text{for} \quad V_0(\psi_{0,1})(\xi) := \frac{n-3}{|\xi|^2}\eta_0'(|\xi|\psi_{0,1}(\xi)), \\ \mathbf{V}_\varepsilon(\mathbf{\Psi}_0)\mathbf{u} &:= \begin{pmatrix} 0 \\ V_\varepsilon(\psi_{0,1})\, u_1 \end{pmatrix} \quad \text{for} \quad V_\varepsilon(\psi_{0,1})(\xi) := \frac{n-3}{|\xi|^2}\left(\eta_\varepsilon'(|\xi|\psi_{0,1}(\xi)) - \eta_0'(|\xi|\psi_{0,1}(\xi))\right), \\ \mathcal{R}_\varepsilon(\mathbf{\Psi}_0)(\xi) &:= \begin{pmatrix} 0 \\ \frac{n-3}{|\xi|^3}\left(\eta_\varepsilon(|\xi|\psi_{0,1}(\xi)) - \eta_0(|\xi|\psi_{0,1}(\xi))\right) \end{pmatrix} \end{aligned}$$

and

$$\widetilde{\mathbf{N}}_\varepsilon(\mathbf{u}) := \begin{pmatrix} 0 \\ \widetilde{N}_\varepsilon(u_1) \end{pmatrix}$$

for

$$\widetilde{N}_\varepsilon(u_1)(\xi) := \frac{n-3}{|\xi|^3}\left(\eta_\varepsilon(|\xi|(\psi_{0,1}(\xi) + u_1(\xi))) - \eta_\varepsilon(|\xi|\psi_{0,1}(\xi)) - \eta_\varepsilon'(|\xi|\psi_{0,1}(\xi))|\xi|u_1(\xi)\right).$$

We are going to formulate Eq. (5.1.10) as a fixed-point problem in suitable function spaces which we discuss in the following.

5.1.3. Function spaces

For $\mathbf{u} = (u_1, u_2), \mathbf{v} = (v_1, v_2) \in C_c^\infty(\mathbb{R}^n) \times C_c^\infty(\mathbb{R}^n)$ and $1 \leq s_1 < s_2$ we set

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathcal{H}^{s_1,s_2}} := \langle u_1, v_1 \rangle_{\dot{H}^{s_1} \cap \dot{H}^{s_2}(\mathbb{R}^n)} + \langle u_2, v_2 \rangle_{\dot{H}^{s_1-1} \cap \dot{H}^{s_2-1}(\mathbb{R}^n)}$$

and define the Hilbert space of radial functions $\mathcal{H}_r^{s_1,s_2}$ as the completion of $C_{c,r}^\infty(\mathbb{R}^n) \times C_{c,r}^\infty(\mathbb{R}^n)$ under the norm induced by $\langle \cdot, \cdot \rangle_{\mathcal{H}^{s_1,s_2}}$,

$$\|\mathbf{u}\|_{\mathcal{H}^{s_1,s_2}} := \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}_{\mathcal{H}^{s_1,s_2}}.$$

In the following, we gather some properties, which are used throughout the chapter. First, we note that for $0 \leq \widetilde{s_1} \leq s_1 < s_2 \leq \widetilde{s_2}$ there is a continuous embedding

$$\dot{H}_r^{\widetilde{s_1}}(\mathbb{R}^n) \cap \dot{H}_r^{\widetilde{s_2}}(\mathbb{R}^n) \hookrightarrow \dot{H}_r^{s_1}(\mathbb{R}^n) \cap \dot{H}_r^{s_2}(\mathbb{R}^n), \tag{5.1.11}$$

see Lemma 4.4 in [39]. In particular, $\mathcal{H}_r^{\widetilde{s_1},\widetilde{s_2}} \hookrightarrow \mathcal{H}_r^{s_1,s_2}$. Moreover, for $m \in \mathbb{N}_0, 0 \leq s_1 < \frac{n}{2}, s_2 > \frac{n}{2} + m$,

$$\dot{H}_r^{s_1}(\mathbb{R}^n) \cap \dot{H}_r^{s_2}(\mathbb{R}^n) \hookrightarrow C_r^m(\mathbb{R}^n). \tag{5.1.12}$$

Hence,

$$\mathcal{H}_r^{s_1,s_2} \hookrightarrow C_r^m(\mathbb{R}^n) \times C_r^{m-1}(\mathbb{R}^n), \tag{5.1.13}$$

for $n \geq 3$, $m \geq 1, 1 \leq s_1 < \frac{n}{2}, s_2 > \frac{n}{2} + m$, where the rightmost space is equipped with the natural topology inherited from $W^{m,\infty}(\mathbb{R}^n) \times W^{m-1,\infty}(\mathbb{R}^n)$, see again Lemma 4.4 in [39]. We further remark that due to Lemma A.1.4 for $n \geq 5$ the blowup solution $\mathbf{\Psi}_0$ defined in Eq. (5.1.7) belongs to $\mathcal{H}_r^{s_1,s_2}$ for all $\frac{n}{2} - 1 < s_1 < s_2$.

5.1.4. Known results for $\varepsilon = 0$

The following results, which we summarize for later reference, have been proven in Propositions 4.1, 6.1 and 6.2 in [39].

Proposition 5.1.1. *Let $n \geq 5$, $\frac{n}{2} - 1 < s < \frac{n}{2}$ and $k > \frac{n}{2} + 1$, $k \in \mathbb{N}$. Then the operators $\widetilde{\mathbf{L}}, \widetilde{\mathbf{L}}_0 : C_{c,r}^\infty(\mathbb{R}^n) \times C_{c,r}^\infty(\mathbb{R}^n) \subset \mathcal{H}_r^{s,k} \to \mathcal{H}_r^{s,k}$ are closable. For the closed operators $(\mathbf{L}, \mathcal{D}(\mathbf{L}))$ and $(\mathbf{L}_0, \mathcal{D}(\mathbf{L}_0))$ we have $\mathcal{D}(\mathbf{L}_0) = \mathcal{D}(\mathbf{L})$ and the following properties hold:*

i) *(Free wave evolution) The operator $\mathbf{L}$ generates a strongly continuous semigroup $(\mathbf{S}(\tau))_{\tau \geq 0}$ of bounded operators on $\mathcal{H}_r^{s,k}$, which satisfies*

$$\|\mathbf{S}(\tau)\mathbf{u}\|_{\mathcal{H}^{s,k}} \leq e^{\left(\frac{n}{2} - 1 - s\right)\tau} \|\mathbf{u}\|_{\mathcal{H}^{s,k}} \tag{5.1.14}$$

for all $\mathbf{u} \in \mathcal{H}_r^{s,k}$ and $\tau \geq 0$.

ii) *(Spectral gap property of* $\mathbf{L}_0$*) There exists* $0 < \widetilde{\omega} < s + 1 - \frac{n}{2}$ *such that*

$$\{\lambda \in \sigma(\mathbf{L}_0) : \operatorname{Re}\lambda \geq -\widetilde{\omega}\} = \{1\}.$$

The point $\lambda = 1$ *is a simple eigenvalue with an explicit eigenfunction*

$$\mathbf{g}_0 := \begin{pmatrix} g_0 \\ 2g_0 + \Lambda g_0 \end{pmatrix}, \quad \text{where} \quad g_0(\xi) = \psi_{0,1}(\xi) + \Lambda\psi_{0,1}(\xi) = \frac{2\sqrt{n-4}}{|\xi|^2 + n - 4}.$$

Moreover, the map $\mathbf{P}_0 : \mathcal{H}^{s,k}_r \to \mathcal{H}^{s,k}_r$ *defined by*

$$\mathbf{P}_0 := \frac{1}{2\pi i}\int_{\partial B_{1/2}(1)} \mathbf{R}_{\mathbf{L}_0}(\lambda)\, d\lambda$$

is a bounded projection with $\operatorname{ran}(\mathbf{P}_0) = \ker(\mathbf{I} - \mathbf{L}_0) = \langle \mathbf{g}_0 \rangle$.

iii) *(Linearized evolution)The operator* $\mathbf{L}_0$ *generates a strongly continuous semigroup* $(\mathbf{S}_0(\tau))_{\tau \geq 0}$ *of bounded operators on* $\mathcal{H}^{s,k}_r$*. Furthermore, for any* $0 < \omega_0 < \tilde{\omega}$ *there is a* $C \geq 1$ *such that*

$$\|\mathbf{S}_0(\tau)(\mathbf{I} - \mathbf{P}_0)\mathbf{u}\|_{\mathcal{H}^{s,k}} \leq Ce^{-\omega_0\tau}\, \|(\mathbf{I} - \mathbf{P}_0)\mathbf{u}\|_{\mathcal{H}^{s,k}} \tag{5.1.15}$$

for all $\mathbf{u} \in \mathcal{H}^{s,k}_r$ *and all* $\tau \geq 0$.

We remark that the exponent s in the above Proposition is chosen to be strictly greater than $\frac{n}{2} - 1$ so that one has exponential decay of the free part $\mathbf{L}$. The fact that 0 does not lie in the spectrum of $\mathbf{L}_0$ will be again a key point for the subsequent analysis.

Now throughout the main parts of the proofs provided in Section 5.1.5 and Section 5.2, we assume $n \geq 5$ and work in $\mathcal{H}^{s,k}_r(\mathbb{R}^n)$ assuming that $(s,k) \in \mathbb{R} \times \mathbb{N}$ satisfy

$$\frac{n}{2} - 1 < s \leq \frac{n}{2} - 1 + \frac{1}{2n-2}, \quad k > n. \tag{5.1.16}$$

5.1.5. Construction of the blowup solution

In the following, we construct for sufficiently small ε a solution $\mathbf{\Phi}_\varepsilon$ to Eq. (5.1.10). By Proposition 5.1.1, the operator $\mathbf{L}_0$ is bounded invertible and thus (5.1.10) can be phrased as a fixed-point problem.
First, we properly define the operators appearing on the right-hand side of the equation. To prove the required Lipschitz bounds, we will make extensive use of Proposition A.2.1 in Appendix A.2 and assume that the exponents $(s,k) \in \mathbb{R} \times \mathbb{N}$ satisfy Eq. (5.1.16). We note that under this assumption, $\mathcal{H}^{s,k}_r \hookrightarrow C^2_r(\mathbb{R}^n) \times C^1_r(\mathbb{R}^n)$ and thus the nonlinearity, the potential and the remainder can be defined pointwise as in Eq. (5.1.10).
In the following we use the notation

$$\mathcal{B}^{s,\ell}_M := \{\mathbf{u} \in \mathcal{H}^{s,\ell}_r : \|\mathbf{u}\|_{\mathcal{H}^{s,\ell}} \leq M\} \subset \mathcal{H}^{s,\ell}_r \quad \text{and} \quad \mathcal{B}_M := \mathcal{B}^{s,k}_M.$$

Lemma 5.1.2. *Let $n \geq 5$ and $(s,k) \in \mathbb{R} \times \mathbb{N}$ satisfy (5.1.16). Then, for any $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \leq 1$, every $M > 0$ and every $\ell \geq k$, we have $\widetilde{\mathbf{N}}_\varepsilon : \mathcal{H}^{s,\ell}_r \to \mathcal{H}^{s,\ell+1}_r$ and the estimate*

$$\|\widetilde{\mathbf{N}}_\varepsilon(\mathbf{u}) - \widetilde{\mathbf{N}}_\kappa(\mathbf{v})\|_{\mathcal{H}^{s,\ell+1}} \lesssim \left(\|\mathbf{u}\|_{\mathcal{H}^{s,\ell}} + \|\mathbf{v}\|_{\mathcal{H}^{s,\ell}}\right) \|\mathbf{u}-\mathbf{v}\|_{\mathcal{H}^{s,\ell}} + \left(\|\mathbf{u}\|^2_{\mathcal{H}^{s,\ell}} + \|\mathbf{v}\|^2_{\mathcal{H}^{s,\ell}}\right) |\varepsilon - \kappa| \tag{5.1.17}$$

holds for all $|\varepsilon|, |\kappa| \leq 1$ and all $\mathbf{u}, \mathbf{v} \in \mathcal{B}^{s,\ell}_M \subset \mathcal{H}^{s,k}_r$.

Proof. We start by showing the Lipschitz bound

$$\|\widetilde{\mathbf{N}}_\varepsilon(\mathbf{u}) - \widetilde{\mathbf{N}}_\varepsilon(\mathbf{v})\|_{\mathcal{H}^{s,\ell+1}} \lesssim \left(\|\mathbf{u}\|_{\mathcal{H}^{s,\ell}} + \|\mathbf{v}\|_{\mathcal{H}^{s,\ell}}\right) \|\mathbf{u}-\mathbf{v}\|_{\mathcal{H}^{s,\ell}} \tag{5.1.18}$$

for all $|\varepsilon| \leq 1$ and all $\mathbf{u}, \mathbf{v} \in \mathcal{B}^{s,\ell}_M$. With the help of (4.1.13) we write

$$\begin{aligned}
&\widetilde{N}_\varepsilon(u_1)(\xi) - \widetilde{N}_\varepsilon(v_1)(\xi) = \\
&\frac{n-3}{|\xi|^3}\Big(\eta_\varepsilon\big(|\xi|(\psi_{0,1}(\xi)+u_1(\xi))\big) - \eta_\varepsilon\big(|\xi|(\psi_{0,1}(\xi)+v_1(\xi))\big) - \eta'_\varepsilon(|\xi|\psi_{0,1}(\xi))|\xi|(u_1(\xi)-v_1(\xi))\Big) \\
&= (n-3)\int_0^1\int_0^1\int_0^1 (u_1(\xi)-v_1(\xi))\big(v_1(\xi)+x(u_1(\xi)-v_1(\xi))\big) \\
&\cdot \big(\psi_{0,1}(\xi) + y\big(v_1(\xi)+x(u_1(\xi)-v_1(\xi))\big)\big) \cdot \eta'''_\varepsilon\left(z|\xi|\big(\psi_{0,1}(\xi)+y(v_1+x(u_1-v_1))\big)\right) dz\, dy\, dx.
\end{aligned}$$

Since $F_\varepsilon(x) := \eta'''_\varepsilon(x)$ satisfies the assumptions of Proposition A.2.1 the estimate (5.1.18) follows from the previous equation and (A.2.3). We now show that

$$\|\widetilde{\mathbf{N}}_\varepsilon(\mathbf{u}) - \widetilde{\mathbf{N}}_\kappa(\mathbf{u})\|_{\mathcal{H}^{s,\ell+1}} \lesssim \|\mathbf{u}\|^2_{\mathcal{H}^{s,\ell}} |\varepsilon - \kappa| \tag{5.1.19}$$

for every $\mathbf{u} \in \mathcal{B}^{s,\ell}_M$ and every $|\varepsilon|, |\kappa| \leq 1$. For this, we use again Eq. (4.1.13) to obtain

$$\begin{aligned}
\widetilde{N}_\varepsilon(u_1)(\xi) - \widetilde{N}_\kappa(u_1)(\xi) = (n-3)\,u_1(\xi)^2 \int_0^1\int_0^1\int_0^1 x\,(\psi_{0,1}(\xi)+yxu_1(\xi)) \\
\cdot \left(\eta'''_\varepsilon\left(z|\xi|\big(\psi_{0,1}(\xi)+yxu_1(\xi)\big)\right) - \eta'''_\kappa\left(z|\xi|\big(\psi_{0,1}(\xi)+yxu_1(\xi)\big)\right)\right) dz\, dy\, dx
\end{aligned}$$

and Eq. (5.1.19) follows from an application of Proposition A.2.1. By combining Eq. (5.1.18) and Eq. (5.1.19) one obtains (5.1.17). The mapping properties of $\widetilde{\mathbf{N}}_\varepsilon$ follow from Eq. (5.1.17) and the fact that $\widetilde{\mathbf{N}}_\varepsilon(\mathbf{0}) = \mathbf{0}$. □

For the potential, we have the following result.

Lemma 5.1.3. *Let $n \geq 5$ and $(s,k) \in \mathbb{R} \times \mathbb{N}$ satisfy (5.1.16). For any $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \leq 1$ and every $\ell \geq k$, $\mathbf{V}_\varepsilon(\mathbf{\Psi}_0) : \mathcal{H}^{s,\ell}_r \to \mathcal{H}^{s,\ell+1}_r$. Furthermore,*

$$\|\mathbf{V}_\varepsilon(\mathbf{\Psi}_0)\mathbf{u} - \mathbf{V}_\kappa(\mathbf{\Psi}_0)\mathbf{u}\|_{\mathcal{H}^{s,\ell+1}} \lesssim |\varepsilon - \kappa|\, \|\mathbf{u}\|_{\mathcal{H}^{s,\ell}}$$

for all $|\varepsilon|, |\kappa| \leq 1$ and all $\mathbf{u} \in \mathcal{H}^{s,\ell}_r$.

Proof. By definition of $\mathbf{V}_\varepsilon(\mathbf{\Psi}_0)$ we have to prove the estimate

$$\|(V_\varepsilon(\psi_{0,1}) - V_\kappa(\psi_{0,1}))\, u_1\|_{\dot{H}^{s-1}\cap\dot{H}^\ell(\mathbb{R}^n)} \lesssim |\varepsilon-\kappa|\, \|u_1\|_{\dot{H}^s\cap\dot{H}^\ell(\mathbb{R}^n)}\,.$$

In a similar way to the previous result one shows with (4.1.12) that

$$\begin{aligned}((V_\varepsilon(\psi_{0,1}) - V_\kappa(\psi_{0,1}))u_1)(\xi) =& \\ (n-3)\psi_{0,1}(\xi)^2 u_1(\xi) \int_0^1\int_0^1 x\,(\eta_\varepsilon'''(|\xi|\psi_{0,1}(\xi)xy) - \eta_\kappa'''(|\xi|\psi_{0,1}(\xi)xy))\,dy\,dx&\end{aligned}$$

holds for every $\xi\in\mathbb{R}^n$. The claim then again follows by an application of Proposition A.2.1. □

Finally, we consider the remainder.

Lemma 5.1.4. *Let $n\geq 5$ and $(s,k)\in\mathbb{R}\times\mathbb{N}$ satisfy (5.1.16). For any $\varepsilon\in\mathbb{R}$ with $|\varepsilon|\leq 1$ and every $\ell\geq k$, we have*

$$\|\mathcal{R}_\varepsilon(\mathbf{\Psi}_0)\|_{\mathcal{H}^{s,\ell+1}} \lesssim 1 \quad \textit{as well as} \quad \|\mathcal{R}_\varepsilon(\mathbf{\Psi}_0) - \mathcal{R}_\kappa(\mathbf{\Psi}_0)\|_{\mathcal{H}^{s,\ell+1}} \lesssim |\varepsilon-\kappa|$$

for all $|\varepsilon|,|\kappa|\leq 1$.

Proof. By the definition of $\mathcal{R}_\varepsilon(\mathbf{\Psi}_0)$ we have to estimate $\xi\mapsto \frac{n-3}{|\xi|^3}\,(\eta_\varepsilon(|\xi|\psi_{0,1}(\xi)) - \eta_\kappa(|\xi|\psi_{0,1}(\xi)))$ in the $\dot{H}^{s-1}\cap\dot{H}^\ell-$norm. To do so we write with the help of (4.1.11)

$$\begin{aligned}&\frac{n-3}{|\xi|^3}\,(\eta_\varepsilon(|\xi|\psi_{0,1}(\xi)) - \eta_\kappa(|\xi|\psi_{0,1}(\xi)))\\ &\quad=(n-3)\psi_{0,1}(\xi)^3\int_0^1\int_0^1\int_0^1 x^2y\,(\eta_\varepsilon'''(|\xi|\psi_{0,1}(\xi)xyz) - \eta_\kappa'''(|\xi|\psi_{0,1}(\xi)xyz))\,dz\,dy\,dx\end{aligned}$$

and conclude the proof by again applying the Schauder estimate from Proposition A.2.1 to the function $F_\varepsilon(x) := \eta_\varepsilon'''(x)$. □

In order to prove gradient blowup as claimed in Theorem 5.0.1, we impose an a priori smallness condition on the perturbation.

Lemma 5.1.5. *Let $n\geq 5$ and $(s,k)\in\mathbb{R}\times\mathbb{N}$ satisfy (5.1.16). There is a $\delta_0^*>0$ such that any $\mathbf{u}=(u_1,u_2)\in\mathcal{B}_{\delta_0^*}$ satisfies*

$$|u_1(0)| < \frac{2}{\sqrt{n-4}} = \psi_{0,1}(0),$$

where

$$\psi_{0,1}(\xi) = 2|\xi|^{-1}\arctan\left(\tfrac{|\xi|}{\sqrt{n-4}}\right).$$

Proof. First, note that $\psi_{0,1}(0)>0$. By Sobolev embedding, there is a $C>0$ such that

$$|u_1(0)| \leq \|u_1\|_{L^\infty(\mathbb{R}^n)} \leq C\|\mathbf{u}\|_{\mathcal{H}^{s,k}} \leq C\delta_0^*$$

for all $\mathbf{u}\in\mathcal{B}_{\delta_0^*}\subset\mathcal{H}_r^{s,k}$. Now choose δ_0^* sufficiently small. □

The next result crucially relies on the characterization of the spectrum of the linearized operator for $\varepsilon = 0$, see Proposition 5.1.1.

Proposition 5.1.6. *Let $n \geq 5$ and $(s,k) \in \mathbb{R} \times \mathbb{N}$ satisfy (5.1.16). There exist $0 < \delta^* \leq \min\{1, \delta_0^*\}$ and $0 < \varepsilon^* \leq 1$ (depending on δ^*) such that for every $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \leq \varepsilon^*$ the map*

$$\mathbf{K}_\varepsilon : \mathcal{B}_{\delta^*} \to \mathcal{B}_{\delta^*}, \quad \mathbf{K}_\varepsilon(\mathbf{\Phi}) := (-\mathbf{L}_0)^{-1}(\mathbf{V}_\varepsilon(\mathbf{\Psi}_0)\mathbf{\Phi} + \mathcal{R}_\varepsilon(\mathbf{\Psi}_0) + \widetilde{\mathbf{N}}_\varepsilon(\mathbf{\Phi})) \tag{5.1.20}$$

is a well-defined contraction. In particular, δ^* and ε^* can be chosen such that

$$\|\mathbf{K}_\varepsilon(\mathbf{\Phi}) - \mathbf{K}_\varepsilon(\mathbf{\Psi})\|_{\mathcal{H}^{s,k}} \leq \frac{1}{2}\|\mathbf{\Phi} - \mathbf{\Psi}\|_{\mathcal{H}^{s,k}} \tag{5.1.21}$$

holds for every $|\varepsilon| \leq \varepsilon^*$ and all $\mathbf{\Phi}, \mathbf{\Psi} \in \mathcal{B}_{\delta^*}$.

Proof. Using the fact that $\mathbf{L}_0 : \mathcal{D}(\mathbf{L}_0) \subset \mathcal{H}^{s,k} \to \mathcal{H}^{s,k}$ is bounded invertible together with the embedding (5.1.11) there exists a constant $C > 0$, such that

$$\|\mathbf{K}_\varepsilon(\mathbf{\Phi})\|_{\mathcal{H}^{s,k}} \leq C\left(\|\mathbf{V}_\varepsilon(\mathbf{\Psi}_0)\Phi\|_{\mathcal{H}^{s,k+1}} + \|\mathcal{R}_\varepsilon(\mathbf{\Psi}_0)\|_{\mathcal{H}^{s,k+1}} + \|\widetilde{\mathbf{N}}_\varepsilon(\mathbf{\Phi})\|_{\mathcal{H}^{s,k+1}}\right)$$

holds for every $\mathbf{\Phi} \in \mathcal{B}_{\delta^*}$ and every $|\varepsilon| \leq \varepsilon^*$ assuming that $0 < \delta^* \leq \min\{1, \delta_0^*\}$ and $0 < \varepsilon^* \leq 1$. We then have by the previous results:

$$\|\mathbf{V}_\varepsilon(\mathbf{\Psi}_0)\mathbf{\Phi}\|_{\mathcal{H}^{s,k+1}} + \|\mathcal{R}_\varepsilon(\mathbf{\Psi}_0)\|_{\mathcal{H}^{s,k+1}} + \|\widetilde{\mathbf{N}}_\varepsilon(\mathbf{\Phi})\|_{\mathcal{H}^{s,k+1}} \leq (C_1\,\varepsilon^* + C_2\,\varepsilon^*/\delta^* + C_3\,\delta^*)\;\delta^*$$

for constants $C_j > 0$, $j \in \{1,2,3\}$ independent of ε^* and δ^*. Now choose $\delta^* < 1/(C_3\,C)$ and $\varepsilon^* = \varepsilon^*(\delta^*)$ sufficiently small to guarantee $C(C_1\,\varepsilon^* + C_2\,\varepsilon^*/\delta^* + C_3\,\delta^*) \leq 1$. Then

$$\|\mathbf{K}_\varepsilon(\mathbf{\Phi})\|_{\mathcal{H}^{s,k}} \leq \delta^*$$

for all $\mathbf{\Phi} \in \mathcal{B}_{\delta^*}$ so that $\mathbf{K}_\varepsilon$ is well-defined on $\mathcal{B}_{\delta^*}$. For the contraction property we take $\mathbf{\Phi}, \mathbf{\Psi} \in \mathcal{B}_{\delta^*}$ and get

$$\begin{aligned}\|\mathbf{K}_\varepsilon(\mathbf{\Phi}) - \mathbf{K}_\varepsilon(\mathbf{\Psi})\|_{\mathcal{H}^{s,k}} &\leq C\left(\|\mathbf{V}_\varepsilon(\mathbf{\Psi}_0)\;(\mathbf{\Phi} - \mathbf{\Psi})\|_{\mathcal{H}^{s,k+1}} + \|\widetilde{\mathbf{N}}_\varepsilon(\mathbf{\Phi}) - \widetilde{\mathbf{N}}_\varepsilon(\mathbf{\Psi})\|_{\mathcal{H}^{s,k+1}}\right)\\ &\leq C\,C_1\,\varepsilon^*\,\|\mathbf{\Phi} - \mathbf{\Psi}\|_{\mathcal{H}^{s,k}} + C\,C_4\,\delta^*\,\|\mathbf{\Phi} - \mathbf{\Psi}\|_{\mathcal{H}^{s,k}}\,,\end{aligned}$$

for $C_4 > 0$. Now choose $\delta^* > 0$ possibly even smaller so that $C\,C_4\,\delta^* < 1/2$ is satisfied. Hence the claim follows by adjusting ε^* and requiring in addition that $C\,C_1\,\varepsilon^* + C\,C_4\,\delta^* \leq 1/2$.

□

This result immediately implies the existence part in the following statement.

Proposition 5.1.7. *Let $n \geq 5$ and $(s,k) \in \mathbb{R} \times \mathbb{N}$ satisfy (5.1.16). Let δ^* and ε^* be as in Proposition 5.1.6. Then for every $|\varepsilon| \leq \varepsilon^*$ there is a unique $\mathbf{\Phi}_\varepsilon \in \mathcal{B}_{\delta^*} \cap \mathcal{D}(\mathbf{L}_0)$ satisfying*

$$\mathbf{L}_0\mathbf{\Phi}_\varepsilon + \mathbf{V}_\varepsilon(\mathbf{\Psi}_0)\mathbf{\Phi}_\varepsilon + \mathcal{R}_\varepsilon(\mathbf{\Psi}_0) + \widetilde{\mathbf{N}}_\varepsilon(\mathbf{\Phi}_\varepsilon) = 0 \tag{5.1.22}$$

in $\mathcal{H}_r^{s,k}$*. Moreover, for every integer* $\ell \geq k$*, there exists a constant* $C_\ell > 0$*, independent of* ε *and* κ*, such that*

$$\|\mathbf{\Phi}_\varepsilon - \mathbf{\Phi}_\kappa\|_{\mathcal{H}^{s,\ell}} \leq C_\ell |\varepsilon - \kappa| \tag{5.1.23}$$

for all $|\varepsilon|, |\kappa| \leq \varepsilon^*$*. In particular,* $\mathbf{\Phi}_\varepsilon \in \bigcap_{\ell \geq k} \mathcal{H}_r^{s,\ell}(\mathbb{R}^n) \subset C_r^\infty(\mathbb{R}^n) \times C_r^\infty(\mathbb{R}^n)$ *and* $\mathbf{\Phi}_\varepsilon$ *solves the equation in a classical sense, i.e.,* $\mathbf{L}_0 \mathbf{\Phi}_\varepsilon = \widetilde{\mathbf{L}}_0 \mathbf{\Phi}_\varepsilon$*.*

Proof. The existence of a unique $\mathbf{\Phi}_\varepsilon \in \mathcal{B}_{\delta^*} \cap \mathcal{D}(\mathbf{L}_0)$ satisfying the Eq. (5.1.22) is a direct consequence of Proposition 5.1.6 and the contraction mapping principle. We now inductively show the Lipschitz bound (5.1.23). For this we take $|\varepsilon|, |\kappa| \leq \varepsilon^*$ and use (5.1.21) to obtain

$$\|\mathbf{\Phi}_\varepsilon - \mathbf{\Phi}_\kappa\|_{\mathcal{H}^{s,k}} = \|\mathbf{K}_\varepsilon(\mathbf{\Phi}_\varepsilon) - \mathbf{K}_\kappa(\mathbf{\Phi}_\kappa)\|_{\mathcal{H}^{s,k}} \leq \frac{1}{2} \|\mathbf{\Phi}_\varepsilon - \mathbf{\Phi}_\kappa\|_{\mathcal{H}^{s,k}} + \|\mathbf{K}_\varepsilon(\mathbf{\Phi}_\kappa) - \mathbf{K}_\kappa(\mathbf{\Phi}_\kappa)\|_{\mathcal{H}^{s,k}}$$

from which $\|\mathbf{\Phi}_\varepsilon - \mathbf{\Phi}_\kappa\|_{\mathcal{H}^{s,k}} \leq 2 \|\mathbf{K}_\varepsilon(\mathbf{\Phi}_\kappa) - \mathbf{K}_\kappa(\mathbf{\Phi}_\kappa)\|_{\mathcal{H}^{s,k}}$ follows. Now, for every $\mathbf{\Phi} \in \mathcal{B}_{\delta^*}$

$$\begin{aligned}\|\mathbf{K}_\varepsilon(\mathbf{\Phi}) - \mathbf{K}_\kappa(\mathbf{\Phi})\|_{\mathcal{H}^{s,k}} \lesssim \|\mathcal{R}_\varepsilon(\mathbf{\Psi}_0) - \mathcal{R}_\kappa(\mathbf{\Psi}_0)\|_{\mathcal{H}^{s,k}} + \|(\mathbf{V}_\varepsilon(\mathbf{\Psi}_0) - \mathbf{V}_\kappa(\mathbf{\Psi}_0))\, \mathbf{\Phi}\|_{\mathcal{H}^{s,k}} \\ + \|\widetilde{\mathbf{N}}_\varepsilon(\mathbf{\Phi}) - \widetilde{\mathbf{N}}_\kappa(\mathbf{\Phi})\|_{\mathcal{H}^{s,k}} \lesssim |\varepsilon - \kappa|\end{aligned}$$

due to Lemma 5.1.2 and Lemma 5.1.3.
Suppose now that the Lipschitz bound from (5.1.23) holds for $\ell \geq k$.
We now define

$$\mathbf{F}_\varepsilon(\mathbf{u}) := \mathbf{V}_\varepsilon(\mathbf{\Psi}_0)\mathbf{u} + \mathcal{R}_\varepsilon(\mathbf{\Psi}_0) + \widetilde{\mathbf{N}}_\varepsilon(\mathbf{u})$$

and from Lemmas 5.1.2, 4.1.15 and 5.1.4 we obtain for every $\ell \geq k$ and every $M > 0$ that there exists a constant $C_{M,\ell} > 0$ such that

$$\|\mathbf{F}_\varepsilon(\mathbf{u}) - \mathbf{F}_\kappa(\mathbf{v})\|_{\mathcal{H}^{s,\ell+1}} \leq C_{M,\ell} \left(\|\mathbf{u} - \mathbf{v}\|_{\mathcal{H}^{s,\ell}} + |\varepsilon - \kappa|\right) \tag{5.1.24}$$

for all $|\varepsilon|, |\kappa| \leq \varepsilon^*$ and all $\mathbf{u}, \mathbf{v} \in \mathcal{H}_r^{s,\ell}$ satisfying $\|\mathbf{u}\|_{\mathcal{H}^{s,\ell}} + \|\mathbf{v}\|_{\mathcal{H}^{s,\ell}} \leq M$.
Only at this point, we will not suppress the dependency of $\mathbf{L}_0$ on the parameters s and k and will therefore denote by $\mathbf{L}_{0,s_1,s_2}$ the closure of $\widetilde{\mathbf{L}}_{0,s_1,s_2}$ in $\mathcal{H}_r^{s_1,s_2}$. Since $\mathbf{L}_{0,s,\ell+1}$ is a restriction of $\mathbf{L}_{0,s,\ell}$ to the subspace $\mathcal{H}_r^{s,\ell+1}$ (see for example [21], p.21, Lemma 3.5 or [39], p.29, Lemma C.1.), we obtain the same result for the resolvent operators, i.e.

$$(-\mathbf{L}_{0,s,\ell})^{-1} \big|_{\mathcal{H}_r^{s,\ell+1}} = (-\mathbf{L}_{0,s,\ell+1})^{-1}.$$

Due to $\mathbf{F}_\varepsilon(\mathbf{\Phi}_\varepsilon) \in \mathcal{H}_r^{s,\ell+1}$ we therefore have

$$\mathbf{\Phi}_\varepsilon = (-\mathbf{L}_{0,s,\ell+1})^{-1}\, \mathbf{F}_\varepsilon(\mathbf{\Phi}_\varepsilon).$$

Subtracting the equations for $\mathbf{\Phi}_\varepsilon$ and $\mathbf{\Phi}_\kappa$ gives

$$\|\mathbf{\Phi}_\varepsilon - \mathbf{\Phi}_\kappa\|_{\mathcal{H}^{s,\ell+1}} \leq \left\|(-\mathbf{L}_{0,s,\ell+1})^{-1}\right\| \|\mathbf{F}_\varepsilon(\mathbf{\Phi}_\varepsilon) - \mathbf{F}_\kappa(\mathbf{\Phi}_\kappa)\|_{\mathcal{H}^{s,\ell+1}}.$$

Since $\mathbf{\Phi}_0 = \mathbf{0}$, we obtain $\|\mathbf{\Phi}_\varepsilon\|_{\mathcal{H}^{s,\ell}} + \|\mathbf{\Phi}_\kappa\|_{\mathcal{H}^{s,\ell}} \leq 2C_\ell\varepsilon^*$, so that we can apply (5.1.24) with $M = 2C_\ell\varepsilon^*$ to obtain together with the induction hypothesis

$$\|\mathbf{\Phi}_\varepsilon - \mathbf{\Phi}_\kappa\|_{\mathcal{H}^{s,\ell+1}} \lesssim \|\mathbf{\Phi}_\varepsilon - \mathbf{\Phi}_\kappa\|_{\mathcal{H}^{s,\ell}} + |\varepsilon - \kappa| \lesssim |\varepsilon - \kappa|,$$

which finishes the induction.
Finally, we show that $\mathbf{L}_0$ acts as a classical differential operator on $\mathbf{\Phi}_\varepsilon$. Since $\mathbf{\Phi}_\varepsilon$ belongs to $\mathcal{D}(\mathbf{L}_0)$ there exists a sequence $(\mathbf{u}_j)_{j\in\mathbb{N}} \subset C^\infty_{c,r}(\mathbb{R}^n) \times C^\infty_{c,r}(\mathbb{R}^n)$ such that

$$\mathbf{u}_j \to \mathbf{\Phi}_\varepsilon \quad \text{and} \quad \widetilde{\mathbf{L}}_0\mathbf{u}_j \to \mathbf{L}_0\mathbf{\Phi}_\varepsilon \quad \text{in} \quad \mathcal{H}^{s,k}_r \quad \text{for} \quad j \to \infty.$$

Now, by the embedding of $\mathcal{H}^{s,k}_r$ into $C^2(\mathbb{R}^n) \times C^1(\mathbb{R}^n)$ we infer by the above convergence of $\mathbf{u}_j$ towards $\mathbf{\Phi}_\varepsilon$ that each term in $\widetilde{\mathbf{L}}_0\mathbf{u}_j$ converges pointwise towards $\widetilde{\mathbf{L}}_0\mathbf{\Phi}_\varepsilon$. Again by Sobolev embedding, the limit $\widetilde{\mathbf{L}}_0\mathbf{u}_j \to \mathbf{L}_0\mathbf{\Phi}_\varepsilon$ holds pointwise, which implies the claim. This last part of the proof shows in fact that $\widetilde{\mathbf{L}}_0$ acts as a classical differential operator on any element in its domain, irrespective of higher regularity. □

Corollary 5.1.8. *Let $n \geq 5$ and $(s,k) \in \mathbb{R}\times\mathbb{N}$ satisfy (5.1.16). Let δ^* and ε^* be as in Proposition 5.1.7. For $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \leq \varepsilon^*$ let $\mathbf{\Phi}_\varepsilon \in \mathcal{B}_{\delta^*}$ be the corresponding solution of Eq. (5.1.22). Then*

$$\mathbf{\Psi}_\varepsilon := \mathbf{\Psi}_0 + \mathbf{\Phi}_\varepsilon \in \bigcap_{\ell\geq s} \dot{H}^\ell_r(\mathbb{R}^n) \times \dot{H}^{\ell-1}_r(\mathbb{R}^n) \subset C^\infty_r(\mathbb{R}^n) \times C^\infty_r(\mathbb{R}^n)$$

solves Eq. (5.1.9) in a classical sense. Furthermore, for every $\ell \geq k$

$$\|\mathbf{\Psi}_\varepsilon - \mathbf{\Psi}_\kappa\|_{\mathcal{H}^{s,\ell}} \lesssim |\varepsilon - \kappa| \tag{5.1.25}$$

for all $|\varepsilon|, |\kappa| \leq \varepsilon^*$.

Finally, we characterize the decay of the solution $\mathbf{\Psi}_\varepsilon = (\psi_{\varepsilon,1}, \psi_{\varepsilon,2})$ at infinity. Corollary 5.1.8 and the generalized Strauss inequality (A.1.3) imply that $|\psi_{\varepsilon,1}(\xi)| \lesssim \langle\xi\rangle^{s-\frac{n}{2}}$ for $s > \frac{n}{2} - 1$ and hence the decay rate is strictly less than 1. In the following, we provide a more refined analysis based on ODE arguments to improve this. This is along the lines of [45].

Proposition 5.1.9. *Let $n \geq 5$ and $(s,k) \in \mathbb{R}\times\mathbb{N}$ satisfy (5.1.16). Let $\mathbf{\Psi}_\varepsilon = (\psi_{\varepsilon,1}, \psi_{\varepsilon,2})$ be the family of profiles constructed in Corollary 5.1.8 for $|\varepsilon| \leq \varepsilon^*$. Then for every $\beta \in \mathbb{N}^n_0$,*

$$|\partial^\beta \psi_{\varepsilon,1}(\xi)| \lesssim \langle\xi\rangle^{-1-|\beta|} \quad \text{and} \quad |\partial^\beta \psi_{\varepsilon,2}(\xi)| \lesssim \langle\xi\rangle^{-2-|\beta|} \tag{5.1.26}$$

for all $\xi \in \mathbb{R}^n$, uniformly in $|\varepsilon| \leq \varepsilon^*$. Furthermore,

$$|\partial^\beta (\psi_{\varepsilon,1}(\xi) - \psi_{\kappa,1}(\xi))| \lesssim |\varepsilon - \kappa|\langle\xi\rangle^{-1-|\beta|} \quad \text{and} \quad |\partial^\beta (\psi_{\varepsilon,2}(\xi) - \psi_{\kappa,2}(\xi))| \lesssim |\varepsilon - \kappa|\langle\xi\rangle^{-2-|\beta|} \tag{5.1.27}$$

for all $\xi \in \mathbb{R}^n$ and $|\varepsilon|, |\kappa| \leq \varepsilon^*$. Moreover, the limits $\lim_{|\xi|\to\infty} |\xi|\psi_{\varepsilon,1}(\xi)$ and $\lim_{|\xi|\to\infty} |\xi|^2(\psi_{\varepsilon,1}(\xi) + \Lambda\psi_{\varepsilon,1}(\xi))$ exist.

Proof. To shorten the notation we define

$$u_\varepsilon(\rho) := \widetilde{\psi}_{\varepsilon,1}(\rho), \quad p_\varepsilon(\rho) := \rho u_\varepsilon(\rho) \quad \text{and} \quad h_\varepsilon(\rho) := p'_\varepsilon(\rho) = u_\varepsilon(\rho) + \rho u'_\varepsilon(\rho) = \widetilde{\psi}_{\varepsilon,2}(\rho).$$

From (5.1.25) and Sobolev embedding we immediately obtain for every $m \in \mathbb{N}_0$ and $R > 0$

$$\sup_{|\varepsilon|\leq\varepsilon^*} \left(\|u_\varepsilon\|_{C^m([0,R])} + \|h_\varepsilon\|_{C^m([0,R])} \right) < \infty \tag{5.1.28}$$

and

$$\|u_\varepsilon - u_\kappa\|_{C^m([0,R])} + \|h_\varepsilon - h_\kappa\|_{C^m([0,R])} \lesssim_{m,R} |\varepsilon - \kappa|. \tag{5.1.29}$$

It therefore remains to establish the decay estimates for large ρ. For this we first define

$$q_\varepsilon(z) := z - \eta_\varepsilon(z) = w_\varepsilon(z) w'_\varepsilon(z)$$

and since w_ε is periodic, smooth and depends smoothly on the parameter ε we have for every $m \in \mathbb{N}_0$

$$\sup_{|\varepsilon|\leq\varepsilon^*} \left\| q_\varepsilon^{(m)} \right\|_{L^\infty(\mathbb{R})} < \infty \quad \text{and} \quad \left\| q_\varepsilon^{(m)} - q_\kappa^{(m)} \right\|_{L^\infty(\mathbb{R})} \lesssim_m |\varepsilon - \kappa|.$$

In particular, $|q_\varepsilon(x) - q_\kappa(y)| \lesssim |x - y| + |\varepsilon - \kappa|$ for every $x, y \in \mathbb{R}$ and $|\varepsilon|, |\kappa| \leq \varepsilon^*$. From the definition of $u_\varepsilon = \widetilde{\psi}_{\varepsilon,1}$ it solves

$$(1-\rho^2) u''_\varepsilon(\rho) + \left(\frac{n-1}{\rho} - 4\rho \right) u'_\varepsilon(\rho) - 2u_\varepsilon(\rho) + \frac{n-3}{\rho^3} \eta_\varepsilon(\rho\, u_\varepsilon(\rho)) = 0. \tag{5.1.30}$$

With this and using $h'_\varepsilon = 2u'_\varepsilon + \rho u''_\varepsilon$, we obtain

$$(1-\rho^2) h'_\varepsilon = (1-\rho^2)(2u'_\varepsilon + \rho u''_\varepsilon) = \left(2\rho - \frac{n-3}{\rho} \right) h_\varepsilon + \frac{n-3}{\rho^2} q_\varepsilon(p_\varepsilon).$$

We now fix $R > 2$, take $\rho \geq R$ and define the following integrating factor

$$\mu(\rho) := \rho^{n-3} (\rho^2 - 1)^{-\frac{n-5}{2}}.$$

A direct calculation gives

$$\frac{\mu'(\rho)}{\mu(\rho)} = \frac{\frac{n-3}{\rho} - 2\rho}{1-\rho^2}.$$

Hence,

$$(\mu h_\varepsilon)'(\rho) = -(n-3) K(\rho) q_\varepsilon(p_\varepsilon(\rho)) \qquad \text{with} \quad K(\rho) := \frac{\mu(\rho)}{\rho^2(\rho^2-1)}. \tag{5.1.31}$$

We have for every $m \in \mathbb{N}_0$ from the definitions

$$\left| \frac{d^m}{d\rho^m} \mu(\rho)^{-1} \right| \lesssim \rho^{-m-2} \qquad \text{and} \quad |K^{(m)}(\rho)| \lesssim \rho^{-m-2}.$$

Integrating (5.1.31) from R to ρ now yields

$$\mu(\rho)h_\varepsilon(\rho) = \mu(R)h_\varepsilon(R) - (n-3)\int_R^\rho K(s)q_\varepsilon(p_\varepsilon)\,ds.$$

And since q_ε is uniformly bounded and $K(s) = \mathcal{O}(s^{-2})$ the integral on the right-hand side converges, so that the limit

$$a_\varepsilon := \lim_{\rho\to\infty} \mu(\rho)h_\varepsilon(\rho) = \mu(R)h_\varepsilon(R) - (n-3)\int_R^\infty K(s)q_\varepsilon(p_\varepsilon)\,ds \tag{5.1.32}$$

exists. The local uniform bound (5.1.28) for h_ε and the uniform bound for q_ε now imply $|a_\varepsilon| \lesssim 1$ with a constant independent of ε.
We now get the following representation for h_ε

$$h_\varepsilon(\rho) = \mu(\rho)^{-1}\left[a_\varepsilon + (n-3)\int_\rho^\infty K(s)q_\varepsilon(p_\varepsilon)\,ds\right]. \tag{5.1.33}$$

Since

$$\left|\int_\rho^\infty K(s)q_\varepsilon(p_\varepsilon)\,ds\right| \lesssim \int_\rho^\infty s^{-2}\,ds \lesssim \rho^{-1}$$

we obtain

$$|h_\varepsilon(\rho)| \lesssim \rho^{-2}.$$

Furthermore, since $p_\varepsilon' = h_\varepsilon$ it follows that

$$|p_\varepsilon(\rho)| \le |p_\varepsilon(R)| + \int_R^\rho |h_\varepsilon(s)|\,ds \lesssim 1. \tag{5.1.34}$$

Thus,

$$|u_\varepsilon(\rho)| \lesssim \frac{|p_\varepsilon(\rho)|}{\rho} \lesssim \rho^{-1}.$$

Moreover, one can show with a similar estimate as in (5.1.34) that p_ε has a finite limit and since $\rho^2\mu(\rho)^{-1} \to 1$ the representation (5.1.33) of h_ε gives $\rho^2 h_\varepsilon(\rho) \to a_\varepsilon$. From this now follow the zeroth-order estimates from (5.1.26) and the existence of the two limits. To show the zeroth-order Lipschitz estimates in (5.1.27) we define

$$P_{\varepsilon,\kappa} := p_\varepsilon - p_\kappa \qquad \text{and} \qquad H_{\varepsilon,\kappa} := h_\varepsilon - h_\kappa.$$

We then obtain with (5.1.31)

$$\left(\mu H_{\varepsilon,\kappa}\right)'(\rho) = -(n-3)K(\rho)Q_{\varepsilon,\kappa}(\rho), \qquad \text{where} \qquad Q_{\varepsilon,\kappa}(\rho) := q_\varepsilon(p_\varepsilon(\rho)) - q_\kappa(p_\kappa(\rho)). \tag{5.1.35}$$

From the estimates for q_ε we have

$$|Q_{\varepsilon,\kappa}(\rho)| \lesssim |\varepsilon-\kappa| + |P_{\varepsilon,\kappa}(\rho)| \lesssim |\varepsilon-\kappa| + M_{\varepsilon,\kappa}(R), \quad \text{where} \quad M_{\varepsilon,\kappa}(R) := \sup_{\rho\geq R} |P_{\varepsilon,\kappa}(\rho)|.$$

Integrating (5.1.35) from R to ρ now gives, using the estimates for μ and K

$$\begin{aligned} |H_{\varepsilon,\kappa}(\rho)| &\leq \frac{\mu(R)}{\mu(\rho)}|H_{\varepsilon,\kappa}(R)| + \frac{n-3}{\mu(\rho)}\int_R^\rho |K(s)|\left(|\varepsilon-\kappa| + M_{\varepsilon,\kappa}(R)\right) ds && (5.1.36)\\ &\lesssim C_R\rho^{-2}|H_{\varepsilon,\kappa}(R)| + C\rho^{-2}\left(|\varepsilon-\kappa| + M_{\varepsilon,\kappa}(R)\right)\int_R^\rho s^{-2}\, ds && (5.1.37)\\ &\lesssim C_R|\varepsilon-\kappa|\rho^{-2} + R^{-1}\left(|\varepsilon-\kappa| + M_{\varepsilon,\kappa}(R)\right)\rho^{-2}, && (5.1.38)\end{aligned}$$

where we obtained the last estimate via the local bounds (5.1.29) from h_ε. On the other hand,

$$P_{\varepsilon,\kappa}(\rho) = P_{\varepsilon,\kappa}(R) + \int_R^\rho H_{\varepsilon,\kappa}(s)\, ds,$$

and since we have, due to (5.1.29), $|P_{\varepsilon,\kappa}(R)| = R|u_\varepsilon(R) - u_\kappa(R)| \lesssim R|\varepsilon-\kappa|$, we obtain together with the preceding estimate

$$M_{\varepsilon,\kappa}(R) \leq C_R|\varepsilon-\kappa| + CR^{-2}\left(|\varepsilon-\kappa| + M_{\varepsilon,\kappa}(R)\right).$$

If we now choose R sufficiently large we obtain $M_{\varepsilon,\kappa}(R) \lesssim |\varepsilon-\kappa|$. Substituting this back into (5.1.36) gives

$$|H_{\varepsilon,\kappa}(\rho)| \lesssim |\varepsilon-\kappa|\rho^{-2}.$$

Using again the local estimate for $P_{\varepsilon,\kappa}$ together with $\sup_{\rho\geq R}|P_{\varepsilon,\kappa}(\rho)| = M_{\varepsilon,\kappa}(R) \lesssim |\varepsilon-\kappa|$ yields

$$|P_{\varepsilon,\kappa}(\rho)| \lesssim |\varepsilon-\kappa|$$

and therefore

$$|u_\varepsilon(\rho) - u_\kappa(\rho)| = \frac{|P_{\varepsilon,\kappa}(\rho)|}{\rho} \lesssim |\varepsilon-\kappa|\rho^{-1}.$$

From the representation (5.1.32) of a_ε and the preceding estimates we then obtain

$$|a_\varepsilon - a_\kappa| \leq \mu(\rho)|H_{\varepsilon,\kappa}(\rho)| + (n-3)\int_\rho^\infty |K(s)||Q_{\varepsilon,\kappa}(s)|\, ds \lesssim |\varepsilon-\kappa|,$$

which shows that the limit is actually Lipschitz continuous with respect to ε as well. To now prove the higher order estimates we now inductively show

$$|h_\varepsilon^{(m)}(\rho)| \lesssim_m \rho^{-m-2} \qquad \text{as well as} \qquad |h_\varepsilon^{(m)}(\rho) - h_\kappa^{(m)}(\rho)| \lesssim_m |\varepsilon-\kappa|\rho^{-m-2}.$$

Since we have just proven the case $m = 0$ we can assume that these estimates hold up to order $m - 1$. Since $p'_\varepsilon = h_\varepsilon$ we obtain for $1 \leq j \leq m$

$$|p_\varepsilon^{(j)}(\rho)| \lesssim_j \rho^{-j-1} \qquad \text{and} \qquad |p_\varepsilon^{(j)}(\rho) - p_\kappa^{(j)}(\rho)| \lesssim_j |\varepsilon - \kappa| \rho^{-j-1}. \tag{5.1.39}$$

The chain rule, together with the uniform bounds for the derivatives of q_ε, yields

$$\left| \frac{d^j}{d\rho^j} q_\varepsilon(p_\varepsilon(\rho)) \right| \lesssim_j \rho^{-j-1} \qquad \text{and} \qquad \left| \frac{d^j}{d\rho^j} \left[q_\varepsilon(p_\varepsilon(\rho)) - q_\kappa(p_\kappa(\rho)) \right] \right| \lesssim_j |\varepsilon - \kappa| \rho^{-j-1}.$$

We now define

$$J_\varepsilon(\rho) := \int_\rho^\infty K(s) q_\varepsilon(p_\varepsilon(s))\, ds.$$

From the estimates for K and the preceding ones we obtain

$$|J_\varepsilon^{(j)}(\rho)| \lesssim_j \rho^{-j-1} \qquad \text{and} \qquad |J_\varepsilon^{(j)}(\rho) - J_\kappa^{(j)}(\rho)| \lesssim_j |\varepsilon - \kappa| \rho^{-j-1}.$$

Using (5.1.33) and the derivative estimates for μ^{-1}, the Leibniz rule yields

$$|h_\varepsilon^{(m)}(\rho)| \lesssim_m \rho^{-m-2}.$$

Together with $|a_\varepsilon - a_\kappa| \lesssim |\varepsilon - \kappa|$ we also obtain

$$|h_\varepsilon^{(m)}(\rho) - h_\kappa^{(m)}(\rho)| \lesssim_m |\varepsilon - \kappa| \rho^{-m-2}.$$

From (5.1.39) we now also have

$$|u_\varepsilon^{(m)}(\rho)| \lesssim_m \rho^{-m-1} \qquad \text{and} \qquad |u_\varepsilon^{(m)}(\rho) - u_\kappa^{(m)}(\rho)| \lesssim_m |\varepsilon - \kappa| \rho^{-m-1},$$

which finishes the proof.

□

By now reversing the coordinate transformations we are ready to prove the first main result of this chapter.

Proof of Theorem 5.0.1. Let $d \geq 3$. Set $n := d + 2$ and let $(s, k) \in \mathbb{R} \times \mathbb{N}$ satisfy (5.1.16). Take δ^* and $\varepsilon^*(\delta^*)$ from Proposition 5.1.7. For $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \leq \varepsilon^*$ let $\mathbf{\Psi}_\varepsilon = (\psi_{\varepsilon,1}, \psi_{\varepsilon,2}) \in C_r^\infty(\mathbb{R}^n) \times C_r^\infty(\mathbb{R}^n)$ be the function obtained in Corollary 5.1.8. Then $\psi_{\varepsilon,1}$ solves

$$(-\Delta + \Lambda^2 + 3\Lambda + 2)\psi_{\varepsilon,1} = N_\varepsilon(\psi_{\varepsilon,1}) \tag{5.1.40}$$

and the radial representative $\widetilde{\psi}_{\varepsilon,1}$ of $\psi_{\varepsilon,1}$ satisfies the nonlinear ODE (5.1.30). Now we define

$$f_\varepsilon(\rho) := \rho \widetilde{\psi}_{\varepsilon,1}(\rho) = \rho \widetilde{\psi}_{0,1}(\rho) + \rho \widetilde{\phi}_{\varepsilon,1}(\rho)$$

where $\widetilde{\phi}_{\varepsilon,1}$ denotes the radial representative of the first component of $\mathbf{\Phi}_\varepsilon$. By setting $\phi_\varepsilon(\rho) := \rho\widetilde{\phi}_{\varepsilon,1}$ we have $f_\varepsilon = f_0 + \phi_\varepsilon$ and f_ε solves

$$(1-\rho^2)f''(\rho) + \left(\frac{d-1}{\rho} - 2\rho\right) f'(\rho) - \frac{d-1}{\rho^2} w_\varepsilon(f(\rho)) w_\varepsilon'(f(\rho)) = 0, \tag{5.1.41}$$

for all $\rho \in (0,\infty)$. Hence,

$$u_\varepsilon^T(t,r) := f_\varepsilon\left(\frac{r}{T-t}\right)$$

provides a solution to Eq. (5.0.1). For the derivative we obtain

$$|\partial_r u_\varepsilon^T(t,0)| = (T-t)^{-1}|\widetilde{\psi}_{\varepsilon,1}(0)|$$

for $0 \le t < T$. In view of Lemma 5.1.5 and our previous assumptions we infer that

$$|\widetilde{\phi}_{\varepsilon,1}(0)| < \frac{2}{\sqrt{n-4}} = \widetilde{\psi}_{0,1}(0),$$

and thus $\widetilde{\psi}_{\varepsilon,1}(0) \neq 0$. In particular, the derivative of u_ε^T blows up as $t \to T^-$. The Lipschitz estimates follow from Proposition 5.1.7 and Sobolev embedding and the decay properties for f_ε are a consequence of Proposition 5.1.9 due to $f_\varepsilon' = \widetilde{\psi}_{\varepsilon,2}$.
The only thing left to show is $0 < f_\varepsilon(\rho) < \pi$ for every $0 < \rho \le 1$. For this we show that every $f \in C^\infty[0,\infty)$ with $f(0) = 0$ and $f'(0) > 0$ satisfying (5.1.41) satisfies $0 < f(\rho) < \pi$ for all $0 < \rho \le 1$. For this we first define

$$\mathcal{E}(\rho) := \frac{1}{2}\rho^2(1-\rho^2)f'(\rho)^2 - \frac{d-1}{2} w_\varepsilon(f(\rho))^2.$$

and if we differentiate this equation and use the profile equation (5.1.41) we obtain

$$\mathcal{E}'(\rho) = \rho(1-\rho^2)f'(\rho)^2 - \rho^3 f'(\rho)^2 + \rho\left(2\rho^2 - (d-1)\right) f'(\rho)^2 = -(d-2)\rho f'(\rho)^2 \le 0.$$

Integrating over $[0,\rho]$ leads to

$$\mathcal{E}(\rho) = -(d-2)\int_0^\rho s f'(s)^2\, ds \le 0.$$

Since $f'(0) > 0$ there exists a neighborhood $[0,\delta)$ with $f'(s) > 0$ for all $s \in [0,\delta)$. We therefore have

$$\mathcal{E}(\rho) < 0 \quad \text{for all} \quad \rho > 0. \tag{5.1.42}$$

We now define

$$\rho_0 := \inf\{\rho > 0 : \ f(\rho) \notin (0,\pi)\} > 0$$

and suppose $\rho_0 \le 1$. By continuity we must have $f(\rho_0) \in \{0,\pi\}$ so that we obtain

$$\mathcal{E}(\rho_0) = \frac{1}{2}\rho_0^2(1-\rho_0^2)f'(\rho_0)^2 \ge 0,$$

which is an immediate contradiction to (5.1.42).

□

Remark 5.1.10. *The preceding argument shows $0 < f_\varepsilon(\rho) < \pi$ for all $0 < \rho \le 1$. For $\rho > 1$, however, we cannot exclude that the profile reaches or crosses the value π, so that the corresponding wave map solution is already understood in the same sense as in Remark 4.0.3.*

5.2. Stability analysis

The aim of this section is to analyze the stability properties of the blowup solution $u_\varepsilon^T \in C^\infty([0,T)\times[0,\infty))$ constructed in Theorem 5.0.1. Again, the starting point will be Eq. (5.1.1) for which we now consider the Cauchy problem

$$\partial_t^2 v - \Delta_x v = \frac{n-3}{|x|^3}\left(|x|v - w_\varepsilon(|x|v)w_\varepsilon'(|x|v)\right) \tag{5.2.1}$$

where the initial data $v(0,x) = v_{0,\varepsilon}(x)$ and $\partial_t v(0,x) = v_{1,\varepsilon}(x)$, $x \in \mathbb{R}^n$ are given by

$$(v_{0,\varepsilon}, v_{1,\varepsilon}) := (v_\varepsilon^1(0,\cdot), \partial_t v_\varepsilon^1(0,\cdot)) + (\varphi_0, \varphi_1) \tag{5.2.2}$$

and v_ε^1 denotes the corresponding blowup solution v_ε^T with blowup time $T = 1$,

$$v_\varepsilon^T(t,x) = \frac{1}{T-t}\psi_\varepsilon\left(\frac{x}{T-t}\right), \quad \psi_\varepsilon(\xi) = |\xi|^{-1} f_\varepsilon(|\xi|).$$

By rewriting the problem as a first order system in similarity coordinates

$$\tau = \log\left(\frac{T}{T-t}\right), \quad \xi = \frac{x}{T-t},$$

for T to be chosen, Eq. (5.2.1) transforms into Eq. (5.1.3), see Section 5.1. By construction, the blowup solution v_ε^T corresponds to the static solution $\mathbf{\Psi}_\varepsilon$ of Eq. (5.1.3),

$$\mathbf{\Psi}_\varepsilon = \begin{pmatrix}\psi_{\varepsilon,1}\\ \psi_{\varepsilon,2}\end{pmatrix}, \text{ with } \psi_{\varepsilon,1} = \psi_\varepsilon \text{ and } \psi_{\varepsilon,2} = \psi_{\varepsilon,1} + \Lambda\psi_{\varepsilon,1}. \tag{5.2.3}$$

Consequently, the initial value problem (5.2.1)-(5.2.2) can be rephrased as

$$\begin{cases}\partial_\tau \Psi(\tau) = \widetilde{\mathbf{L}}\Psi(\tau) + \mathbf{N}_\varepsilon(\Psi(\tau)),\\ \Psi(0) = \mathbf{\Psi}_\varepsilon^{\mathbf{T}} + \varphi^{\mathbf{T}},\end{cases} \tag{5.2.4}$$

for $\mathbf{\Psi}_\varepsilon^{\mathbf{T}} := (T\psi_{\varepsilon,1}(T\cdot), T^2\psi_{\varepsilon,2}(T\cdot))$ and $\varphi^{\mathbf{T}} := (T\varphi_0(T\cdot), T^2\varphi_1(T\cdot))$ the perturbation of the original initial data. We remark that $\widetilde{\mathbf{L}}$ is still the wave operator in similarity coordinates as we introduced it in (5.1.4) and the nonlinearity $\mathbf{N}_\varepsilon$ is defined as in Eq. (5.1.5).
Assuming that the solution to Eq. (5.2.4) will be a small perturbation of $\mathbf{\Psi}_\varepsilon$, we insert the ansatz $\Psi(\tau) = \mathbf{\Psi}_\varepsilon + \Phi_\varepsilon(\tau)$ and get an evolution equation for the (time-dependent) perturbation Φ_ε,

$$\partial_\tau \Phi_\varepsilon(\tau) = \widetilde{\mathbf{L}}\Phi_\varepsilon(\tau) + \mathbf{N}_\varepsilon(\mathbf{\Psi}_\varepsilon + \Phi_\varepsilon(\tau)) - \mathbf{N}_\varepsilon(\mathbf{\Psi}_\varepsilon).$$

A Taylor expansion of $\mathbf{N}_\varepsilon$ around $\mathbf{\Psi}_\varepsilon$ now leads us to the central evolution equation of this section

$$\begin{cases}\partial_\tau \Phi_\varepsilon(\tau) = \widetilde{\mathbf{L}}_\varepsilon\Phi_\varepsilon(\tau) + \widehat{\mathbf{N}}_\varepsilon(\Phi_\varepsilon(\tau)),\\ \Phi_\varepsilon(0) = \varphi^{\mathbf{T}} + \mathbf{\Psi}_\varepsilon^{\mathbf{T}} - \mathbf{\Psi}_\varepsilon.\end{cases} \tag{5.2.5}$$

The linear part $\widetilde{\mathbf{L}}_\varepsilon := \widetilde{\mathbf{L}} + \mathbf{L}'_\varepsilon$ consists of the wave operator $\widetilde{\mathbf{L}}$ and of the potential $\mathbf{L}'_\varepsilon$ which is given by

$$\mathbf{L}'_\varepsilon \mathbf{u} = \begin{pmatrix} 0 \\ V_\varepsilon(\psi_{\varepsilon,1})\, u_1 \end{pmatrix} \quad \text{for} \quad V_\varepsilon(\psi_{\varepsilon,1})(\xi) = \frac{n-3}{|\xi|^2}\eta'_\varepsilon(|\xi|\psi_{\varepsilon,1}(\xi)). \tag{5.2.6}$$

The remaining nonlinearity $\widehat{\mathbf{N}}_\varepsilon$ is of the form

$$\widehat{\mathbf{N}}_\varepsilon(\mathbf{u}) = \begin{pmatrix} 0 \\ \widehat{N}_\varepsilon(u_1) \end{pmatrix}, \tag{5.2.7}$$

with

$$\widehat{N}_\varepsilon(u_1)(\xi) = \frac{n-3}{|\xi|^3}\left(\eta_\varepsilon(|\xi|(\psi_{\varepsilon,1}(\xi)+u_1(\xi))) - \eta_\varepsilon(|\xi|\psi_{\varepsilon,1}(\xi)) - \eta'_\varepsilon(|\xi|\psi_{\varepsilon,1}(\xi))|\xi|u_1(\xi)\right). \tag{5.2.8}$$

Notice that the only trace of the parameter T is in the initial data and that the here defined linearized operator $\widetilde{\mathbf{L}}_\varepsilon$ coincides for $\varepsilon = 0$ with the one we introduced in Section 5.1.4.

5.2.1. Perturbations of the free evolution for small parameters

The free evolution generated by $\widetilde{\mathbf{L}}$ is well understood, see the results summarized in Section 5.1.4. For the linearized flow, the following properties of the perturbation $\mathbf{L}'_\varepsilon$ are crucial. For the rest of the chapter let ε^* denote the constant determined by Proposition 5.1.6.

Proposition 5.2.1. *Let $n \geq 5$ and $(s,k) \in \mathbb{R}\times\mathbb{N}$ satisfy (5.1.16). For every $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \leq \varepsilon^*$ the operator $\mathbf{L}'_\varepsilon : \mathcal{H}^{s,k}_r \to \mathcal{H}^{s,k}_r$ is compact. Furthermore, the family of operators $\mathbf{L}'_\varepsilon$ is Lipschitz continuous in ε, i.e., the inequality*

$$\|\mathbf{L}'_\varepsilon \mathbf{u} - \mathbf{L}'_\kappa \mathbf{u}\|_{\mathcal{H}^{s,k}} \lesssim |\varepsilon - \kappa|\, \|\mathbf{u}\|_{\mathcal{H}^{s,k}}$$

holds for all $\mathbf{u} \in \mathcal{H}^{s,k}_r$ and all $|\varepsilon|, |\kappa| \leq \varepsilon^*$.

Proof. Due to the decay of the blowup solution, see Proposition 5.1.9, $V_\varepsilon(\psi_{\varepsilon,1})$ satisfies the assumptions of Lemma 5.1 in [40] from which the compactness of $\mathbf{L}'_\varepsilon$ follows. For the Lipschitz estimates, we write $V_\varepsilon(\psi_{\varepsilon,1})$ with (4.1.12) in the following way

$$V_\varepsilon(\psi_{\varepsilon,1})(\xi) = (n-3)\psi_{\varepsilon,1}(\xi)^2 \int_0^1 x \int_0^1 \eta'''_\varepsilon(|\xi|\psi_{\varepsilon,1}(\xi)xy)dy\,dx \tag{5.2.9}$$

and conclude by applying the Schauder estimate from Proposition A.2.1 using the Lipschitz continuity of $\psi_{\varepsilon,1}$.

□

By the bounded perturbation theorem we therefore get the following statement, where we first equip $\widetilde{\mathbf{L}}_\varepsilon$ with the domain $\mathcal{D}(\widetilde{\mathbf{L}}_\varepsilon) = \mathcal{D}(\widetilde{\mathbf{L}})$, see Proposition 5.1.1.

Proposition 5.2.2. *Let $n \geq 5$ and $(s,k) \in \mathbb{R} \times \mathbb{N}$ satisfy (5.1.16). For every $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \leq \varepsilon^*$ the operator $\widetilde{\mathbf{L}}_\varepsilon$ is closable and its closure $\mathbf{L}_\varepsilon = \mathbf{L} + \mathbf{L}'_\varepsilon$ with $\mathcal{D}(\mathbf{L}_\varepsilon) = \mathcal{D}(\mathbf{L}) = \mathcal{D}(\mathbf{L}_0)$ generates a strongly continuous semigroup $(\mathbf{S}_\varepsilon(\tau))_{\tau \geq 0}$ of bounded operators on $\mathcal{H}^{s,k}_r$. Furthermore,*

$$\|\mathbf{L}_\varepsilon \mathbf{u} - \mathbf{L}_\kappa \mathbf{u}\|_{\mathcal{H}^{s,k}} \lesssim |\varepsilon - \kappa| \, \|\mathbf{u}\|_{\mathcal{H}^{s,k}}$$

holds for all $\mathbf{u} \in \mathcal{D}(\mathbf{L})$ and all $|\varepsilon|, |\kappa| \leq \varepsilon^*$.

As in the case $\varepsilon = 0$ we do not expect decay of the semigroup due to an exponential instability induced by the time translation symmetry of the original problem. More precisely, we have the following result.

Lemma 5.2.3. *Let $n \geq 5$, $(s,k) \in \mathbb{R} \times \mathbb{N}$ satisfy (5.1.16) and $|\varepsilon| \leq \varepsilon^*$. We then have*

$$\mathbf{L}_\varepsilon \, \mathbf{g}_\varepsilon = \mathbf{g}_\varepsilon, \tag{5.2.10}$$

where $\mathbf{g}_\varepsilon \in C^\infty_r(\mathbb{R}^n) \times C^\infty_r(\mathbb{R}^n) \cap \mathcal{D}(\mathbf{L}_\varepsilon)$ is defined as

$$\mathbf{g}_\varepsilon := \begin{pmatrix} g_{\varepsilon,1} \\ g_{\varepsilon,2} \end{pmatrix} := \begin{pmatrix} \psi_{\varepsilon,1} + \Lambda \psi_{\varepsilon,1} \\ 2\psi_{\varepsilon,1} + 3\Lambda \psi_{\varepsilon,1} + \Lambda^2 \psi_{\varepsilon,1} \end{pmatrix}.$$

Additionally, the eigenfunction $\mathbf{g}_\varepsilon$ depends Lipschitz continuously on the parameter ε in the sense that

$$\|\mathbf{g}_\varepsilon - \mathbf{g}_\kappa\|_{\mathcal{H}^{s,k}} \lesssim |\varepsilon - \kappa|$$

holds for all $|\varepsilon|, |\kappa| \leq \varepsilon^*$.

Proof. We first show for every multi-index $\beta \in \mathbb{N}_0^n$

$$|\partial^\beta g_{\varepsilon,1}(\xi)| \lesssim_\beta \langle \xi \rangle^{-2-|\beta|} \quad \text{and} \quad |\partial^\beta \left(g_{\varepsilon,1}(\xi) - g_{\kappa,1}(\xi) \right)| \lesssim_\beta |\varepsilon - \kappa| \langle \xi \rangle^{-2-|\beta|}$$

as well as

$$|\partial^\beta g_{\varepsilon,2}(\xi)| \lesssim_\beta \langle \xi \rangle^{-3-|\beta|} \quad \text{and} \quad |\partial^\beta \left(g_{\varepsilon,2}(\xi) - g_{\kappa,2}(\xi) \right)| \lesssim_\beta |\varepsilon - \kappa| \langle \xi \rangle^{-3-|\beta|}.$$

From this, together with Lemma 4.5 from [39] and Lemma A.1.4, follows that $\mathbf{g}_\varepsilon$ lies in the domain of $\mathcal{D}(\mathbf{L}_\varepsilon) = \mathcal{D}(\mathbf{L}_0)$ and that it depends Lipschitz continuously on ε with respect to the $\mathcal{H}^{s,k}$-norm.
Recalling Proposition 5.1.9 and its proof we infer that the radial representative of $g_{\varepsilon,1}$ satisfies $\widetilde{g}_{\varepsilon,1} = \widetilde{\psi}_{\varepsilon,2}$ and this implies the estimates for $g_{\varepsilon,1}$.
We now use the same notations from Proposition 5.1.9. Namely,

$$p_\varepsilon(\rho) = \rho u_\varepsilon(\rho) = \rho \widetilde{\psi}_{\varepsilon,1}(\rho), \qquad h_\varepsilon(\rho) = p'_\varepsilon(\rho) = u_\varepsilon(\rho) + \rho u'_\varepsilon(\rho)$$

and h_ε satisfies

$$|h^{(m)}_\varepsilon(\rho)| \lesssim_m \langle \rho \rangle^{-m-2} \qquad \text{and} \qquad |h^{(m)}_\varepsilon(\rho) - h^{(m)}_\kappa(\rho)| \lesssim_m |\varepsilon - \kappa| \langle \rho \rangle^{-m-2}$$

as well as the following ODE

$$(1-\rho^2)h_\varepsilon'(\rho)+\left(\frac{n-3}{\rho}-2\rho\right)h_\varepsilon(\rho)=\frac{n-3}{\rho^2}q_\varepsilon(p_\varepsilon(\rho)),\quad \text{where}\quad q_\varepsilon(z)=w_\varepsilon(z)w_\varepsilon'(z).$$

Multiplication with ρ now gives

$$\rho(1-\rho^2)h_\varepsilon'(\rho)+\left(n-3-2\rho^2\right)h_\varepsilon(\rho)=\frac{n-3}{\rho}q_\varepsilon(p_\varepsilon(\rho))$$

and since $\widetilde{g}_{\varepsilon,2}=2h_\varepsilon+\rho h_\varepsilon'$, it follows that

$$(1-\rho^2)\widetilde{g}_{\varepsilon,2}=2(1-\rho^2)h_\varepsilon+\rho(1-\rho^2)h_\varepsilon'=(5-n)h_\varepsilon+\frac{n-3}{\rho}q_\varepsilon(p_\varepsilon(\rho))$$

and therefore

$$\widetilde{g}_{\varepsilon,2}=\frac{5-n}{1-\rho^2}h_\varepsilon+\frac{n-3}{\rho(1-\rho^2)}q_\varepsilon(p_\varepsilon(\rho)). \tag{5.2.11}$$

Fixing $R>1$ we get for every $j\in\mathbb{N}_0$

$$\left|\frac{d^j}{d\rho^j}\frac{1}{1-\rho^2}\right|\lesssim_{j,R}\rho^{-j-2}\qquad\text{and}\qquad\left|\frac{d^j}{d\rho^j}\frac{1}{\rho(1-\rho^2)}\right|\lesssim_{j,R}\rho^{-j-3}$$

for all $\rho\geq R$. Furthermore, the uniform estimates for the derivatives of p_ε and q_ε from Proposition 5.1.9 gave

$$|q_\varepsilon(p_\varepsilon(\rho))|\lesssim 1\quad\text{and}\quad\left|\frac{d^j}{d\rho^j}q_\varepsilon(p_\varepsilon(\rho))\right|\lesssim_j\rho^{-j-1}\quad\text{for all}\quad j\geq 1.$$

Applying this to (5.2.11) now gives

$$|\widetilde{g}_{\varepsilon,2}^{(m)}(\rho)|\lesssim\rho^{-m-3}.$$

The estimate on every compact interval $[0,R]$ follows from the local C^m-bounds and the Lipschitz estimates follow via a similar argument as in Proposition 5.1.9.
It remains to verify the eigenvalue equation. Since $g_{\varepsilon,2}=2g_{\varepsilon,1}+\Lambda g_{\varepsilon,1}$ the first component of $\widetilde{\mathbf{L}}_\varepsilon\mathbf{g}_\varepsilon=\mathbf{g}_\varepsilon$ is satisfied by definition. The second component is equivalent to

$$\Delta g_{\varepsilon,1}-\Lambda^2 g_{\varepsilon,1}-5\Lambda g_{\varepsilon,1}-6g_{\varepsilon,1}+V_\varepsilon(\psi_{\varepsilon,1})g_{\varepsilon,1}=0.$$

For the radial representative, this equation becomes

$$(1-\rho^2)h_\varepsilon''(\rho)+\left(\frac{n-1}{\rho}-6\rho\right)h_\varepsilon'(\rho)-6h_\varepsilon(\rho)+\frac{n-3}{\rho^2}\eta_\varepsilon'(p_\varepsilon(\rho))h_\varepsilon(\rho)=0. \tag{5.2.12}$$

To show this we start with the profile equation for $u_\varepsilon=\widetilde{\psi}_{\varepsilon,1}$ from (5.1.30)

$$F_\varepsilon[u_\varepsilon]:=(1-\rho^2)u_\varepsilon''+\left(\frac{n-1}{\rho}-4\rho\right)u_\varepsilon'(\rho)-2u_\varepsilon+\frac{n-3}{\rho^3}\eta_\varepsilon(\rho\,u_\varepsilon)=0.$$

One can then check that (5.2.12) is precisely given by

$$\rho \frac{d}{d\rho} F_\varepsilon[u_\varepsilon] + 3F_\varepsilon[u_\varepsilon] = 0,$$

which finishes the proof.

□

For $\varepsilon \neq 0$, the potential is not available in closed form which is however crucial for a direct investigation of the spectral problem and the exclusion of other unstable spectral points. Hence, we resort to perturbative arguments in order to characterize the spectral properties of $\mathbf{L}_\varepsilon$.

Proposition 5.2.4. *Let $n \geq 5$ and $(s,k) \in \mathbb{R} \times \mathbb{N}$ satisfy (5.1.16). For every arbitrary but fixed $0 < \omega_0 < \widetilde{\omega}$ where $\widetilde{\omega}$ is the constant from Proposition 5.1.1, there exists $\varepsilon^{**} > 0$ such that for every $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \leq \varepsilon^{**}$*

$$\sigma(\mathbf{L}_\varepsilon) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < -\omega_0\} \cup \{1\} \tag{5.2.13}$$

holds, $\lambda = 1$ is a simple eigenvalue of $\mathbf{L}_\varepsilon$ and its eigenspace is spanned by $\mathbf{g}_\varepsilon$. Furthermore, for the Riesz projection

$$\mathbf{P}_\varepsilon := \frac{1}{2\pi i} \int_{\partial B_{1/2}(1)} \mathbf{R}_{\mathbf{L}_\varepsilon}(\lambda)\, d\lambda \tag{5.2.14}$$

we have $\operatorname{ran}(\mathbf{P}_\varepsilon) = \ker(\mathbf{I} - \mathbf{L}_\varepsilon) = \langle \mathbf{g}_\varepsilon \rangle$.

Proof. We first show

$$\sigma(\mathbf{L}_\varepsilon) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq -\omega_0\} \subset B_{1/2}(1),$$

where $B_{1/2}(1)$ denotes the open ball in $\mathbb{C}$ of radius $r = \frac{1}{2}$ centered at 1, and that there exists a constant $C_{\omega_0} > 0$ such that the resolvent satisfies

$$\|\mathbf{R}_{\mathbf{L}_\varepsilon}(\lambda)\| \leq C_{\omega_0} \tag{5.2.15}$$

for all $\lambda \in \overline{\mathbb{H}_{-\omega_0}} \setminus B_{1/2}(1)$ and all $|\varepsilon| \leq \varepsilon^{**}$. We argue similarly to [54], Proposition 3.2. Let $|\varepsilon| \leq \varepsilon^*$. To see that $\overline{\mathbb{H}_{-\omega_0}} \setminus B_{1/2}(1)$ belongs to the resolvent set of $\mathbf{L}_\varepsilon$, we notice that due to Proposition 5.1.1 every $\lambda \in \overline{\mathbb{H}_{-\omega_0}} \setminus \{1\}$ belongs to the resolvent set of $\mathbf{L}_0$. Therefore, the identity

$$\lambda - \mathbf{L}_\varepsilon = (\mathbf{I} - (\mathbf{L}_\varepsilon - \mathbf{L}_0)\mathbf{R}_{\mathbf{L}_0}(\lambda))\,(\lambda - \mathbf{L}_0) \tag{5.2.16}$$

holds and it follows that $\lambda - \mathbf{L}_\varepsilon$ is bounded invertible if and only if this is true for $\mathbf{I} - (\mathbf{L}_\varepsilon - \mathbf{L}_0)\mathbf{R}_{\mathbf{L}_0}(\lambda)$. We will prove the invertibility of the latter one by showing that the Neumann series of $(\mathbf{L}_\varepsilon - \mathbf{L}_0)\mathbf{R}_{\mathbf{L}_0}(\lambda)$ converges. For this we split the resolvent in the following way

$$\mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f} = \mathbf{R}_{\mathbf{L}_0}(\lambda)(\mathbf{I} - \mathbf{P}_0)\mathbf{f} + \mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{P}_0\mathbf{f} \tag{5.2.17}$$

where $\mathbf{P}_0$ is the Riesz projection from Proposition 5.1.1. From the decay of the semigroup $(\mathbf{S}_0(\tau))_{\tau\in\mathbb{N}}$ on the stable subspace ran $(\mathbf{I}-\mathbf{P}_0)$, see (5.1.15), we obtain for every $0<\omega<\widetilde{\omega}$ the following bound

$$\|\mathbf{S}_0(\tau)\,(\mathbf{I}-\mathbf{P}_0)\,\mathbf{u}\|_{\mathcal{H}^{s,k}} \lesssim e^{-\omega\,\tau}\,\|(\mathbf{I}-\mathbf{P}_0)\,\mathbf{u}\|_{\mathcal{H}^{s,k}}\,.$$

Since the restriction of $(\mathbf{S}_0(\tau))_{\tau\geq 0}$ to ran $(\mathbf{I}-\mathbf{P}_0)$ equals the semigroup of the restriction of the operator $\mathbf{L}_0$ to ran $(\mathbf{I}-\mathbf{P}_0)$ we can apply [34], p.55, Theorem 1.10 for every $0<\omega<\widetilde{\omega}$ to infer the existence of a constant $M_\omega\geq 1$ such that

$$\|\mathbf{R}_{\mathbf{L}_0}(\lambda)(\mathbf{I}-\mathbf{P}_0)\| \leq \frac{M_\omega}{\operatorname{Re}\lambda+\omega} \tag{5.2.18}$$

holds for every $\lambda\in\mathbb{H}_{-\omega}$. Hence, (by choosing for example $\omega=\frac{\omega_0+\widetilde{\omega}}{2}$ in Eq. (5.2.18)) there exists a constant $C_{\omega_0}>0$ such that

$$\|\mathbf{R}_{\mathbf{L}_0}(\lambda)(\mathbf{I}-\mathbf{P}_0)\| \leq C_{\omega_0}$$

holds for all $\lambda\in\overline{\mathbb{H}_{-\omega_0}}$. For the second term in Eq. (5.2.17) we notice that $\ker(\mathbf{I}-\mathbf{L}_0)=$ ran $(\mathbf{P}_0)=\langle\mathbf{g}_0\rangle$ so that there exists a unique $\mathbf{h}_0\in\mathcal{H}^{s,k}_r$ with

$$\mathbf{P}_0\mathbf{f}=\langle\mathbf{h}_0,\mathbf{f}\rangle_{\mathcal{H}^{s,k}}\,\mathbf{g}_0 \tag{5.2.19}$$

for every $\mathbf{f}\in\mathcal{H}^{s,k}_r$. Since $\mathbf{g}_0$ is an eigenvector of $\mathbf{L}_0$ to the eigenvalue $\lambda=1$ we obtain

$$\mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{P}_0\mathbf{f}=(\lambda-1)^{-1}\langle\mathbf{h}_0,\mathbf{f}\rangle_{\mathcal{H}^{s,k}}\,\mathbf{g}_0. \tag{5.2.20}$$

From these equations we now infer from (5.2.17) the existence of a constant $C_{\omega_0}>0$ with

$$\|\mathbf{R}_{\mathbf{L}_0}(\lambda)\|\leq C_{\omega_0}$$

for every $\lambda\in\overline{\mathbb{H}_{-\omega_0}}\setminus B_{1/2}(1)$. In sight of Proposition 5.2.2 we therefore get

$$\|(\mathbf{L}_\varepsilon-\mathbf{L}_0)\mathbf{R}_{\mathbf{L}_0}(\lambda)\|\lesssim|\varepsilon|$$

for every $\lambda\in\overline{\mathbb{H}_{-\omega_0}}\setminus B_{1/2}(1)$ and every $|\varepsilon|\leq\varepsilon^*$. We now choose $\varepsilon^{**}\leq\varepsilon^*$ small enough so that $\|(\mathbf{L}_\varepsilon-\mathbf{L}_0)\mathbf{R}_{\mathbf{L}_0}(\lambda)\|<1$. Hence, we conclude from (5.2.16) that λ belongs to the resolvent set of $\mathbf{L}_\varepsilon$. Furthermore, we obtain from the same equation that the resolvent has the following representation

$$\mathbf{R}_{\mathbf{L}_\varepsilon}(\lambda)=\mathbf{R}_{\mathbf{L}_0}(\lambda)\,(\mathbf{I}-(\mathbf{L}_\varepsilon-\mathbf{L}_0)\mathbf{R}_{\mathbf{L}_0}(\lambda))^{-1}$$

from which Eq. (5.2.15) follows.
We will now conclude that the spectrum of $\mathbf{L}_\varepsilon$ is contained in a left half-plane except for the unstable eigenvalue $\lambda=1$. From our previous considerations we know that $\partial B_{1/2}(1)$ belongs to the resolvent set of $\mathbf{L}_\varepsilon$. Therefore, the Riesz projection $\mathbf{P}_\varepsilon$ from (5.2.14) is well-defined. We furthermore obtain from the resolvent identity, the Lipschitz continuity of $\mathbf{L}_\varepsilon$ and the uniform bound for the resolvent for every $\lambda\in\partial B_{1/2}(1)$,

$$\|\mathbf{R}_{\mathbf{L}_\varepsilon}(\lambda)-\mathbf{R}_{\mathbf{L}_\kappa}(\lambda)\|\leq\|\mathbf{R}_{\mathbf{L}_\varepsilon}(\lambda)\|\,\|\mathbf{L}_\varepsilon-\mathbf{L}_\kappa\|\,\|\mathbf{R}_{\mathbf{L}_\kappa}(\lambda)\|\lesssim|\varepsilon-\kappa|.$$

Hence $\mathbf{R}_{\mathbf{L}_\varepsilon}$ is Lipschitz continuous with respect to the parameter ε and therefore the same holds true for the Riesz projection $\mathbf{P}_\varepsilon$. Therefore, we know from [44], p.34, Lemma 4.10 that there holds for every $|\varepsilon| \leq \varepsilon^{**}$,

$$\dim \operatorname{ran}(\mathbf{P}_\varepsilon) = \dim \operatorname{ran}(\mathbf{P}_0) = 1.$$

But since we have the inclusion $\langle \mathbf{g}_\varepsilon \rangle \subset \ker(\mathbf{I} - \mathbf{L}_\varepsilon) \subset \operatorname{ran}(\mathbf{P}_\varepsilon)$ equality must hold due to dimensional reasons and there can also not exist any other spectral points of $\mathbf{L}_\varepsilon$ in $B_{1/2}(1)$.

□

Now we are able to characterize the linearized evolution generated by $\mathbf{L}_\varepsilon$.

Proposition 5.2.5. *Let $n \geq 5$ and $(s,k) \in \mathbb{R} \times \mathbb{N}$ satisfy (5.1.16). Then, for every arbitrary but fixed $0 < \omega_0 < \widetilde{\omega}$, where $\widetilde{\omega}$ is the constant from Proposition 5.1.1, and every $|\varepsilon| \leq \varepsilon^{**}$, where ε^{**} is the constant from Proposition 5.2.4, the projection $\mathbf{P}_\varepsilon$ commutes with the respective semigroup $\mathbf{S}_\varepsilon$. In particular,*

$$\mathbf{P}_\varepsilon \, \mathbf{S}_\varepsilon(\tau) = \mathbf{S}_\varepsilon(\tau)\mathbf{P}_\varepsilon = e^\tau \, \mathbf{P}_\varepsilon \tag{5.2.21}$$

for all $\tau \geq 0$. We furthermore have

$$\|\mathbf{S}_\varepsilon(\tau)(\mathbf{I} - \mathbf{P}_\varepsilon)\mathbf{u}\|_{\mathcal{H}^{s,k}} \lesssim e^{-\omega_0 \tau} \, \|(\mathbf{I} - \mathbf{P}_\varepsilon)\mathbf{u}\|_{\mathcal{H}^{s,k}} \tag{5.2.22}$$

as well as

$$\|[\mathbf{S}_\varepsilon(\tau)(\mathbf{I} - \mathbf{P}_\varepsilon) - \mathbf{S}_\kappa(\tau)(\mathbf{I} - \mathbf{P}_\kappa)]\mathbf{u}\|_{\mathcal{H}^{s,k}} \lesssim e^{-\omega_0 \tau} |\varepsilon - \kappa| \, \|\mathbf{u}\|_{\mathcal{H}^{s,k}} \tag{5.2.23}$$

for all $\mathbf{u} \in \mathcal{H}^{s,k}_r, \tau \geq 0$ and $|\varepsilon|, |\kappa| \leq \varepsilon^{**}$.

Proof. Since the semigroup $\mathbf{S}_\varepsilon(\tau)$ commutes with its generator $\mathbf{L}_\varepsilon$ it also commutes with its corresponding resolvent $\mathbf{R}_{\mathbf{L}_\varepsilon}$ and therefore also with the projection $\mathbf{P}_\varepsilon$. Eq. (5.2.21) follows from the fact that $\mathbf{S}_\varepsilon(\tau)\mathbf{P}_\varepsilon\mathbf{f}$ as well as $e^\tau \mathbf{P}_\varepsilon \mathbf{f}$ are solutions of the uniquely solvable Cauchy problem

$$\begin{cases} \partial_\tau \mathbf{u}(\tau) = \mathbf{L}_\varepsilon \mathbf{u}(\tau), \\ \mathbf{u}(0) = \mathbf{P}_\varepsilon \mathbf{f}. \end{cases}$$

Eq. (5.2.22) follows from [54] Theorem A.1 provided that one can show the existence of a constant $M_{\omega_0} > 0$ with

$$\|\mathbf{R}_{\mathbf{L}_\varepsilon}(\lambda)\,(\mathbf{I} - \mathbf{P}_\varepsilon)\| \leq M_{\omega_0} \tag{5.2.24}$$

for all $\lambda \in \overline{\mathbb{H}_{-\omega_0}}$ and $|\varepsilon| \leq \varepsilon^{**}$.
From Proposition 5.2.4 we already know that there exists a constant M_{ω_0} such that

$$\|\mathbf{R}_{\mathbf{L}_\varepsilon}(\lambda)\| \leq M_{\omega_0}$$

holds for every $\lambda \in \overline{\mathbb{H}_{-\omega_0}} \setminus B_{1/2}(1)$ and $|\varepsilon| \leq \varepsilon^{**}$. Now we know that $\mathbf{R}_{\mathbf{L}_\varepsilon}(\lambda)\,(\mathbf{I} - \mathbf{P}_\varepsilon)$ is an analytic map in $\mathbb{H}_{-\omega_0}$ and since the resolvent of the restriction of $\mathbf{L}_\varepsilon$ to the range of $\mathbf{I} - \mathbf{P}_\varepsilon$

coincides with $\mathbf{R}_{\mathbf{L}_\varepsilon}(\lambda)\,(\mathbf{I}-\mathbf{P}_\varepsilon)$ we obtain (5.2.24) also for every λ lying in the compact ball $\overline{B_{1/2}(1)}$.
To prove (5.2.23) let $u \in \mathcal{D}(\mathbf{L}) = \mathcal{D}(\mathbf{L}_\varepsilon) = \mathcal{D}(\mathbf{L}_\kappa)$, choose $\omega_0 < \omega_1 < \widetilde{\omega}$, fix $\tau > 0$ and define

$$F(\sigma) := \mathbf{S}_\varepsilon(\tau-\sigma)\mathbf{S}_\kappa(\sigma)\mathbf{u} \quad \text{for} \quad 0 \le \sigma \le \tau.$$

We obtain

$$\begin{aligned} F'(\sigma) &= -\mathbf{S}_\varepsilon(\tau-\sigma)\mathbf{L}_\varepsilon\mathbf{S}_\kappa(\sigma)\mathbf{u} + \mathbf{S}_\varepsilon(\tau-\sigma)\mathbf{L}_\kappa\mathbf{S}_\kappa(\sigma)\mathbf{u} \\ &= -\mathbf{S}_\varepsilon(\tau-\sigma)(\mathbf{L}_\varepsilon-\mathbf{L}_\kappa)\mathbf{S}_\kappa(\sigma)\mathbf{u} \end{aligned}$$

and integrating over $[0,\tau]$ gives

$$\mathbf{S}_\varepsilon(\tau)\mathbf{u} - \mathbf{S}_\kappa(\tau)\mathbf{u} = F(0) - F(\tau) = \int_0^\tau \mathbf{S}_\varepsilon(\tau-\sigma)(\mathbf{L}_\varepsilon-\mathbf{L}_\kappa)\mathbf{S}_\kappa(\sigma)\mathbf{u}\,d\sigma. \tag{5.2.25}$$

Applying this to $(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u}$ yields

$$\mathbf{S}_\kappa(\tau)(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u} = \mathbf{S}_\varepsilon(\tau)(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u} - \int_0^\tau \mathbf{S}_\varepsilon(\tau-\sigma)(\mathbf{L}_\varepsilon-\mathbf{L}_\kappa)\mathbf{S}_\kappa(\sigma)(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u}\,d\sigma$$

and if we apply $\mathbf{P}_\varepsilon$ to this, we obtain with (5.2.21)

$$\mathbf{P}_\varepsilon\mathbf{S}_\kappa(\tau)(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u} = e^\tau\mathbf{P}_\varepsilon(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u} - \int_0^\tau e^{\tau-\sigma}\mathbf{P}_\varepsilon(\mathbf{L}_\varepsilon-\mathbf{L}_\kappa)\mathbf{S}_\kappa(\sigma)(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u}\,d\sigma.$$

Multiplying by $e^{-\tau}$ we get

$$e^{-\tau}\mathbf{P}_\varepsilon\mathbf{S}_\kappa(\tau)(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u} = \mathbf{P}_\varepsilon(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u} - \int_0^\tau e^{-\sigma}\mathbf{P}_\varepsilon(\mathbf{L}_\varepsilon-\mathbf{L}_\kappa)\mathbf{S}_\kappa(\sigma)(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u}\,d\sigma$$

and from (5.2.22) we get that the left-hand side converges towards 0 as $\tau\to\infty$ and that the integral on the right-hand side converges. We therefore have

$$\mathbf{P}_\varepsilon(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u} = \int_0^\infty e^{-\sigma}\mathbf{P}_\varepsilon(\mathbf{L}_\varepsilon-\mathbf{L}_\kappa)\mathbf{S}_\kappa(\sigma)(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u}\,d\sigma. \tag{5.2.26}$$

We now obtain with (5.2.25)

$$\begin{aligned} \mathbf{S}_\varepsilon(\tau)(\mathbf{I}-\mathbf{P}_\varepsilon)\mathbf{u} - \mathbf{S}_\kappa(\tau)(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u} &= \mathbf{S}_\varepsilon(\tau)(\mathbf{P}_\kappa-\mathbf{P}_\varepsilon)\mathbf{u} + [\mathbf{S}_\varepsilon(\tau)-\mathbf{S}_\kappa(\tau)](\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u} \\ &= \mathbf{S}_\varepsilon(\tau)(\mathbf{P}_\kappa-\mathbf{P}_\varepsilon)\mathbf{u} + \int_0^\tau \mathbf{S}_\varepsilon(\tau-\sigma)(\mathbf{L}_\varepsilon-\mathbf{L}_\kappa)\mathbf{S}_\kappa(\sigma)(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u}\,d\sigma. \end{aligned}$$

If we now use $\mathbf{S}_\varepsilon(\tau) = \mathbf{S}_\varepsilon(\tau)(\mathbf{I}-\mathbf{P}_\varepsilon) + e^\tau \mathbf{P}_\varepsilon$ and (5.2.26) we obtain the following representation formula

$$\begin{aligned}
&\mathbf{S}_\varepsilon(\tau)(\mathbf{I}-\mathbf{P}_\varepsilon)\mathbf{u} - \mathbf{S}_\kappa(\tau)(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u} \\
&\quad = \mathbf{S}_\varepsilon(\tau)(\mathbf{I}-\mathbf{P}_\varepsilon)(\mathbf{P}_\kappa-\mathbf{P}_\varepsilon)\mathbf{u} + \int_0^\tau \mathbf{S}_\varepsilon(\tau-\sigma)(\mathbf{I}-\mathbf{P}_\varepsilon)(\mathbf{L}_\varepsilon-\mathbf{L}_\kappa)\mathbf{S}_\kappa(\sigma)(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u}\,d\sigma \\
&\qquad + e^\tau\left[\mathbf{P}_\varepsilon(\mathbf{P}_\kappa-\mathbf{P}_\varepsilon)\mathbf{u} + \int_0^\tau e^{-\sigma}\mathbf{P}_\varepsilon(\mathbf{L}_\varepsilon-\mathbf{L}_\kappa)\mathbf{S}_\kappa(\sigma)(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u}\,d\sigma\right] \\
&\quad = \mathbf{S}_\varepsilon(\tau)(\mathbf{I}-\mathbf{P}_\varepsilon)(\mathbf{P}_\kappa-\mathbf{P}_\varepsilon)\mathbf{u} + \int_0^\tau \mathbf{S}_\varepsilon(\tau-\sigma)(\mathbf{I}-\mathbf{P}_\varepsilon)(\mathbf{L}_\varepsilon-\mathbf{L}_\kappa)\mathbf{S}_\kappa(\sigma)(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u}\,d\sigma \\
&\qquad + e^\tau\left[\mathbf{P}_\varepsilon(\mathbf{P}_\kappa-\mathbf{I})\mathbf{u} + \int_0^\tau e^{-\sigma}\mathbf{P}_\varepsilon(\mathbf{L}_\varepsilon-\mathbf{L}_\kappa)\mathbf{S}_\kappa(\sigma)(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u}\,d\sigma\right] \\
&\quad = \mathbf{S}_\varepsilon(\tau)(\mathbf{I}-\mathbf{P}_\varepsilon)(\mathbf{P}_\kappa-\mathbf{P}_\varepsilon)\mathbf{u} + \int_0^\tau \mathbf{S}_\varepsilon(\tau-\sigma)(\mathbf{I}-\mathbf{P}_\varepsilon)(\mathbf{L}_\varepsilon-\mathbf{L}_\kappa)\mathbf{S}_\kappa(\sigma)(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u}\,d\sigma \\
&\qquad - e^\tau\int_\tau^\infty e^{-\sigma}\mathbf{P}_\varepsilon(\mathbf{L}_\varepsilon-\mathbf{L}_\kappa)\mathbf{S}_\kappa(\sigma)(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u}\,d\sigma.
\end{aligned}$$

For the first term we obtain with (5.2.22) and the Lipschitz continuity of $\mathbf{P}_\varepsilon$ for every $u \in \mathcal{D}(\mathbf{L})$

$$\|\mathbf{S}_\varepsilon(\tau)(\mathbf{I}-\mathbf{P}_\varepsilon)(\mathbf{P}_\kappa-\mathbf{P}_\varepsilon)\mathbf{u}\|_{\mathcal{H}^{s,k}} \lesssim e^{-\omega_0\tau}|\varepsilon-\kappa|\,\|\mathbf{u}\|_{\mathcal{H}^{s,k}}\,.$$

For the second term we use the Lipschitz continuity of L_ε and get

$$\begin{aligned}
&\left\|\int_0^\tau \mathbf{S}_\varepsilon(\tau-\sigma)(\mathbf{I}-\mathbf{P}_\varepsilon)(\mathbf{L}_\varepsilon-\mathbf{L}_\kappa)\mathbf{S}_\kappa(\sigma)(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u}\,d\sigma\right\| \\
&\lesssim |\varepsilon-\kappa|\int_0^\tau e^{-\omega_1(\tau-\sigma)}e^{-\omega_1\sigma}\,d\sigma\,\|\mathbf{u}\|_{\mathcal{H}^{s,k}} = |\varepsilon-\kappa|\tau e^{-\omega_1\tau}\,\|\mathbf{u}\|_{\mathcal{H}^{s,k}} \lesssim |\varepsilon-\kappa|e^{-\omega_0\tau}\,\|\mathbf{u}\|_{\mathcal{H}^{s,k}}\,.
\end{aligned}$$

For the last term we obtain analogously

$$\begin{aligned}
e^\tau\left\|\int_\tau^\infty e^{-\sigma}\mathbf{P}_\varepsilon(\mathbf{L}_\varepsilon-\mathbf{L}_\kappa)\mathbf{S}_\kappa(\sigma)(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u}\,d\sigma\right\| &\lesssim |\varepsilon-\kappa|e^\tau\int_\tau^\infty e^{-(1+\omega_0)\sigma}\,d\sigma\,\|\mathbf{u}\|_{\mathcal{H}^{s,k}} \\
&\lesssim |\varepsilon-\kappa|e^{-\omega_0\tau}\,\|\mathbf{u}\|_{\mathcal{H}^{s,k}}\,,
\end{aligned}$$

so that the claim follows from the density of $\mathcal{D}(\mathbf{L})$ in $\mathcal{H}^{s,k}_r$.

□

5.2.2. The abstract nonlinear Cauchy problem

In the following, for $(s,k) \in \mathbb{R}\times\mathbb{N}$ satisfying (5.1.16) we fix $\omega := \widetilde{\omega}/2$, where $\widetilde{\omega}$ is the constant from Proposition 5.1.1 and denote by $\overline{\varepsilon} := \varepsilon^{**}(\omega)$ the constant associated via Proposition 5.2.4. First, we state Lipschitz estimates for the nonlinearity $\widehat{\mathbf{N}}_\varepsilon$ defined in Eq. (5.2.7).

Lemma 5.2.6. *Let $n \geq 5$ and $(s,k) \in \mathbb{R} \times \mathbb{N}$ satisfy (5.1.16). Then, for any $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \leq \overline{\varepsilon}$, $\widehat{\mathbf{N}}_\varepsilon : \mathcal{H}^{s,k}_r \to \mathcal{H}^{s,k+1}_r$ and the estimate*

$$\left\| \widehat{\mathbf{N}}_\varepsilon(\mathbf{u}) - \widehat{\mathbf{N}}_\kappa(\mathbf{v}) \right\|_{\mathcal{H}^{s,k+1}} \lesssim \left(\|\mathbf{u}\|_{\mathcal{H}^{s,k}} + \|\mathbf{v}\|_{\mathcal{H}^{s,k}} \right) \|\mathbf{u}-\mathbf{v}\|_{\mathcal{H}^{s,k}} + \left(\|\mathbf{u}\|^2_{\mathcal{H}^{s,k}} + \|\mathbf{v}\|^2_{\mathcal{H}^{s,k}} \right) |\varepsilon - \kappa| \tag{5.2.27}$$

holds for all $|\varepsilon|, |\kappa| \leq \overline{\varepsilon}$ and all $\mathbf{u}, \mathbf{v} \in \mathcal{B}_\delta \subset \mathcal{H}^{s,k}_r$ for $0 < \delta \leq 1$.

Proof. The proof is analogous to the proof of Lemma 5.1.2 using the Schauder estimate from Proposition A.2.1.

□

We now focus on the existence and uniqueness of solutions to the Cauchy problem

$$\begin{cases} \partial_\tau \Phi_\varepsilon(\tau) = \widetilde{\mathbf{L}}_\varepsilon(\Phi_\varepsilon(\tau)) + \widehat{\mathbf{N}}_\varepsilon(\Phi_\varepsilon(\tau)), & \tau \in (0,\infty), \\ \Phi_\varepsilon(0) = \mathbf{u} & \mathbf{u} \in \mathcal{H}^{s,k}_r, \end{cases} \tag{5.2.28}$$

for $\varepsilon \in \mathbb{R}$, $|\varepsilon| \leq \overline{\varepsilon}$ by considering the corresponding integral equation

$$\Phi_\varepsilon(\tau) = \mathbf{S}_\varepsilon(\tau)\mathbf{u} + \int_0^\tau \mathbf{S}_\varepsilon(\tau - \tau')\widehat{\mathbf{N}}_\varepsilon(\Phi_\varepsilon(\tau'))\, d\tau' \quad \text{for all} \quad \tau \geq 0 \quad \text{and} \quad \mathbf{u} \in \mathcal{H}^{s,k}_r. \tag{5.2.29}$$

We introduce the Banach space

$$\mathcal{X} := \{\Phi \in C([0,\infty), \mathcal{H}^{s,k}_r) : \|\Phi\|_{\mathcal{X}} := \sup_{\tau > 0} e^{\omega\tau} \|\Phi(\tau)\|_{\mathcal{H}^{s,k}} < \infty\},$$

as well as

$$\mathcal{X}_\delta := \{\Phi \in \mathcal{X} : \|\Phi\|_{\mathcal{X}} \leq \delta\} = \{\Phi \in C([0,\infty), \mathcal{H}^{s,k}_r) : \|\Phi(\tau)\|_{\mathcal{H}^{s,k}} \leq \delta\, e^{-\omega\tau}, \forall\, \tau > 0\}.$$

Following the standard approach, we introduce the correction term

$$\mathbf{C}(\Phi, \varepsilon, \mathbf{u}) := \mathbf{P}_\varepsilon \left(\mathbf{u} + \int_0^\infty e^{-\tau'}\, \widehat{\mathbf{N}}_\varepsilon(\Phi(\tau'))\, d\tau' \right) \tag{5.2.30}$$

to suppress the exponential growth of the semigroup on the unstable subspace. Consequently, we consider the fixed-point problem

$$\Phi(\tau) = \mathbf{K}(\Phi, \varepsilon, \mathbf{u})(\tau), \tag{5.2.31}$$

where $\mathbf{K}(\Phi, \varepsilon, \mathbf{u})$ is defined as

$$\mathbf{K}(\Phi, \varepsilon, \mathbf{u})(\tau) := \mathbf{S}_\varepsilon(\tau)\left[\mathbf{u} - \mathbf{C}(\Phi, \varepsilon, \mathbf{u})\right] + \int_0^\tau \mathbf{S}_\varepsilon(\tau - \tau')\widehat{\mathbf{N}}_\varepsilon(\Phi(\tau'))\, d\tau'. \tag{5.2.32}$$

This modification stabilizes the evolution as the following result shows:

Proposition 5.2.7. *Let $n \geq 5$, $(s,k) \in \mathbb{R} \times \mathbb{N}$ satisfy (5.1.16). There are constants $0 < \delta_0 < 1$ and $C_0 > 1$ such that for all $0 < \delta \leq \delta_0$, $C \geq C_0$, all $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \leq \overline{\varepsilon}$ and all $\mathbf{u} \in \mathcal{H}^{s,k}_r$ with $\|\mathbf{u}\|_{\mathcal{H}^{s,k}} \leq \frac{\delta}{C}$ there exists a unique function $\Phi_\varepsilon(\mathbf{u}) \in \mathcal{X}_\delta$ such that (5.2.31) holds for all $\tau \geq 0$. Furthermore, the solution map $(\mathbf{u}, \varepsilon) \mapsto \Phi_\varepsilon(\mathbf{u})$ is Lipschitz continuous, i.e.*

$$\|\Phi_\varepsilon(\mathbf{u}) - \Phi_\kappa(\mathbf{v})\|_{\mathcal{X}} \lesssim \|\mathbf{u} - \mathbf{v}\|_{\mathcal{H}^{s,k}} + |\varepsilon - \kappa|$$

for all $\mathbf{u}, \mathbf{v} \in \mathcal{H}^{s,k}_r$ with $\|\mathbf{u}\|_{\mathcal{H}^{s,k}}, \|\mathbf{v}\|_{\mathcal{H}^{s,k}} \leq \frac{\delta}{C}$ and all $|\varepsilon|, |\kappa| \leq \overline{\varepsilon}$.

Proof. First, we show that the map $\mathbf{K}_{(\mathbf{u},\varepsilon)}(\Phi) := \mathbf{K}(\Phi, \varepsilon, \mathbf{u})$ is a well-defined contraction on $\mathcal{X}_\delta$ for all sufficiently large $C > 1$, sufficiently small $\delta > 0$ and all $\mathbf{u} \in \mathcal{H}^{s,k}_r$ with $\|\mathbf{u}\|_{\mathcal{H}^{s,k}} \leq \frac{\delta}{C}$.
For this we will take $\Phi \in \mathcal{X}_\delta$ and $\tau \geq 0$ and notice that we can write $\mathbf{K}_{(\mathbf{u},\varepsilon)}$ in the following way

$$\begin{aligned} &\mathbf{K}_{(\mathbf{u},\varepsilon)}(\Phi)(\tau) \\ &= \mathbf{S}_\varepsilon(\tau)(\mathbf{I} - \mathbf{P}_\varepsilon)\mathbf{u} - \int_\tau^\infty e^{\tau - \tau'} \mathbf{P}_\varepsilon \widehat{\mathbf{N}}_\varepsilon(\Phi(\tau'))\, d\tau' + \int_0^\tau \mathbf{S}_\varepsilon(\tau - \tau')(\mathbf{I} - \mathbf{P}_\varepsilon)\widehat{\mathbf{N}}_\varepsilon(\Phi(\tau'))\, d\tau'. \end{aligned} \tag{5.2.33}$$

From this we obtain with Proposition 5.2.5 and Lemma 5.2.6

$$\left\|\mathbf{K}_{(\mathbf{u},\varepsilon)}(\Phi)(\tau)\right\|_{\mathcal{H}^{s,k}} \lesssim \frac{\delta}{C} e^{-\omega\,\tau} + \delta^2 e^{-2\omega\,\tau} + \delta^2 e^{-\omega\,\tau}(e^{-\omega\,\tau} + 1) \lesssim \left(\frac{1}{C} + \delta\right) \delta e^{-\omega\,\tau}$$

so that we have

$$\left\|\mathbf{K}_{(\mathbf{u},\varepsilon)}(\Phi)(\tau)\right\|_{\mathcal{H}^{s,k}} \leq \delta e^{-\omega\,\tau}$$

if we choose $C \geq C_0$ and $0 < \delta \leq \delta_0$ with C_0 sufficiently large and $\delta_0 > 0$ sufficiently small.
Since the continuity of the mapping $\tau \mapsto \mathbf{K}_{(\mathbf{u},\varepsilon)}(\Phi)(\tau)$ follows from the definition and dominated convergence we conclude that $\mathbf{K}_{(\mathbf{u},\varepsilon)} : \mathcal{X}_\delta \to \mathcal{X}_\delta$ is well-defined.
To show that $\mathbf{K}_{(\mathbf{u},\varepsilon)}$ is a contraction on $\mathcal{X}_\delta$ (for potentially even smaller $\delta > 0$) we take $\Phi, \Psi \in \mathcal{X}_\delta$ and calculate for every $\tau \geq 0$ by using again the representation of $\mathbf{K}_{(\mathbf{u},\varepsilon)}$ from Eq. (5.2.33)

$$\begin{aligned} &\left\|\mathbf{K}_{(\mathbf{u},\varepsilon)}(\Phi)(\tau) - \mathbf{K}_{(\mathbf{u},\varepsilon)}(\Psi)(\tau)\right\|_{\mathcal{H}^{s,k}} \\ &\lesssim \int_\tau^\infty e^{\tau - \tau'} \|\widehat{\mathbf{N}}_\varepsilon(\Phi(\tau')) - \widehat{\mathbf{N}}_\varepsilon(\Psi(\tau'))\|_{\mathcal{H}^{s,k}}\, d\tau' + \int_0^\tau e^{-\omega\,(\tau - \tau')} \|\widehat{\mathbf{N}}_\varepsilon(\Phi(\tau')) - \widehat{\mathbf{N}}_\varepsilon(\Psi(\tau'))\|_{\mathcal{H}^{s,k}}\, d\tau' \\ &\lesssim \left(\delta e^{-2\omega\,\tau} + \delta e^{-\omega\,\tau}(e^{-\omega\,\tau} + 1)\right) \|\Phi - \Psi\|_{\mathcal{X}}. \end{aligned}$$

If we now choose $\delta_0 > 0$ sufficiently small we get

$$\left\|\mathbf{K}_{(\mathbf{u},\varepsilon)}(\Phi) - \mathbf{K}_{(\mathbf{u},\varepsilon)}(\Psi)\right\|_{\mathcal{X}} \leq \frac{1}{2} \|\Phi - \Psi\|_{\mathcal{X}},$$

for all $0 < \delta \leq \delta_0$, $C \geq C_0$ and all $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \leq \bar{\varepsilon}$. The existence of a unique solution in $\mathcal{X}_\delta$ now follows by application of the contraction mapping principle.
What is now left to show is the Lipschitz continuity of the solution map $(\mathbf{u}, \varepsilon) \mapsto \Phi_\varepsilon(\mathbf{u}) \in \mathcal{X}_\delta$.
For this we take $(\mathbf{u}, \varepsilon), (\mathbf{v}, \kappa) \in \mathcal{B}_{\frac{\delta}{C}} \times [-\bar{\varepsilon}, \bar{\varepsilon}]$ and obtain by the previous considerations functions $\Phi_\varepsilon(\mathbf{u}), \Phi_\kappa(\mathbf{v}) \in \mathcal{X}_\delta$ solving

$$\Phi_\varepsilon(\mathbf{u})(\tau) = \mathbf{K}(\Phi_\varepsilon(\mathbf{u})(\tau), \varepsilon, \mathbf{u})(\tau) \quad \text{and} \quad \Phi_\kappa(\mathbf{v})(\tau) = \mathbf{K}(\Phi_\kappa(\mathbf{v})(\tau), \kappa, \mathbf{v})(\tau) \quad \forall\, \tau \geq 0.$$

We now show that

$$\|\mathbf{K}(\Phi_\varepsilon(\mathbf{u}), \varepsilon, \mathbf{u}) - \mathbf{K}(\Phi_\kappa(\mathbf{v}), \kappa, \mathbf{v})\|_{\mathcal{X}} \lesssim \|\mathbf{u} - \mathbf{v}\|_{\mathcal{H}^{s,k}} + |\varepsilon - \kappa|.$$

For this we take $\tau \geq 0$ and estimate the terms in (5.2.33) separately. For the first term we simply get from (5.2.23)

$$\|\mathbf{S}_\varepsilon(\tau)(\mathbf{I} - \mathbf{P}_\varepsilon)\mathbf{u} - \mathbf{S}_\kappa(\tau)(\mathbf{I} - \mathbf{P}_\kappa)\mathbf{v}\| \lesssim \frac{\delta}{C} e^{-\omega\,\tau} |\varepsilon - \kappa| + e^{-\omega\,\tau} \|\mathbf{u} - \mathbf{v}\|_{\mathcal{H}^{s,k}}.$$

For the second term we apply Lemma 5.2.6 and the Lipschitz continuity of $\mathbf{P}_\varepsilon$ to get

$$\begin{aligned}
&\int_\tau^\infty e^{\tau - \tau'} \|\mathbf{P}_\varepsilon\, \widehat{\mathbf{N}}_\varepsilon(\Phi_\varepsilon(\mathbf{u})(\tau')) - \mathbf{P}_\kappa\, \widehat{\mathbf{N}}_\kappa(\Phi_\kappa(\mathbf{v})(\tau'))\|_{\mathcal{H}^{s,k}}\, d\tau' \\
\lesssim\, & |\varepsilon - \kappa| \int_\tau^\infty e^{\tau - \tau'} \|\widehat{\mathbf{N}}_\kappa(\Phi_\kappa(\mathbf{v})(\tau'))\|_{\mathcal{H}^{s,k}}\, d\tau' + \int_\tau^\infty e^{\tau - \tau'} \|\widehat{\mathbf{N}}_\varepsilon(\Phi_\varepsilon(\mathbf{u})(\tau')) - \widehat{\mathbf{N}}_\kappa(\Phi_\kappa(\mathbf{v})(\tau'))\|_{\mathcal{H}^{s,k}}\, d\tau' \\
\lesssim\, & |\varepsilon - \kappa|\, \delta^2\, e^{-2\omega\,\tau} + \delta\, e^{-2\omega\,\tau}\, \|\Phi_\varepsilon(\mathbf{u}) - \Phi_\kappa(\mathbf{v})\|_{\mathcal{X}}
\end{aligned}$$

and for the last term we similarly get

$$\begin{aligned}
&\int_0^\tau \Big\|\mathbf{S}_\varepsilon(\tau - \tau')(\mathbf{I} - \mathbf{P}_\varepsilon)\widehat{\mathbf{N}}_\varepsilon(\Phi_\varepsilon(\mathbf{u})(\tau')) - \mathbf{S}_\kappa(\tau - \tau')(\mathbf{I} - \mathbf{P}_\kappa)\widehat{\mathbf{N}}_\kappa(\Phi_\kappa(\mathbf{v})(\tau'))\Big\|_{\mathcal{H}^{s,k}}\, d\tau' \\
\lesssim\, & |\varepsilon - \kappa| \int_0^\tau e^{-\omega\,(\tau - \tau')} \|\widehat{\mathbf{N}}_\kappa(\Phi_\kappa(\mathbf{v})(\tau'))\|_{\mathcal{H}^{s,k}}\, d\tau' \\
& + \int_0^\tau e^{-\omega\,(\tau - \tau')} \|\widehat{\mathbf{N}}_\varepsilon(\Phi_\varepsilon(\mathbf{u})(\tau')) - \widehat{\mathbf{N}}_\kappa(\Phi_\kappa(\mathbf{v})(\tau'))\|_{\mathcal{H}^{s,k}}\, d\tau' \\
\lesssim\, & |\varepsilon - \kappa|\, \delta^2\, e^{-\omega\,\tau} + \delta\, e^{-\omega\,\tau}\, \|\Phi_\varepsilon(\mathbf{u}) - \Phi_\kappa(\mathbf{v})\|_{\mathcal{X}}.
\end{aligned}$$

With that we obtain

$$\begin{aligned}
\|\Phi_\varepsilon(\mathbf{u}) - \Phi_\kappa(\mathbf{v})\|_{\mathcal{X}} &= \|\mathbf{K}(\Phi_\varepsilon(\mathbf{u}), \varepsilon, \mathbf{u}) - \mathbf{K}(\Phi_\kappa(\mathbf{v}), \kappa, \mathbf{v})\|_{\mathcal{X}} \\
&\lesssim |\varepsilon - \kappa| + \|\mathbf{u} - \mathbf{v}\|_{\mathcal{H}^{s,k}} + \delta\, \|\Phi_\varepsilon(\mathbf{u}) - \Phi_\kappa(\mathbf{v})\|_{\mathcal{X}}.
\end{aligned}$$

For small enough $\delta > 0$ we get $\|\Phi_\varepsilon(\mathbf{u}) - \Phi_\kappa(\mathbf{v})\|_{\mathcal{X}} \lesssim |\varepsilon - \kappa| + \|\mathbf{u} - \mathbf{v}\|_{\mathcal{H}^{s,k}}$ as desired.

□

Now we will go back to considering the specific form of the initial data of the Cauchy problem (5.2.5). For this, we define the following initial data operator

$$\mathbf{U}_\varepsilon(\mathbf{v}, T) := \mathbf{v}^{\mathbf{T}} + \mathbf{\Psi}_\varepsilon^{\mathbf{T}} - \mathbf{\Psi}_\varepsilon := \begin{pmatrix} T v_1(T\cdot) \\ T^2 v_2(T\cdot) \end{pmatrix} + \begin{pmatrix} T\psi_{\varepsilon,1}(T\cdot) - \psi_{\varepsilon,1} \\ T^2 \psi_{\varepsilon,2}(T\cdot) - \psi_{\varepsilon,2} \end{pmatrix}. \tag{5.2.34}$$

Lemma 5.2.8. *Let $n \geq 5$ and $(s,k) \in \mathbb{R} \times \mathbb{N}$ satisfy (5.1.16). Let $0 < \delta \leq \frac{1}{2}$. For every $\varepsilon \in \mathbb{R}$, $|\varepsilon| \leq \overline{\varepsilon}$ the map*

$$T \mapsto \mathbf{U}_\varepsilon(\mathbf{v}, T) : [1-\delta, 1+\delta] \to \mathcal{H}^{s,k}_r$$

is continuous for $\mathbf{v} \in \mathcal{H}^{s,k}_r$. Furthermore, for every $T \in [\frac{1}{2}, \frac{3}{2}]$ the initial data operator can be written as

$$\mathbf{U}_\varepsilon(\mathbf{v}, T) = \mathbf{v}^T + (T-1)\mathbf{g}_\varepsilon + \mathbf{R}_\varepsilon(T), \tag{5.2.35}$$

and there exists a constant $M_\varepsilon > 0$ such that

$$\|\mathbf{R}_\varepsilon(T)\| \leq M_\varepsilon |T-1|^2.$$

Proof. Eq. (5.2.35) follows from Taylor's Theorem applied to the map $[\frac{1}{2}, \frac{3}{2}] \to \mathcal{H}^{s,k}_r, T \mapsto \Psi^T_\varepsilon$ using the fact that

$$\partial_T \begin{pmatrix} T\psi_{\varepsilon,1}(T\cdot) \\ T^2\psi_{\varepsilon,2}(T\cdot) \end{pmatrix} \Big|_{T=1} = \mathbf{g}_\varepsilon.$$

The components $R_{\varepsilon,1}(T)$ and $R_{\varepsilon,2}(T)$ of the remainder term $\mathbf{R}_\varepsilon(T)$ satisfy

$$\|R_{\varepsilon,i}(T)\|_{\dot{H}^{s-(i-1)} \cap \dot{H}^{k-(i-1)}(\mathbb{R}^n)} \lesssim (T-1)^2 \sum_{j=0}^{2} \left\|\Lambda^j \psi_{\varepsilon,i}\right\|_{\dot{H}^{s-(i-1)} \cap \dot{H}^{k-(i-1)}(\mathbb{R}^n)} \quad \text{for} \quad i = 1, 2.$$

The norms on the right-hand side are finite, due to the decay of $\psi_{\varepsilon,i}$ described in Proposition 5.1.9. This determines the constant $M_\varepsilon > 0$. The continuity follows by a standard argument, see for example [39], Lemma 8.2. □

Now, we are in the position to prove the central result of this section.

Theorem 5.2.9. *Let $n \geq 5$ and $(s,k) \in \mathbb{R} \times \mathbb{N}$ satisfy (5.1.16). For any $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \leq \overline{\varepsilon}$, there are constants $0 < \delta_\varepsilon < 1$ and $C_\varepsilon > 1$ such that for all $0 < \delta \leq \delta_\varepsilon$ and all $C \geq C_\varepsilon$ the following statement holds: If $\mathbf{v} \in \mathcal{H}^{s,k}_r$ is real-valued with $\|\mathbf{v}\|_{\mathcal{H}^{s,k}} \leq \frac{\delta}{C^2}$ then there exists a $T_\varepsilon = T_\varepsilon(\mathbf{v}) \in [1 - \frac{\delta}{C}, 1 + \frac{\delta}{C}]$ and a unique $\Phi_\varepsilon \in C([0,\infty); \mathcal{H}^{s,k}_r)$ satisfying*

$$\Phi_\varepsilon(\tau) = \mathbf{S}_\varepsilon(\tau)\mathbf{U}_\varepsilon(\mathbf{v}, T_\varepsilon) + \int_0^\tau \mathbf{S}_\varepsilon(\tau - \tau')\widehat{\mathbf{N}}_\varepsilon(\Phi_\varepsilon(\tau'))\, d\tau' \quad \textit{for all} \quad \tau \geq 0. \tag{5.2.36}$$

Furthermore,

$$\|\Phi_\varepsilon(\tau)\|_{\mathcal{H}^{s,k}} \leq \delta e^{-\omega\tau}, \quad \forall \tau \geq 0.$$

Proof. Let $0 < \delta \leq \delta_0$ and $C \geq C_0 \geq 1$ with δ_0 and C_0 as in Proposition 5.2.7. Let $|\varepsilon| \leq \overline{\varepsilon}$ and $\mathbf{v} \in \mathcal{H}^{s,k}_r$ with $\|\mathbf{v}\|_{\mathcal{H}^{s,k}} \leq \frac{\delta}{C^2}$. Then we obtain from Lemma 5.2.8

$$\begin{aligned} \|\mathbf{U}_\varepsilon(\mathbf{v}, T)\|_{\mathcal{H}^{s,k}} &\lesssim \left\|\mathbf{v}^T\right\|_{\mathcal{H}^{s,k}} + |T-1| \, \|\mathbf{g}_\varepsilon\|_{\mathcal{H}^{s,k}} + \|\mathbf{R}_\varepsilon(T)\|_{\mathcal{H}^{s,k}} \\ &\lesssim \frac{\delta}{C^2} + \frac{\delta}{C} L_\varepsilon + \frac{\delta^2}{C^2} M_\varepsilon \end{aligned}$$

for every $T \in [1-\frac{\delta}{C}, 1+\frac{\delta}{C}]$ where $L_\varepsilon, M_\varepsilon > 0$ are some constants depending on ε. If we now choose δ sufficiently small and C sufficiently large we obtain for every $T \in [1-\frac{\delta}{C}, 1+\frac{\delta}{C}]$ from Proposition 5.2.7 the existence of a unique $\Phi_\varepsilon = \Phi_\varepsilon(\mathbf{v}, T) \in \mathcal{X}_\delta$ which solves

$$\Phi_\varepsilon(\tau) = \mathbf{S}_\varepsilon(\tau)\left[\mathbf{U}_\varepsilon(\mathbf{v}, T) - \mathbf{C}(\Phi_\varepsilon, \varepsilon, \mathbf{U}_\varepsilon(\mathbf{v}, T))\right] + \int_0^\tau \mathbf{S}_\varepsilon(\tau - \tau')\widehat{\mathbf{N}}_\varepsilon(\Phi_\varepsilon(\tau'))\, d\tau'. \quad (5.2.37)$$

Since $\mathbf{C}$ takes values in $\operatorname{ran} \mathbf{P}_\varepsilon = \langle \mathbf{g}_\varepsilon \rangle$ it is enough to show, given $\mathbf{v}$, the existence of a T such that

$$\langle \mathbf{C}(\Phi_\varepsilon(\mathbf{v}, T), \varepsilon, \mathbf{U}_\varepsilon(\mathbf{v}, T)), \mathbf{g}_\varepsilon \rangle_{\mathcal{H}^{s,k}} = 0. \quad (5.2.38)$$

Due to Lemma 5.2.8 and the definition of $\mathbf{C}$ this equation reads as

$$0 = \langle \mathbf{P}_\varepsilon \mathbf{v}^T, \mathbf{g}_\varepsilon \rangle_{\mathcal{H}^{s,k}} + (T-1)\left\|\mathbf{g}_\varepsilon\right\|^2_{\mathcal{H}^{s,k}} + \langle \mathbf{P}_\varepsilon \mathbf{R}_\varepsilon(T), \mathbf{g}_\varepsilon \rangle_{\mathcal{H}^{s,k}} + \langle \mathbf{P}_\varepsilon \int_0^\infty e^{-\tau'} \widehat{\mathbf{N}}_\varepsilon(\Phi_\varepsilon(\tau'))\, d\tau', \mathbf{g}_\varepsilon \rangle_{\mathcal{H}^{s,k}},$$

which can be written as a fixed-point equation for $T \in [1-\frac{\delta}{C}, 1+\frac{\delta}{C}]$,

$$T = 1 - \langle \mathbf{P}_\varepsilon \mathbf{v}^T, \widehat{\mathbf{g}}_\varepsilon \rangle_{\mathcal{H}^{s,k}} - \langle \mathbf{P}_\varepsilon \mathbf{R}_\varepsilon(T), \widehat{\mathbf{g}}_\varepsilon \rangle_{\mathcal{H}^{s,k}} - \langle \mathbf{P}_\varepsilon \int_0^\infty e^{-\tau'} \widehat{\mathbf{N}}_\varepsilon(\Phi_\varepsilon(\tau'))\, d\tau', \widehat{\mathbf{g}}_\varepsilon \rangle_{\mathcal{H}^{s,k}}, \quad (5.2.39)$$

where we have set $\widehat{\mathbf{g}}_\varepsilon = \mathbf{g}_\varepsilon / \left\|\mathbf{g}_\varepsilon\right\|^2_{\mathcal{H}^{s,k}}$. Now we obtain from the assumptions on $\mathbf{v}$, the fact that Φ_ε belongs to $\mathcal{X}_\delta$ and Lemma 5.2.8 as well as Lemma 5.2.6 the following estimate

$$\begin{aligned} &|\langle \mathbf{P}_\varepsilon \mathbf{v}^T, \widehat{\mathbf{g}}_\varepsilon \rangle_{\mathcal{H}^{s,k}}| + |\langle \mathbf{P}_\varepsilon \mathbf{R}_\varepsilon(T), \widehat{\mathbf{g}}_\varepsilon \rangle_{\mathcal{H}^{s,k}}| + |\langle \mathbf{P}_\varepsilon \int_0^\infty e^{-\tau'} \widehat{\mathbf{N}}_\varepsilon(\Phi_\varepsilon(\tau'))\, d\tau', \widehat{\mathbf{g}}_\varepsilon \rangle_{\mathcal{H}^{s,k}}| \\ \lesssim &\frac{\delta}{C^2} L_\varepsilon + \frac{\delta^2}{C^2} M_\varepsilon + \delta^2 N_\varepsilon \end{aligned}$$

for again some constants $L_\varepsilon, M_\varepsilon$ and $N_\varepsilon > 0$. If we now choose $C \geq C_\varepsilon$ and $0 < \delta < \delta_\varepsilon$ with $C_\varepsilon > 1$ sufficiently large and $\delta_\varepsilon < 1$ sufficiently small we get that the right-hand side of (5.2.39) is a continuous mapping from $[1-\frac{\delta}{C}, 1+\frac{\delta}{C}]$ into itself so that we obtain by the fixed-point theorem of Brouwer a $T_\varepsilon \in \left[1-\frac{\delta}{C}, 1+\frac{\delta}{C}\right]$ such that equation (5.2.38) is fulfilled. We therefore conclude that the corresponding solution $\Phi_\varepsilon(\mathbf{v}, T_\varepsilon)$ is a solution to (5.2.36). The claimed uniqueness follows along the lines of the proof of Theorem 5.4 in [41].

□

Now we will show the regularity of the just constructed solution.

Proposition 5.2.10. *Let* $\mathbf{v}$ *satisfy the assumptions of Theorem 5.2.9. If* $\mathbf{v} \in \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ *then the solution* Φ_ε *of Eq.* (5.2.36) *guaranteed by Theorem 5.2.9 is smooth. More precisely,* $\Phi_\varepsilon(\tau)(\xi) = (\phi_{\varepsilon,1}(\tau, \xi), \phi_{\varepsilon,2}(\tau, \xi))$ *with* $\phi_{\varepsilon,i} \in C^\infty([0, \infty) \times \mathbb{R}^n)$. *Furthermore, the components satisfy*

$$\begin{pmatrix} \partial_\tau \phi_{\varepsilon,1} \\ \partial_\tau \phi_{\varepsilon,2} \end{pmatrix} = \begin{pmatrix} \phi_{\varepsilon,2} - \Lambda\, \phi_{\varepsilon,1} - \phi_{\varepsilon,1} \\ \Delta_\xi \phi_{\varepsilon,1} - \Lambda\, \phi_{\varepsilon,2} - 2\phi_{\varepsilon,2} + V_\varepsilon(\psi_{\varepsilon,1})\phi_{\varepsilon,1} + \hat{N}_\varepsilon(\phi_{\varepsilon,1}) \end{pmatrix} \quad (5.2.40)$$

pointwise and

$$\begin{pmatrix}\phi_{\varepsilon,1}(0,\cdot)\\ \phi_{\varepsilon,2}(0,\cdot)\end{pmatrix} = \begin{pmatrix}T_\varepsilon v_1(T_\varepsilon\cdot) + T_\varepsilon\psi_{\varepsilon,1}(T_\varepsilon\cdot) - \psi_{\varepsilon,1}\\ T_\varepsilon^2 v_2(T_\varepsilon\cdot) + T_\varepsilon^2\psi_{\varepsilon,2}(T_\varepsilon\cdot) - \psi_{\varepsilon,2}\end{pmatrix}. \tag{5.2.41}$$

Proof. Assume that all of the constants are chosen such that Theorem 5.2.9 is fulfilled. Then there exists a solution $\Phi_\varepsilon = \Phi_\varepsilon(\mathbf{v}, T_\varepsilon) \in \mathcal{X}_\delta$ satisfying

$$\Phi_\varepsilon(\tau) = \mathbf{S}_\varepsilon(\tau)\mathbf{U}_\varepsilon(\mathbf{v}, T_\varepsilon) + \int_0^\tau \mathbf{S}_\varepsilon(\tau-\tau')\widehat{\mathbf{N}}_\varepsilon(\Phi_\varepsilon(\tau'))\,d\tau' \quad \text{for all} \quad \tau \geq 0.$$

We will now show via an inductive argument that $\Phi_\varepsilon(\tau)$ belongs to $\mathcal{H}^{s,\ell}$ for every $\ell \geq k$ and every $\tau \geq 0$. We already know from Lemma 5.2.6 that $\widehat{\mathbf{N}}_\varepsilon$ maps $\mathcal{H}^{s,\ell}$ into $\mathcal{H}^{s,\ell+1}$ and that the initial data operator $\mathbf{U}_\varepsilon(\mathbf{v}, T_\varepsilon)$ belongs to $\mathcal{H}^{s,\ell}$ for every $\ell \geq k$. Since the restriction of the semigroups are equal to the restricted semigroups, see Lemma C.1 from [39], we can already inductively conclude that $\Phi_\varepsilon(\tau)$ belongs to $\mathcal{H}^{s,\ell}$ for every $\ell \geq k$ and every $\tau \geq 0$. From the Sobolev embedding (5.1.13) we can therefore conclude that $\Phi_\varepsilon(\tau)$ belongs to $C^\infty(\mathbb{R}^n) \times C^\infty(\mathbb{R}^n)$ for every $\tau \geq 0$.
We next show that $\mathbf{U}_\varepsilon(\mathbf{v}, T) \in \mathcal{D}(\mathbf{L}_\varepsilon)$ for $\mathbf{v} \in \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ and $T \in [\frac{1}{2}, \frac{3}{2}]$. Since $\mathcal{D}(\mathbf{L}_\varepsilon) = \mathcal{D}(\mathbf{L}_0)$ and the Schwartz space is contained in $\mathcal{D}(\mathbf{L}_0)$, it remains to consider $\widehat{\psi}_1 := T\psi_{\varepsilon,1}(T\cdot) - \psi_{\varepsilon,1}$ and $\widehat{\psi}_2 := T^2\psi_{\varepsilon,2}(T\cdot) - \psi_{\varepsilon,2}$. We verify that for every $\beta \in \mathbb{N}_0^n$ we have

$$|\partial^\beta \widehat{\psi}_1(\xi)| \lesssim \langle\xi\rangle^{s-\frac{n}{2}-|\beta|-1} \quad \text{and} \quad |\partial^\beta \widehat{\psi}_2(\xi)| \lesssim \langle\xi\rangle^{s-\frac{n}{2}-|\beta|-2} \tag{5.2.42}$$

and apply Lemma 4.5 from [39] as in the proof of Lemma 5.2.3. For this, we first observe that we have for $t > 0$

$$\frac{d}{dt}\left[t\psi_{\varepsilon,1}(t\xi)\right] = \psi_{\varepsilon,1}(t\xi) + \Lambda\psi_{\varepsilon,1}(t\xi) = g_{\varepsilon,1}(t\xi)$$

as well as

$$\frac{d}{dt}\left[t^2\psi_{\varepsilon,2}(t\xi)\right] = 2t\psi_{\varepsilon,2}(t\xi) + t\Lambda\psi_{\varepsilon,2}(t\xi) = tg_{\varepsilon,2}(t\xi).$$

Thus,

$$\widehat{\psi}_1(\xi) = \int_1^T g_{\varepsilon,1}(t\xi)\,dt \qquad \text{and} \qquad \widehat{\psi}_2(\xi) = \int_1^T tg_{\varepsilon,2}(t\xi)\,dt.$$

Since we have $\langle\xi\rangle \leq 2\langle t\xi\rangle \leq 3\langle\xi\rangle$ for $t \in [1/2, 3/2]$ the decay estimates from Lemma 5.2.3 give for every multi-index $\beta \in \mathbb{N}_0^n$

$$|\partial^\beta \widehat{\psi}_1(\xi)| \lesssim_\beta |T-1|\langle\xi\rangle^{-2-|\beta|} \qquad \text{and} \qquad |\partial^\beta \widehat{\psi}_2(\xi)| \lesssim_\beta |T-1|\langle\xi\rangle^{-3-|\beta|},$$

which finishes the proof.

□

Now we are finally able to prove our main stability results, Theorem 5.0.3 and Theorem 5.0.2.

Proof of Theorem 5.0.3. Under the assumption stated in Theorem 5.0.3 choose $\omega = \widetilde{\omega}/2$ and $0 < \overline{\varepsilon}$, depending on ω, as at the beginning of Section 5.2.2
For $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \leq \overline{\varepsilon}$ let $\delta = \delta_\varepsilon$ and $C = C_\varepsilon$ denote the constants from Theorem 5.2.9. Let $(\varphi_0, \varphi_1) \in \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ be radial, real-valued, functions satisfying

$$\|(\varphi_0, \varphi_1)\|_{\dot{H}^s \cap \dot{H}^k(\mathbb{R}^n) \times \dot{H}^{s-1} \cap \dot{H}^{k-1}(\mathbb{R}^n)} < \frac{\delta}{C^2}.$$

Then, by Theorem 5.2.9 and Proposition 5.2.10 there is a $T = T_\varepsilon \in [1 - \frac{\delta}{C}, 1 + \frac{\delta}{C}]$ and unique radial functions $(\varphi_{\varepsilon,1}, \varphi_{\varepsilon,2}) \in C^\infty([0,\infty) \times \mathbb{R}^n) \times C^\infty([0,\infty) \times \mathbb{R}^n)$ solving the initial value problem Eq. (5.2.40) - (5.2.41). Moreover,

$$\|(\varphi_{\varepsilon,1}(\tau,\cdot), \varphi_{\varepsilon,2}(\tau,\cdot))\|_{\mathcal{H}^{s,k}} \leq \delta e^{-\omega\tau},$$

for all $\tau \geq 0$. We set

$$v(t,x) := v_\varepsilon^T(t,x) + \frac{1}{T-t}\varphi_{\varepsilon,1}\left(\log\left(\frac{T}{T-t}\right), \frac{x}{T-t}\right).$$

By construction, $v \in C^\infty([0,T) \times \mathbb{R}^n)$ satisfies Eq. (5.1.1) and

$$(v(0,\cdot), \partial_t v(0,\cdot)) = (v_\varepsilon^1(0,\cdot), \partial_t v_\varepsilon^1(0,\cdot)) + (\varphi_0, \varphi_1).$$

Moreover for $r \in [s,k]$,

$$\|\varphi_{\varepsilon,1}(-\log(T-t) + \log T, \cdot)\|_{\dot{H}^r(\mathbb{R}^n)} \lesssim \|\Phi_\varepsilon(\tau)\|_{\mathcal{H}^{s,k}} \lesssim \delta(T-t)^\omega$$

and

$$\begin{aligned}\|(\partial_0 + \Lambda + 1)\varphi_{\varepsilon,1}(-\log(T-t) + \log T, \cdot)\|_{\dot{H}^{r-1}(\mathbb{R}^n)} &= \|\varphi_{\varepsilon,2}(\log T - \log(T-t), \cdot)\|_{\dot{H}^{r-1}(\mathbb{R}^n)} \\ &\lesssim \|\Phi_\varepsilon(\tau)\|_{\mathcal{H}^{s,k}} \lesssim \delta(T-t)^\omega\end{aligned}$$

by definition and Theorem 5.2.9.

□

Proof of Theorem 5.0.2. By the assumptions of Theorem 5.0.2 the initial data are of the form

$$U_0(x) = U_\varepsilon^1(0,x) + xv_0(|x|), \quad U_1(x) = \partial_t U_\varepsilon^1(0,x) + xv_1(|x|)$$

Consequently, $v_0, v_1 \in C_e^\infty[0,\infty)$ and by setting

$$\varphi_j(y) := v_j(|y|)$$

for $y \in \mathbb{R}^{d+2}$, we obtain radially symmetric, real-valued functions $(\varphi_0, \varphi_1) \in \mathcal{S}(\mathbb{R}^{d+2}) \times \mathcal{S}(\mathbb{R}^{d+2})$. By Proposition A.5 and Remark A.6 of [40] there exists a constant $C > 0$ such that

$$\|(\varphi_0, \varphi_1)\|_{\dot{H}^s \cap \dot{H}^k(\mathbb{R}^{d+2}) \times \dot{H}^{s-1} \cap \dot{H}^{k-1}(\mathbb{R}^{d+2})} \leq C \, \|(\nu_0, \nu_1)\|_{\dot{H}^s \cap \dot{H}^k(\mathbb{R}^d, \mathbb{R}^d) \times \dot{H}^{s-1} \cap \dot{H}^{k-1}(\mathbb{R}^d, \mathbb{R}^d)} \,. \tag{5.2.43}$$

If the Sobolev exponents (s, k) satisfy condition (5.0.6), then (5.0.18) holds for $n := d + 2$. Let $\omega, \bar{\varepsilon}, \delta, M > 0$ be the constants from Theorem 5.0.3. By setting $M_0 := CM$ and requiring

$$\|(\nu_0, \nu_1)\|_{\dot{H}^s \cap \dot{H}^k(\mathbb{R}^d, \mathbb{R}^d) \times \dot{H}^{s-1} \cap \dot{H}^{k-1}(\mathbb{R}^d, \mathbb{R}^d)} \leq \frac{\delta}{M_0},$$

we find that (φ_0, φ_1) satisfy the assumption of Theorem 5.0.3. Hence, there is a $T \in [1 - \delta, 1 + \delta]$ and a unique radial solution $v \in C^\infty([0, T) \times \mathbb{R}^{d+2})$ to (5.0.10). If we set $v(t, \cdot) = \tilde{v}(t, |\cdot|)$ then $\tilde{v}$ solves Eq. (5.0.9) for $t \in [0, T)$ and can be written as

$$\tilde{v}(t, |\cdot|) = \frac{1}{|\cdot|} f_\varepsilon \left(\frac{|\cdot|}{T - t} \right) + \frac{1}{T - t} \tilde{\varphi} \left(\log \left(\frac{T}{T - t} \right), \frac{|\cdot|}{T - t} \right)$$

for $\tilde{\varphi}(t, \cdot) \in C_e^\infty[0, \infty)$ satisfying

$$\begin{aligned} &\|\tilde{\varphi}(-\log(T - t) + \log T, |\cdot|)\|_{\dot{H}^r(\mathbb{R}^{d+2})} \\ &\qquad + \|(\partial_0 + \Lambda + 1)\tilde{\varphi}(-\log(T - t) + \log T, |\cdot|)\|_{\dot{H}^{r-1}(\mathbb{R}^{d+2})} \lesssim \delta (T - t)^\omega \end{aligned} \tag{5.2.44}$$

for all $r \in [s, k]$. We define for $x \in \mathbb{R}^d$, $U(t, x) := x\tilde{v}(t, |x|) \in C^\infty([0, T) \times \mathbb{R}^d, \mathbb{R}^d)$ and find that U can be written as

$$U(t, x) = U_\varepsilon^T(t, x) + \nu \left(t, \frac{x}{T - t} \right),$$

where ν is a corotational function defined via $\nu(t, x) = x \, \tilde{\varphi}(-\log(T - t) + \log T, |x|)$. The inequality from (5.2.44) now implies (5.0.8) by applying Proposition A.5 and Remark A.6 from [40] and the pointwise as well as the local uniform convergence follows immediately from Sobolev embedding.

□

A. Appendix

A.1. Additional properties of homogeneous Sobolev spaces

In this section we gather some additional properties of radial (intersection) homogeneous Sobolev spaces. For a definition of these spaces we refer to the general notations section. The first one is about the Banach algebra property of intersection homogeneous Sobolev spaces.

Lemma A.1.1. *Let $n \geq 5$ and let $0 \leq s_1 < \frac{n}{2} < s_2$. Then $\dot{H}^{s_1}_r(\mathbb{R}^n) \cap \dot{H}^{s_2}_r(\mathbb{R}^n)$ is closed under multiplication, in particular,*

$$\|fg\|_{\dot{H}^{s_1}\cap\dot{H}^{s_2}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{H}^{s_1}\cap\dot{H}^{s_2}(\mathbb{R}^n)} \|g\|_{\dot{H}^{s_1}\cap\dot{H}^{s_2}(\mathbb{R}^n)} \tag{A.1.1}$$

for all $f, g \in \dot{H}^{s_1}_r(\mathbb{R}^n) \cap \dot{H}^{s_2}_r(\mathbb{R}^n)$.

Proof. An elementary proof of this Lemma is given in [41], Appendix A.2. □

To control the L^2-norm (together with polynomial weights) for functions in homogeneous Sobolev spaces one can use Hardy's inequality:

Lemma A.1.2 (Hardy's inequality)**.** *For every $n \in \mathbb{N}$ and $0 \leq s < \frac{n}{2}$ it holds*

$$\big\| |\cdot|^{-s} f \big\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{\dot{H}^s(\mathbb{R}^n)} \tag{A.1.2}$$

for all $f \in \dot{H}^s(\mathbb{R}^n)$.

Proof. See [51], p. 243, Theorem 9.5 or [65], p.334, Lemma A.2. □

The now following version of a generalized Strauss-inequality has been proven for functions in $C^\infty_{c,r}(\mathbb{R}^n)$ in [39], Proposition B.1 (the general statement follows from a density argument together with the embedding from (5.1.11)).

Lemma A.1.3. *Let $n \geq 2, \frac{1}{2} < s < \frac{n}{2}$ and $0 \leq s_1 < \frac{n}{2} < s_2$. Then for every $\beta \in \mathbb{N}^n_0$ with $|\beta| + \frac{n}{2} < s_2$ and $|\beta| + s \geq s_1$*

$$\big\| |\cdot|^{\frac{n}{2}-s} \partial^\beta u \big\|_{L^\infty(\mathbb{R}^n)} \lesssim \|u\|_{\dot{H}^{|\beta|+s}(\mathbb{R}^n)} \tag{A.1.3}$$

for all $u \in \dot{H}^{s_1}_r(\mathbb{R}^n) \cap \dot{H}^{s_2}_r(\mathbb{R}^n)$.

The next Lemma shows that if a radial function has sufficient decay at infinity it belongs to (intersection) radial homogeneous Sobolev spaces.

Lemma A.1.4. *Let $n \geq 2$, $\gamma > 0$ and $f \in C_r^\infty(\mathbb{R}^n)$. Suppose that there exists a constant $A > 0$ such that for every multi-index $\beta \in \mathbb{N}_0^n$*

$$|\partial^\beta f(x)| \lesssim A\langle x\rangle^{-\gamma-|\beta|} \tag{A.1.4}$$

for every $x \in \mathbb{R}^n$ then f belongs to $\dot{H}_r^\sigma(\mathbb{R}^n)$ for every $\sigma \geq 0$ with $\sigma > \frac{n}{2} - \gamma$ and we have

$$\|f\|_{\dot{H}^\sigma(\mathbb{R}^n)} \lesssim A.$$

Proof. To prove this Lemma we will argue along the lines of [41], p.12, Lemma 2.2. Let $\chi \in C_{c,r}^\infty(\mathbb{R}^n)$ be a radial cutoff function with $0 \leq \chi \leq 1$ satisfying

$$\chi(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| \geq 2. \end{cases}$$

For $j \in \mathbb{N}$ define

$$\chi_j(x) := \chi(x/j), \qquad f_j(x) := f(x)\chi_j(x),$$

then $(f_j)_{j\in\mathbb{N}} \subset C_{c,r}^\infty(\mathbb{R}^n)$. Now fix an arbitrary $\tau \geq 0$ with $\tau > \frac{n}{2} - \gamma$. Choose $\ell := \lceil \tau \rceil \in \mathbb{N}_0$ and define $p \in (1, 2]$ by $\frac{1}{p} = \frac{1}{2} + \frac{\ell-\tau}{n}$. Indeed, since $0 \leq \ell - \tau < 1 \leq \frac{n}{2}$, we have $1 < p \leq 2$. The exponents satisfy $\ell - \frac{n}{p} = \tau - \frac{n}{2}$. Consequently, the homogeneous Sobolev embedding (see [65], p. 335) implies

$$\|u\|_{\dot{H}^\tau(\mathbb{R}^n)} \lesssim \|u\|_{\dot{W}^{\ell,p}(\mathbb{R}^n)} \tag{A.1.5}$$

for every $u \in C_c^\infty(\mathbb{R}^n)$. Here, the space $\dot{W}^{\ell,p}(\mathbb{R}^n)$ for $\ell \geq 0$ and $1 \leq p < \infty$ is given by the completion of $C_c^\infty(\mathbb{R}^n)$ under the norm

$$\|u\|_{\dot{W}^{\ell,p}(\mathbb{R}^n)} := \left\|\mathcal{F}^{-1}[|\cdot|^\ell \mathcal{F}u]\right\|_{L^p(\mathbb{R}^n)}$$

and it coincides with the one we introduced in the 'General Notations' section for $p = 2$ due to Plancherel. Furthermore, if $\ell \in \mathbb{N}_0$ is a non-negative integer we also have a representation via partial derivatives in the sense that

$$\|u\|_{\dot{W}^{\ell,p}(\mathbb{R}^n)} \simeq \sum_{|\beta|=\ell} \left\|\partial^\beta u\right\|_{L^p(\mathbb{R}^n)} \tag{A.1.6}$$

holds for every $u \in C_c^\infty(\mathbb{R}^n)$.
Let now $\beta \in \mathbb{N}_0^n$ satisfy $|\beta| = \ell$. By the Leibniz rule we obtain

$$\partial^\beta(f - f_j) = (1 - \chi_j)\partial^\beta f - \sum_{\nu\leq\beta,\, \nu\neq\beta} \binom{\beta}{\nu} \partial^\nu f\, \partial^{\beta-\nu}\chi_j.$$

For the first term, the decay assumptions yield

$$\left\|(1-\chi_j)\partial^\beta f\right\|_{L^p}^p \lesssim A^p \int_{|x|\geq j} \langle x\rangle^{-p(\gamma+\ell)}\,dx.$$

By the choice of τ we have $\frac{n}{p}-\ell-\gamma=\frac{n}{2}-\tau-\gamma<0$ and therefore $p(\gamma+\ell)>n$, so that the integral on the right-hand side converges towards 0 as $j\to\infty$.
Now let $\nu\leq\beta$ with $\nu\neq\beta$. Since χ_j is supported in the annulus $\{x\in\mathbb{R}^n:\ j\leq|x|\leq 2j\}$ we obtain with the decay assumptions

$$\begin{aligned}\left\|\partial^\nu f\,\partial^{\beta-\nu}\chi_j\right\|_{L^p} &\lesssim Aj^{-|\beta-\nu|}\left(\int_j^{2j} r^{-p(\gamma+|\nu|)}r^{n-1}\,dr\right)^{\frac{1}{p}}\\ &\lesssim Aj^{|\nu|-\ell}j^{\frac{n}{p}-\gamma-|\nu|}=Aj^{\frac{n}{2}-\tau-\gamma}\xrightarrow{j\to\infty}0.\end{aligned}$$

Thus, $(f_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in $\dot{W}^{\ell,p}_r(\mathbb{R}^n)$ and one can show with the same estimates from above that we have

$$\sup_{j\in\mathbb{N}}\|f_j\|_{\dot{W}^{\ell,p}(\mathbb{R}^n)}\lesssim A.$$

By the homogeneous Sobolev embedding we obtain

$$\|f_i-f_j\|_{\dot{H}^\tau(\mathbb{R}^n)}\lesssim\|f_i-f_j\|_{\dot{W}^{\ell,p}(\mathbb{R}^n)}\xrightarrow{i,j\to\infty}0,$$

so that $(f_j)_{j\in\mathbb{N}}$ is also a Cauchy sequence in $\dot{H}^\tau_r(\mathbb{R}^n)$. It remains to identify the limit with the given function f. For this we choose exponents σ_- and σ_+ with

$$\frac{n}{2}-\gamma<\sigma_-\leq\sigma\leq\sigma_+\qquad\text{and}\quad\sigma_-<\frac{n}{2}<\sigma_+.$$

Since $(f_j)_{j\in\mathbb{N}}$ is now Cauchy in $\dot{H}^{\sigma_-}_r(\mathbb{R}^n)\cap\dot{H}^{\sigma_+}_r(\mathbb{R}^n)$ it converges against a g in that space with

$$\|g\|_{\dot{H}^{\sigma_-}(\mathbb{R}^n)}+\|g\|_{\dot{H}^{\sigma_+}(\mathbb{R}^n)}\lesssim A.$$

Since f_j converges pointwise towards f and $\dot{H}^{\sigma_-}_r(\mathbb{R}^n)\cap\dot{H}^{\sigma_+}_r(\mathbb{R}^n)$ embeds into $L^\infty(\mathbb{R}^n)$, we have $f=g\in\dot{H}^{\sigma_-}_r(\mathbb{R}^n)\cap\dot{H}^{\sigma_+}_r(\mathbb{R}^n)$ and from the homogeneous Sobolev interpolation

$$\|f\|_{\dot{H}^\sigma(\mathbb{R}^n)}\lesssim\|f\|^\theta_{\dot{H}^{\sigma_-}(\mathbb{R}^n)}\|f\|^{1-\theta}_{\dot{H}^{\sigma_+}(\mathbb{R}^n)}$$

for a suitable $\theta\in[0,1]$, we obtain $\|f\|_{\dot{H}^\sigma(\mathbb{R}^n)}\lesssim A$, which finishes the proof.

□

A.2. Schauder-type estimates for parameter depending nonlinear operators

Here we will show an adaptation of a Schauder-type estimate whose original form can be found in [39], p.26, Proposition A.1.

Proposition A.2.1. *Let $n \geq 5$. For $|\varepsilon| \leq 1$ let $F_\varepsilon \in C^\infty(\mathbb{R})$ be a family of even functions such that for all $\ell \in \mathbb{N}_0$ there exists a constant $C_\ell \geq 0$ such that*

$$|F_\varepsilon^{(\ell)}(x) - F_\kappa^{(\ell)}(y)| \leq C_\ell \, (|\varepsilon - \kappa| + |x - y|) \tag{A.2.1}$$

holds for all $x, y \in \mathbb{R}$ and all $|\varepsilon|, |\kappa| \leq 1$. Then, for every $s \in \mathbb{R}$ and $k \in \mathbb{N}$ that satisfy

$$\frac{n}{2} - 1 < s \leq \frac{n}{2} - 1 + \frac{1}{2(n-1)}, \quad k > n \tag{A.2.2}$$

we have

$$\|u_1 u_2 u_3 \, (F_\varepsilon(|\cdot|v) - F_\kappa(|\cdot|v))\|_{\dot{H}^{s-1} \cap \dot{H}^k(\mathbb{R}^n)} \lesssim |\varepsilon - \kappa| \prod_{i=1}^{3} \|u_i\|_{\dot{H}^s \cap \dot{H}^k(\mathbb{R}^n)} \sum_{j=0}^{k} \|v\|^{2j}_{\dot{H}^s \cap \dot{H}^k(\mathbb{R}^n)} \tag{A.2.3}$$

as well as

$$\begin{aligned} &\|u_1 u_2 u_3 \, (F_\varepsilon(|\cdot|v_1) - F_\varepsilon(|\cdot|v_2))\|_{\dot{H}^{s-1} \cap \dot{H}^k(\mathbb{R}^n)} \\ &\lesssim \prod_{i=1}^{3} \|u_i\|_{\dot{H}^s \cap \dot{H}^k(\mathbb{R}^n)} \, P(\|v_1\|_{\dot{H}^s \cap \dot{H}^k(\mathbb{R}^n)}, \|v_2\|_{\dot{H}^s \cap \dot{H}^k(\mathbb{R}^n)}) \, \|v_1 - v_2\|_{\dot{H}^s \cap \dot{H}^k(\mathbb{R}^n)} \end{aligned} \tag{A.2.4}$$

for all $|\varepsilon|, |\kappa| \leq 1$ and all $u_1, u_2, u_3, v, v_1, v_2 \in \dot{H}^s_r(\mathbb{R}^n) \cap \dot{H}^k_r(\mathbb{R}^n)$ where v, v_1 and v_2 are real-valued and P is a polynomial of degree $\leq 2k+1$.

Proof. We will start with the case where u_1, u_2, u_3, v, v_1 and v_2 belong to $C^\infty_{c,r}(\mathbb{R}^n)$. By repeated application of the fundamental theorem of calculus, see for example [53], there exists for every ε a function $G_\varepsilon \in C^\infty[0, \infty)$ with $G_\varepsilon(x^2) = F_\varepsilon(x)$. By the assumptions on F_ε we obtain for every $\ell \in \mathbb{N}_0$ the existence of a constant $\widetilde{C}_\ell \geq 0$ such that

$$|G_\varepsilon^{(\ell)}(x^2) - G_\kappa^{(\ell)}(y^2)| \leq \widetilde{C}_\ell \, (|\varepsilon - \kappa| + |x - y|) \, .$$

Now we choose s, k according to (A.2.2) and note that by Lemma 5.1.11 it is enough to bound the $\dot{H}^{\lfloor s-1 \rfloor} \cap \dot{H}^k$-norm of the respective left-hand side of (A.2.3) and (A.2.4). Therefore, we can reduce the analysis to estimating

$$\partial^\alpha \left(u_1 u_2 u_3 \left(G_\varepsilon(|\cdot|^2 v^2) - G_\kappa(|\cdot|^2 v^2) \right) \right) \tag{A.2.5}$$

and

$$\partial^\alpha \left(u_1 u_2 u_3 \left(G_\varepsilon(|\cdot|^2 v_1^2) - G_\varepsilon(|\cdot|^2 v_2^2) \right) \right) \tag{A.2.6}$$

in $L^2(\mathbb{R}^n)$ for $|\alpha| \in \{\lfloor s-1 \rfloor, k\}$. For the first term we find that after applying the Leibniz rule and (A.2.1) it suffices to prove a suitable bound for

$$I(x) := x^\gamma \, \partial^{\alpha_1} u_1 \, \partial^{\alpha_2} u_2 \, \partial^{\alpha_3} u_3 \prod_{j=1}^{2\ell} \partial^{\beta_j} v, \tag{A.2.7}$$

where

$$\sum_{i=1}^{3} |\alpha_i| + \sum_{j=1}^{2\ell} |\beta_j| + 2\ell - |\gamma| = |\alpha| \tag{A.2.8}$$

with the condition that $\ell \leq |\alpha|$ and $|\gamma| \leq 2\ell$. This corresponds exactly to the situation considered in [39], Proposition A.1 and arguing along these lines we infer that

$$\|I\|_{L^2(\mathbb{R}^n)} \lesssim \prod_{i=1}^{3} \|u_i\|_{\dot{H}^s \cap \dot{H}^k(\mathbb{R}^n)} \prod_{j=1}^{2\ell} \|v\|_{\dot{H}^s \cap \dot{H}^k(\mathbb{R}^n)} \tag{A.2.9}$$

for $|\alpha| \in \{\lfloor s-1 \rfloor, k\}$. To handle the expression in (A.2.6) we prove the bound

$$\|J\|_{L^2(\mathbb{R}^n)} \lesssim \prod_{i=1}^{3} \|u_i\|_{\dot{H}^s \cap \dot{H}^k(\mathbb{R}^n)} \prod_{j=1}^{2\ell+1} \|v_j\|_{\dot{H}^s \cap \dot{H}^k(\mathbb{R}^n)} \tag{A.2.10}$$

for

$$J(x) := x^\gamma \, \partial^{\alpha_1} u_1 \, \partial^{\alpha_2} u_2 \, \partial^{\alpha_3} u_3 \prod_{j=1}^{2\ell+1} \partial^{\beta_j} v_j \tag{A.2.11}$$

with functions $u_1, u_2, u_3, v_1, \ldots, v_{2\ell+1} \in C^\infty_{c,r}(\mathbb{R}^n)$, and indices $\ell \leq |\alpha|$, $\alpha_1, \alpha_2, \alpha_3, \beta_1, \ldots, \beta_{2\ell+1}, \gamma \in \mathbb{N}_0^n$ with $|\gamma| \leq 2\ell + 1$ satisfying

$$|\alpha_1| + |\alpha_2| + |\alpha_3| + \sum_{j=1}^{2\ell+1} |\beta_j| + 2\ell + 1 - |\gamma| = |\alpha|.$$

We will start with the case $|\alpha| = \lfloor s \rfloor - 1$. Here we define for $i = 2, 3$ and $j = 1, \ldots, 2\ell + 1$

$$a_i := |\alpha_i| + \frac{2\ell + 1 + s - |\alpha|}{2\ell + 3} \quad \text{and} \quad b_j := |\beta_j| + \frac{2\ell + 1 + s - |\alpha|}{2\ell + 3}$$

and get

$$\|J\|_{L^2(\mathbb{R}^n)} \lesssim \big\| |\cdot|^{|\alpha_1| - s} \partial^{\alpha_1} u_1 \big\|_{L^2(\mathbb{R}^n)} \prod_{i=2}^{3} \| |\cdot|^{a_i} \partial^{\alpha_i} u_i \|_{L^\infty(\mathbb{R}^n)} \prod_{j=1}^{2\ell+1} \big\| |\cdot|^{b_j} \partial^{\beta_j} v_j \big\|_{L^\infty(\mathbb{R}^n)}$$

due to $|\gamma| = |\alpha_1| + a_2 + a_3 + \sum_{j=1}^{2\ell+1} b_j - s$. Since $0 \leq s - |\alpha_1| < \frac{n}{2}$ we can use Hardy's inequality from Lemma A.1.2 for the first term and for the rest of the terms we use the generalized Strauss inequality (A.1.3) to obtain (A.2.10). We are allowed to use these inequalities since we have $0 < a_i, b_j < \frac{n-1}{2}$ as well as

$$s \leq \frac{n}{2} - a_i + |\alpha_i| < \frac{n}{2} \quad \text{and} \quad s \leq \frac{n}{2} - b_j + |\beta_j| < \frac{n}{2}. \tag{A.2.12}$$

Since we are in the case where $|\alpha|$ is strictly smaller than s we immediately obtain $0 < a_i, b_j$. For the upper bound on a_i and b_j one has

$$a_i = |\alpha_i| + 1 + \frac{s - \lfloor \frac{n}{2} \rfloor}{2\ell + 3} \leq \frac{n}{2} - 1 + \frac{s - \lfloor \frac{n}{2} \rfloor}{2\ell + 3} \leq \frac{n}{2} - 1 < \frac{n-1}{2}$$

and the same holds true for b_j. Since a_i and b_j are strictly larger than $|\alpha_i|$ and $|\beta_j|$ respectively the upper bound from (A.2.12) is immediate. The most restrictive inequality (for s) is $s \leq \frac{n}{2} - a_i + |\alpha_i|$. This inequality is equivalent to $s + 1 + \frac{s - \lfloor \frac{n}{2} \rfloor}{2\ell+3} \leq \frac{n}{2}$ which can be reformulated as $2(\ell+2)s + 2\ell + 3 \leq (\ell + \frac{3}{2})n + \lfloor \frac{n}{2} \rfloor$. By making a case distinction between odd and even n and using the fact that $0 \leq \ell \leq \lfloor s \rfloor - 1$ this inequality holds for $s \leq \frac{n}{2} - 1 + \frac{1}{2n-2}$.
The case $|\alpha| = k$ will be divided into two sub-cases. First, we assume that the highest derivative in (A.2.11) is of order at least $\frac{n}{2} - 1$. Without loss of generality we assume that this derivative is α_1. We now split the L^2-norm of J into the unit ball and its complement. For $|x| \leq 1$ we get for $a_1 := \min\{1, k - |\alpha_1|\}$

$$|J(x)| \lesssim |x|^{-a_1} \, |\partial^{\alpha_1} u_1| \, |\partial^{\alpha_2} u_2| \, |\partial^{\alpha_3} u_3| \, \prod_{j=1}^{2\ell+1} |\partial^{\beta_j} v_j|.$$

For the first term we use Hardy's inequality and for the rest of the terms we want to use Lemma 5.1.12 to estimate

$$\|\partial^{\alpha_i} u_i\|_{L^\infty(\mathbb{R}^n)} \lesssim \|u_i\|_{\dot{H}^s \cap \dot{H}^k(\mathbb{R}^n)} \quad \text{and} \quad \left\|\partial^{\beta_j} v_j\right\|_{L^\infty(\mathbb{R}^n)} \lesssim \|v_j\|_{\dot{H}^s \cap \dot{H}^k(\mathbb{R}^n)}$$

for every $i = 2, 3$ and $j = 1, \dots, 2\ell + 1$. To see this we consider first the case where $|\alpha_1|$ is strictly smaller than $k - \frac{n}{2}$. Since α_1 is by assumption the highest occurring derivative we also have $|\alpha_i|, |\beta_j| < k - \frac{n}{2}$ so that we can use the embedding. Is $|\alpha_1|$ greater or equal than $k - \frac{n}{2}$ we use the fact that there can only fall at most $k - |\alpha_1|$ derivatives onto u_i and v_j and also that we have chosen $k > n$ so that we obtain

$$|\alpha_i|, |\beta_j| \leq k - |\alpha_1| \leq \frac{n}{2} < k - \frac{n}{2}.$$

Now for the complement of the unit ball we first of all define

$$a_i := \frac{1}{2} + \min\{1, |\alpha_i|\}, \quad b_j := \frac{1}{2} + \min\{1, |\beta_j|\}$$

for every $i = 2, 3$ and $j = 1, \dots, 2\ell + 1$. Since we now have $|\gamma| \leq a_2 + a_3 + \sum_{j=1}^{2\ell+1} b_j - a_1$ due to $|\gamma| \leq \ell + \sum_{j=1}^{2\ell+1} \min\{1, |\beta_j|\}$ we can estimate

$$|J(x)| \lesssim |x|^{-a_1} \, |\partial^{\alpha_1} u_1| \, |x|^{a_2} |\partial^{\alpha_2} u_2| \, |x|^{a_3} |\partial^{\alpha_3} u_3| \, \prod_{j=1}^{2\ell+1} |x|^{b_j} |\partial^{\beta_j} v_j|$$

for all $|x| \geq 1$. The L^2-norm of the first term gets again estimated by Hardy's inequality and then by the $\dot{H}^s \cap \dot{H}^k$-norm of u_1 since we have $s \leq a_1 + |\alpha_1| \leq k$. We estimate the other terms in the $L^\infty-$norm by (A.1.3). We can apply this Lemma due to the fact that we again have $0 < a_i, b_j < \frac{n-1}{2}$. The $\dot{H}^s \cap \dot{H}^k$-estimate then follows from $s \leq \frac{n}{2} - a_i + |\alpha_i| \leq k$ where the upper bound can be seen by making a case distinction for $|\alpha_1|$ strictly greater

than $k - \frac{n-3}{2}$ or smaller than that term. Over all we have finished the proof in the case where one derivative is at least of order $\frac{n}{2} - 1$.
We now assume that all of the derivatives in (A.2.11) are of order strictly smaller than $\frac{n}{2} - 1$, which implies $|\alpha_i|, |\beta_j| \leq \frac{n-3}{2}$ for all $i = 1, 2, 3$ and $j = 1, \dots, 2\ell + 1$. On the unit ball we can immediately estimate

$$|J(x)| \lesssim |x|^{|\alpha_1| - \frac{n-1}{2}} |\partial^{\alpha_1} u_1| \, |\partial^{\alpha_2} u_2| \, |\partial^{\alpha_3} u_3| \prod_{j=1}^{2\ell+1} |\partial^{\beta_j} v_j|$$

and obtain (A.2.10) with Hardy's inequality and the embedding from Lemma 5.1.12.
For the complement of the unit ball we define $\tilde{a}_i := |\alpha_i| + \frac{3}{4}, \tilde{b}_j := |\beta_j| + \frac{3}{4}$ and estimate

$$|J(x)| \lesssim |x|^{|\alpha_1| - \frac{n-1}{2}} |\partial^{\alpha_1} u_1| \, |x|^{\tilde{a}_2} |\partial^{\alpha_2} u_2| \, |x|^{\tilde{a}_3} |\partial^{\alpha_3} u_3| \prod_{j=1}^{2\ell+1} |x|^{\tilde{b}_j} |\partial^{\beta_j} v_j|$$

for all $|x| \geq 1$ since we have $|\gamma| \leq |\alpha_1| - \frac{n-1}{2} + \tilde{a}_2 + \tilde{a}_3 + \sum_{j=1}^{2\ell+1} \tilde{b}_j$. The claim now follows by again applying Hardy's inequality and (A.1.3) due to the fact that we again have $0 < \tilde{a}_i, \tilde{b}_j < \frac{n-1}{2}$ as well as $s \leq \frac{n}{2} - \frac{3}{4} \leq k$. We have therefore shown both inequalities in the case where all the functions belong to $C^\infty_{c,r}(\mathbb{R}^n)$.
The general case now follows via a density argument and the $L^\infty-$embedding of the Sobolev spaces. We will only give the details for the first inequality, the second one can be handled in a similar manner. Take $u_1, u_2, u_3, v \in \dot{H}^s_r(\mathbb{R}^n) \cap \dot{H}^k_r(\mathbb{R}^n)$. Then there exist sequences $(u_{1,j})_{j\in\mathbb{N}}, (u_{2,j})_{j\in\mathbb{N}}, (u_{3,j})_{j\in\mathbb{N}}$ and $(v_j)_{j\in\mathbb{N}}$ in $C^\infty_{c,r}(\mathbb{R}^n)$ which converge to u_1, u_2, u_3 and v in the $\dot{H}^s(\mathbb{R}^n) \cap \dot{H}^k(\mathbb{R}^n)-$norm, respectively. Eqns. (A.2.3) and (A.2.4) imply that

$$\left(u_{1,j} u_{2,j} u_{3,j} \left(F_\varepsilon(|\cdot|v_j) - F_\kappa(|\cdot|v_j)\right)\right)_{j\in\mathbb{N}}$$

forms a Cauchy sequence in $\dot{H}^{s-1}(\mathbb{R}^n) \cap \dot{H}^k(\mathbb{R}^n)$. Due to the $L^\infty-$embedding one can easily show that this sequence converges pointwise to $u_1 u_2 u_3 (F_\varepsilon(|\cdot|v) - F_\kappa(|\cdot|v))$ and therefore also in the $\dot{H}^{s-1} \cap \dot{H}^k(\mathbb{R}^n)-$norm. With this observation (A.2.3) follows for arbitrary $u_1, u_2, u_3, v \in \dot{H}^s_r(\mathbb{R}^n) \cap \dot{H}^k_r(\mathbb{R}^n)$.

□

AI usage declaration

During the preparation of this thesis, the author used ChatGPT-5.6 Sol as a supporting tool. It was consulted for language editing and the reformulation of some paragraphs, as well as for some proof strategies in Propositions 4.1.6, 4.2.7, 5.1.9, 5.2.5 and 5.2.10 and Lemmas 5.2.3 and A.1.4. All suggestions were independently verified and revised by the author before being incorporated into the manuscript. The author assumes full responsibility for the correctness and integrity of the final work.

Bibliography

[1] Johannes Angerer, Sarah Kistner, and Birgit Schörkhuber. *Existence of a stable shrinker for the corotational harmonic map heat flow in higher space dimensions*. 2026. arXiv: 2607.27072 [math.AP]. URL: https://arxiv.org/abs/2607.27072.

[2] John K. Beem, Paul E. Ehrlich, and Kevin L. Easley. *Global Lorentzian geometry*. Second. Vol. 202. Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker, Inc., New York, 1996, pp. xiv+635. ISBN: 0-8247-9324-2.

[3] Arthur L. Besse. *Manifolds all of whose geodesics are closed*. Vol. 93. Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas]. With appendices by D. B. A. Epstein, J.-P. Bourguignon, L. Bérard-Bergery, M. Berger and J. L. Kazdan. Springer-Verlag, Berlin-New York, 1978, pp. ix+262. ISBN: 3-540-08158-5.

[4] Paweł Biernat. "Non-self-similar blow-up in the heat flow for harmonic maps in higher dimensions". In: *Nonlinearity* 28.1 (2015), pp. 167–185. ISSN: 0951-7715,1361-6544. DOI: 10.1088/0951-7715/28/1/167. URL: https://doi.org/10.1088/0951-7715/28/1/167.

[5] Paweł Biernat, Piotr Bizoń, and Maciej Maliborski. "Threshold for blowup for equivariant wave maps in higher dimensions". In: *Nonlinearity* 30.4 (2017), pp. 1513–1522. ISSN: 0951-7715,1361-6544. DOI: 10.1088/1361-6544/aa61ab. URL: https://doi.org/10.1088/1361-6544/aa61ab.

[6] Paweł Biernat and Roland Donninger. "Construction of a spectrally stable self-similar blowup solution to the supercritical corotational harmonic map heat flow". In: *Nonlinearity* 31.8 (2018), pp. 3543–3566. ISSN: 0951-7715,1361-6544. DOI: 10.1088/1361-6544/aabe4c. URL: https://doi.org/10.1088/1361-6544/aabe4c.

[7] Paweł Biernat, Roland Donninger, and Birgit Schörkhuber. "Hyperboloidal similarity coordinates and a globally stable blowup profile for supercritical wave maps". In: *Int. Math. Res. Not. IMRN* 21 (2021), pp. 16530–16591. ISSN: 1073-7928,1687-0247. DOI: 10.1093/imrn/rnz286. URL: https://doi.org/10.1093/imrn/rnz286.

[8] Paweł Biernat, Roland Donninger, and Birgit Schörkhuber. "Stable self-similar blowup in the supercritical heat flow of harmonic maps". In: *Calc. Var. Partial Differential Equations* 56.6 (2017), Paper No. 171, 31. ISSN: 0944-2669,1432-0835. DOI: 10.1007/s00526-017-1256-z. URL: https://doi.org/10.1007/s00526-017-1256-z.

[9] Piotr Bizoń. "Equivariant self-similar wave maps from Minkowski spacetime into 3-sphere". In: *Comm. Math. Phys.* 215.1 (2000), pp. 45–56. ISSN: 0010-3616,1432-0916. DOI: 10.1007/s002200000291. URL: https://doi.org/10.1007/s002200000291.

[10] Piotr Bizoń and Paweł Biernat. "Generic self-similar blowup for equivariant wave maps and Yang-Mills fields in higher dimensions". In: *Comm. Math. Phys.* 338.3 (2015), pp. 1443–1450. ISSN: 0010-3616,1432-0916. DOI: 10.1007/s00220-015-2404-y. URL: https://doi.org/10.1007/s00220-015-2404-y.

[11] Piotr Bizoń, Tadeusz Chmaj, and Zbisław Tabor. "Dispersion and collapse of wave maps". In: *Nonlinearity* 13.4 (2000), pp. 1411–1423. ISSN: 0951-7715,1361-6544. DOI: 10.1088/0951-7715/13/4/323. URL: https://doi.org/10.1088/0951-7715/13/4/323.

[12] Piotr Bizoń and Arthur Wasserman. "Nonexistence of shrinkers for the harmonic map flow in higher dimensions". In: *Int. Math. Res. Not. IMRN* 17 (2015), pp. 7757–7762. ISSN: 1073-7928,1687-0247. DOI: 10.1093/imrn/rnu176. URL: https://doi.org/10.1093/imrn/rnu176.

[13] Thierry Cazenave, Jalal Shatah, and A. Shadi Tahvildar-Zadeh. "Harmonic maps of the hyperbolic space and development of singularities in wave maps and Yang-Mills fields". In: *Ann. Inst. H. Poincaré Phys. Théor.* 68.3 (1998), pp. 315–349. ISSN: 0246-0211. URL: http://www.numdam.org/item?id=AIHPA_1998__68_3_315_0.

[14] Kung-Ching Chang, Wei Yue Ding, and Rugang Ye. "Finite-time blow-up of the heat flow of harmonic maps from surfaces". In: *J. Differential Geom.* 36.2 (1992), pp. 507–515. ISSN: 0022-040X,1945-743X. URL: http://projecteuclid.org/euclid.jdg/1214448751.

[15] Athanasios Chatzikaleas, Roland Donninger, and Irfan Glogić. "On blowup of co-rotational wave maps in odd space dimensions". In: *J. Differential Equations* 263.8 (2017), pp. 5090–5119. ISSN: 0022-0396,1090-2732. DOI: 10.1016/j.jde.2017.06.011. URL: https://doi.org/10.1016/j.jde.2017.06.011.

[16] Bang-Yen Chen. *Pseudo-Riemannian geometry, δ-invariants and applications*. With a foreword by Leopold Verstraelen. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011, pp. xxxii+477. ISBN: 978-981-4329-63-7; 981-4329-63-0. DOI: 10.1142/9789814329644. URL: https://doi.org/10.1142/9789814329644.

[17] Yun Mei Chen and Michael Struwe. "Existence and partial regularity results for the heat flow for harmonic maps". In: *Math. Z.* 201.1 (1989), pp. 83–103. ISSN: 0025-5874,1432-1823. DOI: 10.1007/BF01161997. URL: https://doi.org/10.1007/BF01161997.

[18] Jean-Michel Coron and Jean-Michel Ghidaglia. "Explosion en temps fini pour le flot des applications harmoniques". In: *C. R. Acad. Sci. Paris Sér. I Math.* 308.12 (1989), pp. 339–344. ISSN: 0249-6291.

[19] Ovidiu Costin, Roland Donninger, and Irfan Glogić. "Mode stability of self-similar wave maps in higher dimensions". In: *Comm. Math. Phys.* 351.3 (2017), pp. 959–972. ISSN: 0010-3616,1432-0916. DOI: 10.1007/s00220-016-2776-7. URL: https://doi.org/10.1007/s00220-016-2776-7.

[20] Ovidiu Costin, Roland Donninger, and Xiaoyue Xia. "A proof for the mode stability of a self-similar wave map". In: *Nonlinearity* 29.8 (2016), pp. 2451–2473. ISSN: 0951-7715,1361-6544. DOI: 10.1088/0951-7715/29/8/2451. URL: https://doi.org/10.1088/0951-7715/29/8/2451.

[21] Elek Csobo, Irfan Glogić, and Birgit Schörkhuber. "On blowup for the supercritical quadratic wave equation". In: *Anal. PDE* 17.2 (2024), pp. 617–680. ISSN: 2157-5045,1948-206X. DOI: 10.2140/apde.2024.17.617. URL: https://doi.org/10.2140/apde.2024.17.617.

[22] Roland Donninger. "On stable self-similar blowup for equivariant wave maps". In: *Comm. Pure Appl. Math.* 64.8 (2011), pp. 1095–1147. ISSN: 0010-3640,1097-0312. DOI: 10.1002/cpa.20366. URL: https://doi.org/10.1002/cpa.20366.

[23] Roland Donninger. "Spectral theory and self-similar blowup in wave equations". In: *Bull. Amer. Math. Soc. (N.S.)* 61.4 (2024), pp. 659–685. ISSN: 0273-0979,1088-9485. DOI: 10.1090/bull/1845. URL: https://doi.org/10.1090/bull/1845.

[24] Roland Donninger and Irfan Glogić. "On the existence and stability of blowup for wave maps into a negatively curved target". In: *Anal. PDE* 12.2 (2019), pp. 389–416. ISSN: 2157-5045,1948-206X. DOI: 10.2140/apde.2019.12.389. URL: https://doi.org/10.2140/apde.2019.12.389.

[25] Roland Donninger and Frederick Moscatelli. *Blowup stability of wave maps without symmetry*. 2026. arXiv: 2601.19516 [math.AP]. URL: https://arxiv.org/abs/2601.19516.

[26] Roland Donninger and Frederick Moscatelli. "Mode stability of self-similar wave maps without symmetry in higher dimensions". In: *Trans. Amer. Math. Soc. Ser. B* 13 (2026), pp. 441–476. ISSN: 2330-0000. DOI: 10.1090/btran/258. URL: https://doi.org/10.1090/btran/258.

[27] Roland Donninger and Matthias Ostermann. "On stable self-similar blowup for corotational wave maps and equivariant Yang-Mills connections". In: *arXiv preprint arXiv:2409.14733* (2024).

[28] Roland Donninger, Birgit Schörkhuber, and Peter C. Aichelburg. "On stable self-similar blow up for equivariant wave maps: the linearized problem". In: *Ann. Henri Poincaré* 13.1 (2012), pp. 103–144. ISSN: 1424-0637,1424-0661. DOI: 10.1007/s00023-011-0125-0. URL: https://doi.org/10.1007/s00023-011-0125-0.

[29] Roland Donninger, Birgit Schörkhuber, and Alexander Wittenstein. "Stable blowup for supercritical wave maps into perturbed spheres". In: *J. Funct. Anal.* 291.9 (2026), Paper No. 111601. ISSN: 0022-1236,1096-0783. DOI: 10.1016/j.jfa.2026.111601. URL: https://doi.org/10.1016/j.jfa.2026.111601.

[30] Roland Donninger and David Wallauch. "Optimal blowup stability for supercritical wave maps". In: *Adv. Math.* 433 (2023), Paper No. 109291, 86. ISSN: 0001-8708,1090-2082. DOI: 10.1016/j.aim.2023.109291. URL: https://doi.org/10.1016/j.aim.2023.109291.

[31] Roland Donninger and David Wallauch. "Optimal blowup stability for three-dimensional wave maps". In: *Anal. PDE* 18.4 (2025), pp. 895–962. ISSN: 2157-5045,1948-206X. DOI: 10.2140/apde.2025.18.895. URL: https://doi.org/10.2140/apde.2025.18.895.

[32] James Eells Jr. and J. H. Sampson. "Harmonic mappings of Riemannian manifolds". In: *Amer. J. Math.* 86 (1964), pp. 109–160. ISSN: 0002-9327,1080-6377. DOI: 10.2307/2373037. URL: https://doi.org/10.2307/2373037.

[33] James Eells and Luc Lemaire. *Selected topics in harmonic maps*. Vol. 50. CBMS Regional Conference Series in Mathematics. Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1983, pp. v+85. ISBN: 0-8218-0700-5. DOI: 10.1090/cbms/050. URL: https://doi.org/10.1090/cbms/050.

[34] Klaus-Jochen Engel and Rainer Nagel. *One-parameter semigroups for linear evolution equations*. Vol. 194. Graduate Texts in Mathematics. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt. Springer-Verlag, New York, 2000, pp. xxii+586. ISBN: 0-387-98463-1.

[35] Huijun Fan. "Existence of the self-similar solutions in the heat flow of harmonic maps". In: *Sci. China Ser. A* 42.2 (1999), pp. 113–132. ISSN: 1006-9283,1862-2763. DOI: 10.1007/BF02876563. URL: https://doi.org/10.1007/BF02876563.

[36] Andreas Gastel. "Singularities of first kind in the harmonic map and Yang-Mills heat flows". In: *Math. Z.* 242.1 (2002), pp. 47–62. ISSN: 0025-5874,1432-1823. DOI: 10.1007/s002090100306. URL: https://doi.org/10.1007/s002090100306.

[37] Tej-Eddine Ghoul, Slim Ibrahim, and Van Tien Nguyen. "Construction of type II blowup solutions for the 1-corotational energy supercritical wave maps". In: *J. Differential Equations* 265.7 (2018), pp. 2968–3047. ISSN: 0022-0396,1090-2732. DOI: 10.1016/j.jde.2018.04.058. URL: https://doi.org/10.1016/j.jde.2018.04.058.

[38] J. Ginibre and G. Velo. "The Cauchy problem for the O(N), $\mathbf{C}$P($N-1$), and $G_{\mathbf{C}}(N, p)$ models". In: *Ann. Physics* 142.2 (1982), pp. 393–415. ISSN: 0003-4916,1096-035X. DOI: 10.1016/0003-4916(82)90077-X. URL: https://doi.org/10.1016/0003-4916(82)90077-X.

[39] Irfan Glogić. "Globally stable blowup profile for supercritical wave maps in all dimensions". In: *Calc. Var. Partial Differential Equations* 64.2 (2025), Paper No. 46, 34. ISSN: 0944-2669,1432-0835. DOI: 10.1007/s00526-024-02901-7. URL: https://doi.org/10.1007/s00526-024-02901-7.

[40] Irfan Glogić. "Stable blowup for the supercritical hyperbolic Yang-Mills equations". In: *Adv. Math.* 408.part B (2022), Paper No. 108633, 52. ISSN: 0001-8708. DOI: 10.1016/j.aim.2022.108633. URL: https://doi.org/10.1016/j.aim.2022.108633.

[41] Irfan Glogić, Sarah Kistner, and Birgit Schörkhuber. "Existence and stability of shrinkers for the harmonic map heat flow in higher dimensions". In: *Calc. Var. Partial Differential Equations* 63.4 (2024), Paper No. 96, 33. ISSN: 0944-2669,1432-0835. DOI: 10.1007/s00526-024-02707-7. URL: https://doi.org/10.1007/s00526-024-02707-7.

[42] Chao Hao Gu. "On the Cauchy problem for harmonic maps defined on two-dimensional Minkowski space". In: *Comm. Pure Appl. Math.* 33.6 (1980), pp. 727–737. ISSN: 0010-3640,1097-0312. DOI: 10.1002/cpa.3160330604. URL: https://doi.org/10.1002/cpa.3160330604.

[43] Frédéric Hélein. *Harmonic maps, conservation laws and moving frames*. Second. Vol. 150. Cambridge Tracts in Mathematics. Translated from the 1996 French original, With a foreword by James Eells. Cambridge University Press, Cambridge, 2002, pp. xxvi+264. ISBN: 0-521-81160-0. DOI: 10.1017/CBO9780511543036. URL: https://doi.org/10.1017/CBO9780511543036.

[44] Tosio Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Reprint of the 1980 edition. Springer-Verlag, Berlin, 1995, pp. xxii+619. ISBN: 3-540-58661-X.

[45] Otared Kavian and Fred B. Weissler. "Finite energy self-similar solutions of a nonlinear wave equation". In: *Comm. Partial Differential Equations* 15.10 (1990), pp. 1381–1420. ISSN: 0360-5302,1532-4133. DOI: 10.1080/03605309908820730. URL: https://doi.org/10.1080/03605309908820730.

[46] J. Krieger, W. Schlag, and D. Tataru. "Renormalization and blow up for charge one equivariant critical wave maps". In: *Invent. Math.* 171.3 (2008), pp. 543–615. ISSN: 0020-9910,1432-1297. DOI: 10.1007/s00222-007-0089-3. URL: https://doi.org/10.1007/s00222-007-0089-3.

[47] John M. Lee. *Introduction to Riemannian manifolds*. Second. Vol. 176. Graduate Texts in Mathematics. Springer, Cham, 2018, pp. xiii+437. ISBN: 978-3-319-91754-2; 978-3-319-91755-9.

[48] John M. Lee. *Introduction to smooth manifolds*. Second. Vol. 218. Graduate Texts in Mathematics. Springer, New York, 2013, pp. xvi+708. ISBN: 978-1-4419-9981-8.

[49] Fanghua Lin and Changyou Wang. *The analysis of harmonic maps and their heat flows*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008, pp. xii+267. ISBN: 978-981-277-952-6; 981-277-952-3. DOI: 10.1142/9789812779533. URL: https://doi.org/10.1142/9789812779533.

[50] Charles W. Misner. "Harmonic maps as models for physical theories". In: *Phys. Rev. D (3)* 18.12 (1978), pp. 4510–4524. ISSN: 0556-2821. DOI: 10.1103/PhysRevD.18.4510. URL: https://doi.org/10.1103/PhysRevD.18.4510.

[51] Camil Muscalu and Wilhelm Schlag. *Classical and multilinear harmonic analysis. Vol. I*. Vol. 137. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2013, pp. xviii+370. ISBN: 978-0-521-88245-3.

[52] Barrett O'Neill. *Semi-Riemannian geometry*. Vol. 103. Pure and Applied Mathematics. With applications to relativity. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983, pp. xiii+468. ISBN: 0-12-526740-1.

[53] Matthias Ostermann. "A characterization of the subspace of radially symmetric functions in Sobolev spaces". In: *Commun. Contemp. Math.* 27.3 (2025), Paper No. 2450018, 15. ISSN: 0219-1997,1793-6683. DOI: 10.1142/S0219199724500184. URL: https://doi.org/10.1142/S0219199724500184.

[54] Matthias Ostermann. "Stable blowup for focusing semilinear wave equations in all dimensions". In: *Trans. Amer. Math. Soc.* 377.7 (2024), pp. 4727–4778. ISSN: 0002-9947,1088-6850. DOI: 10.1090/tran/9069. URL: https://doi.org/10.1090/tran/9069.

[55] Igor Rodnianski and Jacob Sterbenz. "On the formation of singularities in the critical O(3) σ-model". In: *Ann. of Math. (2)* 172.1 (2010), pp. 187–242. ISSN: 0003-486X,1939-8980. DOI: 10.4007/annals.2010.172.187. URL: https://doi.org/10.4007/annals.2010.172.187.

[56] Jalal Shatah. "Weak solutions and development of singularities of the SU(2) σ-model". In: *Comm. Pure Appl. Math.* 41.4 (1988), pp. 459–469. ISSN: 0010-3640,1097-0312. DOI: 10.1002/cpa.3160410405. URL: https://doi.org/10.1002/cpa.3160410405.

[57] Jalal Shatah and Michael Struwe. *Geometric wave equations*. Vol. 2. Courant Lecture Notes in Mathematics. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1998, pp. viii+153. ISBN: 0-9658703-1-6; 0-8218-2749-9.

[58] Jalal Shatah and A. Shadi Tahvildar-Zadeh. "On the Cauchy problem for equivariant wave maps". In: *Comm. Pure Appl. Math.* 47.5 (1994), pp. 719–754. ISSN: 0010-3640,1097-0312. DOI: 10.1002/cpa.3160470507. URL: https://doi.org/10.1002/cpa.3160470507.

[59] Jacob Sterbenz and Daniel Tataru. "Regularity of wave-maps in dimension $2 + 1$". In: *Comm. Math. Phys.* 298.1 (2010), pp. 231–264. ISSN: 0010-3616,1432-0916. DOI: 10.1007/s00220-010-1062-3. URL: https://doi.org/10.1007/s00220-010-1062-3.

[60] Michael Struwe. "On the evolution of harmonic mappings of Riemannian surfaces." In: *Commentarii mathematici Helvetici* 60 (1985), pp. 558–581. URL: http://eudml.org/doc/140031.

[61] Michael Struwe. "On the evolution of harmonic maps in higher dimensions". In: *J. Differential Geom.* 28.3 (1988), pp. 485–502. ISSN: 0022-040X,1945-743X. URL: http://projecteuclid.org/euclid.jdg/1214442475.

[62] Atsushi Tachikawa. "Rotationally symmetric harmonic maps from a ball into a warped product manifold". In: *Manuscripta Math.* 53.3 (1985), pp. 235–254. ISSN: 0025-2611,1432-1785. DOI: 10.1007/BF01626399. URL: https://doi.org/10.1007/BF01626399.

[63] Terence Tao. "Global regularity of wave maps. I. Small critical Sobolev norm in high dimension". In: *Internat. Math. Res. Notices* 6 (2001), pp. 299–328. ISSN: 1073-7928,1687-0247. DOI: 10.1155/S1073792801000150. URL: https://doi.org/10.1155/S1073792801000150.

[64] Terence Tao. "Global regularity of wave maps. II. Small energy in two dimensions". In: *Comm. Math. Phys.* 224.2 (2001), pp. 443–544. ISSN: 0010-3616,1432-0916. DOI: 10.1007/PL00005588. URL: https://doi.org/10.1007/PL00005588.

[65] Terence Tao. *Nonlinear dispersive equations*. Vol. 106. CBMS Regional Conference Series in Mathematics. Local and global analysis. Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006, pp. xvi+373. ISBN: 0-8218-4143-2. DOI: 10.1090/cbms/106. URL: https://doi.org/10.1090/cbms/106.

[66] Daniel Tataru. "Local and global results for wave maps. I". In: *Comm. Partial Differential Equations* 23.9-10 (1998), pp. 1781–1793. ISSN: 0360-5302,1532-4133. DOI: 10.1080/03605309808821400. URL: https://doi.org/10.1080/03605309808821400.

[67] Daniel Tataru. "On global existence and scattering for the wave maps equation". In: *Amer. J. Math.* 123.1 (2001), pp. 37–77. ISSN: 0002-9327,1080-6377. URL: http://muse.jhu.edu/journals/american_journal_of_mathematics/v123/123.1tataru.pdf.

[68] Neil Turok and David Spergel. "Global texture and the microwave background". In: *Phys. Rev. Lett.* 64 (23 June 1990), pp. 2736–2739. DOI: 10.1103/PhysRevLett.64.2736. URL: https://link.aps.org/doi/10.1103/PhysRevLett.64.2736.

[69] Max Weissenbacher, Herbert Koch, and Roland Donninger. *Mode stability of blow-up for wave maps in the absence of symmetry*. 2025. arXiv: 2503.02632 [math.AP]. URL: https://arxiv.org/abs/2503.02632.

[70] Alexander Wittenstein. *Stable blowup for the harmonic map heat flow into perturbed spheres*. 2026. arXiv: 2608.03332 [math.AP]. URL: https://arxiv.org/abs/2608.03332.