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Note

A Fenchel-Moreau conjugation-based approach for computing the pre-kernel/pre-nucleolus ⋆,⋆⋆

Holger I. Meinhardt

Institute of Operations Research, Karlsruhe Institute of Technology (KIT), Englerstr. 11, Building: 11.40, Karlsruhe, 76128, Baden-Württemberg, Germany

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ABSTRACT

Computing the (pre-)nucleolus of a coalitional game with transferable utility (TU game) is generally NP-hard. Therefore, no universal polynomial-time algorithm exists. However, polynomial-time algorithms exist for specific game classes. The most general approach in the literature is based on the ellipsoid method and Maschler's approximation scheme. In contrast, we propose polynomial-time solvability by using a Fenchel-Moreau conjugation-based approach from convex analysis that applies the method of Meinhardt (2013) to compute a pre-kernel element. This method not only allows the computation of the pre-nucleolus for game classes with a singleton pre-kernel with a runtime complexity of $O(n^3)$. It also works for game classes with a set-valued pre-kernel whenever the pre-nucleolus is a vertex or a separated point within the pre-kernel set. Finally, we focus on the replication of the pre-nucleolus in the sense of Meinhardt (2024) for a class of related games to demonstrate the versatility of the Fenchel-Moreau conjugation-based approach while referring to group algebra.

1. Introduction

Computing the (pre-)nucleolus of a coalitional game with transferable utility (TU game) is, in general, NP-hard, whereas polynomial-time algorithms exist for specific classes such as assignment games (cf. Solymosi and Raghavan, 1994), matching games (cf. Kern and Paulusma, 2003), or standard tree games (cf. Megiddo, 1978; Granot et al., 1996). However, Faigle et al. (2001) provided a polynomial-time algorithm for computing the nucleolus for the relatively large class of games having a sole intersection point of the pre-kernel with the core. Their proposed computation procedure relies on the ellipsoid method to solve the associated least-cores and on Maschler's scheme for approximating the (pre-)kernel. Yet, it is well-known that the ellipsoid method encounters a major drawback, namely, the unbounded complexity in a real number model. This provokes numerical instability issues with floating-point arithmetic, causing the ellipsoid algorithm to fail or converge to an incorrect solution (see Meinhardt (2025) for the details). In order

⋆ DEDICATED TO THE MEMORY OF THEO DRIESSEN. His brilliant mind, in the prime of his creative and human life, was crushed by its heavy burden in the dark night and finally silenced as a result of his Parkinson's disease (July 6, 2026).

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E-mail address: holger.meinhardt@partner.kit.edu

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to overcome the negative side effects of the ellipsoid method, we suggest an alternative approach for computing the pre-nucleolus by relying on a Fenchel-Moreau conjugation-based approach from convex analysis, which can identify the pre-nucleolus when the pre-kernel is a single point. In contrast, for game classes where the pre-kernel is set-valued, this approach finds the pre-nucleolus whenever it is a vertex or a separated point within the pre-kernel set. However, if the pre-nucleolus is not such an exposed point and belongs to the relative interior of the pre-kernel set, the algorithm generally cannot compute it. To illustrate this approach, we resume the method of Meinhardt (2013) to compute a pre-kernel element. Though it is a specialized and highly optimized algorithm for the pre-kernel, it assures a runtime complexity of $O(n^3)$ for computing the pre-nucleolus whenever it is an exposed point of the pre-kernel set or the pre-kernel is single-valued and the Faigle et al. (2001) assumption is satisfied, which indicates a polynomial-time algorithm for this class of games. Finally, we focus on the replication of the pre-kernel in the sense of Meinhardt (2024) for a class of related games to demonstrate the versatility of the approach based on the Fenchel-Moreau conjugation with reference to group algebra.

The remainder of the paper is organized as follows: In Section 2, we introduce some basic notations and definitions to investigate the pre-kernel and the pre-nucleolus. Then, Section 3 provides the concept of the indirect function and gives a dual pre-kernel representation in terms of solution sets. In the next step, the notion of lexicographically smallest most effective coalitions is introduced in order to identify payoff equivalence classes on the domain of the objective function from which a pre-kernel element can be determined. Section 4 reviews our Fenchel-Moreau conjugation method for finding a pre-kernel element (cf. Meinhardt, 2013). We establish its usefulness for certain game classes in the computation of the pre-nucleolus by means of polynomial runtime complexity. In particular, we demonstrate, relying on the example of a weighted majority game with six players and a disconnected pre-kernel, that its computation is possible within a single iteration step, instead of the theoretically expected upper bound of 14 iterations. Finally, we focus in Section 5 on the replication of the pre-nucleolus in the sense of Meinhardt (2024) for a class of related games to demonstrate the power of the Fenchel-Moreau-based approach. The paper closes with some concluding remarks.

2. Some preliminaries

A cooperative game with transferable utility is a pair $\langle N, v\rangle$, where N is the non-empty finite player set $N := \{1, 2, \dots, n\}$, and v is the characteristic function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) := 0$. A player i is an element of N, and a coalition S is an element of the power set of 2^N. The real number $v(S) \in \mathbb{R}$ is called the value or worth of a coalition $S \in 2^N$. Let S be a coalition, the number of members in S will be denoted by $s := |S|$. We assume throughout that $v(N) > 0$ and $n \geq 2$ is valid. In addition, we identify a cooperative game by the vector $v := (v(S))_{S\subseteq N} \in \mathcal{G}^n = \mathbb{R}^{2^n}$, if no confusion can arise. Finally, the relevant game space for our investigation is defined by $\mathcal{G}(N) := \{v \in \mathcal{G}^n \mid v(\emptyset) = 0 \wedge v(N) > 0\}$.

If $\mathbf{x} \in \mathbb{R}^n$, we write $x(S) := \sum_{k\in S} x_k$ for every $S \in 2^N$ with $x(\emptyset) := 0$. The set of vectors $\mathbf{x} \in \mathbb{R}^n$, which satisfies the efficiency principle $v(N) = x(N)$ is called the **pre-imputation set** and it is defined by $\mathcal{I}^*(N, v) := \{\mathbf{x} \in \mathbb{R}^n \mid x(N) = v(N)\}$, or more concisely as $\mathcal{I}^*(v)$, where an element $\mathbf{x} \in \mathcal{I}^*(v)$ is called a pre-imputation. The set of pre-imputations, which satisfies, in addition, the **individual rationality property** $x_k \geq v(\{k\})$ for all $k \in N$ is called the **imputation set** $\mathcal{I}(N, v)$.

The **core** of a game $\langle N, v\rangle$ is a set-valued solution that is constituted by the imputations satisfying the individual rationality property as well as the coalitional rationality property, i.e. the core of a game $v \in \mathcal{G}^n$ is given by $\mathcal{C}(N, v) := \left\{\mathbf{x} \in \mathcal{I}(N, v) \mid x(N) = v(N) \text{ and } x(S) \geq v(S)\ \forall\ S \subset N\right\}$. The core of a n-person game may be empty. Whenever it is non-empty, we have some incentive for mutual cooperation in the grand coalition.

Given a vector $\mathbf{x} \in \mathcal{I}^*(v)$, we define the **excess** of coalition S with respect to the pre-imputation $\mathbf{x}$ in the game $\langle N, v\rangle$ by $e^v(S, \mathbf{x}) := v(S) - x(S)$.

Take a game $v \in \mathcal{G}^n$. For any pair of players $i, j \in N, i \neq j$, the **maximum surplus** of player i over player j with respect to any pre-imputation $\mathbf{x} \in \mathcal{I}^*(v)$ is given by the maximum excess at $\mathbf{x}$ over the set of coalitions containing player i but not player j, thus $s_{ij}(\mathbf{x}, v) := \max_{S\in\mathcal{G}_{ij}} e^v(S, \mathbf{x})$, where $\mathcal{G}_{ij} := \{S \mid i \in S \text{ and } j \notin S\}$. The set of all pre-imputations $\mathbf{x} \in \mathcal{I}^*(v)$ that balances the maximum surpluses for each distinct pair of players $i, j \in N, i \neq j$ is called the **pre-kernel** of the game v, and is defined by

$$\mathcal{PK}(N, v) := \left\{\mathbf{x} \in \mathcal{I}^*(v) \mid s_{ij}(\mathbf{x}, v) = s_{ji}(\mathbf{x}, v) \quad \text{for all } i, j \in N, i \neq j\right\}. \tag{1}$$

The pre-kernel has the advantage of addressing a stylized bargaining process, in which the figure of argumentation is a **pairwise equilibrium procedure** of claims while relying on best arguments, that is, the coalitions that will best support the claim. The pre-kernel solution characterizes all those imputations in which all pairs of players $i, j \in N, i \neq j$ are in equilibrium with respect to their claims.

In order to define the pre-nucleolus of a game $v \in \mathcal{G}^n$, take any $\mathbf{x} \in \mathbb{R}^n$ to define a 2^n-tuple vector $\theta(\mathbf{x})$ whose components are the excesses $e^v(S, \mathbf{x})$ of the 2^n coalitions $S \subseteq N$, arranged in decreasing order, that is, $\theta_i(\mathbf{x}) := e^v(S_i, \mathbf{x}) \geq e^v(S_j, \mathbf{x}) =: \theta_j(\mathbf{x})$ if $1 \leq i \leq j \leq 2^n$. Ordering the so-called complaint or dissatisfaction vectors $\theta(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ by the lexicographic order $\leq_L$ on $\mathbb{R}^n$, we shall write $\theta(\mathbf{x}) <_L \theta(\mathbf{y})$ if $\exists$ an integer $1 \leq k \leq 2^n$, such that $\theta_i(\mathbf{x}) = \theta_i(\mathbf{y})$ for $1 \leq i < k$ and $\theta_k(\mathbf{x}) < \theta_k(\mathbf{y})$. Furthermore, we write $\theta(\mathbf{x}) \leq_L \theta(\mathbf{y})$ if either $\theta(\mathbf{x}) <_L \theta(\mathbf{y})$ or $\theta(\mathbf{x}) = \theta(\mathbf{y})$. Now the pre-nucleolus $\mathcal{PN}(N, v)$ over the pre-imputations set $\mathcal{I}^*(v)$ is defined by

$$\mathcal{PN}(N, v) = \left\{\mathbf{x} \in \mathcal{I}^*(N, v) \mid \theta(\mathbf{x}) \leq_L \theta(\mathbf{y})\ \forall\ \mathbf{y} \in \mathcal{I}^*(N, v)\right\}. \tag{2}$$

The **pre-nucleolus** of any game $v \in \mathcal{G}^n$ is non-empty as well as a unique point, and it is referred to as $\nu(v)$ if the game context is clear from the contents or $\nu(N, v)$. Recall that the pre-nucleolus always belongs to the pre-kernel set. Hence, $\nu(N, v) \in \mathcal{PK}(N, v)$.

In addition, we want to discuss some important game properties. A game $v \in \mathcal{G}^n$ is said to be **convex** if it satisfies

$$v(S \cup \{i\}) - v(S) \leq v(S \cup \{i, j\}) - v(S \cup \{j\}) \qquad \text{if } S \subseteq N\backslash\{i, j\}. \tag{3}$$

This subclass of games has been introduced by Shapley (1971), and we denote it by $\mathcal{CV}$. Convex games have a non-empty core, and the Shapley value is the center of gravity of the extreme points of the core (cf. Shapley, 1971). Moreover, we say that a TU game $\langle N, v\rangle$ is **average-convex**, whenever its characteristic function $v : 2^N \to \mathbb{R}$ satisfies the subsequent condition (cf. Iñarra and Usategui, 1993):

$$\sum_{i\in S}\Big[v(S) - v(S\backslash\{i\})\Big] \leq \sum_{i\in S}\Big[v(T) - v(T\backslash\{i\})\Big] \qquad \text{for all } S \subseteq T \subseteq N, S \neq \emptyset. \tag{4}$$

Let us denote this class of games by $\mathcal{ACV}$. Note that convex games satisfy the average-convexity condition by summing up, over all $i \in S$, the valid convexity conditions $v(S) - v(S\backslash\{i\}) \leq v(T) - v(T\backslash\{i\})$. Thus, the class of convex TU games is included in the class of average-convex TU games, viz., $\mathcal{CV}^n \subset \mathcal{ACV}^n$ is satisfied.

3. A dual pre-kernel representation

Now, we present an alternative approach to characterizing the pre-kernel in terms of the so-called indirect function, introduced by Martinez-Legaz (1996). The indirect function of a characteristic function game – which is a Fenchel-Moreau generalized conjugation – provides the same information as the n-person cooperative game with transferable utility under consideration. In this section, we review some crucial results extensively studied in Meinhardt (2013, Chap. 5 & 6) as the building blocks to investigate the computational aspect of the pre-kernel.

The **convex conjugate** or **Fenchel transform** $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$ (where $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$) of a convex function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ (cf. Rockafellar, 1970, Section 12) is defined by

$$f^*(\mathbf{x}^*) = \sup_{\mathbf{x}\in\mathbb{R}^n}\{\langle\, \mathbf{x}^*, \mathbf{x}\,\rangle - f(\mathbf{x})\} \qquad \forall \mathbf{x}^* \in \mathbb{R}^n.$$

Observe that the Fenchel transform f^* is the point-wise supremum of affine functions $p(\mathbf{x}^*) = \langle\, \mathbf{x}, \mathbf{x}^*\,\rangle - \mu$ such that $(\mathbf{x}, \mu) \in (\mathcal{C}\times\mathbb{R}) \subseteq (\mathbb{R}^n \times \mathbb{R})$, whereas $\mathcal{C}$ is a convex set. Thus, the Fenchel transform f^* is again a convex function.

We can generalize the definition of a Fenchel transform (cf. Martinez-Legaz, 1996) by introducing a fixed non-empty subset $\mathcal{K}$ of $\mathbb{R}^n$, then the conjugate of a function $f : \mathcal{K} \to \overline{\mathbb{R}}$ is $f^c : \mathbb{R}^n \to \overline{\mathbb{R}}$, given by

$$f^c(\mathbf{x}^*) = \sup_{\mathbf{x}\in\mathcal{K}}\{\langle\, \mathbf{x}^*, \mathbf{x}\,\rangle - f(\mathbf{x})\} \qquad \forall \mathbf{x}^* \in \mathbb{R}^n,$$

which is also known as the **Fenchel-Moreau conjugation**.

A vector $\mathbf{x}^*$ is said to be a subgradient of a convex function f at a point $\mathbf{x}$, if

$$f(\mathbf{z}) \geq f(\mathbf{x}) + \langle\, \mathbf{x}^*, \mathbf{z} - \mathbf{x}\,\rangle \qquad \forall \mathbf{z} \in \mathbb{R}^n.$$

The set of all subgradients of f at $\mathbf{x}$ is called the subdifferentiable of f at $\mathbf{x}$ and it is defined by

$$\partial f(\mathbf{x}) := \{\mathbf{x}^* \in \mathbb{R}^n \mid f(\mathbf{z}) \geq f(\mathbf{x}) + \langle\, \mathbf{x}^*, \mathbf{z} - \mathbf{x}\,\rangle \quad (\forall \mathbf{z} \in \mathbb{R}^n)\}.$$

The set of all subgradients $\partial f(\mathbf{x})$ is a closed convex set, which could be empty or may consist of just one point. The multivalued mapping $\partial f : \mathbf{x} \mapsto \partial f(\mathbf{x})$ is called the subdifferential of f.

Theorem 1. *The indirect function $\pi : \mathbb{R}^n \to \mathbb{R}$ of any n-person TU game is a non-increasing polyhedral convex function such that*

(i) $\partial\pi(\mathbf{x}) \cap \{-1, 0\}^n \neq \emptyset \qquad \forall \mathbf{x} \in \mathbb{R}^n$,
(ii) $\{-1, 0\}^n \subset \bigcup_{\mathbf{x}\in\mathbb{R}^n} \partial\pi(\mathbf{x})$, *and*
(iii) $\min_{\mathbf{x}\in\mathbb{R}^n}\ \pi(\mathbf{x}) = 0$.

Conversely, if $\pi : \mathbb{R}^n \to \mathbb{R}$ satisfies $(i) - (iii)$ then there exists a unique n-person TU game $\langle N, v\rangle$ having π as its indirect function, its characteristic function is given by

$$v(S) = \min_{\mathbf{x}\in\mathbb{R}^n}\left\{\pi(\mathbf{x}) + \sum_{k\in S} x_k\right\} \qquad \forall\, S \subseteq N. \tag{5}$$

Hence, the indirect function provides the same information as the characteristic function. For the details, we refer the reader to Martinez-Legaz (1996).

According to the above result, the arising **indirect function** $\pi : \mathbb{R}^n \to \mathbb{R}_+$ is given by

$$\pi(\mathbf{x}) = \max_{S\subseteq N}\left\{v(S) - \sum_{k\in S} x_k\right\} \qquad \forall \mathbf{x} \in \mathbb{R}^n. \tag{6}$$

This approach allows us to propose a more efficient and simple method to determine a pre-kernel element (cf. Meinhardt, 2013).

A pre-imputation that comprises from a vector $\mathbf{x}$ the possibility of compensation between a pair of players $i, j \in N, i \neq j$, is denoted as $\mathbf{x}^{i,j,\delta} = (x_k^{i,j,\delta})_{k\in N} \in \mathbb{I}^*(N, v)$, with $\delta \geq 0$, which is given by $\mathbf{x}^{i,j,\delta}_{N\backslash\{i,j\}} = \mathbf{x}_{N\backslash\{i,j\}}$, $x_i^{i,j,\delta} = x_i - \delta$ and $x_j^{i,j,\delta} = x_j + \delta$.

By the next Lemma, which is due to Meseguer-Artola (1997), we shall establish that the indirect function π of game v can be related to the maximum surpluses.

Lemma 1. *Let $\langle N, v\rangle$ be an n-person cooperative game with side-payments. Let π and s_{ij} be the associated indirect function and the maximum surplus of player i against player j, respectively. If $\epsilon \in \mathbb{R}$ and $\mathbf{x} \in \mathcal{I}^{\epsilon}(v)$, then the equality:*

$$s_{ij}(\mathbf{x}, v) = \pi(\mathbf{x}^{\,i,j,\delta}) - \delta$$

holds for every $i, j \in N$, $i \neq j$, and for every $\delta \geq \delta_1(\mathbf{x}, v)$, where:

$$\delta_1(\mathbf{x}, v) := \max_{k \in N, S \subset N \setminus \{k\}} |v(S \cup \{k\}) - v(S) - x_k|.$$

A characterization of the pre-kernel in terms of the indirect function is due to Meseguer-Artola (1997). Here, we present this representation for the trivial coalition structure $\mathcal{B} = \{N\}$ only since our algorithm evaluates a pre-kernel element for the grand coalition. Thus, for our purpose, it is enough to concentrate on the trivial coalition structure while restating the following result:

Proposition 1. *For a TU game with indirect function π, a pre-imputation $\mathbf{x} \in \mathcal{I}^{*}(N, v)$ is in the pre-kernel of $\langle N, v\rangle$ if, and only if, for every $i, j \in N$ $i < j$, and some $\delta \geq \delta_1(v, \mathbf{x})$, one has*

$$\pi(\mathbf{x}^{\,i,j,\delta}) = \pi(\mathbf{x}^{\,j,i,\delta}).$$

For non-trivial coalition structures, we refer the reader to the book of Meinhardt (2026).

Meseguer-Artola (1997) was the first who recognized that based on the result of Proposition 1 a pre-kernel element can be derived as a solution of an overdetermined system of nonlinear equations. Every overdetermined system can be equivalently expressed as a minimization problem. The set of global minima coalesces with the pre-kernel set.[1] For the trivial coalition structure $\mathcal{B} = \{N\}$, the overdetermined system of nonlinear equations is given by

$$\begin{cases} f_{ij}(\mathbf{x}) = 0 & \forall i, j \in N, i < j \\ f_0(\mathbf{x}) = 0 \end{cases} \tag{7}$$

where, for some $\delta \geq \delta_1(\mathbf{x}, v)$,

$$f_{ij}(\mathbf{x}) := \pi(\mathbf{x}^{\,i,j,\delta}) - \pi(\mathbf{x}^{\,j,i,\delta}) \qquad \forall i, j \in N, i < j, \tag{7-a}$$

and

$$f_0(\mathbf{x}) := \sum_{k \in N} x_k - v(N). \tag{7-b}$$

For any overdetermined system, an equivalent minimization problem is associated such that the set of global minima coincides with the solution set of the system. The solution set of such a minimization problem is the set of payoff vectors $\mathbf{x}$ which minimizes the function

$$h(\mathbf{x}) := \sum_{\substack{i,j \in N \\ i<j}} (f_{ij}(\mathbf{x}))^2 + (f_0(\mathbf{x}))^2 \geq 0 \qquad \forall\, \mathbf{x} \in \mathbb{R}^n. \tag{8}$$

For further details, see Meinhardt (2013, Chap. 5 & 6). Then, one can establish the subsequent result:

Corollary 1. *For a TU game $\langle N, v\rangle$ with indirect function π, it holds that*

$$h(\mathbf{x}) = \sum_{\substack{i,j \in N \\ i<j}} (f_{ij}(\mathbf{x}))^2 + (f_0(\mathbf{x}))^2 = \min_{\mathbf{y} \in \mathcal{I}^{*}(N,v)} h(\mathbf{y}) = 0, \tag{9}$$

if, and only if, $\mathbf{x} \in \mathcal{PK}(N, v)$.

Therefore, the pre-kernel set can be characterized as the solution set of function h. This result will be used to propose a comprehensible and more efficient routine to seek a pre-kernel element. Nevertheless, notice that the objective function h of type (8) is not suitable to implement an efficient and simple method to single out a pre-kernel element. This is due to the fact that the structural form of objective function h is not well-behaved. The function is neither convex nor quadratic, nor it is differentiable everywhere. Apparently, it is possible to apply on function (8) a modified *Steepest Descent Method* to get a pre-kernel point. This was the proposed method by Meseguer-Artola (1997). In contrast, the proposed methods presented by Meinhardt (2013, Algorithms 8.1.1–8.3.1) rely on the idea to identify smaller and more tractable optimization problems, from which a pre-kernel element can be determined. This preferred method is applicable by the observation that the objective function h is composed of a finite family of quadratic and convex functions (cf. Meinhardt (2013, Proposition 6.2.2)). The proposed algorithms subsequently solve convex optimization problems, which converge in a finite number of iterations to a pre-kernel point. We shall reexamine some crucial properties of the pre-kernel to identify a partition on the payoff space, that is, a partition of the domain of function h into payoff equivalence classes, from which a unique convex and

[1] Of course, an overdetermined system may have an empty solution set. However, there is no loss of generality in implicitly assuming here that the set of global minima is non-empty. Since the above overdetermined system transforms into a collection of non-homogeneous systems of linear equations of the form $\mathbf{Q}\,\mathbf{x} = -\mathbf{a}$. As we observe in a while, the matrix $\mathbf{Q} \in \mathbb{R}^{n^2}$ is expressed by $\mathbf{Q} = 2 \cdot \mathbf{E}\,\mathbf{E}^{\top}$, and the associated column vector $\mathbf{a}$ is obtained by $\mathbf{E}\,\vec{\alpha}$. Hence, the vector $\mathbf{a}$ belongs to the span of $\{\mathbf{q}_1, \ldots, \mathbf{q}_n\}$. This implies that the rank of the augmented matrix, denoted as $\mathbf{C} := (\mathbf{Q}|\mathbf{a})$, is equal to the rank of matrix $\mathbf{Q}$, which assures the non-emptiness of the set of global minima.

quadratic function can be obtained. As we understand these details, we discuss the method that is employed by the Algorithm 3.1 to find an element from the pre-kernel set.

To identify a partition of the domain of function h into payoff equivalence classes, we first define the set of **most effective** or **significant coalitions** for each pair of players $i, j \in N, i \neq j$ at the payoff vector $\mathbf{x}$ by

$$\mathcal{C}_{ij}(\mathbf{x}) := \left\{ S \in \mathcal{G}_{ij} \;\middle|\; s_{ij}(\mathbf{x}, v) = e^{v}(S, \mathbf{x}) \right\}. \tag{10}$$

When we gather for all pairs of player $i, j \in N, i \neq j$ all these coalitions that support the claim of a specific player over some other players, we have to consider the concept of the collection of most effective or significant coalitions w.r.t. $\mathbf{x}$, which we define as in Maschler et al. (1979, p. 315) by

$$\mathcal{C}(\mathbf{x}) := \bigcup_{\substack{i,j \in N \\ i \neq j}} \mathcal{C}_{ij}(\mathbf{x}). \tag{11}$$

From the set of most effective or significant coalitions of a pair of players $i, j \in N, i \neq j$ at the payoff vector $\mathbf{x}$ the smallest cardinality over the set of most effective coalitions is defined as

$$\Phi_{ij}(\mathbf{x}) := \min \left\{ |S| \;\middle|\; S \in \mathcal{C}_{ij}(\mathbf{x}) \right\}. \tag{12}$$

Gathering all these sets having the smallest cardinality for all pairs of players $i, j \in N, i \neq j$, we end up with

$$\Psi_{ij}(\mathbf{x}) := \left\{ S \in \mathcal{C}_{ij}(\mathbf{x}) \,\middle|\, \Phi_{ij}(\mathbf{x}) = |S| \right\}. \tag{13}$$

With respect to an arbitrary payoff vector $\mathbf{x}$, the set of coalitions of smallest cardinality $\Psi_{ij}(\mathbf{x})$, which is minimized w.r.t. the lexicographical order $<_L$, is determined by

$$\mathcal{S}_{ij}(\mathbf{x}) := \left\{ S \in \Psi_{ij}(\mathbf{x}) \,\middle|\, S <_L T \text{ for all } S \neq T \in \Psi_{ij}(\mathbf{x}) \right\} \quad \forall i, j \in N, i \neq j. \tag{14}$$

We denote this set as the **lexicographically smallest most effective coalitions** w.r.t. $\mathbf{x}$ related to the pair of players $i, j \in N, i \neq j$.

After these preparative definitions, we are in a position to specify the set of lexicographically smallest most effective coalitions w.r.t. $\mathbf{x}$ through

$$\mathcal{S}(\mathbf{x}) := \bigcup_{\substack{i,j \in N \\ i \neq j}} \mathcal{S}_{ij}(\mathbf{x}). \tag{15}$$

This set will be denoted more concisely as the set of **lexicographically smallest coalitions**. Given the correspondence $\mathcal{S}$ on the payoff space, we say that two payoff vectors $\mathbf{x}$ and $\mathbf{y}$ are equivalent w.r.t. the binary relation $\sim$ iff $\mathcal{S}(\mathbf{x}) = \mathcal{S}(\mathbf{y})$. This binary relation induces a partition on the payoff space. Having identified payoff equivalence classes, we can select an arbitrary payoff vector to get a unique quadratic and convex function. To see this, select the payoff vector $\mathbf{x}$ from the payoff equivalence class $[\vec{\gamma}]$, where $\vec{\gamma}$ is the representative vector of $[\vec{\gamma}]$, then we get the set $\mathcal{S}(\mathbf{x})$, from which a rectangular matrix $\mathbf{E}$ can be constructed through $\mathbf{E}_{ij} := (\mathbf{1}_{S_{ji}} - \mathbf{1}_{S_{ij}}) \in \mathbb{R}^n$, $\forall i, j \in N, i < j$, and $\mathbf{E}_0 := -\mathbf{1}_N \in \mathbb{R}^n$. Let $q = \binom{n}{2} + 1$, combining these q-column vectors, we can construct matrix $\mathbf{E}$ as a $(n \times q)$-matrix in $\mathbb{R}^{n \times q}$, which is given by

$$\mathbf{E} := [\mathbf{E}_{1,2}, \ldots, \mathbf{E}_{n-1,n}, \mathbf{E}_0] \in \mathbb{R}^{n \times q}. \tag{16}$$

A matrix $\mathbf{Q} \in \mathbb{R}^{n^2}$ can now be expressed as $\mathbf{Q} = 2 \cdot \mathbf{E}\,\mathbf{E}^{\top}$, a column vector $\mathbf{a}$ as $2 \cdot \mathbf{E}\,\vec{\alpha} \in \mathbb{R}^n$. Whereas vector $\vec{\alpha} \in \mathbb{R}^q$ is specified through $\alpha_{ij} := (v(S_{ij}) - v(S_{ji})) \in \mathbb{R}$ $\forall i, j \in N, i < j$ and $\alpha_0 := v(N)$. Finally, the scalar α is given by $\|\vec{\alpha}\|^2$, whereas $\mathbf{E} \in \mathbb{R}^{n \times q}, \mathbf{E}^{\top} \in \mathbb{R}^{q \times n}$ and $\vec{\alpha} \in \mathbb{R}^q$.

From vector $\vec{\gamma}$, the set (15) is constructed and then matrix $\mathbf{Q}$, column vector $\mathbf{a}$, and scalar α are induced, from which a quadratic and convex function can be specified through

$$h_{\gamma}(\mathbf{x}) = (1/2) \cdot \langle\, \mathbf{x}, \mathbf{Q}\,\mathbf{x} \,\rangle + \langle\, \mathbf{x}, \mathbf{a} \,\rangle + \boldsymbol{\alpha} \qquad \mathbf{x} \in \mathbb{R}^n, \tag{17}$$

In view of Proposition 6.2.2 Meinhardt (2013), function h, as defined by (8), is composed of a finite family of quadratic and convex functions of type (17). For the details, we refer the interested reader to Meinhardt (2013, Chap. 5 & 6). In accordance with Theorem 7.3.1 by Meinhardt (2013, p. 137) a dual representation of the pre-kernel is obtained as a finite union of convex and restricted solution sets $M(h_{\gamma_k}, \overline{[\vec{\gamma}_k]})$ of a quadratic and convex function h_{γ_k}, that is,

$$\mathcal{PK}(N, v) = \bigcup_{k \in \mathcal{J}'} M(h_{\gamma_k}, \overline{[\vec{\gamma}_k]}), \tag{18}$$

where $\mathcal{J}'$ is a finite index set such that $\mathcal{J}' := \{k \in \mathcal{J} \mid g(\vec{\gamma}_k) = 0\}$. In addition, $g(\vec{\gamma}_k) = 0$ is the minimum value of a minimization problem under constraints of function h_{γ_k} over the closed payoff set $\overline{[\vec{\gamma}_k]}$. The solution sets $M(h_{\gamma_k}, \overline{[\vec{\gamma}_k]})$ are convex. Taking the finite union of convex sets gives us a non-convex set if $|\,\mathcal{J}'\,| \geq 2$. Hence, the pre-kernel set is generically a non-convex set. For the class of convex games and three-person games we have $|\,\mathcal{J}'\,| = 1$, which implies that the per-kernel must be a singleton.

For this purpose, we consider a mapping that sends a point $\vec{\gamma}$ to a point $\vec{\gamma}_\circ \in M(h_{\vec{\gamma}})$ through

$$\Gamma(\vec{\gamma}) := -\left(\mathbf{Q}^{\dagger}\ \mathbf{a}\right)(\vec{\gamma}) = -\left(\mathbf{Q}_{\vec{\gamma}}^{\dagger}\ \mathbf{a}_{\vec{\gamma}}\right) = \vec{\gamma}_\circ \in M(h_{\vec{\gamma}}) \qquad \forall \vec{\gamma} \in \mathbb{R}^n, \tag{19}$$

where $\mathbf{Q}_{\vec{\gamma}}$ and $\mathbf{a}_{\vec{\gamma}}$ are the matrix and the column vector induced by vector $\vec{\gamma}$, respectively. Notice that matrix $\mathbf{Q}_{\vec{\gamma}}^{\dagger}$ is the pseudo-inverse of matrix $\mathbf{Q}_{\vec{\gamma}}$. In addition, the set $M(h_{\vec{\gamma}})$ is the solution set of function $h_{\vec{\gamma}}$. Under a regime of orthogonal projection, this mapping induces a cycle-free method to evaluate a pre-kernel point for any class of TU games. We restate here Algorithm 8.1.1 of Meinhardt (2013) in a more succinctly written form through Method Algorithm 3.1

Algorithm 3.1: Procedure to seek a pre-kernel element.

Data: Arbitrary TU Game $\langle N, v \rangle$, and a payoff vector $\vec{\gamma}_0 \in \mathbb{R}^n$.
Result: A payoff vector s.t. $\vec{\gamma}_{k+1} \in \mathcal{PK}(N, v)$.
begin
0 $k \longleftarrow 0, \quad \mathcal{S}(\vec{\gamma}_{-1}) \longleftarrow \emptyset$
1 Select an arbitrary starting point $\vec{\gamma}_0$
if $\vec{\gamma}_0 \notin \mathcal{PK}(N, v)$ **then** Continue
else Stop
2 Determine $\mathcal{S}(\vec{\gamma}_0)$
if $\mathcal{S}(\vec{\gamma}_0) \neq \mathcal{S}(\vec{\gamma}_{-1})$ **then** Continue
else Stop
repeat
3 **if** $\mathcal{S}(\vec{\gamma}_k) \neq \emptyset$ **then** Continue
else Stop
4 Compute $\mathbf{E}_k$ and $\vec{\alpha}_k$ from $\mathcal{S}(\vec{\gamma}_k)$ and v
5 Determine $\mathbf{Q}_k$ and $\mathbf{a}_k$ from $\mathbf{E}_k$ and $\vec{\alpha}_k$
6 Calculate by Formula (19) $\mathbf{x}$
7 $k \longleftarrow k + 1$
8 $\vec{\gamma}_{k+1} \longleftarrow \mathbf{x}$
9 Determine $\mathcal{S}(\vec{\gamma}_{k+1})$
until $\mathcal{S}(\vec{\gamma}_{k+1}) = \mathcal{S}(\vec{\gamma}_k)$
end

Meinhardt (2013, Theorem 8.1.2) establishes that this iterative procedure converges toward a pre-kernel point. In view of Meinhardt (2013, Theorem 9.2.1), we even know that at most $\binom{n}{2} - 1$-iteration steps are sufficient to successfully terminate the search process. However, we have strong empirical evidence that generically a maximum of $n + 1$-iteration steps are needed to determine an element from the pre-kernel set.

Example 1. To present the reader with a sketch of how the algorithm works, we focus on a step-by-step procedure for computing a pre-kernel element for a weighted majority game that we have borrowed from the literature (see Kopelowitz, 1967 or Stearns, 1968). Consider a **six-person** weighted majority game based on the following parameters: $[16; 2, 4, 4, 5, 6, 7]$. This game has a disconnected pre-kernel, which consists of the two points $\{(0, 0, 0, 0, 1, 1)/2, (0, 1, 1, 1, 1, 1)/5\}$, whereas the last point is also the pre-nucleolus of the game. Set $k = 0$ and take $\vec{\gamma}_1 = \mathbf{x}_0 = (0, 1, 0, 0, 0, 0) \in [\vec{\gamma}_1]$. From the excess vector $exc(\mathbf{x}_0)$, we now get the subsequent set of lexicographically smallest coalitions for each pair of players:

$$\mathcal{S}(\mathbf{x}_0) = \{\{1,3,4,5\},\{1,4,5,6\},\{1,3,5,6\},\{1,3,4,6\},\{1,3,4,5\},\dots,$$
$$\{2,4,6\},\dots,\{3,4,6\},\dots,\{3,4,6\},\dots,\{3,5,6\},\dots,\{3,4,6\},\dots,\{3,4,6\}\},$$

whereas the generically lexicographical order of pairs gives the order of the pairs of players in $\mathcal{S}(\mathbf{x}_0)$. We restrict ourselves to the pairs: $\{(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(3,1),(4,1),(5,1),(6,1),(6,5)\}$. Then, matrix $\mathbf{E}$ is defined by $\mathbf{E} := [\mathbf{E}_{1,2}, \dots, \mathbf{E}_{5,6}, \mathbf{E}_0] \in \mathbb{R}^{6\times 16}$. This induces the following quadratic objective function:

$$h^0_{\gamma_1}(\mathbf{x}) = (1/2) \cdot \langle\, \mathbf{x}, \mathbf{Q}_0\, \mathbf{x} \,\rangle + \langle\, \mathbf{x}, \mathbf{a}_0 \,\rangle + \alpha_0$$
$$= (1/2) \cdot \mathbf{x}^T \cdot \begin{pmatrix} 10 & 1 & 2 & 1 & 6 & -5 \\ 1 & 6 & -3 & 0 & 1 & 1 \\ 2 & -3 & 9 & 0 & 0 & -1 \\ 1 & 0 & 0 & 9 & -5 & 1 \\ 6 & 1 & 0 & -5 & 12 & -3 \\ -5 & 1 & --1 & 1 & -3 & 7 \end{pmatrix} \mathbf{x} + \mathbf{x}^T \cdot \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix} + 1.$$

Solving the linear system $\mathbf{Q}_0\, \mathbf{x} = -\mathbf{a}_0$ yields: $\mathbf{x}_1 = (0, 1, 1, 1, 1, 1)/5 \in [\vec{\gamma}_2]$.

This solution is the pre-nucleolus of the weighted majority game, which is reached in only one iteration step, instead of the theoretically expected upper bound of $\binom{6}{2} - 1 = 15 - 1 = 14$ iterations. Thus, we observe that even for games with a non-single-valued pre-kernel, the pre-nucleolus can be computed. However, it must be a vertex or a separated point within the solution set of the

pre-kernel. Whenever it is located within the relative interior of the pre-kernel set, the algorithm will, in general, fail to compute the pre-nucleolus of a TU game.

By contrast, starting with the imputation vertex $\vec{\gamma}_3 = \mathbf{x}_0 = (0,0,0,0,0,1) \in [\vec{\gamma}_3]$ yields after two iterative steps to the second pre-kernel element of the game rather than the theoretical upper bound of $15 - 1 = 14$ iteration steps.

This example is in full extent discussed in Meinhardt (2013, Sec. 8.5). In particular, there is illustrated of how one can derive from a set of lexicographically smallest most effective coalitions $\mathcal{S}(\mathbf{x})$ a rectangular matrix $\mathbf{E}$, which is conceived through $\mathbf{E}_{ij} := (\mathbf{1}_{S_{ji}} - \mathbf{1}_{S_{ij}}) \in \mathbb{R}^n,\ \forall i,j \in N, i < j$, and $\mathbf{E}_0 := -\mathbf{1}_N \in \mathbb{R}^n$. The associated column vector $\mathbf{a}$ is obtained by $\mathbf{E}\ \vec{\alpha} \in \mathbb{R}^n$, whereas the vector $\vec{\alpha}$ is given by $\alpha_{ij} := (v(S_{ji}) - v(S_{ij})) \in \mathbb{R}$, for all $i,j \in N, i < j$, and $\alpha_0 := v(N)$. This information is enough to set up a system of linear equations $\mathbf{E}_k^{\top}\ \mathbf{x}_{k+1} + \vec{\alpha}_k = \mathbf{0}$ to get a solution vector $\mathbf{x}_{k+1}$ for the iteration step k. Formulating the complete quadratic optimization problem is unnecessary. #

4. Pre-kernel computation by the indirect function representation

The Fenchel-Moreau-based approach of Algorithm 3.1, though originally conceived to compute a pre-kernel element, can, as a byproduct, also be applied to compute the pre-nucleolus for all game classes for which it is known that the pre-kernel is single-valued. That is to say, not only for convex games, but also for almost-convex games or veto-rich games, to name just a few important game classes with a single pre-kernel point. Furthermore, as we saw in the example above, the Fenchel-Moreau-based method offers far greater flexibility in computing the pre-nucleolus than simply limiting itself to the class of games with a single-valued pre-kernel. Nevertheless, for set-valued pre-kernel solutions, the pre-nucleolus must lie in an exposed position in order to be successfully computed.

To illustrate the key features of Algorithm 3.1, we take into account the fact that only $n(n-1)/2 + 1$ constraints are required, rather than considering a linear problem with an exponentially long set of equations, as in the ellipsoid method. This drastically reduces the order of the set of constraints from $O(2^n)$ to $O(n^2)$, allowing the overdetermined system of equations to be reformulated as a continuous but non-differentiable minimization problem. Furthermore, the domain of the associated objective function can be divided into a finite set of payoff equivalence classes, i.e., convex sets. Within each of these equivalence classes, the complex, non-differentiable objective function that characterizes the pre-kernel simplifies to a regular quadratic and convex function, which allows us to iteratively solve a system of quadratic convex minimization problems associated with these classes, rather than solving a single, complex, non-convex problem. Thus, using the payoff equivalence class, the originally difficult problem is reduced to a finite sequence of well-behaved convex quadratic minimization problems that are solvable by linear algebra. It is shown that the Fenchel-Moreau-based approach offers a significant simplification of the characterization of the pre-nucleolus for the class of games with a single-valued pre-kernel. At each iteration step, a solution vector for a quadratic optimization problem of type (17) is determined, thereby simultaneously reducing all differences in the surpluses between all player pairs. Therefore, it is sufficient to solve a linear system of equations of the form $\mathbf{Q}\,\mathbf{x} = -\mathbf{a}$ with infinite precision via the mapping in (19), offering a worst-case runtime complexity of $O(n^3)$.

Furthermore, under a regime of orthogonal projections, the search space for updating the payoff vector is filtrated in such a way that convergence to a pre-kernel element occurs cycle-free. Thus, the application of orthogonal projections ensures that, at each step, the updated payoff vector is shifted within a subspace corresponding to the current equivalence class toward a balanced state (in which the surpluses are equalized). The imposed filtration of the vector space of excess configurations means that, instead of an infinite number of transfers, only a finite number of at most $(\binom{n}{2} - 1)$ side payments need to be carried out between the pair of players to successfully complete the search process (cf. Meinhardt, 2013, Theorem 9.2.1). Here, each iteration step on the payoff space is translated into the vector space of excess configurations, while the dimension of the search space is increased by at least one. Thus, not the payoff vector space of dimension n is filtrated, but rather the basis vectors of the vector space of balanced excesses are completed, until finally the vector of maximum surpluses is embedded within the vector subspace of balanced excesses of dimension $q = \binom{n}{2} + 1$. Typically, for an n-player game, this process requires at most $(n+1)$ iterations, instead of the $\binom{n}{2} - 1$ steps needed in the worst case. Therefore, fast convergence is guaranteed, which represents a significant improvement over iterative methods such as the Maschler approximation method, which may exhibit unpredictable or slower convergence.

The Fenchel-Moreau-based approach to computing the pre-nucleolus of a game with a single-valued pre-kernel avoids relying on the least-core as a search space that contains the pre-nucleolus. This eliminates the need for the ellipsoid method or the minimization of submodular functions. Instead, the method focuses directly on the balancedness condition of the maximum surpluses, as a consequence, no intermediate bound is required as a pre-nucleolus catcher. It converges directly to a pre-kernel element without first having to restrict the search to a smaller, bounded set. This means that the approach is more efficient, as it involves a specialized and highly optimized algorithm for the pre-kernel that can also be applied to the pre-nucleolus calculation for the specified game classes. It utilizes a dual characterization based on the Fenchel-Moreau conjugacy, but applies it in a structured manner, thereby avoiding the complexity and potential inefficiency of more general convex optimization methods of this type. Therefore, this method is not a universally applicable sequential QP procedure. It is a highly tailored algorithm developed specifically for the pre-kernel computation. The strength of this approach lies in the specific dual characterization achieved through the indirect function – an increasing polyhedral convex function – which enables us to identify the structure of the pre-kernel.

In addition, – similar to the work of Faigle et al. (2001) – it must be ensured that all maximum surpluses $s_{ij}(\mathbf{x}, v)$ across all distinct pairs of players for every vector $\mathbf{x} \in \mathbb{R}^n$ can be computed by an algorithm in polynomial time. It is then also possible to determine the set of the lexicographically smallest most significant coalitions (14) on the quotient space $dom\, h/\sim$ in polynomial time. Finding the smallest or largest element in an unsorted array requires a linear runtime with respect to the size of the array. Yet, the assumption regarding the maximum surpluses conceals a crucial logical weakness that will reveal itself as a logical flaw as soon as it proves to

be invalid. This is similar to the assumption introduced by Faigle et al. (2001) to overcome the **NP**-hardness problem in calculating a pre-kernel point or the pre-nucleolus using their method. Nevertheless, we do not consider the assumption regarding the maximum surpluses $s_{ij}(\mathbf{x}, v)$ to be critical. This is because the practical relevance of their calculation was demonstrated by our MATLAB software tool `MatTuGames` (cf. Meinhardt, 2025a) for large n values up to 35, whereby the physical limits of our computer resources – with 3 TB of main memory – were successfully exploited within a reasonable amount of time.

A particular advantage is that no oracle queries are required; this is already achieved by the polynomial-time algorithm for determining all possible maximum surpluses $s_{ij}(\mathbf{x}, v)$ for the allocation $\mathbf{x}$. Therefore, there is absolutely no need to compute submodular functions or coalition values via an oracle. Furthermore, the model operates with infinite precision rather than arbitrary precision. Consequently, neither knowledge of the encoding length of the largest number in the model nor knowledge of the total number of bits required to encode the problem is necessary. In our view, a key advantage over the algorithms discussed in the literature is that no restriction is required to bound the search space using a pre-nucleolus catcher. Thus, there is no need to evaluate any additional structure of the problem or to query a separation oracle. It should be obvious that exploiting too much structure of a problem must have a negative impact on an algorithm's performance. In this context, there is also no need to make use of any form of combinatorial properties – such as those of extended polymatroids – or structural properties of a game class, such as convexity. This is because the scheme of orthogonal projection naturally filters the search space and limits the number of iteration steps required to converge to the pre-nucleolus. Hence, Algorithm 3.1 reduces to an algorithm that maps a point $\vec{\gamma}$ via the Formula (19) to a point $\vec{\gamma}_\circ \in M(h_{\vec{\gamma}})$, i.e., to the solution set of a quadratic optimization problem. This can be achieved by solving a system of linear equations with a runtime complexity of $O(n^3)$, which suggests an algorithm running in polynomial time.

5. Replicating the pre-nucleolus

As we have seen above, the Fenchel-Moreau-based algorithm can also be applied to the calculation of the pre-nucleolus, provided that it is located in an exposed position or if it is known that the pre-kernel is a single point, i.e., coincides with the pre-nucleolus. And this is despite the fact that it was originally designed as a highly specialized algorithm specifically for computing the pre-kernel. This appears to significantly limit the applicability of this method in the analysis of the pre-nucleolus. Meinhardt (2024), however, has demonstrated that dual characterization with reference to group algebra opens up theoretical possibilities that allow a single-valued pre-kernel element of a default game to be replicated as its sole pre-kernel element for a set of related games. In this context, replication means demonstrating that a specific solution point from one game can also be found as a solution in another, but related, game, even if the parameters of the original game are modified.

Roughly speaking, this study explores the stability of a pre-kernel point, i.e., the pre-nucleolus, as a bargaining outcome for a related game. Variation bounds in the game parameters are identified without destroying the pre-kernel properties. The pre-kernel element of the default game remains invariant against a change in the game parameter space. Note that such a related game need not have the same game properties as the original game, though it is established that it possesses the same single-valued pre-kernel.

Apart from the single-valuedness of the pre-kernel for the default game, one of these specific conditions leverages the fact that the payoff equivalence class containing the unique pre-kernel point must be full-dimensional, which means that the payoff equivalence class has a nonempty interior. This non-empty interior condition of the payoff equivalence class containing the unique pre-kernel point makes it possible to inscribe a full-dimensional ellipsoid into the payoff equivalence class, from which a null space in the game space can be identified. These variation bounds define the region within the game space in which the pre-kernel element remains invariant under the group action on the set of equivalent ordered bases. This means that within this region, the game parameters can be varied without affecting the pre-kernel properties of this payoff vector. It follows that these bounds specify a redistribution of bargaining power among the coalitions, while the selected pre-imputation remains a pre-kernel point. Even when the values of the maximum surpluses are varied, the set of most effective coalitions remains unchanged by the parameter change. This suggests that a bargaining outcome referring to this specific pre-kernel point remains stable in the face of variations in the game parameter space. Hence, a set of related games can be determined that are linearly independent and possess the selected pre-kernel element of the default game as well as a pre-kernel point. And a fortiori, as its pre-nucleolus.

In particular, the group action of the positive general linear group on the set of ordered bases (bargaining situations) ensures the replication of the pre-kernel element through:

I. The linear mapping of a pre-kernel element into a specific vector subspace of balanced excesses $\mathcal{E}$ consists of a single point.
II. Hence, there is no other pre-kernel element in a payoff equivalence class that belongs to the same class of ordered bases, i.e., the bargaining situation is located in the same orbit. This is reflecting an equivalent bargaining situation, which is transversal to $\mathcal{E}$.
III. There can not exist any other non-transversal vector subspace of balanced excesses $\mathcal{E}_1$ in which a linear transformation of a pre-kernel element can be mapped.
IV. The single pre-kernel element of the default game also preserves the pre-nucleolus property for the related games. This excludes the possibility that there must exist at least a second pre-kernel point, namely the pre-nucleolus.

Meinhardt (2024) generalized these results. It was established there that the pre-kernel must be a single point even on the convex hull encompassing the default game and the related games in the game space, and that it is identical to the element determined by the default game. Furthermore, the pre-kernel correspondence restricted to this convex subset in the game space must be single-valued and consequently continuous. Hence, the pre-nucleolus is an invariant (stable) solution on a subdomain of the game space following the axiomatization of the pre-kernel.

Table 1
Four-person average-convex game v.

Game	$\{1,2\}$	$\{2,3\}$	$\{1,2,3\}$	$\{1,2,4\}$	$\{2,3,4\}$	$\{1,2,3,4\}$
v[a,b,c]	3	4	6	6	4	12

[a] Otherwise, $v(S) = 0$.
[b] Pre-Kernel and Pre-Nucleolus: $(8, 11, 9, 8)/3$.
[c] v is average-convex but not convex.

Table 2
List of Games with the same single-valued Pre-Kernel (Nucleolus)[a] as v.

$\mu = -0.5$ Game	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{3\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$	$\{4\}$
v	0	0	3	0	0	4	6	0
v_1	–5/44	1/367	137/47	–10/83	4/19	171/44	645/104	–5/44
v_2	15/68	–3/46	95/31	1/72	1/3	287/68	293/46	15/68
v_3	–1/9	–7/30	215/79	–14/73	–7/17	35/9	427/74	–1/9
v_4	0	–11/100	153/46	2/15	2/15	4	373/59	0
v_5	0	11/100	73/23	–2/15	–2/15	4	278/45	0
v_6	0	11/100	73/23	–2/15	–2/15	4	278/45	0
v_7	0	11/100	73/23	–2/15	–2/15	4	278/45	0
v_8	0	–1/30	200/67	2/49	2/49	4	143/23	0
v_9	0	–1/30	200/67	2/49	2/49	4	143/23	0
v_{10}	0	–1/30	200/67	2/49	2/49	4	143/23	0
v_{11}	0	–1/30	200/67	2/49	2/49	4	143/23	0

$\mu = -0.5$ Game	$\{1,4\}$	$\{2,4\}$	$\{1,2,4\}$	$\{3,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	N	ACV [b]	SCV [c]	CR [d]
v	0	0	6	0	0	4	12	Y	Y	Y
v_1	–25/124	–4/47	488/83	–8/29	1/25	587/158	12	N	Y	Y
v_2	27/77	2/31	433/72	13/84	24/79	197/47	12	N	Y	Y
v_3	–5/32	–22/79	424/73	–31/116	–27/70	184/47	12	N	Y	Y
v_4	–5/78	–4/23	92/15	9/89	1/217	120/31	12	N	Y	Y
v_5	–17/39	4/23	88/15	–9/89	–55/109	128/31	12	N	Y	Y
v_6	5/78	–15/46	88/15	–9/89	–1/217	127/35	12	N	Y	Y
v_7	5/78	4/23	88/15	–9/89	–1/217	128/31	12	Y	Y	Y
v_8	1/54	–1/67	296/49	–31/77	–18/55	69/19	12	N	Y	Y
v_9	1/54	–1/67	296/49	4/41	–18/55	219/53	12	N	Y	Y
v_{10}	1/54	–1/67	296/49	4/41	5/29	69/19	12	N	Y	Y
v_{11}	1/54	–1/67	296/49	4/41	5/29	219/53	12	Y	Y	Y

[a] Pre-Kernel and Pre-Nucleolus: $(8, 11, 9, 8)/3$.
[b] ACV: Average-Convex Game.
[c] SCV: Semi-Convex Game.
[d] CR: Game with non-empty Core.
[e] Note: Computation performed with MatTuGames (cf. Meinhardt, 2025a).

Example 2. Next, we demonstrate the replication result of a single-valued pre-kernel. By doing so, we consider a four-person average-convex game that is not convex. This game is specified by Table c.

Further inspection reveals that this game has a veto-player, who is player two. A property that is much easier to test than average-convexity, especially for large values of n. Hence, apart from average-convexity, the game is even a veto-rich game, which gives an additional cause that the pre-kernel must coalesce with the pre-nucleolus (cf. Arin and Feltkamp, 1997), whereas its outcome is given by the point $\nu(v) = \mathcal{PK}(v) = (8, 11, 9, 8)/3$. Note that average-convex games are zero-monotonic. For this class of TU games, the pre-kernel coincides with the kernel, respectively, the pre-nucleolus with the nucleolus. Removing duplicates, the set of lexicographically smallest most significant coalitions $\mathcal{S}(\nu(v)) = \{\{1\}, \{3\}, \{4\}, \{2,3\}, \{1,2,4\}\}$ is balanced with weights given by $\{1,1,1,1,1\}/2$. This imputation is even an interior point of the payoff equivalence class composed of this collection. To see this, select an $\epsilon > 0$ sufficiently small and let $\mathbf{z} \in \mathbb{R}^4$ s.t. $z(N) = 0$; then one can establish that $\mathcal{S}(\nu(v) + \epsilon\,\mathbf{z}) = \mathcal{S}(\nu(v))$ holds. Thus, the non-empty interior condition is valid. Due to Meinhardt (2024, Theorem 4.1), a redistribution of the bargaining power among coalitions can be obtained while supporting the imputation $(8, 11, 9, 8)/3$ still as a pre-kernel element for a set of related games. In order to get a null space $\mathcal{N}_{\mathcal{W}}$, we set the parameter μ to -0.5. In this case, the rank of matrix $\mathcal{W}$ must be equal to 4, and we could derive at most 11 linearly independent games that replicate the point $(8, 11, 9, 8)/3$ as a pre-kernel element. Moreover, Meinhardt (2024, Theorem 4.4) even states that this point is also the sole pre-kernel point; hence, the pre-kernel coincides with the pre-nucleolus for these games.

From Table c, we realize that the pre-nucleolus is stable against an induced variation in the game parameter. More precisely, the pre-nucleolus is invariant against a parameter variation within the specific interval of $\mu \in (-0.5, 1.3)$. To get an idea about the underlying bargain stability, we just mention here that the convex hull of the collection of games $\{v, \ldots, v_{11}\}$ form a subset in the

game space – the game setting has not changed, and with it not the figure of argumentation during the bargaining process; only the bargaining power of the coalitions has been changed without affecting the outcome of the game. Within this convex set, each TU game has exactly the above element as its sole pre-kernel point. This implies that we have identified a stable bargaining scenario where a settlement of an agreement is not problematic while referring to the principles of distributive justice (axioms) related to the pre-kernel. The agreement based on these principles remains stable after having varied the parameter set within the specified range. Therefore, the selected agreement point cannot be obstructed by the principles that describe the pre-kernel within this subset of the game space. Though the figure of argumentation has not referred to the norms of distributive arbitration based on the nucleolus, the nucleolus as a bargaining outcome is stable due to its coincidence with the pre-kernel, and the incentive to cooperate remains valid. #

To the best of our knowledge, such a stability analysis of a bargaining outcome has so far been conducted by researchers only for linear solution concepts such as the Shapley value (see Algaba et al. (2020, Chap. 6 & 7)), but not for nonlinear solutions such as the pre-kernel and the pre-nucleolus. However, as we have illustrated, due to the nonlinearity of these solution concepts, this kind of stability analysis requires a significantly broader and more sophisticated set of mathematical tools than is the case with the Shapley value.

6. Synopsis and potential of the Fenchel-Moreau conjugation-based approach

The Fenchel-Moreau-based algorithm employs convex analysis – specifically, Fenchel-Moreau transforms – to dualize the characteristic function. Particularly, it transforms the characteristic function v of a cooperative game into its convex conjugate π, an indirect function of an n-person TU game that provides the same information. This allows, turning non-smooth excess functions into piecewise-smooth, convex and subdifferentiable structures, which simplifies the characterization of the pre-kernel. Specifically, it transforms the piecewise linear, non-convex, and ambiguous structure of maximum surpluses into an overdetermined system of equations, which is equivalent to a minimization problem. The associated objective function is transformed into the composite of a finite family of quadratic convex functions, allowing to find elements of the pre-kernel by iteratively minimizing a family of quadratic functions. This even guarantees polynomial-time solvability when a similar assumption to that made by Faigle et al. (2001) is imposed.

Besides a dual pre-kernel characterization, it offers a vast theoretical advantage as it eases the analysis of the pre-kernel and related solutions considerably. From this simplification of its characterization, a replication result follows: all games derived from a default game with a single-valued pre-kernel must also have this point as their sole pre-kernel point. To get this, one chooses a default game with a single-valued pre-kernel and satisfying the non-empty interior condition of a payoff equivalence class, then a full-dimensional ellipsoid is inscribable within the payoff equivalence class from which a null space in the game space can be identified. Through these variation bounds, the range within the game space is specified, leaving the pre-kernel element invariant under the group action on the set of equivalent ordered bases. This means that, within this range, the game parameters can be varied without affecting the pre-kernel properties of this payoff vector. Hence, a single-valued pre-kernel is replicable under the non-empty interior condition. This invokes the following implications for the pre-nucleolus. The default pre-nucleolus of the original game is a stable bargaining outcome.

Finally, we observe some potential in applying this method to machine learning. In particular, on the technique of feature attribution. This is a technique for explaining a specific model prediction by quantifying how much each input feature, such as a data variable, contributes to a particular outcome. Features, like data, can be treated as players, and an algorithm based on a cooperative game theoretical solution concept assigns credits to them. The most common approach for feature assignment or data evaluation in machine learning based on game theory is the Shapley value. Although its computation is **NP**-hard, polynomial-time algorithms for the Shapley value are only known when symmetry is established. Then, a computation of $O(n^m)$ is feasible, where m is the number of types involved in a homogeneous group game (see, for instance, Hernández-Lamoneda and Sánchez-Sánchez, 2015; Yan and Procaccia, 2021; Kiang, 2026). In contrast, the proposed Fenchel-Moreau conjugation-based approach offers a polynomial-time algorithm for computing even non-symmetric pre-kernel elements of $O(n^3)$. Hence, approximating a pre-kernel element would now be a viable alternative to problems of credit assignment with hundreds or thousands of data variables (players) in machine learning.

Informed consent

This study did not involve human participants, and informed consent was not required.

Ethics approval

This study did not involve human participants, and ethics approval was not required.

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Declaration of competing interest

I, Holger Ingmar Meinhardt, acting as Researcher for the Karlsruhe Institute of Technology (KIT), hereby declare that I do not have any actual, potential, or perceived conflict of interest in relation to the submitted manuscript. I have no financial, personal, or professional interests that could be perceived as impacting my objectivity or integrity. I also confirm that I will not for myself or others, seek or accept any gifts or benefits from any supplier/partner involved. I promise to immediately inform Elsevier if any conflict of interest arises.

Data availability

No data was used for the research described in the article.

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