Expected Hitting Times for Diffusion on p-Adic Arithmetic Schemes

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Abstract

In this article, a Cheeger inequality for integral Laplacian operators on the p-adic analytic manifold given by the integral points of an n-dimensional p-adic integral scheme having a differential n-form ω with poles is proven under the condition that the divisor ω is strictly normal crossing, and that the integral kernel function is a negative power of the geodetic distance on the manifold. A study of the growth of the spectrum is undertaken in the case of large distance powers, including the number of eigenvalues below a given threshold depending on the heat kernel estimates by Bendikov and his collaborators. The Cauchy problem is solved, and a Feynman-Kac formula with bounded potentials established. This culminates in two-sided estimates of the expected hitting time of a fibre of the reduction map depending on the zero and pole orders of the differential form present in the scheme.

1 Introduction

Physical processes like diffusion can exhibit, through their behaviour, properties of the underlying geometry of the state space. In the case of an underlying algebraic structure, the question of how much time one can expect to hit a certain area containing certain types of singularities is of interest, as this can lead to physical experiments designed to explore an algebraically defined manifold, and collect information about its local structure, provided some estimates about expected hitting times for local neighbourhoods can be found depending on the algebraic information contained in such a local neighbourhood. The case of an integral scheme $\mathfrak{X}$ defined over a non-archimedean local field K is of interest, because of the singularities potentially ocurring in its special fibre. The fact that its integral points form a p-adic analytic manifold $\mathfrak{X}(O_K)$, where O_K is the local ring of K, allow p-adic analytic methods recently developped mainly by the author, and partly by his collaborator, to give birth to estimates on expected times for hitting fibres of the natural reduction map $\mathfrak{X}(O_K) \to \mathfrak{X}(\mathbb{F}_q)$ to the points of the scheme $\mathfrak{X}$ defined over the residue field $\mathbb{F}_q$ of K, depending on singularity information contained in the fibre. More precisely, under the condition that there exists an algebraic differential n-form with poles and zeros on $\mathfrak{X}$, where n is the dimension of $\mathfrak{X}$, whose

divisor satisfies the strictly normal crossing property, the theory of Laplacian integral operators using a geodetic-distance kernel function on p-adic analytic manifolds can be extended towards solving the task of finding such estimates via the solution of an associated Dirichlet problem.

Boundary value problems on p-adic domains have been studied quite recently, prominent examples being given by Kochubei and Antoniouk [1] for Neumann boundary value problems, Kochubei [22] for Dirichlet problems, of which the diffusion on a p-adic ball is already addressed in his book [20], cf. also [21]. Neumann boundary value problems on compact p-adic analytic manifolds are studied by the author in [6], where they are used in order to develop a zero searching method for functions on the manifold. This method is extended in [7] to schemes of the mentioned type, and includes branched covering maps.

In order to be able to estimate expected hitting times, a spectral analysis of a generator $\mathscr{L}$ of a Markov process is helpful, as examples from the literature show [26, 24]. Exit times and their probabilities on the p-adic number field $\mathbb{Q}_p$ are studied in [36, 29]. But the point in our case is that a Green kernel for the solution of the associated Dirichlet problem, expressed in terms the spectrum of $\mathscr{L}$, can be employed. If the operator $\mathscr{L}$ is enhanced with a potential, then Feynman-Kac formula come into bearing, cf. [36] among other articles by D. Weisbart on a path-space approach towards this end. Most prominent results obtained by studying path spaces are his scaling limit theorems, cf. [37, 28]. Schrödinger equations over the p-adics with non-local potentials are investigated in [34]

This article continues the work on extracting number-theoretic information via stochastic processes with the latest result being the ability to hear Tamagawa factors from the spectrum of a p-adic Laplacian on a p-adic abelian variety [5], and sits beside recent work on p-adic advection-diffusion [10] as well as continuous-time random walks on the unit interval transported from the p-adic unit disc, which is in particular useful for visualisation purposes of p-adic processes [9].

The only requirements from algebraic geometry required for this task is the concept of a strictly normal crossing $\mathbb{Q}$-divisor which is leisurely introduced in this article. The book [25] is a helpful companion for deepening one's knowledge in related algebraic geometry. In the previous article [7], the essentially self-adjoint operator $\Delta^{\alpha}_{\omega^{1/r}}$ was studied. Here, it turns out that it is already closed due to the existence of an orthonormal basis of the L^2-space $L^2(\mathfrak{X}(O_K), \mu_\omega)$ consisting of locally constant eigenfunctions of $\Delta^{\alpha}_{\omega^{1/r}}$, cf. Proposition 4.7. Hence, the Dirichlet Laplacian $\bar{\Delta}^{\alpha}_{\omega^{1/r}}$ used in *loc. cit.* coincides with $\Delta^{1/r}_{\omega}$.

Another distinction needs to be highlighted, in order to not become a cause of confusion: An algebraic differential n-form ω on an n-scheme $\mathfrak{X}$ will in very many cases have zeros, whereas an analytic differential n-form can be found on $\mathfrak{X}(O_K)$ that has no zeros, because of compactness. This result can be attributed to Serre in [31] for general compact p-adic

analytic manifolds. Such an analytic differential n-form gives rise to the so-called *canonical measure* ν_{can}, and e.g. Oesterlé in [27] makes heavy use of this. But also ω gives rise to a measure μ_ω by generalising the process from form to measure by Weil [35, Chapter 2.2], since the zero set of ω has canonical measure zero, cf. [38, Lemma 3.1]. Oesterlé proved a strong continuity relation between μ_ω and ν_{can} which we exploit in this article. What is "canonical" about the canonical measure is that it naturally comes from an integral structure on $\mathfrak{X}(O_K)$ for free, as $\mathfrak{X}$ is an O_K-scheme.

The first main result of this article is a Cheeger inequality for the integral operator

$$\Delta^{\alpha}_{\omega^{1/r}} u(x) = \int\limits_{\mathfrak{X}(O_K)} d_\omega(x,y)^{-\alpha}(u(x)-u(y))\, d\mu_\omega(y)$$

for test functions $u\colon \mathfrak{X}(O_K) \to \mathbb{R}$. Here, d_ω is geodetic distance on the compact p-adic analytic manifold $\mathfrak{X}(O_K)$ endowed with a finite atlas compatible with the canonical integral structure on the manifold, and μ_ω is the measure on $\mathfrak{X}(O_K)$ coming from an algebraic differential form ω with poles on $\mathfrak{X}$ as

$$\mu_\omega(A) = \int_A |f_U(x)|^{1/r}\, |dx| \ ,$$

where ω is given on a chart U of $\mathfrak{X}(O_K)$ containing a Borel set $A \subset \mathfrak{X}(O_K)$ as

$$\omega|_U = f_U\, dx$$

with f a rational function, and where $|dx|$ is the Haar measure (coming from dx) on K^n via the chart map $U \to K^n$. The natural number r is meant to be sufficiently large that the measure μ_ω is finite. This is possible according to [19, Proposition 3.5]. Extending the definition of Cheeger constant from graph theory as in [15] to a Cheeger constant h_α given for $\Delta^{\alpha}_{\omega^{1/r}}$, the theorem is stated as:

Theorem 4.2. *It holds true that*

$$\lambda_1 \geq \frac{1}{2} h_\alpha^2 \, ,$$

where λ_1 is the spectral gap of $\Delta^{\alpha}_{\omega^{1/r}}$.

Notice that in [7, Corollary 4.8], a spectral gap of $\Delta^{\alpha}_{\omega^{1/r}}$ was already established for $\alpha > 1$. This result is followed by technical results, after which the number of eigenvalues of $\Delta^{\alpha}_{\omega^{1/r}}$ below a given threshold is determined via a heat kernel estimate by Bendikov and coauthors. This estimate is taken from [16, Theorem 5.1]. The statement:

Theorem 4.11. *Assume $\alpha > 1$. Then the spectral mass function $M(\lambda)$ associated with $\Delta^{\alpha}_{\omega^{1/r}}$ satisfies*

$$M(\lambda) \leq eC\lambda^{1/(\alpha-1)}$$

for $\lambda > 1$, and some $C > 0$.

The Cauchy problem for $\Delta^{\alpha}_{\omega^{1/r}}$ over $L^2(\mathfrak{X}(O_K), \mu_\omega)$ is addressed:

Theorem 4.12. *The Cauchy problem associated with $\Delta^{\alpha}_{\omega^{1/r}}$ on $L^2(\mathfrak{X}(O_K), \mu_\omega)$ with continuous initial condition $u_0 \operatorname{dom}(\Delta^{\alpha}_{\omega^{1/r}})$ has a unique solution*

$$u(t,x) = \int_{\mathfrak{X}(O_K)} u_0(y) p_t(x,y)\, d\mu_\omega(y)$$

with $u(\cdot, x)$ continuously differentiable for each $x \in \mathfrak{X}(O_K)$, and $u(t,\cdot) \in L^2(\mathfrak{X}(O_K), \mu_\omega)$, where $p_t(\cdot,\cdot)$ is an L^2-heat kernel function on $\mathfrak{X}(O_K)^2$. The semigroup acting on $C(\mathfrak{X}(O_K))$ has the Feller property, and thus is attached to a strong Markov process having càdlàg sample paths.

After proving a general Feynman-Kac formula for stochastic processes valued in $\mathfrak{X}(O_K)$ (Theorem 5.1), the finiteness of expected hitting times is proven as solutuions of a certain Dirichlet problem for

$$\mathscr{L} = \Delta^{\alpha}_{\omega^{1/r}} + V$$

with $V\colon \mathfrak{X}(O_K) \to \mathbb{R}$ a continuous potential bounded from below. This is Theorem 5.2. This culminates in two-sided bounds of expected hitting times of a fibre $\rho^{-1}(\bar{a})$ of the reduction map

$$\rho\colon \mathfrak{X}(O_K) \to \mathfrak{X}(\mathbb{F}_q)$$

in terms of the ball of canonical measure q^{-n} having minimal μ_ω-volume (lower bound), and the μ_ω-volume of $\mathfrak{X}(O_K)$ (upper bound). The latter is meaningful, too, because it depends on the distribution of all possible zero and pole orders of ω along the singularities of the special fibre $\mathfrak{X}(\mathbb{F}_q)$ of the scheme $\mathfrak{X}$. We view this as a first result, as the bounds express singularities in the scheme which are independent of the particular fibre $\rho^{-1}(\bar{a})$. A natural continuation of this research is to investigate fibre-specific two-sided bounds.

The following section recalls facts about geodetic distance, and Laplacian integral operators on p-adic analytic manifolds using it, from the author's previous work. Section 3 shows how these notions can be extended to $\mathfrak{X}(O_K)$, where $\mathfrak{X}$ is a smooth, separeted O_K-scheme of finite type and dimension n, endowed with an algebraic differential n-form with zeros and poles, together with the strictly normal crossing condition. Section 4 is subdivided into four subsection: 4.1 Cheeger inequalities, 4.2 spectral growth, 4.3 spectral mass, and 4.4 Cauchy problem. The last section 5 contains two subsections: 5.1 Feynman-Kac formula, and 5.2 expected time to hitting singularities.

2 Diffusion on p-adic analytic manifolds

We recall here the approach and some results of [4] about diffusion on p-adic analytic manifolds. The description here is to separate the state space into two kinds: a collection of small

balls disjointly covering the p-adic analytic manifold M, with a Vladimirov-type diffusion on each of these, and a diffusion as defined in graph theory on the face poset associated with the (connected!) nerve complex given by a finite equalising atlas on M. A brief intro into some of these concepts is now given.

Let A be the weighted adjacency matrix of a finite undirected graph. Then the Laplacian is defined as

$$L = D - A\,,$$

where A is the adjacency matrix of a weighted graph structure on the the poset associated with the nerve complex $N(\mathcal{A})$, where $\mathcal{A}$ is a finite equalising atlas chosen on the compact p-adic analytic manifold M. By compactness, M has a finite atlas, and the equalising property means that the transition maps between chart images in K^n of overlapping charts in $\mathcal{A}$ take balls to balls of equal radius. This property follows from O_K-compatibility, a requirement intimately related with the existence of a nowhere vanishing differential n-form ω on M, where n is the dimension of M, cf. [31, Théorème (2)]. This intimate relationship is explained e.g. in [17, Section I.1], where it is even called a *Riemannian structure*, whereas in [12, §3.2], it is named *R-structure*, where in our case $R = O_K$, the integral ring of a non-archimedean local field K. We also say *integral structure* to an O_K-structure, as the ring O_K is fixed, here. More on p-adic analytic manifolds can be found in the books [35, 18, 30, 32].

An integral structure Λ defines a Radon measure μ_Λ on M, and this is helpful in defining the *geodetic distance* on the triple $(M, \mathcal{A}, \omega)$. Details on this can be found in [4, §3.2]. The required equalising property comes for free, if one restricts to an O_K-compatible atlas $\mathcal{A}$, i.e. if all transition maps between overlapping charts in $\mathcal{A}$ are O_K-analytic, since in this case, all balls in K^n are taken to balls of equal radius. The geodetic distance itself is introduced in [4, §3.2] as

$$d_g(x,y) = \begin{cases} \mu_\Lambda(x \wedge y), & x, y \text{ contained in a ball of } M \\ \min\limits_{\gamma\colon\, x \rightsquigarrow y} \mu_\Lambda(U_\gamma(x,y)), & x \wedge y \text{ does not exist in } \mathfrak{B}(M,\mathcal{A}) \end{cases}, \tag{1}$$

where $\mathfrak{B}(M, \mathcal{A})$ is the poset whose points are the balls of M and the compact-open subsets of M associated with the simplices of the nerve complex $N(\mathcal{A})$ of the atlas $\mathcal{A}$. The *join*

$$x \wedge y \in \mathfrak{B}(M, \mathcal{A})$$

of $x \neq y$ in M is the smallest ball in M containing both x and y, if it exists. Of course, if $x = y \in M$, then we define $d_g(x, y) = 0$. This is a genuine distance function, if the nerve complex $N(\mathcal{A})$ is connected [4, Lemma 3.3]. Therefore, connectivity of $N(\mathcal{A})$ is assumed throughout this article.

Before defining a Laplacian integral operator on M, notice that the graph-theoretic closure and boundary can be extended for any kernel function

$$w\colon M \times M \to \mathbb{R}_+$$

as

$$\begin{aligned}\delta_w\Omega &= \{y \in M \setminus \Omega \mid \exists\, x \in \Omega\colon w(x,y) \neq 0\}\\ \mathrm{cl}_w\,\Omega &= \Omega \cup \delta_w\Omega\,,\end{aligned}$$

and these are the w-boundary and the w-closure of Ω in M w.r.t. the kernel function w, respectively.

In the present case, a kernel function based on the geodetic distance $d_\mu(x,y)$ will be used. It is obtained by replacing in (1) the measure μ_Λ with a more general Radon measure μ on M. More precisely, this Radon measure will depend on an *algebraic* differential n-form on $M = \mathfrak{X}(O_K)$ with both zeros and poles, where $\mathfrak{X}$ is an O_K-scheme of a certain quite general kind. The following section will give a more detailed account on this.

Denote the face σ from the face poset $|N(\mathcal{A})|$, whose associated compact open U_σ is the smallest one containing $x \in M$, as $\sigma(x)$. Thus define the following kernel function:

$$w(x,y) = d_\mu(x,y)^{-\alpha} \tag{2}$$

for $x, y \in M$ and $\alpha > 0$. Obtain now the integral operator:

$$\mathcal{L}u(x) = \int_M w(x,y)(u(x) - u(y))\, d\mu(y)$$

as a diffusion Laplacian for functions $u\colon M \to \mathbb{C}$.

Remark 2.1. *Since the face poset $|N(\mathcal{A})|$ is in a natural way a finite T_0-space, this operator here in a sense is related to the one from [9]. However, in that article, the operator would be advection on $|N(\mathcal{A})|$, whereas here it is symmetric.*

The diffusion equation of interest is

$$\frac{\partial}{\partial t}u(t,x) + \mathcal{L}_\Omega u(t,x) = 0\,,$$

and in particular its Cauchy problem, but specialised to the situation of an $\mathfrak{X}(O_K)$-scheme and the Radon measure on $\mathfrak{X}(O_K)$ induced by a differential form ω on $\mathfrak{X}$ with zeros and poles. Also, the ability to address Dirichlet problems related to such a Laplacian integral operator are the theme for the following sections. In [9], such Dirichlet problems were considered for related advection-diffusion processes on finite state spaces with finite rooted p-regular trees attached to the vertices of a finite topological T_0-space.

3 Operators on SNCD schemes

Let in the following $\mathfrak{X}$ be an O_K-scheme. By this we mean a morphism $\mathfrak{X} \to \mathrm{Spec}(O_K)$ of schemes which is smooth, separated, and of finite type of dimension n. The set $\mathfrak{X}(O_K)$ of

O_K-rational points of $\mathfrak{X}$ bears a natural structure of a compact p-adic analytic n-manifold as an open inside the space $\mathfrak{X}(K)$. Let $\mathfrak{U}$ be an open subscheme of $\mathfrak{X}$, and let

$$\omega \in \Gamma\left(\mathfrak{U}, \left(\Omega^n_{\mathfrak{X}/O_K}\right)^{\otimes r}\right) \tag{3}$$

be an r-pluricanonical n-form on $\mathfrak{X}$ with poles outside $\mathfrak{U}$. It gives rise to a measure $\mathrm{mod}(\omega)^{\frac{1}{r}}$ on $\mathfrak{X}(O_K)$ as

$$\int_U d\mathrm{mod}(\omega)^{\frac{1}{r}} = \int_U |g_U(x)|^{\frac{1}{r}} \, |dx| \ ,$$

where on the chart $U \subset \mathfrak{X}(O_K)$ the differential form ω is presented as

$$\omega = g_U \, dx^{\otimes r}$$

for some O_K-analytic function $g_U \colon U \to O_K$.

Assumption. *It is assumed that* $\mathrm{div}(\omega)$ *is a strictly normal crossing divisor as a* $\mathbb{Q}$*-divisor over* $\mathrm{Spec}(O_K)$*. Further, it is assumed that* $r >> 0$*.*

Under this assumption, we can write

$$\mathrm{div}(\omega) = \sum_{i=1}^{s} a_i \mathfrak{D}_i \, ,$$

where the irreducible pieces $\mathfrak{D}_1, \ldots, \mathfrak{D}_s$ are smooth over O_K. Define

$$\mathfrak{D}_J = \bigcap_{i \in J} \mathfrak{D}_i \, , \quad \mathfrak{D}_J^\circ = \mathfrak{D}_J \setminus \bigcup_{j \in \{1,\ldots,s\} \setminus J} \mathfrak{D}_J$$

for $J \subset \{1, \ldots, s\}$, and the understanding that $\mathfrak{D}_\emptyset = \mathfrak{X}$. The assumption $r >> 0$ plays out as follows:

Lemma 3.1. *It holds true that*

$$\mathrm{mod}(\omega)^{\frac{1}{r}}(\mathfrak{X}(O_K)) = \frac{1}{q^n} \sum_{J \subset \{1,\ldots,s\}} |\mathfrak{D}_J^\circ(\mathbb{F}_q)| \prod_{j \in J} \frac{q-1}{q^{(a_j/r)+1} - 1} \, ,$$

if $a_1, \ldots, a_s > -r$*. In particular, the measure* $\mathrm{mod}(\omega)^{\frac{1}{r}}$ *is finite for* $r >> 0$*.*

Proof. [19, Proposition 3.5]. □

We begin with an adaptation of the small wavelet eigenvalue formula for O_K-schemes $\mathfrak{X}$ with a pluricanonical differential form ω satisfying our assumptions.

Endowing $\mathfrak{X}(O_K)$ with the Radon measure $\mathrm{mod}(\omega)^{\frac{1}{r}}$ is equivalent to specifying the Radon-Nikodym derivative

$$\left\| \omega^{\frac{1}{r}} \right\| := \frac{d\mathrm{mod}(\omega)^{\frac{1}{r}}}{d\nu_{\mathrm{can}}} \, ,$$

where ν_{can} is the *canonical measure* on $\mathfrak{X}(O_K)$. This is the unique Radon measure satisfying constancy on the fibres of the maps

$$\mathfrak{X}(O_K) \to \mathfrak{X}(O_K/\pi O_K)$$

for $n \geq 1$ with fibre measure q^{-nk} for $k \in \mathbb{N}$. Here, we assume that O_K^n has Haar measure equal to 1. The existence of the Radon-Nikodym derivative is given for $r >> 0$ by the following theorem:

Theorem 3.2 (Oesterlé 1984)**.** *It holds true that*

$$\int_A d\mathrm{mod}(\omega)^{\frac{1}{r}} = \int_A \left\|\omega^{\frac{1}{r}}(x)\right\| \, d\nu_{\mathrm{can}}(x)$$

for any Borel set $A \subseteq \mathfrak{X}(O_K)$.

Proof. Cf. [27, Theorem 2.4]. □

Remark 3.3. *An abstract existence proof of the Radon-Nikodym derivative $\left\|\omega^{1/r}\right\|$ can also be found in [7, Lemma 3.3]. The relationship to the previous section is that the canonical measure ν_{can} is of the form μ_Λ for a suitable integral structure Λ on the compact p-adic analytic n-manifold $M = \mathfrak{X}(O_K)$. The reason is that ν_{can} is given by a nowhere vanishing* analytic *differential n-form on $\mathfrak{X}(O_K)$, and an integral structure giving rise to this measure can be obtained, as explained in e.g. [12, Section 3.3]. In particular, the zeros and poles of ω are all encoded in the distribution function $\left\|\omega^{\frac{1}{r}}(x)\right\|$ of the measure $\mathrm{mod}(\omega)^{\frac{1}{r}}$ w.r.t. the canonical measure.*

The following notation for the measure on schemes will be used:

$$\mu_\omega := \mathrm{mod}(\omega)^{\frac{1}{r}} ,$$

where ω is a pluricanonical form as in (3) on $\mathfrak{X}$ satisfying the Assumption above.

Let the kernel functions $w(x, y)$ with $x, y \in \mathfrak{X}(O_K)$ be defined similarly as in (2), only that the distance $d_\omega(x, y)$ is now being used. Using [38, Lemma 3.1], one readily sees that the zero sets of μ_ω are all of codimension at least one. Hence, the measure μ_ω is non-vanishing on all open subsets of $\mathfrak{X}(O_K)$, and finite by the Assumption above.

Remark 3.4. *We remark that since compact open subsets of $\mathfrak{X}(O_K)$ have positive μ_ω-measure, the notion of geodetic distance can be defined using the measure μ_ω in (1).This then is also a distance function even in the case that μ_ω does have zeros.*

The instantiation of the operator $\mathcal{L}$ in the previous section to the case $M = \mathfrak{X}(O_K)$ and $\mu = \mu_\omega$ will be denoted as $\Delta^\alpha_{\omega^{1/r}}$, written out as

$$\Delta^\alpha_{\omega^{1/r}} u(x) = \int_{\mathfrak{X}(O_K)} d_\omega(x, y)^{-\alpha} (u(x) - u(y)) \, d\mu_\omega(y)$$

for $x \in \mathfrak{X}(O_K)$, $\alpha \in \mathbb{R}$, and $u\colon \mathfrak{X}(O_K) \to \mathbb{C}$. Notice that this is the kernel function

$$w(x,y) = d_\omega(x,y)^{-\alpha}$$

with $\alpha \in \mathbb{R}$.

In this general setting, something can be said about the spectrum of $\Delta^\alpha_{\omega^{1/r}}$:

Proposition 3.5 (small wavelet eigenvalue, pluricanonical)**.** *The wavelets ψ with small support in $\mathfrak{X}(O_K)$ are eigenfunctions of $\Delta^\alpha_{\omega^{1.r}}$ with corresponding eigenvalue*

$$\lambda_\psi = \int_{\mathfrak{X}(O_K)\setminus B_\ell(z)} d_\omega(z,y)^{-\alpha}\, d\mu_\omega(y) + \left(1 - q^{-n}(1+(-1)^n)\right) \mu_\omega(B_\ell(z))^{1-\alpha/n}\,,$$

where $\operatorname{supp}(\psi) = B_\ell(z) \subset \mathfrak{X}(O_K)$ *with $\ell \in \mathbb{Z}$ is a sufficiently small ball not containing a zero or pole of ω.*

Proof. The Radon-Nikodym derivative $\left\|\omega^{1/r}\right\|$ for $r >> 0$ exists due to Theorem 3.2. It is locally constant outside of the support of $\operatorname{div}(\omega)$ by the algebraicity of the pluricanonical form ω, as can be seen from its local description (3). Hence, the proof of the small wavelet eigenvalue formula in [11, Proposition 3.3] generalises to this case, which yields the asserted formula. Notice that the statement of Proposition 3.3 in [11] omits the factor

$$(1 - q^{-n}(1+(-1)^n)$$

in the second term, whereas its proof does not. In both cases, there is a typo in the exponent of the ball volume. The proof of [8, Lemma 2.6] observes that the kernel function $w_\Omega(x,y)$ needs to be ultrametric only for $x, y \in B_\ell(z)$, the wavelet support. The reason lies in generalising the Kozyrev wavelet formula [23, Theorem 3]. What is also needed, is the vanishing mean of a wavelet ψ, and that is the reason for the support of ψ having to be sufficiently small, in order for ψ to be an eigenfunction of $\mathscr{L}$. □

4 Spectral asymptotics

This section is devoted to a study of the spectrum of $\Delta^\alpha_{\omega^{1/r}}$ for $\alpha >> 1$. Due to the fact that the distribution function of the measure μ_ω w.r.t. the canonical measure on $\mathfrak{X}(O_K)$ is not locally constant, this study is somewhat more involved than otherwise.

4.1 Cheeger inequalities

The Cheeger constant from graph theory inspires the following definition:

Definition 4.1. *The quantitiy*

$$h_\alpha = \inf\left\{ \frac{\mu_{(\alpha)}(\partial_w \Omega)}{\mu_\omega(\Omega)} \mid 0 < \mu_\omega(\Omega) \leq \frac{1}{2}\mu_\omega(\operatorname{cl}_w \Omega) \right\}$$

with

$$\mu_{(\alpha)}(\partial_w\Omega) = \int_\Omega \int_{\delta_w\Omega} d_\omega(x,y)^{-\alpha} d\mu_\omega(x)\, d\mu_\omega(y)$$

and

$$\partial_w\Omega = \Omega \times \delta_w\Omega$$

for $\Omega \subseteq \mathfrak{X}(O_K)$ measurable, is the Cheeger constant *for the operator $\mathscr{L}_\alpha$.*

Theorem 4.2 (Cheeger inequality)**.** *It holds true that*

$$\lambda_1 \geq \frac{h_\alpha^2}{2}$$

for $\alpha > 0$.

Proof. First, the lower bound for the spectral gap.

Let ϕ be a real-valued eigenfunction of $\Delta^\alpha_{\omega^{1/r}}$ having eigenvalue λ_1. Then define the sets

$$V[\phi]^+ = \{x \in \mathfrak{X}(O_K) \mid \phi(x) \geq 0\}, \quad V[\phi]^- = \{x \in \mathfrak{X}(O_K) \mid \phi(x) < 0\}\,.$$

W.l.o.g. assume that $\mu_\omega(V[\phi]^+) \leq \frac{1}{2}\mu_\omega(\mathfrak{X}(O_K))$. Otherwise replace ϕ with $-\phi$. Using the Gauss-Green formula [6, eq. (2)], obtain that

$$\lambda_1\langle\phi,\phi_+\rangle = \langle\mathscr{L}_\alpha\phi,\phi_+\rangle = \frac{1}{2}\iint\limits_{\mathfrak{X}(O_K)^2} d_\omega(x,y)^{-\alpha}(\phi(x)-\phi(y))(\phi_+(x)-\phi_+(y))\, d\mu_\omega(x)\, d\mu_\omega(y)\,.$$

Using that

$$(\phi(x)-\phi(y))(\phi_+(x)-\phi_+(y)) \geq (\phi_+(x)-\phi_+(y))^2\,,$$

it now follows that

$$\lambda_1 \geq R_\alpha[\phi_+]\,,$$

whereby $\phi_+ \neq 0$, as otherwise $\langle\phi,1\rangle = 0$ would imply that $\phi_- = 0$, which is impossible since $\phi \neq 0$. Hence, it suffices to show that

$$R_\alpha[\phi_+] \geq \frac{h_\alpha^2}{2}\,.$$

Since

$$\mu_\omega(\{\phi_+ > 0\}) \leq \mu_\omega(V[\phi]^+) \leq \frac{1}{2}\mu_\omega(\mathfrak{X}(O_K))\,,$$

it is possible to apply [7, Lemma 4.4] to ϕ_+^2, and this yields

$$\iint\limits_{\mathfrak{X}(O_K)^2} d_\omega(x,y)^{-\alpha}(\phi_+(x)^2-\phi_+(y)^2)\, d\mu_\omega(x)\, d\mu_\omega(y) \geq h_\alpha \int\limits_{\mathfrak{X}(O_K)} \phi_+(x)^2\, d\mu_\omega(x) \tag{4}$$

The following estimate:

$$\begin{aligned}
&\iint_{\mathfrak{x}(O_K)^2} d_\omega(x,y)^{-\alpha}\left(\phi_+(x)^2-\phi_+(y)^2\right)\,d\mu_\omega(x)\,d\mu_\omega(y)\\
&\qquad=\frac{1}{2}\iint_{\mathfrak{x}(O_K)^2} d_\omega(x,y)^{-\alpha}\left|\phi_+(x)^2-\phi_+(y)^2\right|\,d\mu_\omega(x)\,d\mu_\omega(y)\\
&\qquad=\frac{1}{2}\iint_{\mathfrak{x}(O_K)^2} d_\omega(x,y)^{-\frac{1}{2}\alpha}\left|\phi_+(x)-\phi_+(y)\right|\,d\omega(x,y)^{-\frac{1}{2}\alpha}\left|\phi_+(x)-\phi_-(y)\right|\,d\mu_\omega(x)\,d\mu_\omega(y)\\
&\qquad\le\left(\frac{1}{2}\iint_{\mathfrak{x}(O_K)^2} d_\omega(x,y)^{-\alpha}(\phi_+(x)-\phi_+(y))^2\,d\mu_\omega(x)\,d\mu_\omega(y)\right.\\
&\qquad\qquad\left.\times\frac{1}{2}\iint_{\mathfrak{x}(O_K)^2} d_\omega(x,y)^{-\alpha}(\phi_+(x)+\phi_+(y))^2\,d\mu_\omega(x)\,d\mu_\omega(y)\right)^{\frac{1}{2}},
\end{aligned}$$

where the last inequality is the Cauchy-Schwarz inequality. Using the identity

$$\frac{1}{2}(a+b)^2\le a^2+b^2\,,$$

it now follows that

$$\begin{aligned}
&\iint_{\mathfrak{x}(O_K)^2} d_\omega(x,y)^{-\alpha}\left|\phi_+(x)^2-\phi_+(y)^2\right|\,d\mu_\omega(x)\,d\mu_\omega(y)\\
&\qquad\le\left(\iint_{\mathfrak{x}(O_K)^2} d_\omega(x,y)^{-\alpha}\left|\phi_+(x)-\phi_+(y)\right|^2\,d\mu_\omega(x)\,d\mu_\omega(y)\right.\\
&\qquad\qquad\left.\times\iint_{\mathfrak{x}(O_K)^2}\left(\phi_+(x)^2+\phi_+(y)^2\right)\,d\mu_\omega(x)\,d\mu_\omega(y)\right)^{\frac{1}{2}}\\
&\qquad=\left(2\iint_{\mathfrak{x}(O_K)^2} d_\omega(x,y)^{-\alpha}(\phi_+(x)-\phi_+(y))^2\,d\mu_\omega(x)\,d\mu_\omega(y)\right.\\
&\qquad\qquad\left.\times\iint_{\mathfrak{x}(O_K)^2} d_\omega(x,y)^{-\alpha}\phi_+(x)^2\,d\mu_\omega(x)\,d\mu_\omega(y)\right)^{\frac{1}{2}}.
\end{aligned}$$

Together with (4), this yields

$$h_\alpha \int\limits_{\mathfrak{X}(O_K)} \phi_+(x)^2 \, d\mu_\omega(x) \leq \left(2 \iint\limits_{\mathfrak{X}(O_K)^2} d_\omega(x,y)^{-\alpha} \, |\phi_+(x) - \phi_+(y)|^2 \right)^{\frac{1}{2}} \cdot \left(\int\limits_{\mathfrak{X}(O_K)} \phi_+(x)^2 \, d\mu_\omega(x) \right)^{\frac{1}{2}},$$

from which the asserted spectral lower bound follows by taking the square. □

Lemma 4.3. *It holds true that*

$$h_\alpha \geq \frac{1}{2} \mu_\omega(\mathfrak{X}(O_K))^{1-\alpha}$$

for $\alpha \geq 0$.

Proof. Observe that

$$d_\omega(x,y)^{-\alpha} \geq \operatorname{diam}_\omega(\mathfrak{X}(O_K))^{-\alpha} \geq \mu_\omega(\mathfrak{X}(O_K))^{-\alpha}$$

for $x, y \in \mathfrak{X}(O_K)$ and $\alpha \geq 0$. Hence,

$$h_\alpha \geq \mu_\omega(\mathfrak{X}(O_K))^{-\alpha} \, \frac{1}{2} \mu_\omega(\mathfrak{X}(O_K))$$

for any $\Omega \subset \mathfrak{X}(O_K)$ such that $0 < \mu_\omega(\Omega) \leq \frac{1}{2} \mu_\omega(\mathfrak{X}(O_K))$. This implies the assertion. □

Corollary 4.4. *Let* $\alpha \geq 0$. *Then*

$$\lambda_1 \geq \frac{1}{8} \mu_\omega(\mathfrak{X}(O_K))^{2-\alpha}$$

holds true.

Proof. This is an immediate consequence of Theorem 4.2 and Lemma 4.3. □

Remark 4.5. *In [7, Proposition 4.7], the spectral gap is bounded from below in terms of the largest pole order of* ω. *Here, it is given in terms of the Cheeger constant* h_α, *and its lower bound in terms of the volume of* $\mathfrak{X}(O_K)$.

By identifying balls in $\mathfrak{X}(O_K)$ with points, one obtains a finite quotient graph as follows: Let $\sim_k$ be the equivalence relation on $\mathfrak{X}(O_K)$ which identifies the fibres of the reduction map

$$\rho_k \colon \mathfrak{X}(O_K) \to \mathfrak{X}(O_K/\pi^k O_K)$$

with points. Notice that these fibres are balls in $\mathfrak{X}(O_K)$ for each $k \in \mathbb{N}$. Denote the quotient set as

$$X_k = \mathfrak{X}(O_K)/\sim_k ,$$

and endow it with the pushforward measure

$$\mu_\omega([x]_k) := \mu_\omega\left(\{y \in \mathfrak{X}(O_K) \mid y \sim_k x\}\right) ,$$

and the descended distance

$$d_\omega([x]_k, [y]_k) := \begin{cases} d_\omega(x,y), & [x]_k \neq [y]_k \\ 0, & \text{otherwise} \end{cases}$$

for $x, y \in \mathfrak{X}(O_K)$. This is well-defined, because the geodetic distance is locally the p-adic norm on K^n. The graph structure on X_k is now given via the p-adic analytic manifold $(\mathfrak{X}(O_K), \mathcal{A})$ as follows: namely, X_k is the vertex set of the complete weighted graph with edge weight

$$w([x]_k, [y]_k) = d_\omega([x]_k, [y]_k)$$

for $[x]_k \neq [y]_k$, and vertex measure μ_ω. This finite graph will be denoted as

$$G_k = (X_k, w, \mu_\omega) ,$$

and generalises in some sense the kind of graphs studied in [15]. Denote the corresponding graph Laplacian as L_k^α, and now we can define its Cheeger constant as

$$h_{G_k} = \inf\left\{ \frac{\mu_\alpha(\partial_w \Omega)}{\mu_\omega(\Omega)} \mid 0 < \mu_\omega(\Omega) \leq \frac{1}{2}\mu_\omega(X) \right\},$$

where this time Ω runs through all subsets of X_k satisfying the condition. Of course, we have that

$$\mu_\omega(X) = \mu_\omega(\mathfrak{X}(O_K))$$

holds true. The measure μ_α is defined accordingly as

$$\mu_\alpha(B \times C) = \sum_{v \in B} \sum_{w \in C} d_\omega(v,w)^{-\alpha} \mu_\omega(v)\mu_\omega(w)$$

for $B, C \subseteq X_k$. Since we have not found a reference, we will now state the Cheeger inequalities for the spectral gap of L_k^α in terms of the Cheeger constant defined above. It proof is a slight generalisation of [15, Lemma 3.2 and Theorem 3.3], and we will only give a proof of the upper bound, because it is needed in following sections.

Theorem 4.6 (Cheeger Inequalities for weighted measure graphs)**.** *It holds true that*

$$2h_{G_k} \geq \lambda_{k,1} \geq \frac{1}{2} h_{G_k}^2 ,$$

where $\lambda_{k,1}$ is the spectral gap of the Laplacian matrix L_k^α.

Proof. The upper bound (Cf. the proof of [15, Lemma 3.2]).

Let $\Omega \subset X_k$ be such that the infimum defining h_{G_k} is attained. The function

$$f(x) = \begin{cases} 1, & x \in \Omega \\ -a, & x \notin \Omega \end{cases}$$

with

$$a = \frac{\mu_\omega(\Omega)}{\mu_\omega(X_k \setminus \Omega)} \leq 1$$

satisfies

$$f \perp 1_X \,.$$

Since $\lambda_{k,1}$ as the spectral gap minimises Rayleigh quotients, we have

$$\lambda_{k,1} \leq R_{G_k}[f] = \frac{\langle L_k^\alpha f, f\rangle}{\langle f, f\rangle} \,,$$

and thus it suffices to prove

$$R_{G_k}[f] \leq 2h_{G_k} \,.$$

This is now seen as follows:

$$\langle f, f\rangle = \sum_{v\in X} f(v)^2 \mu_\omega(v) = \mu_\omega(\Omega) + a^2\mu_\omega(X_k \setminus \Omega) = (1+a)\mu_\omega(\Omega) \,,$$

and

$$\begin{aligned} \langle L_k^\alpha f, f\rangle &= \frac{1}{2}\sum_{v,w} (f(v) - f(w))^2 \, \mu_\alpha(v,w) \\ &= \sum_{v\in\Omega}\sum_{w\in X_k\setminus\Omega} (f(x) - f(y))^2 \mu_\alpha(v,w) \\ &= (1+a)^2 \sum_{v\in\Omega}\sum_{w\in X_k\setminus\Omega} \mu_\alpha(v,w) \\ &= (1+a)^2 \mu_\alpha(\partial_w\Omega) \,, \end{aligned}$$

where we write

$$\mu_\alpha(v,w) := \mu_\alpha(\{v\}\times\{w\}) = d_\omega(v,w)^{-\alpha}\mu_\omega(v)\mu_\omega(w)$$

for $v, w \in X_k$. It follows that

$$R_{G_k}[f] \leq \frac{(1+a)^2\mu_\alpha(\partial_w\Omega)}{(1+a)\mu_\omega(\Omega)} = (1+a)h_{G_k} \leq 2h_{G_k} \,,$$

as asserted.

The proof of the lower bound imitates the proof of [15, Theorem 3.3] in the same way. □

4.2 Spectral growth

Proposition 4.7. *All eigenfunctions ϕ of $\Delta^{\alpha}_{\omega^{1/r}}$ with eigenvalue λ_ϕ are locally constant. Furthermore, they are eigenvectors of some L^{α}_{k} for some $k \in \mathbb{N}$ with the same eigenvalue λ_ϕ.*

Proof. First, we show that $\Delta^{\alpha}_{\omega^{1/r}}$ has a set of eigenfunctions which span $\mathcal{D}(\mathfrak{X}(O_K))$. Namely, all eigenfunctions ϕ of L^{α}_{k} are also eigenfunctions of $\Delta^{\alpha}_{\omega^{1/r}}$, because

$$\begin{aligned}
&\int\limits_{\mathfrak{X}(O_K)} w(x,y)(\phi(x)-\phi(y))\,d\mu_\omega(y) \\
&\qquad = \sum_{C\neq B_k(x)} d_\omega(B_k(x),C)^{-\alpha}(\phi(B_k(x))-\phi(C))\mu_\omega(C) \\
&\qquad = L^{\alpha}_{k}\phi(B_k(x)) = \lambda_{\phi,k}\phi(x)\,,
\end{aligned}$$

whence

$$\lambda_\phi = \lambda_{\phi,k}$$

holds true. Since the eigenfunctions of L^{α}_{k} for $k \in \mathbb{N}$ together span $\mathcal{D}(\mathfrak{X}(O_K))$, the first assertion follows by density of $\mathcal{D}(\mathfrak{X}(O_K))$ in $L^2(\mathfrak{X}(O_K),\mu_\omega)$. □

Lemma 4.8. *Let $\phi\colon \mathfrak{X}(O_K)\to\mathbb{R}$ be an eigenfunction of $\Delta^{\alpha}_{\omega^{1/r}}$ with eigenvalue λ_ϕ. Then*

$$\lambda_\phi \geq \frac{1}{8}\mu_\omega(\mathrm{supp}(\phi))^{1-\alpha}$$

for $\alpha > 1$.

Proof. In the beginning of the proof of [7, Proposition 4.7] that

$$\lambda_\phi \geq R_\alpha[\phi_+]\,,$$

which, by the way, generalises a corresponding statement in the proof of [15, Theorem 3.3]. Further, we have

$$R_\alpha[\phi_+] \geq \frac{1}{8}\mu_\omega(\mathrm{supp}\,\phi)^{1-\alpha}$$

for $\alpha > 1$, according to [7, Lemma 4.6]. □

Theorem 4.9. *Given $x \in \mathfrak{X}(O_K)\setminus\rho_k^{-1}(\bar{a})$ for some $k\in\mathbb{N}$, the following sum converges:*

$$\sum_{\phi\colon\,\lambda_\phi>0}\lambda_\phi^{-1}\phi(x)\int\limits_{\rho_k^{-1}(\bar{a})}\phi(y)\,d\mu_\omega(y)\,,$$

if $\alpha >> 1$, where it is assumed that ϕ runs through the non-constant part of the onb of $L^2(\mathfrak{X}(O_K),\mu_\omega)$, given as a nested sequence of eigenvectors of L^{α}_{ℓ} for $\ell\in\mathbb{N}$.

Proof. Lemma 4.8 says that that

$$\lambda_\phi \geq \frac{1}{8}\mu_\omega(\operatorname{supp}\phi)\,.$$

In the worst case, there are no more than

$$q^{n\ell}\,|\mathfrak{X}(\mathbb{F}_q)|$$

eigenfunctions ϕ of L^α_ℓ, for which

$$\rho_k^{-1}(\bar{a}) \cap \operatorname{supp}\phi \neq \emptyset\,.$$

The condition $\|\phi\|^2 = 1$ for eigenfunction ϕ of L^α_ℓ means that

$$\max|\phi|^2 \leq \mu_\omega(\operatorname{supp}\phi)^{-1}\,.$$

Since $X_\ell \subset X'_\ell$ for $\ell \leq \ell'$, it follows that if

$$\phi \perp \mathcal{D}(X_\ell) \quad \Rightarrow \quad \operatorname{supp}\phi \subset B \in \mathfrak{B}(X_\ell)\,,$$

because

$$L^\alpha_{\ell+1} = L^\alpha_\ell + \sum_{B\in X_{\ell+1}} L^\alpha_\ell\,,$$

and we assumed our onb of $L^2(\mathfrak{X}(O_K),\mu_\omega)$ as a nested sequence of the eigenfunction onb's of the L^α_ℓ with $\ell \in \mathbb{N}$. This means that

$$cq^{n\ell} \leq \max|\phi|^2 \leq Cq^{n\ell}$$

for some $C, c > 0$, and in total there are no more than

$$q^{n\ell}\,|\mathfrak{X}(\mathbb{F}_q)|$$

of summands for fixed ℓ in the expression of concern, named I, here. It follows that

$$\begin{aligned} I &\leq 8\sum_{\ell\in\mathbb{N}}(C'q^{n\ell})^{1-\alpha}Cq^{2n\ell}\,|\mathfrak{X}(O_K)| \\ &= 8C\,|\mathfrak{X}(O_K)|\sum_{\ell\in\mathbb{N}}(C')^{1-\alpha}(q^{n\ell})^{3-\alpha} < \infty \end{aligned}$$

for $\alpha >> 1$, since for ϕ an eigenfunction of L^α_ℓ used here, we have

$$\mu_\omega(\operatorname{supp}\phi) \leq C'q^{n\ell}$$

with $\ell \in \mathbb{N}$. This proves the assertion. □

Lemma 4.10. *Given an eigenfunction ϕ of L_k^α with $\alpha > 1$, the following inequality holds true for the eigenvalue λ_ϕ of $\Delta^\alpha_{\omega^{1/r}}$ associated with ϕ:*

$$\lambda_\phi \leq C' \mu_\omega(B_{k,\min})^{-\alpha}$$

with $B_{k,\min} \subset X_k$ the ball with minimal μ_ω-volume, and $C' > 1$.

Proof. We have

$$\begin{aligned}
\lambda_\phi \phi(x) = R_\alpha[\phi] &= \sum_{B \neq C} w(B,C)(\phi(B) - \phi(C))^2 \mu_\omega(B)\mu_\omega(C) \\
&= \sum_{\substack{B \neq C \\ \text{in } \operatorname{supp}_k \phi}} w(B,C)(\phi(B) - \phi(C))^2 \mu_\omega(B)\mu_\omega(C) \\
&+ \sum_{B \in \operatorname{supp}_k \phi} \sum_{C \in \delta_w \operatorname{supp}_k \phi} w(B,C)\phi(B)^2 \mu_\omega(B)\mu_\omega(C) \\
&\leq R_\phi[\phi] + \sum_{B \in \operatorname{supp}_k \phi} \deg_\omega(B)\phi(B)^2 \mu_\omega(B) \\
&= R_\phi[\phi] + \deg_w(B_{\max}) \,,
\end{aligned}$$

where $R_\phi[\cdot]$ is the Rayleigh quotient for the confined operator

$$L_k^\phi u(B) = \sum_{C \in \operatorname{supp}_k \phi} w(B,C)(\phi(B) - \phi(C))\mu_\omega(C) \,,$$

$\deg_w(B)$ is the degree

$$\deg_w(B) = \sum_{C \neq B} w(B,C)\mu_\omega(C) \,,$$

and

$$\operatorname{supp}_k \phi = \operatorname{supp} \phi / \sim_k$$

with $\sim_k$ the equivalence relation identifying balls to points in X_k. Now,

$$\begin{aligned}
\langle \phi, \phi \rangle_{L^2(X_k,\mu_\omega)} R_\phi[\phi] &= \sum_{\substack{B \neq C \\ \text{in } \operatorname{supp}_k \phi}} d_\omega(B,C)^{-\alpha}(\phi(B) - \phi(C))^2 \mu_\omega(B)\mu_\omega(C) \\
&\leq \sum_{\substack{B \neq C \\ \text{in } \operatorname{supp}_k \phi}} d_\omega(B,C)^{-\alpha}(\phi(B) + \phi(C))^2 \mu_\omega(B)\mu_\omega(C) \\
&\leq 2\mu_\omega(B_{k,\min})^{-\alpha} \,,
\end{aligned}$$

where $B_{k,\min}$ is the ball in X_k of minimal μ_ω-volume. Also,

$$\deg_\omega(B_{\max}) = \sum_{C \neq B_{\max}} d_\omega(B_{\max}, C)^{-\alpha} \mu_\omega(C) \leq \mu_\omega(B_{k,\min})^{-\alpha} \,.$$

Together, this yields

$$\lambda_\phi \leq C' \mu_\omega(B_{k,\min})^{-\alpha}$$

with

$$C' = 1 + \frac{2}{\langle \phi, \phi \rangle_{L^2(X_k,\mu_\omega)}} > 1,$$

as asserted. □

4.3 Spectral Mass

The kernel function of the spectral projection operator E_λ onto the $[0,\lambda]$ portion of the spectrum of $\Delta^\alpha_{\omega^{1/r}}$ is

$$K_\lambda(x,y) = \sum_{\phi\colon \lambda_\phi \leq \lambda} \phi(x)\overline{\phi(y)}\,,$$

where ϕ is taken from an orthonormal basis of $\mathrm{dom}(\mathcal{E})$ consisting of eigenfunctions of $\Delta^\alpha_{\omega^{1/r}}$ for $\alpha > 1$. The spectral mass function is

$$M(\lambda) = \mu_\omega(\mathfrak{X}(O_K))^{-1} \int_{\mathfrak{X}(O_K)} K_\lambda(x,x)\, d\mu_\omega(x)$$

for $\lambda \geq 0$, and one sees that

$$\mu_\omega(\mathfrak{X}(O_K))M(\lambda) = \left\{\lambda' \in \mathrm{Spec}(\Delta^\alpha_{\omega^{1/r}}) \mid \lambda' \leq \lambda\right\} =: N(\lambda) < \infty$$

for $\alpha > 1$, as the spectrum in this case is unbounded from above and discrete according to [7, Corollary 3.9].

Theorem 4.11. *Assume $\alpha > 1$. There exists $C > 0$ such that*

$$M(\lambda) \leq eC\, \lambda^{1/(\alpha-1)}$$

for $\lambda > 0$.

Proof. Recapitulate the proof of [7, Theorem 3.7] which is just a straightforward adaptation of the proof of [6, Theorem 5.6]: First, notice that the metric space $(\mathfrak{X}(O_K), d_\omega)$ endowed with the Radon measure μ_ω is separable, proper, β-regular for $\beta = 1$, and locally ultrametric, according to [7, Lemma 3.6]. This then allows to apply locally the heat kernel estimate in [15, Theorem 5.1]. That means to say, since $\mathfrak{X}(O_K)$ is a finite disjoint union of copies of O_K^n, that the heat kernel estimate

$$ct^{1/(1-\alpha)}\left(1 + \frac{d_\omega(x,y)}{t^{1/(\alpha-1)}}\right)^{-\alpha} \leq p^B_t(x,y) \leq Ct^{1/(1-\alpha)}\left(1 + \frac{d_\omega(x,y)}{t^{1/(\alpha-1)}}\right)^{-\alpha}$$

for some $C > c > 0$ applies to each of these finitely many balls $B \subset \mathfrak{X}(O_K)$. Since the process generated by $\Delta^\alpha_{\omega^{1/r}}$ induces a continuous-time Random Walk on the finite quotient space obtained by identifying each of these finite balls with a single point, the heat kernel

$p_t(x,y)$ decomposes into the sum of the $p_t^B(x,y)$ plus the heat kernel for the finite discrete set, the same heat kernel estimate applies to $p_t(x,y)$. Hence, it follows that

$$ct^{1/(1-\alpha)} \leq p_t(x,x) \leq Ct^{1/(1-\alpha)}$$

for $\alpha > 1$. Then, using

$$\Phi(t) = t^{1/(1-\alpha)},$$

the asserted inequality now follows from [33, eq. (2.23)]. □

Given a real-valued function $\phi\colon \mathfrak{X}(O_K) \to \mathbb{R}$, define

$$\phi_+ = \begin{cases} \phi, & \phi > 0 \\ 0, & \text{otherwise} \end{cases}$$

as the *positive part* of ϕ.

4.4 Cauchy Problem

Recall from [7, Theorem 3.7] that the semigroup $\exp\left(-t\Delta^{\alpha}_{\omega^{1/r}}\right)$ is ultracontractive with rate function

$$\gamma(t) = Ct^{1/(\alpha-1)}$$

for some $C > 0$, and all $t > 0$, in the case $\alpha > 1$. This allows to solve the Cauchy problem:

Theorem 4.12 (Cauchy Problem)**.** *The Cauchy problem*

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) & = \Delta^{\alpha}_{\omega^{1/r}}u(t,x) \\ u(0,x) & = u_0(x) \in \operatorname{dom}(\Delta^{\alpha}_{\omega^{1/r}}) \cap C(\mathfrak{X}(O_K),\mathbb{R}) \end{cases} \tag{5}$$

has a unique solution $u(t,\cdot) \in L^2(\mathfrak{X}(O_K),\mu_\omega)$ for $t \geq 0$, such that $u(\cdot,x)$ is continuously differentiable for each $x \in \mathfrak{X}(O_K)$. The solution is given by

$$u(t,x) = \int_{\mathrm{cl}_w\,\Omega} u_0(y)p_t(x,y)\,d\mu_\omega(y)$$

for a heat kernel function $p_t(\cdot,\cdot) \in L^2(\mathfrak{X}(O_K)^2, \mu_\omega \wedge \mu_\omega)$ for $t > 0$. The semigroup acting on $C(\mathfrak{X}(O_K))$ has the Feller property, and thus is attached to a strong Markov process having càdlàg sample paths.

Proof. According to [7, Theorem 3.7], the semigroup $\exp(-t\Delta^{\alpha}_{\omega^{1/r}})$ is ultracontractive for $\alpha > 1$. In other words, each operator $\exp(-t\Delta_{\omega^{1/r}})$ is a bounded map $L^2(\mathfrak{X}(O_K),\mu_\omega) \to L^\infty(\mathfrak{X}(O_K))$. Since $L^2(\mathfrak{X}(O_K),\mu_\omega) \subset L^1(\mathfrak{X}(O_K),\mu_\omega)$ for $t > 0$, it follows by the Dunford-Pettis criterion for kernel functions [13][Theorem 2.2.5] that there exists a kernel function $p_t(\cdot,\cdot) \in L^\infty(\mathfrak{X}(O_K)^2)$ such that

$$\exp\left(-t\Delta^{\alpha}_{\omega^{1/r}}\right)u_0(x) = \int_{\mathfrak{X}(O_K)} p_t(x,y)u_0(y)\,d\mu_\omega(y)$$

for almost all $x \in \mathrm{cl}_w\,\Omega$ and $u \in L^\infty(\mathrm{cl}_w\,\Omega) \cap L^2(\mathrm{cl}_w\,\Omega, \mu_\omega)$. This proves the existence of the solution the Cauchy problem (5).

In order to see that $p_t(\cdot,\cdot) \in L^2(\mathfrak{X}(O_K)^2, \mu_\omega \wedge \mu_\omega)$, observe that

$$e^{-z} < \frac{1}{z}$$

for $z > 0$, and this now implies that

$$\|p_t(x,y)\|_{L^2} = \sum_\phi e^{-t\lambda_\phi} \leq \sum_{k\in\mathbb{N}}^{\infty} \exp\left(-tCq^{n(\alpha+1)k}\right) = \sum_{k\in\mathbb{N}}^{\infty} e^{-tC}\left(e^{q^{n(\alpha+1)}}\right)^{-k} < \infty\,,$$

where the inequality uses the bound

$$\lambda_\phi \geq \frac{1}{8}\mu_\omega(\mathrm{supp}\,\phi)^{1-\alpha} \geq C(q^{n\alpha})^k$$

from Lemma 4.8, together with the worst-case multiplicity q^{nk} for an eigenfunction ϕ in L^α_k from a nested onb of $L^2(\mathfrak{X}(O_K), \mu_\omega)$.

The semigroup $\exp\left(-t\Delta^\alpha_{\omega^{1/r}}\right)$ is readily seen to be a C_0-semigroup for $t \geq 0$. Uniqueness of the solution to (5) now follows with [2, Proposition 2.2.3]. The semigroup is clearly positive and a contraction for $t \geq 0$. The Feller property now follows, if it is also right-continous. This can be seen as follows: let $f \in C(\mathfrak{X}(O_K), \mathbb{R})$. Then

$$\begin{aligned}\left\|\exp\left(-t\Delta^\alpha_{\omega^{1/r}}\right) f - f\right\|_\infty &= \left\|\sum_{\phi\colon\,\lambda_\phi>0} (e^{-t\lambda_\phi}-1) f_\phi \phi\right\|_\infty \\ &\leq C \sum_{\phi\colon\,\lambda_\phi>0} \left|e^{-t\lambda_\phi}-1\right|\end{aligned}$$

for some $C > 0$ and all $t > 0$. Clearly, this expression tends toward zero as $t \to 0$. This shows that the semigroup $\exp\left(-t\Delta^\alpha_{\omega^{1/r}}\right)$ is a Feller semigroup on $C(\mathfrak{X}(O_K)$, and hence is associated with a strong Markov process having càdlàg sample paths, because $\mathfrak{X}(O_K)$ is compact and separable, cf. [14, Theorem 4.2.7]. □

5 Expected Hitting Times

This section is meant to first introduce Feynman-Kac formulae, and to prove the finiteness of expected hitting times with and without bounded potentials. The last subsection gives bounds for expected hitting times in terms of the zeros and poles of ω contained in a given fibre $\rho^{-1}(\bar{a})$ of the reduction map.

5.1 Feynman-Kac formulae via Martingales

Let $(\Sigma, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\geq 0})$ be a filtered abstract probability space. Let $X\colon \mathbb{R}_+ \times \Sigma \to \mathfrak{X}(O_K)$ be a stochastic process generated by an integral operator $\mathscr{L}$. Define its associated *Dynkin process* as

$$M[\mathscr{L}, 0]_t(f) = f(X_t) - f(X_0) - \int_0^t (\mathscr{L} f)(X_s)\, ds$$

for $f \in \mathrm{dom}(\mathscr{L})$. In order to be more specific about operator and the space it is acting on, let

$$V\colon \Omega \to \mathbb{R}$$

be continuous and bounded from below. The operator of interest here is

$$\mathscr{L} = \mathscr{L}_\Omega + V$$

acting on $L^2(\mathrm{cl}_w\, \Omega, \mu_\omega)$, where again $\Omega \subset \mathfrak{X}(O_K)$ is an open domain with scheme $\mathfrak{X}$ satisfying the Assumption in the previous section.

A general Dirichlet problem with $g, V\colon \Omega \to \mathbb{R}$, $h\colon \delta_w\Omega \to \mathbb{R}$ continuous functions is given as follows:

$$\begin{cases} (-\mathscr{L}_\Omega u)(x) + V(x)u(x) = g(x), & x \in \Omega \\ u(x) = h(x), & x \in \delta_w\Omega\,. \end{cases} \tag{6}$$

Define the *first hitting time* as

$$\tau_A = \inf\{t > 0 \mid X_t \in A\}$$

for $A \subset \mathfrak{X}(O_K)$. Since [14, Proposition 2.1.5] is applicable, this random variable is a stopping time. Using hitting times of the w-boundary of an open $\Omega \subset \mathfrak{X}(O_K)$, one formulation of a Feynman-Kac formula on the integral points of O_K-schemes is thus:

Theorem 5.1 (Feynman-Kac Formula)**.** *Let X_t be the continuous-time Markov process associated with $\mathscr{L}_\Omega$. The associated Martingale problem has as its unique solution X_t. If $u \in \mathrm{dom}(\mathscr{L}_\Omega)$ solves the Dirichlet problem (6), and*

$$\mathbb{E}_x\left[\tau_{\delta_w\Omega} \exp\left(\inf_{x\in\Omega} V(x)\tau_{\delta_w\Omega}\right)\right] < \infty \tag{7}$$

for $x \in \Omega$, then

$$u(x) = \mathbb{E}_x\left[h(X_{\tau_{\delta_w\Omega}}) \exp\left(-\int_0^{\tau_{\delta_w\Omega}} V(X_s)\, ds\right) + \int_0^{\tau_{\delta_w\Omega}} g(X_t) \exp\left(-\int_0^t V(X_s)\, ds\right)\, dt\right]$$

for $x \in \Omega$.

Proof. Following the proof of [3, Theorem 7.15], we can say that the following process

$$M[\mathscr{L}_\Omega, V]_t(f) = e^{-\int_0^t V(X_s)\,ds} f(X_t) + \int_0^t e^{-\int_0^s V(X_r)\,dr} \left[V(X_s)f(X_s) - (\mathscr{L}_\Omega f)(X_s)\right] ds$$

is a martingale, if and only if $M[\mathscr{L}_\Omega, 0]_t(f)$ is a martingale, cf. [3, Lemma 5.35]. The latter is the Dynkin process associated with $\mathscr{L}_\Omega$. And $\mathscr{L}_\Omega$ is the infinitesimal generator of a Feller process according to Theorem 4.12. It is well-known that the Dynkin process associated with a Feller process is a martingale. Hence, $M[\mathscr{L}_\Omega, V]_t(f)$ is a martingale. Doob's Optional Stopping Theorem, cf. [3, Theorem 3.67], applies here due to condition (7). This now proves that the solution $u(x)$ has the asserted form. □

5.2 Expected Time to Hitting Singularities

Recall that the Green kernel associated with $\Delta^\alpha_{\omega^{1/r}}$ can be written as the distribution associated with

$$G(x,y) = \int_0^\infty p_t(x,y)\,dt\,,$$

where $p_t(x,y)$ is the L^2-heat kernel found in Theorem 4.12. Take the expression

$$G(x,y) = \sum_\phi \lambda_\phi^{-1} \phi(x)\overline{\phi(y)} \tag{8}$$

for $x, y \in \mathrm{cl}_w\,\Omega$, as it comes by formally integrating the heat kernel expanded as

$$p_t(x,y) = \sum_\phi e^{-t\lambda_\phi} \phi(x)\overline{\phi(y)}\,,$$

where ϕ runs through an onb of $L^2(\mathfrak{X}(O_K), \mu_\omega)$ consisting of eigenfunctions of $\bar{\Delta}^\alpha_{\omega^{1/r}}$. In the following, we will investigate expected hitting times as solutions of suitable Dirichlet problems. The set $\Omega \subset \mathfrak{X}(O_K)$ will be understood as an open subset.

Theorem 5.2 (Expected hitting time with potential)**.** *Assume $\alpha >> 1$. The following Dirichlet Problem:*

$$\begin{cases} -\Delta^\alpha_{\omega^{1/r}} u(x) + V(x)u(x) = 1, & x \in \Omega \\ u(x) = 0, & x \in \delta_w\Omega \end{cases} \tag{9}$$

has a unique solution $u \in \mathrm{dom}(\Delta^\alpha_{\omega^{1/r}})$ given by

$$u(x) = \begin{cases} \mathbb{E}_x\left[\tau_{\delta_w\Omega} \exp\left(-\int_0^t V(X_s)\,ds\right) dt\right], & x \in \Omega \\ 0, & x \in \delta_w\Omega \end{cases}.$$

In particular, the expected exit time $\mathbb{E}_x[\tau_{\delta_w\Omega}]$ is finite in the case that $V = 0$.

Proof. *Case* $V = 0$. Existence and uniqueness of a weak solution of (9) can be shown, as this is a special case of [8, Theorem 3.3]. Using the expansion (8) of the Green function $G(x, y)$, obtain that the expression

$$\mathbb{E}_x[\tau_{\delta_w\Omega}] = \int_{\delta_w\Omega} G(x,y)\, d\mu_\omega(y) = \sum_\phi \lambda_\phi^{-1}\phi(x) \int_{\delta_w\Omega} \overline{\phi(y)}\, d\mu_\omega(y)\,,$$

which is finite, according to Theorem 4.9. Now, extend this function by zero on $\delta_w\Omega$, and obtain thus an actual solution of (9). This proves the assertions in the case $V = 0$.

Case of general $V(x)$. First, observe that

$$\mathbb{E}_x\left[\tau_{\delta_w\Omega} \exp\left(\inf_{x\in\Omega} V(x)\tau_{\delta_w\Omega}\right)\right] < \infty$$

for $x \in \Omega$, because $\mathbb{E}_x[\tau_{\delta_w\Omega}] < \infty$, according to the previous case, and because V is bounded. The existence of a solution of (9) now follows from the existence of the Green kernel

$$G_{\Omega,V}(x, d\mu_\omega(y)) = \mathbb{E}_x\left[\int_0^{\tau_{\delta_w\Omega}} \exp\left(-\int_0^t V(X_s)\, ds\right) 1_{X_t\in d\mu_\omega(y)}\, dt\right] \tag{10}$$

for $x, y \in \mathfrak{X}(O_K)$, due to Theorem 5.1 which says that the Dynkin process associated with $\mathscr{L} = \Delta^\alpha_{\omega^{1/r}} + V$ is a martingale, and this implies that $\mathscr{L}$ generates a Feller process whose Green kernel has the given form (10). Since now a solution $u \in \mathrm{dom}(\Delta^\alpha_{\omega^{1/r}})$ of (9) can be obtained via integrating over the Green kernel like this:

$$u(x) = \int_{\delta_w\Omega} G_{\Omega,V}(x, d\mu_\omega(y))\,,$$

it follows from Theorem 5.1 that $u(x)$ has the asserted form for $x \in \mathrm{cl}_w\,\Omega$. □

In order to apply the results so far to the singularties of $\mathfrak{X}(\mathbb{F}_q)$, let

$$\rho\colon \mathfrak{X}(O_K) \to \mathfrak{X}(\mathbb{F}_q)$$

be the reduction map. The μ_ω-measure of a fibre $\rho^{-1}(\bar a)$ of a point $\bar a \in \mathfrak{X}(\mathbb{F}_q)$ is

$$\mu_\omega(\rho^{-1}(\bar a)) = q^{-n} \prod_{j\in J(a)} \frac{q-1}{q^{\epsilon_j}-1}\,, \tag{11}$$

where

$$\epsilon_j = \frac{a_j}{r} + 1 > 0\,,$$

and the set $J(a) \subset \{1, \ldots, s\}$ is the unique set such that

$$\bar a \in \mathfrak{D}^\circ_{J(a)}(\mathbb{F}_q)$$

for $a \in \mathfrak{X}(O_K)$ reducing to $\bar{a}$ via the reduction map ρ.

Let $a, x \in \mathfrak{X}(O_K)$ such that $\rho(x) \neq \rho(a)$. Then define the sets

$$\bar{N}(x,a) = \{\phi \mid \lambda_\phi > 0 \text{ and } a, x \in \operatorname{supp}(\phi)\},$$
$$N(x,a) = \left\{\phi \in \bar{N}(x,a) \mid \rho^{-1}(\bar{a}) \sqcup \{x\} \subset \operatorname{supp}(\phi)\right\},$$

where ϕ is taken from an onb of $L^2(\mathfrak{X}(O_K), \mu_\omega)$ of eigenfunctions of $\Delta^\alpha_{\omega^{1/r}}$, given as a nested set of eigenfunctions of L^α_k with $k \in \mathbb{N}$.

Theorem 5.3 (Bounds for expected times of hitting fibres)**.** *Assume that $x \in \mathfrak{X}(O_K) \setminus \rho^{-1}(\bar{a})$, and $\alpha >> 1$. Then there exist constants $C, c > 0$ such that the expected hitting time $\mathbb{E}_x[\tau_{\rho^{-1}(\bar{a})}]$, with $a \in \mathfrak{X}(O_K)$, $\rho(a) = \bar{a} \in \mathfrak{X}(\mathbb{F}_q)$, satisfies*

$$\begin{aligned} c \prod_{j \in J_{\min}} \left[\frac{q-1}{q^{\epsilon_j}-1}\right]^{-\alpha} \sum_{k \in \mathbb{N}} q^{-nk\alpha} \sum_{\phi \in N_k(x,a)} \phi(x) \int_{\rho^{-1}(\bar{a})} \overline{\phi(y)}\, d\mu_\omega(y) \\ \leq \mathbb{E}_x[\tau_{\rho^{-1}(\bar{a})}] \\ \leq C \left[\sum_{J \subseteq I} |\mathfrak{D}^\circ_J(\mathbb{F}_q)| \prod_{j \in J} \frac{q-1}{q^{\epsilon_j}-1}\right]^{2\alpha-2} \sum_{\phi \in \bar{N}(x,a)} \phi(x) \int_{\rho^{-1}(\bar{a})} \overline{\phi(y)}\, d\mu_\omega(y), \end{aligned}$$

where

$$N_k = \left\{\phi \mid \phi \text{ is in the } L^\alpha_k\text{-part, but not in the } L^\alpha_{k-1}\text{-part, of the nested onb}\right\},$$

and with $J_{\min} \subset \{1, \ldots, s\}$ such that the product

$$\prod_{j \in J_{\min}} \frac{q-1}{q^{\epsilon_j}-1} \tag{12}$$

is minimal.

Proof. The first inequality follows from

$$\mathbb{E}_x[\tau_{\rho^{-1}(\bar{a})}] = \sum_{\phi \in \bar{N}(x,a)} \lambda_\phi^{-1} \phi(x) \int_{\rho^{-1}(\bar{a})} \overline{\phi(y)}\, d\mu_\omega(y),$$

and

$$\lambda_\phi \leq \mu_\omega(B_{k,\min})^{-\alpha} = C\left(q^{-nk} F_{\min}\right)^{-\alpha},$$

according to Lemma 4.10, and with

$$F_{\min} = \prod_{j \in J_{\min}} \frac{q-1}{q^{\epsilon_j}-1}.$$

The second asserted inequality follows from

$$\lambda_\phi \geq \lambda_1 \geq \frac{1}{8} \mu_\omega(\mathfrak{X}(O_K))^{2-2\alpha}$$

for $\alpha > 1$, according Corollary 4.4, together with Lemma 3.1. □

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