Characterizations of classes of lifetime distributions generalizing the NBUE class.

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Abstract

We introduce a new class of lifetime distributions exhibiting a notion of positive aging, called the $\mathcal{M}$-class, which is strongly related to the well known $\mathcal{L}$-class. It is shown that distributions in the $\mathcal{M}$-class can not have an undesirable property recently observed in an example of an $\mathcal{L}$-class distribution by Klar (2002). Moreover, it is shown how these and related classes of life distributions can be characterized by expected remaining lifetimes after a family of random times, thus extending the notion of NBUE. We give examples of $\mathcal{M}$-class distributions by using simple sufficient conditions, and we derive reliability bounds for distributions in this class.

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1 Introduction.

Notions of positive aging play an important role in reliability theory, survival analysis and other fields. Therefore an abundance of classes of distributions describing aging have been considered in the literature; see e.g. Barlow and Proschan (1975) or Kijima (1997) for an overview. One of the more commonly used classes is the so called $\mathcal{L}$-class. It was introduced by Klefsjö (1983) as a large class of distributions exhibiting a notion of aging containing most of previously known classes like IFR, DMRL, NBU, NBUE or HNBUE. However, Klar (2002) gives an example of a distribution within that class having an infinite third moment and a hazard rate that tends to zero as time approaches infinity. This example leads to serious doubts whether the $\mathcal{L}$-class should be considered as a reasonable notion of positive aging.

In this paper we present a new class of life distributions, called the $\mathcal{M}$-class. It is defined similarly as the $\mathcal{L}$-class; only the ordering of the Laplace transforms is replaced by the ordering of the moment generating functions. Theorem 3.1 shows that the $\mathcal{M}$-class is a more reasonable notion of aging, which does not have the undesirable property of the $\mathcal{L}$-class mentioned above. Moreover, in Lemma 3.1 and Theorem 3.6 we show that the $\mathcal{L}$-class and the $\mathcal{M}$-class as well as other classes of life distributions can be characterized by expected remaining lifetimes after a family of random times, thus extending the notion of NBUE. In Section 4 we describe sufficient conditions for the $\mathcal{M}$-class, which yield many examples for distributions in the $\mathcal{M}$-class. Finally, a reliability bound for distributions in the $\mathcal{M}$-class is determined in Section 5. Before stating these results we will recall some facts about characteristics of lifetime distributions and about stochastic orderings in Section 2.

2 Preliminaries.

2.1 Characteristics of Lifetime Distributions.

Let X be a non-negative random variable describing the lifetime of some device. Throughout the paper it is assumed that X has a positive finite mean μ. The distribution function of X is denoted by F, and $\bar{F}(t) = 1 - F(t) = P(X > t)$ is the survival function. The recursively defined sequences

$$F^{(n+1)}(t) = \int_0^t F^{(n)}(x)\mathrm{d}x \quad \text{and } \bar{F}^{(n+1)}(t) = \int_t^\infty \bar{F}^{(n)}(x)\mathrm{d}x,$$

where $F^{(1)} = F$ and $\bar{F}^{(1)} = \bar{F}$, are also of interest. Note that $\bar{F}^{(n)}$ is finite whenever EX^{n-1} is finite, since it can be shown by partial integration that

$$\bar{F}^{(n+1)}(t) = \frac{1}{n!}E(X-t)_+^n,$$

where $x_+ = \max\{x, 0\}$.

If F is absolutely continuous with density f,

$$r(t) = \frac{f(t)}{\bar{F}(t)}$$

is the hazard rate or failure rate function. By F_t, we denote the distribution function of the residual lifetime after time t, i.e.

$$F_t(x) = P(X \le t + x | X > t) = \frac{F(t+x) - F(t)}{\bar{F}(t)}.$$

The corresponding densities and hazard rates are called f_t and r_t, respectively, whenever they exist, and X_t shall be a random variable with distribution function F_t. The function $\mathrm{MRL}(t) = EX_t$ is called the *mean residual life function*. Note that

$$\bar{F}_t(x) = \frac{\bar{F}(t+x)}{\bar{F}(t)} \quad \text{and hence } \mathrm{MRL}(t) = \frac{\bar{F}^{(2)}(t)}{\bar{F}(t)}.$$

If F is absolutely continuous, then the density and hazard rate of F_t are

$$f_t(x) = \frac{f(t+x)}{\bar{F}(t)} \quad \text{and} \quad r_t(x) = r(t+x),$$

respectively. The function

$$F_e(x) = \int_0^x \frac{\bar{F}_X(t)}{\mu} \mathrm{d}t \tag{2.1}$$

is a distribution function, which is called the equilibrium distribution. It plays an important role in renewal theory, see e.g. Feller (1971). It has mean $\mu_e = E(X^2)/(2EX)$, and its hazard rate r_e is given by

$$r_e(t) = \frac{\bar{F}_X(t)}{\int_t^\infty \bar{F}_X(x)\mathrm{d}x} = \frac{1}{EX_t} = \frac{1}{\mathrm{MRL}(t)},$$

so that the hazard rate r_e of the equilibrium distribution is the reciprocal of the mean residual life function. Moreover, $\bar{F}_e(t) = \bar{F}^{(2)}(t)/\mu$ and hence the identity

$$\bar{F}_e(t) = \frac{\mathrm{MRL}(t)\bar{F}(t)}{\mu} \tag{2.2}$$

holds. Let $m(t) = E\left(e^{tX}\right)$ and m_e denote the moment generating function of X and of its equilibrium distribution, respectively. Since

$$(m(t)-1)/t \;=\; \int_0^\infty e^{tx}\bar{F}(x)dx \;=\; \mu\int_0^\infty e^{tx}d\bar{F}_e(x),$$

we obtain

$$m_e(t) \;=\; \frac{m(t)-1}{\mu\, t}, \tag{2.3}$$

which is finite if and only if $m(t) < \infty$.

2.2 Stochastic Orders.

Many classes of lifetime distributions are characterized by stochastic orders. Most of these stochastic orders can be generated by a class $\mathcal{F}$ of real valued functions in the following way: Two random variables X and Y are said to be comparable with respect to the integral stochastic order $\leq_{\mathcal{F}}$ if and only if $Ef(X) \leq Ef(Y)$ holds for all functions $f \in \mathcal{F}$. For a general treatment of this type of stochastic orders we refer to Müller (1997) or Müller and Stoyan (2002). Important examples are

- the usual stochastic order $X \leq_{st} Y$, if $Ef(X) \leq Ef(Y)$ for all increasing functions;
- the convex order $X \leq_{cx} Y$, if $Ef(X) \leq Ef(Y)$ for all convex functions;

- the increasing convex order $X \leq_{icx} Y$, if $Ef(X) \leq Ef(Y)$ for all increasing convex functions; and

- the increasing concave order $X \leq_{icv} Y$, if $Ef(X) \leq Ef(Y)$ for all increasing concave functions.

Many details and applications of these orders can be found in the monographs Müller and Stoyan (2002) and Shaked and Shanthikumar (1994).

These orders are generated by functions which are characterized by the sign of the first or of the first and second derivative. This can be generalized to higher derivatives. The order generated by functions with non-negative derivatives up to order s is denoted by $\leq_{s-icx}$ and the order generated by functions with alternating signs of the derivatives up to order s is denoted by $\leq_{s-icv}$. Alternative definitions are possible as follows. It is

$$X \leq_{s-icv} Y \quad \text{if } F_X^{(s)}(t) \leq F_Y^{(s)}(t) \ \text{ for all } t \geq 0$$

and

$$X \leq_{s-icx} Y \quad \text{if } \bar{F}_X^{(s)}(t) \leq \bar{F}_Y^{(s)}(t) \ \text{ for all } t \geq 0.$$

These orders have been considered e.g. by Denuit, Lefevre, and Shaked (1998), Fishburn (1980) and Rolski (1976).

Even weaker orders are obtained by considering only exponential functions.

Definition 2.1. Two non-negative random variables are said to be comparable with respect to

- Laplace transform order (written $X \leq_{Lt} Y$), if $Ee^{-tX} \geq Ee^{-tY}$ for all $t > 0$;

- moment generating function order (written $X \leq_{mgf} Y$), if $Ee^{t_0 Y}$ is finite for some $t_0 > 0$ and $Ee^{tX} \leq Ee^{tY}$ for all $t > 0$.

Note that $X \leq_{s-icv} Y$ implies $X \leq_{Lt} Y$, and $X \leq_{s-icx} Y$ implies $X \leq_{mgf} Y$. Of course $\leq_{mgf}$ only makes sense if the moment generating function exists for at least some values. Therefore we added the requirement that $Ee^{t_0 Y}$ is finite for some $t_0 > 0$. Notice that the domain where the function $t \mapsto Ee^{tY}$ is finite is an interval, and that this function is increasing and infinitely differentiable there. Indeed, it is analytic and thus determined by its Taylor series, the coefficients of which are just the moments of the distribution, see Lukacs (1970).

The Laplace transform order $\leq_{Lt}$ is well known in the literature. It was studied by Reuter and Riedrich (1981), Alzaid, Kim, and Proschan (1991) and Denuit (2001), among others. In can be considered as a limit of the $\leq_{s-icv}$-orders for $s \to \infty$. Indeed, Reuter and Riedrich (1981) have shown that it is generated by the set of functions having derivatives of any order, alternating in sign. It is not true, however, that $X \leq_{Lt} Y$ implies $X \leq_{s-icv} Y$ for some s; see Fishburn (1980) for additional conditions ensuring that implication.

In contrast to $\leq_{Lt}$, the moment generating function order $\leq_{mgf}$ was hardly ever considered before. It is mentioned cursorily in the actuarial literature (see e.g. Denuit (2001) and Kaas and Hesselager (1995)). Notice that Shaked and Shanthikumar (1994) use the notion of moment generating function order and the symbol $\leq_{mgf}$ as an equivalent to the Laplace transform order.

It seems to be an open problem to characterize the maximal generator of $\leq_{mgf}$, i.e. the maximal class of functions f, for which $X \leq_{mgf} Y$ implies $Ef(X) \leq Ef(Y)$. It is *not* true that this maximal generator contains all functions with non-negative derivatives of all orders. That class of functions characterizes the so called moment order $\leq_{mom}$, which can be defined as follows.

Definition 2.2. Two non-negative random variables are said to be comparable with respect to moment order (written $X \leq_{mom} Y$), if $EX^k \leq EY^k$ for all $k = 1, 2, \ldots$.

The moment order is considered in Shaked and Shanthikumar (1994), p. 103ff. Obviously, it is stronger than the moment generating function order. The connection between the two orders can also be seen from the following result.

Theorem 2.1. *$X \leq_{mgf} Y$ holds if and only if*

$$\sum_{i=0}^{\infty} \frac{t^i}{(i+1)!} EX^i \;\leq\; \sum_{i=0}^{\infty} \frac{t^i}{(i+1)!} EY^i$$

for all $t > 0$.

Proof. A simple integration by parts shows that $X \leq_{mgf} Y$ holds if and only if $\int e^{tx} \bar{F}_X(x) \mathrm{d}x \leq \int e^{tx} \bar{F}_Y(x) \mathrm{d}x$ for all $t > 0$. Using the Taylor series expansion of e^{tx} and exchanging integration and summation yields the result. $\square$

The following example demonstrates that $\leq_{mom}$ is strictly stronger than $\leq_{mgf}$.

Example 2.1. Let X be a random variable with

$$P(X = 1) = 0.204, \; P(X = 2) = 0.678 \; \text{ and } \; P(X = 10) = 0.118.$$

Further, assume that Y is exponentially distributed with $EY = EX = 2.74$. Then $X \leq_{mgf} Y$, but $EX^3 > EY^3$; thus $X \not\leq_{mom} Y$.

Müller and Stoyan (2002) consider $\leq_{Lt}$ for arbitrary real valued random variables. Using this extension $X \leq_{mgf} Y$ is equivalent to $-X \geq_{Lt} -Y$. Therefore $\leq_{mgf}$ shares all properties of $\leq_{Lt}$ mentioned there. We recall here the most interesting properties of $\leq_{mgf}$ without proofs.

Theorem 2.2. *a) $\leq_{mgf}$ is closed under convolution, i.e., if $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ are independent and non-negative with $X_i \leq_{mgf} Y_i$ for $i = 1, \ldots, n$, then*

$$\sum_{i=1}^{n} X_i \leq_{mgf} \sum_{i=1}^{n} Y_i.$$

b) Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be i.i.d. non-negative with $X_i \leq_{mgf} Y_i$; and M and N independent of (X_i) and (Y_i) with $M \leq_{mgf} N$. Then

$$\sum_{i=1}^{M} X_i \leq_{mgf} \sum_{i=1}^{N} Y_i.$$

c) $\leq_{mgf}$ is closed under mixtures, i.e., if $F_\theta \leq_{mgf} G_\theta$ for all θ in some set Θ and if H is a distribution on Θ, then

$$\int_\Theta F_\theta H(\mathrm{d}\theta) \leq_{mgf} \int_\Theta G_\theta H(\mathrm{d}\theta).$$

d) $\leq_{mgf}$ is invariant under scale transformation, i.e.,

$$X \leq_{mgf} Y \quad \textit{implies} \quad \alpha X \leq_{mgf} \alpha Y$$

for all $\alpha > 0$.

Similar results for the ordering $\leq_{mom}$ can be found in Shaked and Shanthikumar (1994), 3.B.4.

3 The $\mathcal{L}$-class and the $\mathcal{M}$-class

Klefsjö (1983) introduced the $\mathcal{L}$-class of life distributions consisting of all distributions F with mean μ_F such that $F \geq_{Lt} \mathrm{Exp}(1/\mu_F)$. The $\mathcal{L}$-class is the largest of the well known classes of life distributions. It is larger than IFR, NBU, NBUE, DMRL, HNBUE, etc. However, it is so large that it contains distributions which can hardly be said to exhibit positive aging. Klar (2002) gives an example of a distribution in the $\mathcal{L}$-class with an infinite third moment. Indeed, in this example $\mathrm{MRL}(t) \to \infty$ and $r(t) \to 0$ for $t \to \infty$. This contradicts our intuitive feeling of aging. Therefore we suggest as an alternative the so called $\mathcal{M}$-class, which is obtained by replacing the Laplace transform order by the moment generating function order.

Definition 3.1. a) A non-negative random variable X with mean $\mu = EX$ and distribution function F is said to be in the $\mathcal{M}$-class if $F \leq_{mgf} \mathrm{Exp}(1/\mu)$, where $\mathrm{Exp}(1/\mu)$ denotes an exponential distribution with mean μ.

b) X is said to be in the $\mathcal{LM}$-class, if it is in the $\mathcal{L}$-class and in the $\mathcal{M}$-class.

c) If the reversed inequality is true in a), X is said to be in the $\bar{\mathcal{M}}$-class.

Note that X is in the $\mathcal{M}$-class if and only if

$$Ee^{tX} = \int_0^\infty e^{tx} F(\mathrm{d}x) \leq \frac{1}{1-\mu t} \quad \text{for all } 0 \leq t < 1/\mu, \tag{3.1}$$

or, equivalently,

$$\int_0^\infty \bar{F}(x)e^{tx}\mathrm{d}x \leq \frac{\mu}{1-\mu t} \quad \text{for all } 0 \leq t < 1/\mu. \tag{3.2}$$

For the $\mathcal{LM}$-class the inequalities in (3.1) and (3.2) must hold for all $-\infty < t < 1/\mu$. Since the moment generating function is known explicitly for many parametric families of distributions, these conditions are often easy to check.

Note that Ee^{tX} must be finite for all $t < 1/\mu$. This is not possible if $r(t) \to 0$ for $t \to \infty$. Indeed, the long run average of the hazard rate, or more precisely

$$\liminf_{x\to\infty} \frac{1}{x}\int_0^x r(t)\mathrm{d}t \;=\; -\limsup_{x\to\infty} \frac{\log \bar{F}(x)}{x},$$

must be larger than $1/\mu$ for any distribution in the $\mathcal{M}$-class, as the following result reveals.

Theorem 3.1. *If F is a distribution in the $\mathcal{M}$-class with mean μ, then*

$$-\limsup_{x\to\infty} \frac{\log \bar{F}(x)}{x} \geq \frac{1}{\mu}. \tag{3.3}$$

Proof. It is known that the moment generating function has a singularity at the point

$$\sigma = -\limsup_{x\to\infty} \frac{\log \bar{F}(x)}{x},$$

see e.g. Widder (1941) or Lukacs (1970). For a distribution in the $\mathcal{M}$-class the m.g.f. must be finite for all $0 \leq t < 1/\mu$, and thus it must be $\sigma \geq 1/\mu$. □

We immediately get the following corollary.

Corollary 3.1. *If F is an absolutely continuous distribution in the $\mathcal{M}$-class with hazard rate r and mean μ, then*

$$\limsup_{t\to\infty} r(t) \geq \frac{1}{\mu}. \tag{3.4}$$

In contrast to these properties, it follows from results in Kotlyar and Khomenko (1993) that an $\mathcal{L}$-class distribution with mean μ and a continuous hazard rate r fulfills

$$\lim_{x\to 0} r(x) \leq \frac{1}{\mu}.$$

Rolski (1975) introduced the notion of harmonic new better than used in expectation (HNBUE). One possible characterization of this property is as follows. It holds for a distribution F with mean μ if and only if $F \leq_{cx} \mathrm{Exp}(1/\mu)$. Thus we immediately have the following result.

Theorem 3.2. *The $\mathcal{M}$-class and the $\mathcal{LM}$-class contain all HNBUE distributions.*

As an immediate corollary we obtain that property (3.3) in particular holds for HNBUE distributions. To the best of our knowledge even that result would be new.

If F has mean μ, the requirement $F \le_{s-icx} \mathrm{Exp}(1/\mu)$ for some $s > 2$ leads to a class of life distributions that could be called HNBUE(s). It lies between HNBUE distributions and the $\mathcal{M}$-class. Deshpande, Kochar, and Singh (1986) considered the case $s = 3$. Another class of distributions between HNBUE(s) and the $\mathcal{M}$-class is obtained by requiring $F \le_{mom} \mathrm{Exp}(1/\mu)$. Example 2.1 shows that this class is strictly smaller than the $\mathcal{M}$-class.

The $\mathcal{M}$-class has the following properties.

Theorem 3.3. *a) The $\mathcal{M}$-class is closed under convolution, i.e., if X and Y are independent and in the $\mathcal{M}$-class, then $X + Y \in \mathcal{M}$, too.*

b) The $\mathcal{M}$-class is invariant under scale transformation, i.e., if $X \in \mathcal{M}$ then $\alpha X \in \mathcal{M}$ for all $\alpha > 0$.

Proof. a) follows from Theorem 2.2 a), taking into account that the convolution of two exponential distributions is IFR and hence in the $\mathcal{M}$-class. b) is a consequence of Theorem 2.2 d) and the fact that αY has an exponential distribution, whenever Y is exponential. $\square$

The next theorem delivers an abundance of examples of life distributions in the $\mathcal{M}$-class. It also shows that the $\mathcal{M}$-class is much larger than the HNBUE class. Note that distributions in the $\mathcal{L}$-class (and hence also distributions in any smaller class like IFR, NBU, NBUE, HNBUE etc.) cannot have point masses in the origin. Thus the $\mathcal{M}$-class seems to be the only notion of positive aging, which includes distributions with point masses at the origin.

Theorem 3.4. *The $\mathcal{M}$-class contains all random variables X with $P(a \le X \le b) = 1$ and $EX \ge (a+b)/2$ for some $0 \le a < b$.*

Proof. a) First suppose that $P(X = a) = p = 1 - P(X = b)$ for some $p \le 1/2$. It follows from equation (3.1) that this distribution is in the $\mathcal{M}$-class iff

$$f_p(x) := \frac{1}{1 - (pa + (1-p)b)x} - pe^{ax} - (1-p)e^{bx} \ge 0 \tag{3.5}$$

for $0 \le x < 1/(pa + (1-p)b)$. A Taylor series expansion yields $f_p(x) = \sum_{k=0}^{\infty} a_k(p)x^k$ with

$$a_k(p) = (pa + (1-p)b)^k - p\frac{a^k}{k!} - (1-p)\frac{b^k}{k!}.$$

Thus it is sufficient to show that $a_k(p) \ge 0$ for all $p \le 1/2$ and all $k = 0, 1, 2, \ldots$ For $p = 1/2$ this is easy to see, and for $p < 1/2$ notice that for the derivative $a_k'(p)$ it holds

$$\begin{aligned} a_k'(p) &= k(pa+(1-p)b)^{k-1}(a-b) - \frac{a^k}{k!} + \frac{b^k}{k!} \\ &\le \frac{1}{k!}(b^k - a^k) - k(b-a)\Big(\frac{a+b}{2}\Big)^{k-1} \\ &= (b-a)\sum_{i=0}^{k-1} a^i b^{k-1-i}\Big(\frac{1}{k!} - \frac{k}{2^{k-1}}\Big) \\ &\le 0. \end{aligned}$$

Thus $a_k(p) \ge 0$ for all $p \le 1/2$ and all $k = 0, 1, 2, \ldots$.

b) Now assume that $P(a \le X \le b) = 1$ and $EX \ge (a+b)/2$. Let

$$P(Y = a) = \frac{b - EX}{b - a} = 1 - P(Y = b).$$

It is well known that $X \le_{cx} Y$ (see e.g. Müller and Stoyan (2002), Example 1.10.5), and hence $X \le_{mgf} Y$, too. According to part a) $Y \le_{mgf} \mathrm{Exp}(1/EX)$ and thus by transitivity $X \le_{mgf} \mathrm{Exp}(1/EX)$, i.e. X is in the $\mathcal{M}$-class. □

Notice that the condition on the mean describes an intuitive notion of aging, namely that the item is more likely to fail near the end of its potential lifetime than at its beginning. The bound $(a+b)/2$ for the mean is sharp in the case $a = 0$.

All symmetric distributions with bounded support fulfill the assumption of Theorem 3.4. Thus we have the following corollary.

Corollary 3.2. *Any symmetric life distribution is contained in the $\mathcal{M}$-class.*

Lin (1998) has shown that a distribution F is in the $\mathcal{L}$-class if and only if $F \ge_{Lt} F_e$. A similar result holds for the $\mathcal{M}$-class.

Theorem 3.5. *A distribution F belongs to the $\mathcal{M}$-class if and only if $F \ge_{mgf} F_e$.*

Proof. By (2.3), $m_e(t) \le m(t)$ for all $t > 1/\mu$. Therefore,

$$\begin{aligned} F \ge_{mgf} F_e \quad &\Leftrightarrow \quad m_e(t) \le m(t), \quad 0 \le t < 1/\mu \\ &\Leftrightarrow \quad m(t) \le (1-\mu t)^{-1}, \quad 0 \le t < 1/\mu, \end{aligned}$$

which holds if and only if F is in the $\mathcal{M}$-class. □

Remark 3.1. The above reasoning shows that $F \in \mathcal{M}$ is equivalent to

$$m_e(t) \quad \le \quad (1-\mu t)^{-1}, \quad 0 \le t < 1/\mu. \tag{3.6}$$

Therefore, if $F \in \mathcal{M}$ and $Var(X) = (EX)^2$ (and, consequently, $\mu_e = \mu$), then $F_e \in \mathcal{M}$. However, for arbitrary $\mathcal{M}$-class distributions, $Var(X) \le (EX)^2$ or $\mu_e \le \mu$. Hence, (3.6) does not imply $F_e \in \mathcal{M}$ in general (see Example 4.2 below).

The next lemma shows that any stochastic comparison of a distribution F with its equilibrium distribution F_e has an interesting interpretation in terms of the mean residual life function.

Lemma 3.1. *Let F be a life distribution with mean μ and equilibrium distribution F_e, and f some increasing right-continuous function. Then*

$$\int f \mathrm{d}F \geq \int f \mathrm{d}F_e \tag{3.7}$$

holds if and only if

$$\int \mathrm{MRL}\ \mathrm{d}G \leq \mu, \tag{3.8}$$

where G is a distribution with

$$\mathrm{d}G = \frac{\bar{F}\mathrm{d}f}{\int \bar{F}\mathrm{d}f}.$$

Proof. We can assume without loss of generality that $f(0) = 0$. Partial integration then yields that $\int f\mathrm{d}F = \int \bar{F}\mathrm{d}f$ and similarly for F_e. Using (2.2), it follows that (3.7) is equivalent to

$$\int \bar{F}\mathrm{d}f \geq \int \frac{\mathrm{MRL}\ \bar{F}}{\mu}\mathrm{d}f.$$

This can be rewritten as (3.8). □

Lemma 3.1 has a nice interpretation. If some device has a random lifetime X with distribution F, then $\int f\mathrm{d}F \geq \int f\mathrm{d}F_e$ holds if and only if the expected remaining lifetime EX_Y of the device after some random time Y (independent of X) with distribution G is smaller than the original expected lifetime $\mu = EX$.

Thus the condition $F \geq_{\mathfrak{F}} F_e$ for some class $\mathfrak{F}$ of increasing right-continuous functions always leads to a notion of aging generalizing the notion of NBUE. Notice that the well known result that NBUE holds if and only if $F \geq_{st} F_e$ can easily be recovered from Lemma 3.1.

For the $\mathcal{L}$-, $\mathcal{M}$- and the $\mathcal{LM}$-class we immediately get the following characterization.

Theorem 3.6. *Let X be a random lifetime with distribution F and mean μ. Then*

a) X is in the $\mathcal{L}$-class, if the expected remaining lifetime $EX_Y \leq \mu$ for all random time points Y that are independent of X and have a density of the form

$$f_t(x) = \frac{\bar{F}(x)e^{tx}}{\int \bar{F}(x)e^{tx}\mathrm{d}x}, \quad x > 0, \tag{3.9}$$

for some $t < 0$.

b) X is in the $\mathcal{M}$-class, if $EX_Y \leq \mu$ for all Y that are independent of X and have a density of the form (3.9) for some $0 < t < 1/\mu$.

c) X is in the $\mathcal{LM}$-class, if $EX_Y \leq \mu$ for all Y that are independent of X and have a density of the form (3.9) for some $-\infty < t < 1/\mu$.

This results gives interesting insight into the difference between the $\mathcal{L}$-class and the $\mathcal{M}$-class. Notice that the densities f_t are increasing in t with respect to the likelihood ratio order $\leq_{lr}$, and hence their means are increasing functions of t. (Recall that $f \leq_{lr} g$ holds if g/f is increasing and that this implies $f \leq_{st} g$; see Müller and Stoyan (2002).)

Thus a distribution is in the $\mathcal{L}$-class, if the expected remaining lifetimes EX_Y are smaller than the original expected lifetime EX for a family of random time points Y with small expected values $EY \leq E(X^2)/(2EX)$; whereas in the characterization of the $\mathcal{M}$-class these time points have large means $EY \geq E(X^2)/(2EX)$. This explains why the $\mathcal{L}$-class contains life distributions where the mean residual life function $\mathrm{MRL}(t)$ goes to infinity for $t \to \infty$, whereas this is not possible for the $\mathcal{M}$-class.

As a further example of the comparison of F and F_e, consider the class of distributions with $F \geq_{s-icx} F_e$ for $s \geq 1$. If X has distribution function F, this is equivalent to the requirement

$$E(X-t)_+^{s-1} \quad \geq \quad E(X_e - t)_+^{s-1} \tag{3.10}$$

for all $t \geq 0$. Using

$$\begin{aligned} E(X_e - t)_+^{s-1} &= \int_0^\infty (x-t)_+^{s-1} dF_e(x) \\ &= 1/\mu \int_t^\infty (x-t)^{s-1} \bar{F}(x)dx \\ &= E(X-t)_+^s/(\mu s), \end{aligned}$$

and introducing Bondesson's functions

$$M_n(t) \quad = \quad \frac{E(X-t)_+^n}{nE(X-t)_+^{n-1}}$$

for integer $n \geq 1$ and for all t such that $F(t) < 1$ (see Stein and Dattero (1999)), we can write (3.10) as

$$M_s(t) \quad \leq \quad \mu$$

for all t such that $F(t) < 1$. Now consider the generalized mean remaining lifetime

$$\mathrm{MRL}_n(t) \quad = \quad \frac{\int_t^\infty \bar{F}^{(n)}(x)dx}{\bar{F}^{(n)}(t)},$$

introduced by Fagiuoli and Pellerey (1993). Since $M_n = \mathrm{MRL}_n$ (Theorem 2 in Stein and Dattero (1999)), a further equivalent form of (3.10) is

$$\mathrm{MRL}_s(t) \quad \leq \quad \mu \tag{3.11}$$

for all t such that $F(t) < 1$. Since, for $s = 1$, (3.11) is the definition of NBUE, the aging classes under discussion could be termed s-NBUE.

4 Examples of $\mathcal{M}$-class distributions

Many life distributions like the gamma or the Weibull distribution with shape parameter greater than 1 are HNBUE and, hence, also belong to the $\mathcal{M}$- as well to the $\mathcal{L}$-class. To obtain more examples, we use the following theorem (Kaas and Hesselager (1995), Theorem 2.3) which gives sufficient conditions for the s-increasing convex order.

A real-valued function ϕ on $[0, \infty)$ is said to have n sign changes (denoted as $S[\phi] = n$) if there exists a disjoint partition $I_1 < I_2 < \ldots < I_{n+1}$ of $[0, \infty)$ such that ϕ has opposite signs on subsequent intervals I_j and I_{j+1} and $\int_{I_j} \phi(t)dt \neq 0$ for all j.

Theorem 4.1. *Let U and V be positive random variables with distribution functions G and H, respectively. Further, let $EU^j = EV^j, j = 1, \ldots, s-2$ and $EU^{s-1} \leq EV^{s-1}$. Then each of the following conditions is sufficient for $U \leq_{s-icx} V$.*

(i) $S[G - H] \leq s - 1$ *with* $\lim_{t\to\infty} G(t) - H(t) \geq 0$.

(ii) $S[\log(g/h)] \leq s$ *with* $\lim_{t\to\infty} h(t) - g(t) \geq 0$*, where G and H are assumed to be absolutely continuous with densities g and h, respectively.*

As in the case of the HNBUE class, the coefficient of variation is not greater than 1 for distributions in the $\mathcal{M}$-class. However, the exponential distribution is not characterized within the $\mathcal{M}$-class by the property that the coefficient of variation is one.

Example 4.1. Consider a random variable X with $P(X = 0) = P(X = 1) = 1/2$. Since $Var(X) = (EX)^2 = 1/4$, X is not HNBUE. However, by Theorem 3.4 or Theorem 4.1, X belongs to the $\mathcal{M}$-class.

The moment generating function of the lognormal distribution does not exist for $t > 0$. Hence, the lognormal distribution does not belong to the $\mathcal{M}$-class. Note that for all lognormal distributions the hazard rate is zero at $t = 0$, increases to a maximum and then decreases, approaching zero at $t \to \infty$. Therefore, inequality (3.4) is not satisfied.

Other well known life distributions are the inverse Gaussian and the Birnbaum-Saunders distribution. There, the membership in the $\mathcal{M}$-class depends on the shape parameters.

Example 4.2. Let $IG(\mu, \lambda)$ denote the inverse Gaussian distribution with parameters $\mu > 0$ and $\lambda > 0$, which has density

$$f(x) = \sqrt{\frac{\lambda}{2\pi}}\, x^{-3/2} \exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right), \qquad x > 0.$$

The mean and variance of $IG(\mu, \lambda)$ are μ and μ^3/λ, respectively.

Proposition 4.1. *Let $X \sim IG(\mu, \lambda)$ with $\lambda \geq 2\mu$ and $Y \sim \exp(1/\mu)$. Then $X \leq_{3-icx} Y$.*

Proof. By rescaling if necessary, we may take $\mu = 1$, so $X \sim IG(1, \lambda)$ and $Y \sim \exp(1)$. The moment generating function of X is $m(t) = \exp\left(\lambda\left(1 - \sqrt{1 - 2t/\lambda}\right)\right)$, which exists for all $t < 1$ if $\lambda \geq 2$. For some constant c, we have

$$\log \frac{f_Y(x)}{f_X(x)} \;=\; c + \frac{3}{2}\log x + \frac{\lambda}{2x} - \left(1 - \frac{\lambda}{2}\right) x,$$

which tends to $+\infty$ for $x \to \infty$ and has no more than three sign changes (see Kaas and Hesselager (1995), p. 197). By Theorem 4.1(ii), the assertion follows. □

Consequently, each $IG(\mu, \lambda)$-distribution with $\lambda \geq 2\mu$ belongs to the $\mathcal{M}$-class. On the other hand, $IG(\mu, \lambda) \in \mathcal{L}$ for $\lambda \geq \mu$; see Klar (2002). Chhikara and Folks (1989) show that the hazard rate of $IG(\mu, \lambda)$ increases from zero at time $t = 0$ until it attains a maximum at some critical time and then decreases to the non-zero asymptotic value $\lambda/2\mu^2$. Hence, inequality (3.4) holds for $\lambda \geq 2\mu$, but not for $\mu \leq \lambda < 2\mu$.

Now consider the case $\mu = 1, \lambda = 2$. The expected value of the equilibrium distribution is $\mu_e = 3/4$, but m_e does not exist for $t \geq 1$. Hence, $F_e \notin \mathcal{M}$.

Example 4.3. The Birnbaum-Saunders distribution $BS(\gamma, \delta)$ with parameters $\gamma, \delta > 0$ has density

$$f(x) \;=\; \frac{\exp(\gamma^{-2})}{2\gamma\sqrt{2\pi\delta}}\, x^{-3/2}(x+\delta)\exp\left\{-\frac{1}{2\gamma^2}\left(\frac{x}{\delta} + \frac{\delta}{x}\right)\right\}, \qquad x > 0;$$

see Johnson, Kotz, and Balakrishnan (1995), p. 651. The mean and variance of $BS(\gamma, \delta)$ are $\delta(\gamma^2/2 + 1)$ and $\delta^2\gamma^2(5\gamma^2/4 + 1)$, respectively.

Since the second parameter is a scale parameter, it follows from Theorem 3.3 that we can without loss of generality take the means to be equal to 1. Hence, let $X \sim BS(\gamma, (\gamma^2/2 + 1)^{-1})$ and $Y \sim \exp(1)$. Now,

$$\log \frac{f_Y(x)}{f_X(x)} \;=\; c + \frac{2 - 3\gamma^2}{4\gamma^2}x + \left(\gamma^2(\gamma^2+2)x\right)^{-1} + \frac{3}{2}\log x - \log\left(x(\gamma^2+2) + 2\right)$$

which tends to $+\infty$ for $x \to \infty$ only if $\gamma^2 \leq 2/3$. This function has no more than three sign changes; see Klar (2002). By Theorem 4.1(ii), we have $X \leq_{3-icx} Y$ for $\gamma^2 \leq 2/3$. Therefore, the Birnbaum-Saunders distribution belongs to the $\mathcal{M}$-class, provided $\gamma \leq \sqrt{2/3}$.

As the inverse Gaussian distribution, the Birnbaum-Saunders distribution belongs to the $\mathcal{L}$-class provided the coefficient of variation does not exceed 1 (i.e. $\gamma \leq 1$). Since $\lim_{t\to\infty} r(t) = 1/(2\delta\gamma^2)$, $\lim_{t\to\infty} r(t) \geq 1/\mu$ only for $\gamma^2 \leq 2/3$. Consequently, $X \notin \mathcal{M}$ for $\sqrt{2/3} < \gamma \leq 1$.

5 An explicit reliability bound for the $\mathcal{M}$-class

Theorem 5.1. *Let $F \in \mathcal{M}$ with mean μ. Then the following bound holds:*

$$\bar{F}(t) \;\leq\; \frac{t}{\mu} \exp\left(1 - \frac{t}{\mu}\right) \quad \text{for all} \quad t > \mu.$$

Proof. Using

$$\int_0^\infty e^{sx} F(dx) \;\geq\; \int_t^\infty e^{sx} F(\mathrm{d}x) \;=\; e^{st}\bar{F}(t) \quad (s > 0),$$

we obtain for distributions in the $\mathcal{M}$-class

$$e^{st}\bar{F}(t) \;\leq\; m_F(s) \;\leq\; (1 - s\mu)^{-1} \quad (0 < s < 1/\mu).$$

Therefore,

$$\bar{F}(t) \;\leq\; \inf_{0<s<1/\mu} \frac{e^{-st}}{1 - s\mu} \;=\; \frac{e^{-s_0 t}}{1 - s_0\mu},$$

where $s_0 = (t - \mu)/(t\mu)$. □

Remark 5.1. (i) The upper bound μ/t, which applies to any survival function with mean μ, is better than the bound in Theorem 5.1 in the range $\mu < t < 3.51\mu$.

(ii) Klefsjö (1982) obtained the sharp upper bound $e^{1-t/\mu}$ $(t > \mu)$ for survival functions in the HNBUE class. For the $\mathcal{L}$-class, the upper bound

$$\bar{F}(t) \leq \big((t/\mu)^2 - 2t/\mu + 2\big)^{-1}$$

holds, which is sharp for $t/\mu \geq 2 + \sqrt{2}$; see Klar (2002).

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