Testing exponentiality against the $\mathcal{L}$-class of life distributions

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Abstract

This paper studies two classes of tests for exponentiality against the nonparametric class $\mathcal{L}$ of life distributions introduced by Klefsjö (1983a). The test statistics are integrals of the suitably weighted difference between the empirical Laplace transform of given data and the Laplace transform of a fitted exponential distribution. Both classes of tests are related to the first nonzero component of Neyman's smooth test for exponentiality. We derive the limit distributions of the test statistics in case of a general underlying distribution and the local approximate Bahadur efficiency of the tests against several parametric families of alternatives to exponentiality. The results of a simulation study corroborate the theoretical findings.

Key words: Exponential distribution, life distribution, Laplace transform, $\mathcal{L}$-class, goodness-of-fit test, local approximate Bahadur efficiency.

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1 Introduction

A distribution function F with support $[0, \infty)$ and finite mean $\mu = \int_0^\infty \overline{F}(x)dx$, where $\overline{F} = 1 - F$, is said to belong to the $\mathcal{L}$-class of life distributions ($F \in \mathcal{L}$)

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if

$$\int_0^\infty e^{-sx}\overline{F}(x)dx \;\; \geq \;\; \frac{\mu}{1+s\mu} \quad \text{for all } s \geq 0. \tag{1}$$

The class $\mathcal{L}$ was introduced by Klefsjö (1983a). By means of the Laplace transform $L_F(s) = E_F e^{-sX}$, (1) can be restated as

$$L_F(s) \;\leq\; L(s, 1/\mu) \quad \text{for all } s \geq 0, \tag{2}$$

where $L(s,\lambda) = \lambda/(\lambda+s)$ denotes the Laplace transform of the exponential distribution with distribution function $F(t,\lambda) = 1 - \exp(-\lambda t)$ for $t \geq 0$ (see, e.g., Lin (1998), Theorem 2). From (2), a distribution belongs to the $\mathcal{L}$-class if it dominates the exponential distribution with the same mean in the Laplace transform order (Stoyan (1983), p. 22). If the reversed inequality holds in (1) then F belongs to the so-called $\overline{\mathcal{L}}$-class of distributions (see Klefsjö (1983a)). The $\mathcal{L}$-class is strictly larger than the harmonic new better than used in expectation (HNBUE) class of life distributions , satisfying $\int_t^\infty \overline{F}(x)dx \leq \mu \exp(-t/\mu)$ for every $t \geq 0$ (Rolski (1975)).

Klefsjö (1983b) seems to be the first who considered tests of exponentiality against HNBUE alternatives; for further tests see Klar (2000) and the references cited therein. Chaudhuri (1997) proposed a test of exponentiality against the $\mathcal{L}$-class of distributions. Defining

$$\varphi_\epsilon(F) \;\; = \;\; \sup\left\{ \int_0^\infty e^{-sx}\overline{F}(x)dx - \frac{\mu}{1+s\mu} : 0 \leq s \leq F^{-1}(1-\epsilon) \right\},$$

where $0 < \epsilon < 1$ is a (small) fixed number, he used $D_{n,\epsilon} = n^{1/2}\overline{X}_n^{-1}\varphi_\epsilon(F_n)$ as a test statistic. Here $\overline{X}_n = n^{-1}\sum_{j=1}^n X_j$ is the mean of a random sample $X_1,\ldots,X_n$ of size n from F, and $F_n(x) = n^{-1}\sum_{j=1}^n \mathbf{1}\{X_j \leq x\}$ is the empirical distribution function. An alternative expression for $D_{n,\epsilon}$ is

$$D_{n,\epsilon} \;\; = \;\; n^{1/2}\overline{X}_n^{-1} \sup\left\{ \frac{1}{s}\left(\frac{1}{1+s\overline{X}_n} - \frac{1}{n}\sum_{j=1}^n \exp(-sX_j) \right) : 0 \leq s \leq X_{(m)} \right\},$$

where $m = \lceil n(1-\epsilon) \rceil$ is the smallest integer exceeding $n(1-\epsilon)$, and $X_{(1)} < \ldots < X_{(n)}$ are the order statistics of $X_1,\ldots,X_n$. Note that the computational formula (2.3) in Chaudhuri (1997) seems to be in doubt since it contains $F^{-1}(1-\epsilon)$, which is unknown.

A disadvantage of this approach is that a test of exponentiality based on $D_{n,\epsilon}$ is not consistent against each alternative from the class $\mathcal{L}$ since the test statistic does not consider the upper tail of the Laplace transforms. Moreover, only rough approximations of the quantiles of the asymptotic null distribution of $D_{n,\epsilon}$ are available. Finally, there is only very limited empirical evidence on the power of the test.

This paper proposes tests for exponentiality against the class $\mathcal{L}$ that do not share these deficiencies. To this end, let

$$L_n(t) \quad = \quad \int_0^\infty e^{-tX} dF_n(x) \quad = \quad \frac{1}{n}\sum_{i=1}^n e^{-tX_i}$$

denote the empirical Laplace transform of $X_1, \ldots, X_n$. In view of (2), it seems natural to base a test of

$$H_0: \; F \in \mathcal{E} = \{F(\cdot, \lambda), \; \lambda > 0\}$$

against the alternative

$$H_1: \; F \in \mathcal{L} \quad \text{and} \quad F \notin \mathcal{E}$$

on the empirical counterpart $L_n(x) - L(x, 1/\overline{X}_n)$ of $L_F(x) - L(x, 1/\mu)$. A first class of test statistics proposed is $(T_{n,a})_{a>0}$, where

$$T_{n,a} \quad = \quad \overline{X}_n \int_0^\infty \Big(L_n(t) - L(t, 1/\overline{X}_n)\Big) \exp(-a\overline{X}_n t) \; dt, \tag{3}$$

and a is a positive constant. Since $L(x) - L(x, 1/\mu)$ is nonpositive for alternatives from the class $\mathcal{L}$, H_0 is rejected for large *negative* values of $T_{n,a}$. Similarly, a test of exponentiality against $\overline{\mathcal{L}}$-class alternatives has an upper rejection region. Using the formula $\int_0^\infty \exp(-at)/(1+t)\, dt = e^a E_1(a)$, where $E_1(a) = \int_a^\infty \exp(-t)/t \, dt$ is the exponential integral, $T_{n,a}$ takes the form

$$T_{n,a} \quad = \quad \frac{1}{n}\sum_{j=1}^n \frac{1}{Y_j + a} \; - \; e^a E_1(a), \tag{4}$$

where $Y_j = X_j/\overline{X}_n,\ 1 \le j \le n$.

A second class of test statistics is $(\tilde{T}_{n,a})_{a>0}$, where

$$\begin{aligned}\tilde{T}_{n,a} &= \overline{X}_n \int_0^\infty \Big(L_n(t) - L(t, 1/\overline{X}_n)\Big)\,(1+\overline{X}_n t)\ \exp(-a\overline{X}_n t)\ dt \qquad (5)\\ &= \frac{1}{n}\sum_{j=1}^n \left(\frac{1}{Y_j+a} + \frac{1}{(Y_j+a)^2}\right) - \frac{1}{a}.\end{aligned}$$

Note that the only distinction between $T_{n,a}$ and $\tilde{T}_{n,a}$ is the different weight function, which is $\overline{X}_n \exp(-a\overline{X}_n t)$ in (3) and $\overline{X}_n(1+\overline{X}_n t)\exp(-a\overline{X}_n t)$ in (5). Variants of $T_{n,a}$ and $\tilde{T}_{n,a}$, which result from (3) and (5) by squaring the difference $L_n(t) - L(t, 1/\overline{X}_n)$, have been studied in Henze (1993) and Henze and Meintanis (2000), respectively, as omnibus tests for exponentiality.

The paper is organized as follows. In Section 2 we state the asymptotic behavior of the statistics $T_{n,a}$ and $\tilde{T}_{n,a}$ as $a \to \infty$ and derive their limit distributions in case of a general underlying distribution. A test for exponentiality rejecting H_0 for large negative values of $T_{n,a}$ or $\tilde{T}_{n,a}$ is seen to be consistent against each fixed alternative from the class $\mathcal{L}$. Moreover, we give an example of a non-exponential continuous distribution that belongs to the class $\mathcal{L}$ but is not HNBUE. Section 3 is devoted to the calculation of local approximate Bahadur efficiencies of the proposed tests of exponentiality with respect to five families of alternative distributions from the class $\mathcal{L}$. In Section 4 we present the results of a simulation study that corroborates the theoretical findings.

2 Asymptotic distributions, Consistency

Our first result shows that, letting the parameter a figuring in (3) and (5) tend to infinity, both $T_{n,a}$ and $\tilde{T}_{n,a}$, when suitably scaled, approach the same limit, which is connected with some well-known statistics for testing exponentiality.

2.1 Proposition *For fixed n, we have*

$$T_{n,\infty} \equiv \lim_{a\to\infty} a^3\, T_{n,a} = \lim_{a\to\infty} a^3\, \tilde{T}_{n,a} = \frac{1}{n}\sum_{j=1}^n Y_j^2 - 2.$$

PROOF. The proof follows the same lines as the proof of Theorem 2.1 in Baringhaus et al. (2000) and will thus be omitted. ■

Notice that $\sqrt{n}\,T_{n,\infty}/2$ is the first nonzero component of Neyman's smooth test of fit for exponentiality (see, e.g., Koziol (1987)); it is asymptotically most powerful for testing H_0 against the linear failure rate distribution (Doksum and Yandell (1984)). Up to one-to-one transformations, $T_{n,\infty}$ coincides with Greenwood's statistic $G_n = 1/n^2 \sum_{j=1}^{n} Y_j^2$ (Greenwood (1946)) and with the sample coefficient of variation $CV_n = S_n/\overline{X}_n$, where $S_n^2 = n^{-1}\sum_{j=1}^{n}(X_j - \overline{X}_n)^2$ denotes the sample variance.

It is well-known that $\sqrt{n}\,T_{n,\infty}/2$ has a limiting unit normal distribution under H_0; hence, the asymptotic null distribution of $\sqrt{n}\,T_{n,\infty}$ is $\mathcal{N}(0,4)$. The next theorem gives the asymptotic distribution of $T_{n,a}$ for $0 < a < \infty$. Since the representation of $T_{n,a}$ in (4) shows that $T_{n,a}$ is scale-invariant, we assume $\mu = 1$ in the following.

2.2 Theorem *Assume $X_1, \ldots, X_n$ is a random sample of a nonnegative nondegenerate random variable X with finite second moment. Then, as $n \to \infty$,*

$$\sqrt{n}\,(T_{n,a} - E\,T_{n,a}) \xrightarrow{\mathcal{D}} \mathcal{N}(0,\sigma^2), \qquad (6)$$

where

$$\sigma^2 = E\left(\kappa_1(X-1) + \frac{1}{X+a} - \mu_1\right)^2 \qquad (7)$$

and

$$\kappa_1 = E\left[\frac{X}{(X+a)^2}\right], \qquad \mu_1 = E\left[\frac{1}{X+a}\right]. \qquad (8)$$

Under H_0, we have

$$\sqrt{n}\,T_{n,a} \xrightarrow{\mathcal{D}} \mathcal{N}(0,\sigma_0^2), \qquad (9)$$

where

$$\sigma_0^2 = (2a+1)e^a E_1(a) - (a^2+2a+2)e^{2a}E_1^2(a) + \frac{1-a}{a}. \qquad (10)$$

PROOF: Notice that

$$\begin{aligned}\sqrt{n}\,(T_{n,a} - E\,T_{n,a}) &= \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(\frac{1}{Y_j + a} - E\left[\frac{1}{Y_1 + a}\right]\right) \\ &= U_{n,1} + U_{n,2} + U_{n,3},\end{aligned}$$

where

$$\begin{aligned}U_{n,1} &= \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(\frac{1}{Y_j + a} - \frac{1}{X_j + a}\right), \\ U_{n,2} &= \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(\frac{1}{X_j + a} - \mu_1\right), \\ U_{n,3} &= \sqrt{n}\left(\mu_1 - E\left[\frac{1}{Y_1 + a}\right]\right).\end{aligned}$$

A Taylor expansion of the function $g(t) = 1/(t + a)$ around $t = X_j$ yields

$$U_{n,1} = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}(X_j - 1)\kappa_1 + o_P(1),$$

whence, by the Central limit theorem and Slutsky's lemma, $U_{n,1} + U_{n,2} \stackrel{\mathcal{D}}{\longrightarrow} \mathcal{N}(0, \sigma^2)$, where σ^2 is given in (7). To complete the proof of (6) and (9), we show that the nonrandom sequence $U_{n,3}$ tends to zero. To this end, observe that

$$\frac{1}{Y_1 + a} - \frac{1}{X_1 + a} = R_{n,1} - R_{n,2},$$

where

$$R_{n,1} = \frac{(\overline{X}_n - 1)X_1}{(X_1 + a)^2}, \qquad R_{n,2} = \frac{(\overline{X}_n - 1)^2 a X_1}{(X_1 + a)^2(X_1 + a\overline{X}_n)}.$$

Now,

$$\sqrt{n}\;ER_{n,1} = \frac{1}{\sqrt{n}}\left(E\left[\frac{\left(\sum_{j=2}^{n} X_j - (n-1)\right)X_1}{(X_1 + a)^2}\right] + E\left[\frac{(X_1 - 1)X_1}{(X_1 + a)^2}\right]\right),$$

which tends to zero as $n \to \infty$ since the first expectation on the right-hand side vanishes. To show $\lim \sqrt{n}E[R_{n,2}] = 0$, note that

$$\begin{aligned}0 \leq \sqrt{n}R_{n,2} &= \frac{1}{\sqrt{n}}\left(\sqrt{n}(\overline{X}_n - 1)\right)^2 \frac{a}{(X_1 + a)^2}\,\frac{X_1}{X_1 + a\overline{X}_n} \\ &\leq \frac{1}{\sqrt{n}}\left(\sqrt{n}(\overline{X}_n - 1)\right)^2 \frac{1}{a}\end{aligned}$$

and $E\left(\sqrt{n}(\overline{X}_n - 1)\right)^2 = Var(X) < \infty$.

The formula (10) for σ^2 in case of H_0 follows from straightforward calculations. ■

The next result states the asymptotic distribution of $\tilde{T}_{n,a}$. The proof follows the reasoning given above and will thus be omitted.

2.3 Theorem *Assume $X_1, \ldots, X_n$ is a random sample of a nonnegative non-degenerate random variable X with finite second moment. Then, as $n \to \infty$,*

$$\sqrt{n}\left(\tilde{T}_{n,a} - E\,\tilde{T}_{n,a}\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tilde{\sigma}^2),$$

where

$$\tilde{\sigma}^2 = E\left(\frac{1}{X+a} - \mu_1 + \frac{1}{(X+a)^2} - \mu_2 + (X-1)(\kappa_1 + \kappa_2)\right)^2,$$

where, in addition to μ_1 and κ_1 defined in (8),

$$\mu_2 = E\left[\frac{1}{(X+a)^2}\right], \qquad \kappa_2 = 2E\left[\frac{X}{(X+a)^3}\right].$$

Under H_0, we have

$$\sqrt{n}\,\tilde{T}_{n,a} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tilde{\sigma}_0^2),$$

where

$$\tilde{\sigma}_0^2 = (2/a - 1/6)e^a E_1(a) - e^{2a} E_1^2(a) + (a^2 - 7a + 2)/(6a^3). \qquad (11)$$

Regarding consistency of the tests that reject the hypothesis of exponentiality for large negative values of $T_{n,a}$ or $\tilde{T}_{n,a}$, we have the following result.

2.4 Theorem *Let $0 < a < \infty$, $\alpha \in (0,1)$, and let $z_n(\alpha)$ denote the α-quantile of $T_{n,a}$ under H_0. Under a fixed alternative distribution from H_1, we have*

$$\lim_{n\to\infty} P(T_{n,a} < z_n(\alpha)) = 1,$$

i.e., a one-sided test (lower rejection region) based on $T_{n,a}$, $0 < a < \infty$, is consistent against each alternative from the class $\mathcal{L}$. Likewise, a test for exponentiality rejecting H_0 for large negative values of $\tilde{T}_{n,a}$ is consistent against each alternative from the class $\mathcal{L}$.

Proof. It follows from Theorem 2.2 and its proof that $z_n(\alpha) = O(1/\sqrt{n})$. Moreover, Theorem 2.2 implies

$$T_{n,a} \xrightarrow{P} E\left[\frac{1}{X_1 + a}\right] - e^a E_1(a) \tag{12}$$

as $n \to \infty$. Since the function $t \mapsto 1/(t+a)$ is completely monotone (see Stoyan (1983)), the stochastic limit in (12) is negative under each alternative distribution from the class $\mathcal{L}$ (see Theorem 2.1 of Bhattacharjee (1999)), proving the assertion for $T_{n,a}$. Regarding consistency of the test of H_0 based on $\tilde{T}_{n,a}$, Theorem 2.3 implies

$$\tilde{T}_{n,a} \xrightarrow{P} E\left[\frac{1}{X_1 + a}\right] + E\left[\frac{1}{(X_1 + a)^2}\right] - \frac{1}{a}$$

as $n \to \infty$. Since the function $t \mapsto 1/(t+a) + 1/(t+a)^2$ is completely monotone, the rest of the argument follows the lines above. ■

Klefsjö (1986) pointed out that a test based on CV_n (or, equivalently, on $T_{n,\infty}$) is consistent against HNBUE alternatives, since the exponential distribution is characterised within the HNBUE class by the fact that the coefficient of variation (CV) equals 1. Bhattacharjee and Sengupta (1996) gave an example of a two-point distribution with $CV = 1$ that belongs to the $\mathcal{L}$-class. Hence, a test based on CV_n (or on $T_{n,\infty}$) is not consistent for testing H_0 against the wider $\mathcal{L}$-class. The following proposition provides an example of a continuous distribution different from the exponential distribution which is in the $\mathcal{L}$-class and satisfies $CV = 1$ (and, hence, is not HNBUE).

2.5 Proposition *Let $IG(\mu, \lambda)$ denote the inverse Gaussian distribution with parameters $\mu > 0$ and $\lambda > 0$, which has the density*

$$f(x;\mu,\lambda) = \sqrt{\frac{\lambda}{2\pi}}\, x^{-3/2} \exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right), \qquad x > 0.$$

a) If $\lambda \geq \mu$, then $IG(\mu, \lambda) \in \mathcal{L}$.

b) For $\lambda < \mu$, $IG(\mu, \lambda)$ belongs neither to $\mathcal{L}$ nor to $\overline{\mathcal{L}}$.

Proof: The Laplace transform of $IG(\mu, \lambda)$ is

$$L_{IG}(s) = \exp\left[\frac{\lambda}{\mu}\left(1 - \sqrt{1 + \frac{2\mu^2 s}{\lambda}}\right)\right], \quad s \geq 0.$$

Writing $v(s) = (1 + 2\mu^2 s/\lambda)^{1/2}$, we have $L_{IG}^{-1}(s) = \exp\left(\lambda(v(s) - 1)/\mu\right)$. For $\lambda \geq \mu$,

$$\begin{aligned} \left(L_{IG}^{-1}(s)\right)' &= \lambda v'(s)/\mu \exp\left(\lambda(v(s) - 1)/\mu\right) \\ &= \mu v^{-1}(s) \exp\left(\lambda(v(s) - 1)/\mu\right) \\ &\geq \mu v^{-1}(s) \exp\left(v(s) - 1\right). \end{aligned}$$

The inequality $\exp(u) > 1 + u \, (u \neq 0)$ yields $\left(L_{IG}^{-1}(s)\right)' > \mu$ for $s > 0$. Using $L_{IG}^{-1}(0) = 1$, one obtains $L_{IG}^{-1}(s) - 1 > \mu s$ for $s > 0$; hence

$$L_{IG}(s) < \frac{1}{1 + \mu s} \quad \text{for} \quad s > 0,$$

which is assertion a). To prove b), note that expectation and variance of $IG(\mu, \lambda)$ are μ and μ^3/λ, respectively. If $\lambda < \mu$ then $CV = \sqrt{\mu/\lambda} > 1$, and consequently $IG(\mu, \lambda) \notin \mathcal{L}$ (see Bhattacharjee and Sengupta (1996)). On the other hand, $IG(\mu, \lambda) \notin \overline{\mathcal{L}}$, since $L_{IG}(s) < 1/(1 + \mu s)$ for s large enough. ∎

By Proposition 2.5 a), the inverse Gaussian distribution $IG(\mu, \mu)$, $\mu > 0$, having Laplace transform $\exp\left(1 - \sqrt{1 + 2\mu s}\right)$ for $s \geq 0$, is a continuous distribution with $CV = 1$ that belongs to the $\mathcal{L}$-class. Chhikara and Folks (1989) show that the hazard rate of $IG(\mu, \lambda)$ increases from zero at time $t = 0$ until it attains a maximum at some critical time and then decreases to the non-zero asymptotic value $\lambda/2\mu^2$.

3 Local approximate Bahadur efficiency

In this section, we investigate the efficiency of the test for exponentiality that rejects H_0 for large negative values of $T_{n,a}$ against several one-parametric families of distributions from the class $\mathcal{L}$. In each case, the parameter space, denoted by Θ, is some subinterval of $(0,\infty)$. To stress the dependence of probabilities and expectations on the true underlying parameter $\vartheta \in \Theta$, we write P_ϑ and $E_\vartheta[\cdot]$, respectively. Depending on the specific alternative family, the unit exponential distribution corresponds either to the parameter value $\vartheta_0 = 1$ or to the value $\vartheta_0 = 0$. For reasons of mathematical tractability, our measure of efficiency is the local approximate Bahadur slope (see , e.g., Nikitin (1995), p. 10).

To this end, write $S_{n,a} = -\sqrt{n}T_{n,a}/\sigma_0$, where σ_0^2 is defined in (10), and let $\Phi(t) = \int_{-\infty}^t 1/\sqrt{2\pi}\,\exp(-x^2/2)dx$ be the standard normal distribution function. Putting $F_n(t,\vartheta) = P_\vartheta(S_{n,a} \le t)$, Theorem 2.2 yields $\lim_{n\to\infty} F_n(t,\vartheta_0) = \Phi(t),\ t \in \mathbb{R}$. Letting $L_{n,a}^* = 1 - \Phi(S_{n,a})$, we will prove that

$$-\frac{1}{n}\log L_{n,a}^* \ \stackrel{P}{\longrightarrow}\ \frac{1}{2}c^*(\vartheta,a)\ >\ 0 \quad \text{under}\ \ P_\vartheta, \tag{13}$$

where

$$c^*(\vartheta,a)\ =\ \left[\frac{1}{\sigma_0}\left\{E_\vartheta\left(\frac{1}{\frac{X}{\mu(\vartheta)}+a}\right)\ -\ E_{\vartheta_0}\left(\frac{1}{X+a}\right)\right\}\right]^2$$

and $\mu(\vartheta) = E_\vartheta[X]$. The function $c^*(\cdot,a)$ is called the approximate Bahadur slope of the sequence $(S_{n,a})$ of test statistics (see Nikitin (1995), p. 10).

To prove (13), notice that, by (12),

$$\frac{S_{n,a}}{\sqrt{n}}\ \stackrel{P}{\longrightarrow}\ -\frac{1}{\sigma_0}\left\{E_\vartheta\left(\frac{1}{\frac{X}{\mu(\vartheta)}+a}\right)\ -\ E_{\vartheta_0}\left(\frac{1}{X+a}\right)\right\}$$

under P_ϑ. Since this stochastic limit is positive for $\vartheta \ne \vartheta_0$, $S_{n,a} \to \infty$ P_ϑ-stochastically ($\vartheta \ne \vartheta_0$). Moreover, using $1-\Phi(x) \sim \varphi(x)/x$ as $x \to \infty$, where φ is the density of the standard normal distribution, some algebra gives

$$\begin{aligned}-\frac{1}{n}\log L_{n,a}^* \quad &\sim\quad -\frac{1}{n}\log\left(\varphi(S_{n,a})\frac{1}{S_{n,a}}\right)\\ &=\quad \frac{1}{2}\,\frac{S_{n,a}^2}{n}\ +\ o_{P_\vartheta}(1)\\ &=\quad \frac{1}{2}\,c^*(\vartheta,a)\ +\ o_{P_\vartheta}(1),\end{aligned}$$

proving (13).

We now consider the local behavior of $c^*(\vartheta, a)$ as $\vartheta \to \vartheta_0$, assuming the one-parametric family of alternative distributions to be sufficiently regular to allow a Taylor expansion of order two of $c^*(\vartheta, a)$ with respect to ϑ. Moreover, differentiation of $E_\vartheta[1/(X/\mu(\vartheta)) + a]$ may be done under the integral sign. These assumptions hold for each of the five families of distributions considered later in this section. After straightforward calculations, one obtains

$$c^*(\vartheta, a) \sim \frac{l_a^2}{\sigma_0^2}(\vartheta - \vartheta_0)^2 \qquad \text{as } \vartheta \to \vartheta_0,$$

where

$$l_a = \mu'(\vartheta_0)\,((1+a)\,e^a E_1(a) - 1) + \int_0^\infty \frac{1}{x+a} \frac{\partial}{\partial\vartheta} f(x,\vartheta)\Big|_{\vartheta=\vartheta_0} dx,$$

and σ_0^2 is given in (10).

Our measure of asymptotic local efficiency of $T_{n,a}$ is

$$e_{F_\vartheta}(T_{n,a}) = \frac{l_a^2}{\sigma_0^2}.$$

We conjecture that $e_{F_\vartheta}(T_{n,a})$ is equal to the asymptotic Pitman efficiency

$$\lim_{n\to\infty} \left[\frac{d}{d\vartheta} E_\vartheta(T_{n,a})\Big|_{\vartheta=\vartheta_0} \right]^2 \left(\sigma_0^2\right)^{-1}. \tag{14}$$

However, computing the limit figuring in (14) seems to be unfeasible.

We have calculated $e_F(T_{n,a})$ for linear failure rate, Makeham, Pareto, Weibull and gamma alternatives. These are given by the distribution functions

$$\begin{aligned}
F_\vartheta^{(1)}(x) &= 1 - \exp\left(-\left(x + \vartheta x^2/2\right)\right) \quad \text{for } x \ge 0, \vartheta \ge 0,\\
F_\vartheta^{(2)}(x) &= 1 - \exp\left(-\left(x + \vartheta\left(x + e^{-x} - 1\right)\right)\right) \quad \text{for } x \ge 0, \vartheta \ge 0,\\
F_\vartheta^{(3)}(x) &= 1 - (1+\vartheta x)^{-1/\vartheta} \quad \text{for } x \ge 0, \vartheta \ge 0,\\
F_\vartheta^{(4)}(x) &= 1 - \exp\left(-x^\vartheta\right) \quad \text{for } x \ge 0, \vartheta > 0,\\
F_\vartheta^{(5)}(x) &= \Gamma(\vartheta)^{-1} \int_0^x t^{\vartheta-1} e^{-t}\, dt \quad \text{for } x \ge 0, \vartheta > 0,
\end{aligned}$$

respectively. For $F_\vartheta^{(1)}, F_\vartheta^{(2)}$ and $F_\vartheta^{(3)}$, H_0 corresponds to $\vartheta = \vartheta_0 = 0$, and for $F_\vartheta^{(4)}$ and $F_\vartheta^{(5)}$, we have $\vartheta_0 = 1$.

Calculations give the efficiencies

$$\begin{aligned} e_{F^{(1)}}(T_{n,a}) &= e_{F^{(3)}}(T_{n,a}) \\ &= \Big((a^2/2 + 2a + 1)\, e^a\, E_1(a) - (a+3)/2\Big)^2 / \sigma_0^2 \end{aligned}$$

for $a > 0$, where $\sigma_0^2\ (= \sigma_0^2(a))$ is given in (10). Notice that $e_{F^{(1)}}$ is an increasing function of a with maximum value $e_{F^{(1)}}(T_{n,\infty}) = 1$. This result is not particularly surprising since, as mentioned above, the test based on $T_{n,\infty}$ is asymptotically most powerful for testing H_0 against the linear failure rate distribution.

Next, we have

$$e_{F^{(2)}}(T_{n,a}) = \Big((a+3)\, e^a\, E_1(a)/2 - 2\, e^{2a}\, E_1(2a) - 1/2\Big)^2 / \sigma_0^2$$

for $a > 0$. $e_{F^{(2)}}$ has a maximum value at $a^* = 1.51$ with $e_{F^{(2)}}(T_{n,a^*}) = 0.083$. This value is approximately 1/12, which is the Pitman efficiency of the asymptotically most powerful test of exponentiality against the Makeham distribution (Doksum and Yandell (1984)).

For the modified statistic $\tilde{T}_{n,a}$, one obtains the efficiencies

$$\begin{aligned} e_{F^{(1)}}(\tilde{T}_{n,a}) &= e_{F^{(3)}}(\tilde{T}_{n,a}) \\ &= ((a+2)\, e^a\, E_1(a) - (a+1)/a)^2 / \tilde{\sigma}_0^2 \end{aligned}$$

and

$$e_{F^{(2)}}(\tilde{T}_{n,a}) = \Big(e^a\, E_1(a)/2 - 2\, e^{2a}\, E_1(2a) + 1/(2a)\Big)^2 / \tilde{\sigma}_0^2$$

for $a > 0$, where $\tilde{\sigma}_0^2$ is given in (11). Just like $e_{F^{(1)}}(T_{n,a})$, $e_{F^{(1)}}(\tilde{T}_{n,a})$ is an increasing function of a with maximum value $e_{F^{(1)}}(\tilde{T}_{n,\infty}) = 1$. $e_{F^{(2)}}$ has a maximum at $a^* = 1.95$ with $e_{F^{(2)}}(\tilde{T}_{n,a^*}) = 0.082$. Figure 1 shows the local approximate Bahadur efficiencies of $T_{n,a}$ and $\tilde{T}_{n,a}$ against LFR and Makeham

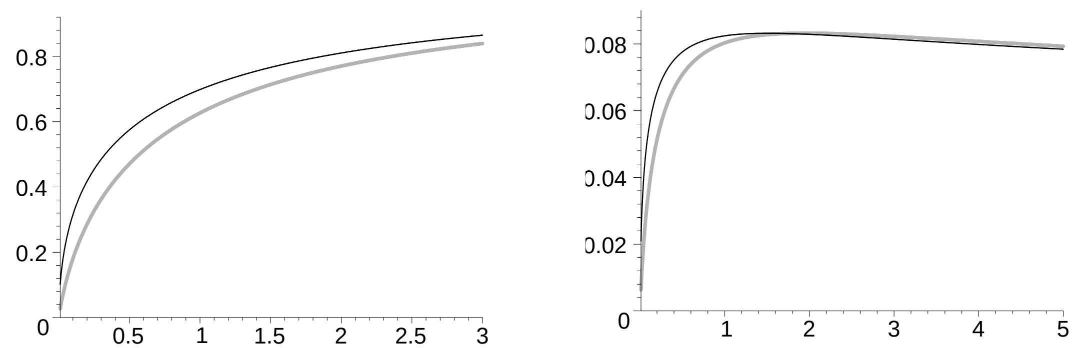

Figure 1: Local approximate Bahadur efficiency of $T_{n,a}$ (black) and $\tilde{T}_{n,a}$ (grey) against LFR (left) and Makeham alternatives (right)

alternatives.

Instead of giving the expressions for $e_{F^{(4)}}$ and $e_{F^{(5)}}$ which are quite complex, Figure 2 displays the efficiencies of $T_{n,a}$ and $\tilde{T}_{n,a}$ against Weibull and Gamma alternatives for $a \in (0,3)$. The maximum value of $e_{F^{(4)}}(T_{n,a})$ is 1.56 at $a = 0.38$; $e_{F^{(5)}}(T_{n,a})$ attains its maximum value of 0.62 at $a = 0.092$. The maximum value 1.55 of $e_{F^{(4)}}(\tilde{T}_{n,a})$ is attained for $a = 0.66$; $e_{F^{(5)}}(\tilde{T}_{n,a})$ attains its maximum value of 0.61 at $a = 0.22$.

4 Simulations

This section presents the results of two Monte Carlo studies. The first simulation study was conducted in order to obtain critical points of the statistics under discussion. Tables 1 to 4 show the p-quantiles of $T^*_{n,a}$, $\tilde{T}^*_{n,a}$ and $T^*_{n,\infty}$ under exponentiality for several sample sizes and $p = 0.05, 0.10, 0.90$ and 0.95, respectively. The asterisk indicates that the statistics have been scaled, that is, we considered $T^*_{n,a} = \sqrt{n}T_{n,a}/\sigma_0$, $\tilde{T}^*_{n,a} = \sqrt{n}\tilde{T}_{n,a}/\tilde{\sigma}_0$ and $T^*_{n,\infty} = \sqrt{n}T_{n,\infty}/2$. Thus, each of the statistics has a limit standard normal distribution under the hypothesis. The weight parameters for $T^*_{n,a}$ and $\tilde{T}^*_{n,a}$ were chosen to be $a = 0.1, 0.5, 1, 3, 5$ and 10. The entries in Tables 1 to 4 are based on 100000

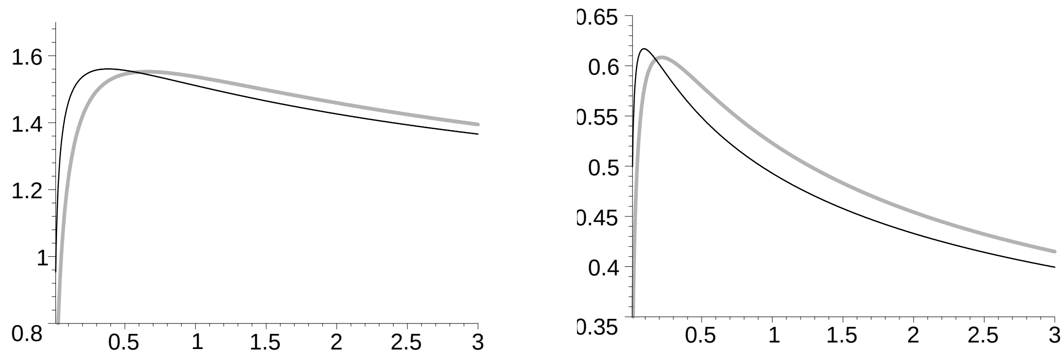

Figure 2: Local approximate Bahadur efficiency of $T_{n,a}$ (black) and $\tilde{T}_{n,a}$ (grey) against Weibull (left) and gamma alternatives (right)

n	$T^*_{n,a}$						$\tilde{T}^*_{n,a}$						$T^*_{n,\infty}$
	0.1	0.5	1.0	3.0	5.0	10.0	0.1	0.5	1.0	3.0	5.0	10.0	–
10	-1.55	-1.69	-1.67	-1.52	-1.44	-1.32	-1.33	-1.68	-1.68	-1.56	-1.46	-1.33	-1.09
20	-1.60	-1.68	-1.68	-1.59	-1.51	-1.41	-1.48	-1.68	-1.69	-1.61	-1.53	-1.44	-1.22
50	-1.63	-1.69	-1.68	-1.62	-1.58	-1.52	-1.55	-1.67	-1.68	-1.64	-1.59	-1.52	-1.35
100	-1.64	-1.67	-1.66	-1.63	-1.60	-1.55	-1.59	-1.67	-1.67	-1.64	-1.62	-1.57	-1.44
200	-1.63	-1.66	-1.66	-1.64	-1.62	-1.58	-1.61	-1.67	-1.67	-1.64	-1.63	-1.60	-1.49
500	-1.64	-1.66	-1.66	-1.64	-1.64	-1.61	-1.62	-1.65	-1.66	-1.65	-1.63	-1.61	-1.55
1000	-1.65	-1.66	-1.66	-1.65	-1.64	-1.63	-1.62	-1.64	-1.66	-1.64	-1.64	-1.62	-1.57

Table 1: Empirical 5%-quantiles of $T^*_{n,a}$, $\tilde{T}^*_{n,a}$ and $T^*_{n,\infty}$ based on 100000 replications

replications; here, we always used $\lambda = 1$.

The speed of convergence to the asymptotic values is high for small values of a; for larger values of the weight parameter and for the limiting case $T^*_{n,\infty}$, convergence is quite slow. The finite sample quantiles are not symmetric around 0.

A second simulation study has been conducted to examine the dependence of the power of the tests on the weight function. As alternative distributions

n	$T^*_{n,a}$						$\tilde{T}^*_{n,a}$						$T^*_{n,\infty}$
	0.1	0.5	1.0	3.0	5.0	10.0	0.1	0.5	1.0	3.0	5.0	10.0	–
10	-1.34	-1.42	-1.41	-1.33	-1.26	-1.16	-1.20	-1.42	-1.43	-1.35	-1.27	-1.18	-0.98
20	-1.34	-1.40	-1.39	-1.33	-1.29	-1.22	-1.27	-1.38	-1.39	-1.35	-1.30	-1.23	-1.08
50	-1.32	-1.36	-1.36	-1.33	-1.31	-1.27	-1.29	-1.34	-1.36	-1.35	-1.31	-1.27	-1.15
100	-1.30	-1.33	-1.35	-1.33	-1.31	-1.28	-1.29	-1.33	-1.34	-1.33	-1.32	-1.28	-1.20
200	-1.31	-1.32	-1.32	-1.32	-1.30	-1.29	-1.28	-1.32	-1.32	-1.31	-1.30	-1.29	-1.23
500	-1.29	-1.30	-1.32	-1.31	-1.30	-1.29	-1.29	-1.30	-1.31	-1.31	-1.31	-1.30	-1.26
1000	-1.29	-1.29	-1.30	-1.31	-1.29	-1.29	-1.29	-1.30	-1.31	-1.31	-1.30	-1.28	-1.27

Table 2: Empirical 10%-quantiles of $T^*_{n,a}$, $\tilde{T}^*_{n,a}$ and $T^*_{n,\infty}$ based on 100000 replications

n	$T^*_{n,a}$						$\tilde{T}^*_{n,a}$						$T^*_{n,\infty}$
	0.1	0.5	1.0	3.0	5.0	10.0	0.1	0.5	1.0	3.0	5.0	10.0	–
10	1.14	1.04	0.99	0.88	0.82	0.75	1.19	1.07	1.02	0.90	0.84	0.76	0.58
20	1.18	1.13	1.09	1.04	1.00	0.96	1.23	1.14	1.10	1.06	1.02	0.97	0.82
50	1.23	1.19	1.17	1.14	1.13	1.13	1.26	1.21	1.17	1.16	1.14	1.12	1.04
100	1.24	1.21	1.21	1.20	1.19	1.19	1.27	1.22	1.20	1.19	1.20	1.18	1.13
200	1.25	1.23	1.23	1.23	1.23	1.22	1.28	1.24	1.24	1.23	1.24	1.23	1.20
500	1.26	1.26	1.26	1.26	1.25	1.24	1.27	1.26	1.26	1.25	1.25	1.24	1.25
1000	1.27	1.26	1.26	1.26	1.26	1.26	1.27	1.26	1.26	1.25	1.27	1.25	1.27

Table 3: Empirical 90%-quantiles of $T^*_{n,a}$, $\tilde{T}^*_{n,a}$ and $T^*_{n,\infty}$ based on 100000 replications

from the $\mathcal{L}$-class, we used the Weibull, Gamma and Linear failure rate distribution with scale parameter 1 and shape parameter ϑ, denoted by $W(\vartheta)$, $\Gamma(\vartheta)$ and $LFR(\vartheta)$, respectively. Furthermore, the inverse Gaussian distribution $IG(1,\lambda)$ with $\lambda = 1, 1.2, 1.5$ and 2.0 was chosen; these values of λ correspond to a coefficient of variation of 1, 0.91, 0.82 and 0.71, respectively. Finally, we took the Pareto distribution $Par(\vartheta)$ with scale parameter 1 and shape parameter ϑ as an alternative from the class $\overline{\mathcal{L}}$.

We used routines of the IMSL-library to obtain Weibull and Gamma

n	$T^*_{n,a}$						$\tilde{T}^*_{n,a}$						$T^*_{n,\infty}$
	0.1	0.5	1.0	3.0	5.0	10.0	0.1	0.5	1.0	3.0	5.0	10.0	–
10	1.58	1.45	1.42	1.33	1.27	1.20	1.71	1.51	1.43	1.35	1.28	1.22	1.02
20	1.63	1.52	1.50	1.48	1.46	1.43	1.69	1.56	1.53	1.49	1.46	1.44	1.34
50	1.64	1.58	1.58	1.57	1.58	1.58	1.68	1.60	1.58	1.57	1.58	1.58	1.56
100	1.64	1.60	1.59	1.60	1.62	1.63	1.67	1.62	1.61	1.59	1.61	1.62	1.63
200	1.65	1.62	1.62	1.62	1.63	1.65	1.68	1.63	1.60	1.62	1.63	1.66	1.68
500	1.64	1.62	1.63	1.64	1.63	1.65	1.66	1.63	1.64	1.64	1.63	1.65	1.70
1000	1.64	1.62	1.62	1.64	1.65	1.65	1.67	1.63	1.63	1.64	1.65	1.65	1.68

Table 4: Empirical 95%-quantiles of $T^*_{n,a}$, $\tilde{T}^*_{n,a}$ and $T^*_{n,\infty}$ based on 100000 replications

random numbers and the inversion method to generate random numbers from the LFR and Pareto distribution. Inverse Gaussian random variates were generated by the 'transformations with multiple roots method' of Michael et al. (1976).
The first six columns of Tables 5, 6 and 7 show power estimates of the tests based on $T^*_{n,a}$ for $a = 0.1, 0.5, 1, 3, 5, 10$ for $n = 20, 50$ and $n = 100$, respectively. The next six columns give the corresponding results of the tests based on $\tilde{T}^*_{n,a}$. The last column contains the results of $T^*_{n,\infty}$. All entries are the percentages of 10000 Monte Carlo samples that resulted in rejection of H_0, rounded to the nearest integer. The nominal level of the test is $\alpha = 0.05$.

The main conclusions that can be drawn from the simulation results are the following:

1. The tests based on $T^*_{n,a}$ and $\tilde{T}^*_{n,a}$ behave fairly similar, whereby the power of the tests depends to a certain extent on a.

   Against Weibull alternatives, $T^*_{n,0.5}$ and $\tilde{T}^*_{n,1}$ perform best. Similarly, $T^*_{n,0.1}$ and $\tilde{T}^*_{n,0.5}$ are most powerful against Gamma distributions. Large values of a are best suited to safeguard against LFR and Pareto alternatives. $\tilde{T}^*_{n,0.1}$ outperforms all tests under consideration in case of an inverse Gaussian distribution.

Alternative	$T^*_{n,a}$						$\tilde{T}^*_{n,a}$						$T^*_{n,\infty}$
	0.1	0.5	1.0	3.0	5.0	10.0	0.1	0.5	1.0	3.0	5.0	10.0	–
$Exp(1)$	5	5	5	5	5	5	5	5	5	5	5	5	5
$W(1.2,1)$	22	23	22	21	21	21	20	23	23	22	21	21	20
$W(1.5,1)$	62	67	67	65	64	63	55	66	67	65	64	62	61
$W(1.8,1)$	89	93	93	93	92	92	82	92	93	93	92	92	90
$\Gamma(1.5,1)$	34	33	31	29	28	28	32	34	32	29	28	27	26
$\Gamma(2.0,1)$	69	68	65	60	59	57	65	69	67	61	59	57	54
$\Gamma(2.5,1)$	91	90	88	83	82	80	87	91	89	84	82	80	77
$LFR(0.5)$	14	17	18	18	19	19	13	16	17	18	18	18	18
$LFR(1.0)$	21	26	27	29	29	30	18	24	27	28	29	29	29
$LFR(2.0)$	32	39	42	43	43	44	26	37	41	43	43	43	43
$LFR(3.0)$	38	49	51	53	53	54	32	45	50	53	53	53	53
$IG(1,1.0)$	49	32	26	20	18	17	57	39	29	20	18	17	15
$IG(1,1.2)$	65	47	38	30	28	26	73	54	42	31	28	25	23
$IG(1,1.5)$	84	68	59	48	45	42	89	75	64	49	45	41	38
$IG(1,2.0)$	97	90	83	73	70	67	99	93	87	75	71	67	62
$Par(0.25)$	19	25	27	28	29	29	15	23	26	28	29	28	28
$Par(0.50)$	43	53	56	56	56	56	36	50	54	56	56	56	54
$Par(0.75)$	66	74	76	76	76	75	58	72	75	76	76	75	73
$Par(1.00)$	82	87	87	87	86	85	75	86	87	87	87	85	84

Table 5: Empirical power of the tests based on $T^*_{n,a}$, $\tilde{T}^*_{n,a}$ and $T^*_{n,\infty}$, $\alpha = 0.05$, $n = 20$, 10000 replications

2. The results for the 'local' alternatives $W(1.2, 1), \Gamma(1.5, 1), LFR(0.5)$ and $Par(0.25)$ are in good agreement with the local approximate Bahadur efficiencies (see Figures 1 and 2). On the whole, the behaviour is similar for 'distant' alternatives; for example, the power of the tests against LFR alternatives is always increasing in a.

3. $T^*_{n,\infty} = \sqrt{n}T_{n,\infty}/2$ rejects the $IG(1, 1)$-distribution in 15% of all cases, irrespective of the sample size. Indeed, it follows from Theorem 4.2 in Henze and Klar (1996) that $T^*_{n,\infty}$ has a limiting centered normal distribution with variance 9/4 under an $IG(1, 1)$-distribution. Consequently, the test for exponentiality based on $T^*_{n,\infty}$ will reject H_0 in 14%

Alternative	$T^*_{n,a}$						$\tilde{T}^*_{n,a}$						$T^*_{n,\infty}$
	0.1	0.5	1.0	3.0	5.0	10.0	0.1	0.5	1.0	3.0	5.0	10.0	–
$Exp(1)$	5	5	5	5	5	5	5	5	5	5	5	5	6
$W(1.2, 1)$	42	44	44	42	41	39	37	44	44	42	41	39	38
$W(1.5, 1)$	95	97	97	97	96	95	90	97	97	97	96	95	94
$W(1.8, 1)$	100	100	100	100	100	100	100	100	100	100	100	100	100
$\Gamma(1.5, 1)$	68	65	62	56	54	51	65	67	64	57	54	52	48
$\Gamma(2.0, 1)$	98	97	97	94	92	91	97	98	97	94	93	91	87
$\Gamma(2.5, 1)$	100	100	100	100	99	99	100	100	100	100	99	99	98
$LFR(0.5)$	25	32	35	38	39	38	20	30	34	38	39	39	39
$LFR(1.0)$	40	52	57	61	62	62	31	49	55	61	62	62	62
$LFR(2.0)$	58	74	79	82	83	83	44	70	77	82	83	83	83
$LFR(3.0)$	69	84	88	91	91	91	54	81	86	90	91	91	91
$IG(1, 1.0)$	87	54	40	25	21	18	97	68	47	26	22	18	15
$IG(1, 1.2)$	97	79	65	46	40	35	100	89	73	49	41	36	30
$IG(1, 1.5)$	100	96	89	75	69	63	100	99	93	78	70	64	56
$IG(1, 2.0)$	100	100	99	96	94	90	100	100	100	97	94	91	86
$Par(0.25)$	32	43	46	49	50	50	24	39	44	49	50	50	48
$Par(0.50)$	73	83	85	86	86	85	61	80	84	86	86	85	83
$Par(0.75)$	93	97	97	97	97	97	87	96	97	97	97	97	96
$Par(1.00)$	99	100	100	100	100	100	97	99	100	100	100	100	99

Table 6: Empirical power of the tests based on $T^*_{n,a}$, $\tilde{T}^*_{n,a}$ and $T^*_{n,\infty}$, $\alpha = 0.05$, $n = 50$, 10000 replications

of all cases for large n, if the nominal level is 0.05 and the underlying distribution is $IG(1, 1)$.

4. If nothing is known about the $\mathcal{L}$-class ($\overline{\mathcal{L}}$-class) alternative, the tests based on $T^*_{n,1}$ or $\tilde{T}^*_{n,1}$ can be recommended since they distribute their power more evenly over the range of alternatives.

A basic drawback of omnibus goodness-of-fit tests is that their power is fairly poor except for a rather small set of alternatives (see, e.g., Janssen (1995)). Therefore, if one has some knowledge about the class of distribu-

Alternative	$T^*_{n,a}$						$\tilde{T}^*_{n,a}$						$T^*_{n,\infty}$
	0.1	0.5	1.0	3.0	5.0	10.0	0.1	0.5	1.0	3.0	5.0	10.0	–
$Exp(1)$	5	5	5	5	5	5	5	5	5	5	5	5	5
$W(1.2,1)$	68	72	71	68	67	65	61	71	72	69	67	65	62
$W(1.5,1)$	100	100	100	100	100	100	100	100	100	100	100	100	100
$W(1.8,1)$	100	100	100	100	100	100	100	100	100	100	100	100	100
$\Gamma(1.5,1)$	93	91	89	83	80	77	91	92	90	84	81	77	72
$\Gamma(2.0,1)$	100	100	100	100	100	100	100	100	100	100	100	100	99
$\Gamma(2.5,1)$	100	100	100	100	100	100	100	100	100	100	100	100	100
$LFR(0.5)$	39	54	59	64	65	65	29	49	56	63	64	65	65
$LFR(1.0)$	63	80	85	89	89	90	47	75	82	88	89	89	90
$LFR(2.0)$	85	96	97	98	99	99	69	93	97	98	98	99	99
$LFR(3.0)$	92	99	99	100	100	100	79	98	99	100	100	100	100
$IG(1,1.0)$	99	78	57	32	26	21	100	90	68	35	27	21	15
$IG(1,1.2)$	100	97	88	64	56	48	100	99	93	68	57	48	38
$IG(1,1.5)$	100	100	99	93	88	82	100	100	100	95	89	83	72
$IG(1,2.0)$	100	100	100	100	100	99	100	100	100	100	100	99	96
$Par(0.25)$	48	64	68	72	73	72	35	58	65	72	73	73	71
$Par(0.50)$	93	97	98	98	98	98	84	96	98	98	98	98	97
$Par(0.75)$	100	100	100	100	100	100	99	100	100	100	100	100	100
$Par(1.00)$	100	100	100	100	100	100	100	100	100	100	100	100	100

Table 7: Empirical power of the tests based on $T^*_{n,a}$, $\tilde{T}^*_{n,a}$ and $T^*_{n,\infty}$, $\alpha = 0.05$, $n = 100$, 10000 replications

tions which may occur, it is reasonable to use tests that are well adapted to detect the possible alternatives. On the other hand, the example of the $IG(\mu,\mu)$-distribution which may be a reasonable life distribution shows that it may be even more dangerous to overly restrict the set of possible alternatives.

In this respect, the proposed tests for exponentiality against the $\mathcal{L}$-class, which is the largest of the commonly used classes of life distributions, seem to be a good compromise.

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