How Well Does a Power Law Fit to a Diabatic Boundary-Layer Wind Profile?

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Abstract

Two types of wind profile laws are frequently used for the atmospheric surface-layer: the theoretically derived logarithmic profile with corrections for non-neutral thermal stratification and the empirically derived power law. Due to its mathematical simplicity the power law is widely used. This study investigates in which situations the power law is a good approximation to the logarithmic profile. In extension to existing studies not only the slope of the logarithmic and the power law profiles but also the curvatures should coincide. The roughness and stratification conditions for which such a coincidence is possible are calculated analytically. For neutral and unstable conditions slope and curvature of a power law profile cannot coincide with that of the logarithmic profile. This can only happen under certain circumstances in a stably stratified flow. The practical result of this sudy is that the power law offers a good fit to the logarithmic profile for slightly stable conditions and for very smooth surfaces only. Thus the power law profile provides a good description of the wind profile over the sea but not over rough terrain.

Introduction

The vertical wind profile in the atmospheric boundary-layer (ABL) depends on several parameters such as surface roughness, changes in surface roughness, orography, thermal stratification of the air, and the undisturbed wind speed above the ABL. Two types of description of the wind profile are frequently used for the lower part of the ABL, the surface layer or Prandtl-layer: the logarithmic wind profile together with correction functions for atmospheric stability (Businger et al. [1], Dyer [2], henceforward called logarithmic profile for simplicity) and the power law (Davenport [3]). The logarithmic profile is derived from a physically based similarity theory and usually depends on the three parameters: friction velocity u_*, the ratio of height above ground to roughness length z/z_0, and the ratio of height to the Monin-Obukhov-length z/L_*. The power law is empirically derived and depends on two parameters: the wind speed in the anemometer height $u(z_A)$ and an exponent n. This exponent varies with surface roughness and atmospheric stability.

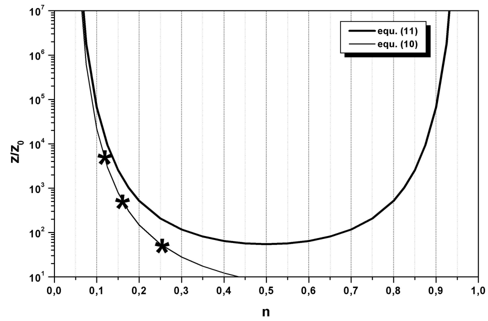

Fig. 1: Exponent n of the power law from equ. (10) (thin line) and from equ. (11) (bold line). The three asterisks mark the position of the examples shown in Fig. 2.

The choice of the suitable way of describing the wind profile is often made by practical arguments. Although today computer resources set nearly no limits any more , the power law is often chosen due to its mathematical simplicity. It is often claimed that both descriptions lead more or less to the same results. We will investigate here theoretically how close the logarithmic profile can be described by a power law. This is not a new issue. Already in Sedefian [4] a relation was derived theoretically how the power law exponent n depends on z/z_0 and z/L_*. This was done by equating the slopes of a logarithmic profile and a power law. As long as the height range over which the two profiles should match is small the solution given by Sedefian is practical and sufficient. One will always find a power law with an exponent n that fits to a given logarithmic profile in a given height.

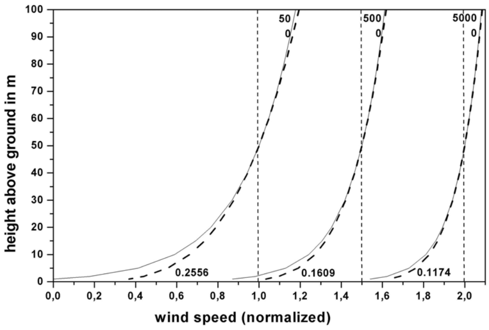

Fig. 2: Three logarithmic wind profiles for neutral stratification ($z/L_*=0$) and their approximation by power law profiles. The middle pair of profiles has been shifted by 0.5, the rightmost pair by 1.0 to the right. The two numbers at the top of the profiles give z/z_0 and z/L_*, the number at the bottom the exponent n of the respective power law.

But today's tasks in wind engineering (the construction of large wind turbines and the design of high buildings) often require the extrapolation of the wind profile over considerable height intervals. Fore these purposes the two types of description are only equivalent if it is possible to find a power law that fits to the logarithmic profile in slope and curvature over the respective range. The following investigation will demonstrate that this is possible only for certain combinations of surface roughness and atmospheric stability in stably stratified boundary-layer flow.

Basic Profile Equations

The power law is usually formulated:

$$u(z) = u(z_A)\,(z/z_A)^n \quad (1)$$

with the anemometer height z_A and the exponent n. The logarithmic law reads for neutral stability:

$$u(z) = (u_*/\kappa)\,\ln(z/z_0) \quad (2)$$

with the friction velocity u_*, the von Kármán constant $\kappa = 0.4$. For non-neutral stratification (2) is modified to:

$$u(z) = (u_*/\kappa)\,(\ln(z/z_0) - \psi(z/L_*)) \quad (3)$$

with

$$\psi(z/L_*) = \begin{cases} \ln((1+x^2)/2((1+x)/2)^2) - 2\,\mathrm{arctg}(x) + \pi/2 & \text{for } z/L_* < 0 \\ -4.7\, z/L_* & \text{for } z/L_* \geq 0 \end{cases} \quad (4)$$

and $x = (1 - 15\, z/L_*)^{1/4}$. The values for the two constants 4.7 and 15 (we use here the values given by Businger [5]) vary slightly in the literature. The following comparison of the power law with the logarithmic profiles does not depend on the exact values of these constants. The Monin-Obukhov-length L_* is defined by:

$$L_* = - \rho c_p \Theta u_*^3 / (\kappa g H_0) \quad (5)$$

with air density ρ, the specific heat of air c_p, the potential temperature Θ, the Earth's gravity acceleration g, and the surface turbulent heat flux H_0.

In the next section we will consider the most simple case of neutrally stratified flow by comparing (1) and (2). After this, in the subsequent section, we will analyse the general case of non-neutrally stratified flow by comparing (1) and (3).

Comparison of the Two Profile Laws for Neutral Stratification

Two wind profiles are identical if they have equal slope and curvature in all heights. They are nearly identical in a small height interval if they have equal slope in the center of the height interval. The height interval where we find an approximate sameness of the two profiles would be larger if not only the slope but also the curvature is identical in the center of this interval. While we can always find parameter sets that make the slopes of (1) and (2) identical at a given height it is not guaranteed that the curvature can be made equal, too. For the investigation of the possibility whether this can happen we need the mathematical formulation of the slope and the curvature of (1) and (2).

The slope of the logarithmic wind profile under neutral stratification (2) is given by:

$$\partial u/\partial z = (1/\kappa)\,(u_*/z) = \ln^{-1}(z/z_0)\, u(z) / z \quad (6)$$

and the curvature of the logarithmic profile follows by taking the second derivative of (2):

$$\partial^2 u/\partial^2 z = - (1/\kappa)\,(u_*/z^2) = - \ln^{-1}(z/z_0)\, u(z) / z^2 \quad (7)$$

The slope of the power law (1) by differentiating yields:

$$\partial u/\partial z = u(z_A) / z_A\, n\, (z/z_A)^n / (z / z_A) = n\, u(z_A)\, (z/z_A)^n / z = n\, u(z) / z \quad (8)$$

and the curvature of the power laws reads:

$$\partial^2 u/\partial^2 z = n\,(n-1)\, u(z_A)\, (z/z_A)^n / z^2 = n\,(n-1)\, u(z) / z^2 \quad (9)$$

Equating the slopes (6) and (8) delivers:

$$n = \ln^{-1}(z/z_0) \quad (10)$$

which equals the formulation given by Sedefian [4] in the limit of neutral stratification. (10) means that the exponent n decreases with height for a given roughness length z_0. The height in which the slopes of the two wind profiles (1) and (2) should be equal – this is usually the anemometer height $z = z_A$ – has therefore to be specified a priori. The dependence of n on height is the stronger the smaller the ratio z/z_0 is (see Fig. 1). Due to this fact the dependence of n with height is stronger for complex terrain where the roughness length z_0 is large and it can nearly be neglected for water surfaces with very small roughness lengths. In order to see whether we can find an exponent n so that both the slope and the curvature agree in a given height we must equate the formulas for the curvature of the two profiles (7) and (9). This yields the relation:

$$n\,(n-1) = - \ln^{-1}(z/z_0) \quad (11)$$

For z/z_0 < 54.6 equation (11) has no solution at all (Figure 1). For z/z_0 = 0.25 it has one solution (n = 0.5) and for z/z_0 < 0.25 it has two solutions of which we always take the smaller one. This solution is approaching the solution of equation (10) asymptotically for z/z_0 against infinity. Therefore a power law with equal slope and curvature as the logarithmic profile can only exist in the limit n against zero. Thus, for neutral stratification, a power law with equal slope and curvature that fits the logarithmic profile over a larger height range cannot be constructed. The use of (10) for calculating an exponent n of a power wind profile that is an approximation to the logarithmic wind profile is the better the larger z/z_0 is, i.e. the smoother the surface is. For complex terrain on the other hand, the power law with an exponent n given by (10) is not a good approximation to the true wind profile. This is demonstrated in Fig. 2 where we present wind profiles computed from (1) and (2) for three different height-to-roughness ratios z/z_0. The height where the profiles should be identical is chosen to be 50 m and the wind profiles have been normalized to the wind speed in this height. The wind speed difference between the logarithmic profile and the power law profile at 100 m height is 1.3% for z/z_0 = 50 (power law exponent n = 0.2556 from (10)) and 0.3% for z/z_0 =5000 (n = 0.1174). The relative difference between the two profiles at 10 m height is 11.2% and 2.0% respectively.

Comparison of the Two Formulations for Non-Neutral Stratification

Usually – except for very strong winds – the atmosphere is not stratified neutrally. For non-neutral stratification the slope of the logarithmic profile (3) is determined by:

$$\partial u/\partial z = \begin{cases} (\ln(z/z_0) - \psi(z/L_*))^{-1}\, u(z)\, (1/x) / z & \text{for } z/L_* < 0 \\ (\ln(z/z_0) + 4.7\, z/L_*)^{-1}\, u(z)\, (1 + 4{,}7\, z/L_*) / z & \text{for } z/L_* > 0 \end{cases} \tag{12}$$

and its curvature by:

$$\partial^2 u/\partial^2 z = \begin{cases} -\, (\ln(z/z_0) - \psi(z/L_*))^{-1}\, u(z)\, (1/x)\, (1/z^2)\, (1 + (z/\, x)\, \partial x/\partial z) & \text{for } z/L_* < 0 \\ -\, (\ln(z/z_0) + 4.7\, z/L_*)^{-1}\, u(z)\, 1/z^2 & \text{for } z/L_* > 0 \end{cases} \tag{13}$$

The expression $(z/\, x)\, \partial x/\partial z$ equals $-3.75\, z/L_*\, (1/x^4)$. Slope and curvature of the power law (1) do not depend explicitly on stratification and thus remain unchanged. Looking for equal slopes in non-neutrally stratified flow now requires the investigation of the possible identity of (8) and (12). We get:

$$n = \begin{cases} (\ln(z/z_0) - \psi(z/L_*))^{-1}\, (1/x) & \text{for } z/L_* < 0 \\ (\ln(z/z_0) + 4.7\, z/L_*)^{-1}\, (1 + 4.7\, z/L_*) & \text{for } z/L_* > 0 \end{cases} \tag{14}$$

which are exactly the equations found by Sedefian [4]. From (14) it is obvious that n is smaller with unstable stratification than with a neutral one, but turns out larger with stable stratification, because x and the expression in brackets containing z/L_* are both larger than unity.

We had seen from Fig. 2 that the neutral logarithmic profile is always steeper (in the manner we have plotted the Figure, steeper means that wind speed is increasing less with height) than a power law profile fitted to it in the height $z = z_A$. As the logarithmic profile for unstable stratification is even steeper than the one for neutral stratification we do not expect a match with the power law profile for unstable stratification. But for stable stratification, the slope of the logarithmic profile is smaller than for neutral conditions and a fit may become possible. We therefore equate the curvatures from equations (9) and (13) yielding:

$$n\,(n-1) = \begin{cases} -\, (\ln(z/z_0) - \psi(z/L_*))^{-1}\, (1/x)\, (1 - 3.75\, z/L_*\, (1/x^4)) & \text{for } z/L_* < 0 \\ -\, (\ln(z/z_0) + 4.7\, z/L_*)^{-1} & \text{for } z/L_* > 0 \end{cases} \tag{15}$$

Now, for stable stratification – in contrast to the neutral stratification above and to unstable conditions – we have the possibility to define conditions in which the lower equation of (14) and equation (15) can be valid simulataneously. For such a power law profile which has equal slope and curvature in the height $z = z_A$ the following condition must hold:

$$n = 1 - (1 + 4.7\, z/L_*)^{-1} \tag{16}$$

In contrast to the neutral case it is possible to find an exponent n, but this exponent depends on the static stability z/L_* of the flow. Because both parameters that determine the value of n in (14) or in (15), the roughness parameter z/z_0 and the stability parameter z/L_*, contain the same height z these two parameters cannot be chosen independently. The possible values for n in the phase space spanned by z/z_0 and z/L_* can be found by either equating the lower equation from (14) and (16) or by equating the lower equation from (15) and (16):

$$\ln(z/z_0) = 2 + 1 / (4.7\, z/L_*) \tag{17}$$

Fig. 3 illustrates the solutions from equations (14), (15), and (17). An evaluation of (17) demonstrates that the stabiliy of the atmosphere must increase with increasing roughness and decreasing anemeometer height in order to find a power law profile with the same slope and curvature as the logarithmic profile. The curved thin lines from the lower left to the upper right represent the solution of equation (14), the lines with the maximum just left of $z/L_* = 0$ the solution of equation (15) (please note that the lowest line is the one for n = 0.5, and that the lines for n = 0.3 and n = 0.7 are identical), and the thick line marks the solution of (17). As designed the thick curve goes through the points where solutions from (14) and (15) are identical.

Fig. 4 displays three examples of wind profiles for non-neutral stratification, one for unstable conditions and a large roughness length, one which lies exactly on the curve from equation (17) so that slope and curvature coincide simultaneously, and one for very stable conditions. For a roughness length of z_0 = 0.023 m (z/z_0 = 2173) and a Monin-Obukhov-length of L_* = 1500 m (z/L_* = 0.0333) a power law profile with n = 0,15 has equal slope and curvature at $z = z_A$ = 50 m as the logarithmic profile. At z = 100 m the two profiles only differ by 0.1%, at 10 m by 0.9%. This is an even better fit than the fit for the neutral wind profile with z/z_0 = 5000 in Fig. 2. For the two profiles under unstable conditions the respective deviations at 100 m and at 10 m are 4.5% and 89.9%, for the two profiles under very stable conditions these deviations are -3.5% and -14.0%.

Conclusions

We have extended the analysis by Sedefian [4] and shown that only for certain conditions in stably stratified boundary-layer flow it is possible to find a power law profile that has the same slope and curvature as a logarithmic wind profile and thus fits the logarithmic profile almost perfectly over a wide height range. The respective combinations of roughness and Monin-Obukhov-length for which this good fit is possible have been derived analytically. In a purely neutrally stratified boundary-layer this perfect fit is not possible although the fit becomes the better the smoother the surface is. The worst fit occurs for unstable conditions and high roughness lengths.

For high wind speeds which are most favourable for wind energy conversion the stratification of the boundary-layer usually becomes nearly neutral. The above considerations then show that only for very smooth terrain (offshore and near the coasts) the power law is a good approximation to the real surface-layer wind profile. Extrapolations of the wind profile above the height of the surface layer (80 to 100 m) by either laws (1) or (3) should be made with very great care because these laws are valid for the surface layer only (Emeis [6]).

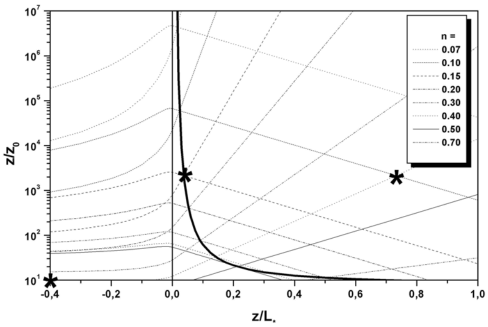

Fig. 3: Solution of the equations (14), (15), and (17) in the phase space spanned by the roughness parameter z/z_0 and the stratification parameter z/L_*. Thin lines from lower left to upper right (calculated from (14)) indicate for different exponents n when a logarithmic profile and a power law profile have equal slopes, thin lines from left to lower right (calculated from (15)) indicate for different exponents n when a logarithmic profile and a power law profile have equal curvatures, the thick line (calculated from (17)) runs through the points where the solutions from (14) and (15) are equal. The three asterisks mark the position of the examples shown in Fig. 4.

Due to the fact that the atmosphere is usually stably stratified in the mean it becomes obvious from the above calculations why the power law approach has been so successful in many cases.

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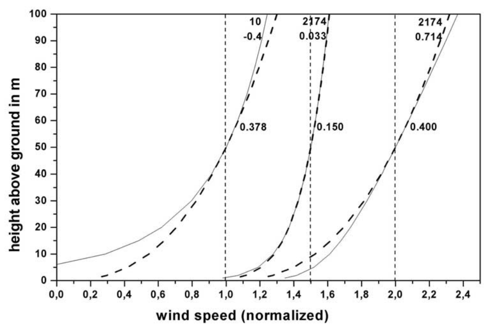

Fig. 4: Three logarithmic wind profiles for non-neutral stratification ($z/L_* \neq 0$) and their approximation by power law profiles. The middle pair of profiles has been shifted by 0.5, the rightmost pair by 1.0 to the right. The two numbers at the top of the profiles give z/z_0 and z/L_*, the number in the middle of the profiles the exponent n of the respective power law.