Hedging general claims in jump diffusion models

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Abstract

A stochastic control problem of Schweizer is considered for simultaneous control of claims and investment risk. Examples with diffusion and with jump-processes are given. Furthermore, for the continuous case a partial differential equation is derived for optimal hedging of constant liabilities.

1. Introduction and Summary

We consider the problem of hedging a general claim - a liability - in an incomplete market with sources of randomness which are diffusions or pure jump processes. This problem has been studied extensively by Schweizer (see [4] and [5]), who gave a complete solution to this problem in the discrete time case, and in the continuous time case under the condition that the mean-variance tradeoff is deterministic. In Hipp [3] the form of the optimal hedging strategy is given for the case that the Girsanov martingale density (the minimal martingale density) admits a short Itô representation. Examples, simulations, and a pde for the computation of the optimal hedging strategy for this case can be found in Hipp [2]. In this paper we shall give several examples in which either the liability or the tradeable asset contains a jump process. Furthermore, a pde will be given for the general continuous time case (i.e. without a short Itô representation) for the optimal hedging strategy when the liability is a constant. For the example of a linear drift and a power diffusion term, we present the formula and plot for the residual risk and give simulations for the comparison of locally optimal and globally optimal hedging strategies. For further results in the case in which a short Itô representation is not possible see Delbaen and Schachermayer [1].

We consider a very simple model in which two stochastic processes are given:

$$\begin{aligned} dX(t) &= \alpha dt + \beta dW(t) + \gamma dN(t),\ X(0) = x_0 \\ dL(t) &= adt + bdW(t) + cdN(t) + ddV(t) + edM(t),\ L(0) = l_0, \end{aligned}$$

where $X(t)$ is square integrable and models the asset price process to hedge with; $X(t)$ might equally well be multivariate. We also assume $L(t)$ to be square integrable, $L(T)$ is the liability to be hedged. The processes V, W are independent standard Wiener, the processes N, M are independent of V, W and orthogonal. The functions α, β, γ and $a, ..., e$ are assumed to be predictable, smooth, and depending on t, $X(t)$, $L(t)$, and on claim sizes observed until time t. Our process $(X(t), L(t))$ is Markovian and describes an incomplete market in two respects: it is incomplete since it involves compound Poisson processes, and is incomplete since the quantity $L(t)$ is not traded. Our aim is to find a predictable process $\theta(t)$ such that

$$G_t(\theta) = \int_0^t \theta(s) dX(s)$$

is defined and square integrable for all t, and such that

$$E\left(L(T) - G_T(\theta)\right)^2$$

is minimized. For applications in insurance, $L(t)$ would be (estimated) premium income up to time t minus (estimated) cost for claims notified up to time t. The process $X(t)$ would be the process of market prices of financial assets and/or insurance futures or options. In order not to overload the paper, we shall not discuss the problems concerning choice of T, choice of objective function, statistical problems and implementation in the real world. For real world implementation, the discrete time approach seems suitable since in this case we have optimal hedging strategies in the general case; however, these optimal solutions are quite complicated, they cannot be communicated. Hence maybe a continuous time solution which can be communicated might be useful. In fact, the examples with jump processes considered in this paper, are quite intuitive. On the other hand, we believe that the continuous case ($\gamma = 0$ and/or $c = e = 0$) does not solve the problem for insurance liabilities since i) diffusion approximations for the claims process are bad when large claims are possible, ii) the fluctuation in premium income can be modelled much better by jump processes, and iii) the residual risk will usually be large if a jump process is hedged by a continuous paths process.

In the following, we need the concepts of a minimal martingale density, the variance optimal martingale density, the short Itô representation, and the mean-variance tradeoff process. Let $\mathbb{F}(t)$, $0 \le t \le T$, be the natural filtration of our process $(X(t), L(t))$.

$U \in L_2$ is a *martingale density* (for $X(t)$, $0 \le t \le T$,) if $EU = 1$ and

$$E\left(U\left(X(t) - X(s)\right) | \mathbb{F}(s)\right) = 0, \;\; 0 < s < t < T.$$

U admits a *short Itô representation* if

$$U = u_0 + \int_0^T u(t) dX(t).$$

Let $\theta_1(t)$ be the solution of our hedging problem for the constant liability $L(T)=1$. The *variance optimal martingale density* is

$$Z_1(T)=(1-G_T(\theta_1))/E\left((1-G_T(\theta_1))\right).$$

Notice that $Z_1(T)$ admits a short Itô representation and is a martingale density, it is in fact the unique martingale density admitting a short Itô representation.

Let $Y_1, Y_2, \ldots$ be the jump sizes of $N(t)$ which are assumed to be iid and independent of W, V, and the underlying counting processes. Write μ_k for the $k-$th moment of Y_1. The drift of X equals $\alpha+\lambda\gamma\mu_1$, the local variance of X is $\beta^2+\lambda\gamma^2\mu_2$. The mean-variance tradeoff process is the predictable process

$$\kappa=\frac{(\alpha+\lambda\gamma\mu_1)^2}{\beta^2+\lambda\gamma^2\mu_2}.$$

Let A be the predictable process defined by

$$A=\frac{\alpha+\lambda\gamma\mu_1}{\beta^2+\lambda\gamma^2\mu_2}.$$

The minimal martingale density $Z(T)$ is the final value of the process $Z(t)$ given by

$$dZ(t)=-AZ(t-)(\beta dW(t)+\gamma d\widetilde{N}(t))$$

with $\widetilde{N}(t)=N(t)-\lambda\mu_1 t$. The martingale density is of the Girsanov type, but in the case with jumps it looks a bit different: If $0=T_0<T_1\ldots$ are the jump times and if $N_0(t)$ is the counting process corresponding to $N(t)$, then with

$$B(t,u)=\exp\left(-\int_t^u A\beta dW-\frac{1}{2}\int_t^u A^2\beta^2\right)+\int_t^u A\gamma\lambda\mu_1 ds$$

we have

$$Z(T)=B(T_{N_0(T)},T)\prod_{k=1}^{N_0(T)}\left(B(T_{k-1},T_k)-A(T_k)\gamma(T_k)Y_k\right).$$

Under the following condition

$$\exp\left(-\int_0^T \kappa(t)dt\right) \text{ is independent of } W(t),\ 0\le t\le T \qquad (1.1)$$

we have a simple expression for $Z_1(T)$:

$$Z_1(T)=Z(T)\exp\left(-\int_0^T \kappa(t)dt\right).$$

This implies that under (1.1) the optimal strategy $\theta_1(t)$ has the form

$$\theta_1(t) = A(t)\,(1 - G_t(\theta_1))$$

and the residual risk equals

$$E\,(1 - G_t(\theta_1))^2 = E\exp\left(-2\int_0^T \kappa(t)dt\right).$$

This does not solve our problem, since we are still left with the question of existence and admissibility of $\theta_1(t)$. But at least we can try to compute $\theta_1(t)$ with the above formula and try to verify its admissibility.

For general liabilities $L(T)$ we shall need the following *Föllmer-Schweizer decomposition* for $L(T)$:

$$L(T) = L_0 + \int_0^T g(t)dX(t) + R(T),$$

where $g(t)$ is predictable, and $R(t)$ is a martingale which is orthogonal to X. We shall also use the name *intrinsic value process* (introduced by Schweizer) for the process

$$\widehat{L}(t) = L_0 + \int_0^t g(s)dX(s) + R(t).$$

We can now state our first theorem:

Theorem 1.1. *Under assumption (1.1) the optimal hedging strategy for $L(T)$ has the following form:*

$$\theta_L(t) = g(t) + A(t)\left(\widehat{L}(t) - G_{t-}(\theta_L)\right).$$

2. The continuous case

We shall now have a somewhat closer look at the case $\gamma = c = e = 0$. In this situation, $Z_1(t) > 0$, $0 \leq t \leq T$ (see Delbaen and Schachermayer [1]). If $Z(T)$ admits a short Itô representation

$$Z(T) = z_0 + \int_0^T z(t)dX(t), \tag{2.1}$$

then $Z_1(T) = Z(T)$ and $\theta_1(t) = -z(t)$. Let $\mathbb{P}^*$ be the equivalent martingale measure with density $Z_1(T)$. Then there exists a standard Wiener process V_1 which - under $\mathbb{P}^*$ - is orthogonal to X. If (2.1) holds, then $V = V_1$. We obtain our next theorem:

Theorem 2.1. *If $\beta > 0$, if $L(T)$ admits a Föllmer-Schweizer decomposition*

$$L(T) = L_0 + \int_0^T g(t)dX(t) + \int_0^T h(t)dV_1(t)$$

and if the adjusted intrinsic value process is

$$\widehat{L}(t) = L_0 + \int_0^t g(s)dX(s) + \int_0^t h(s)dV_1(s),$$

then the optimal hedging strategy $\theta_L(t)$ for $L(T)$ has the form

$$\theta_L(t) = g(t) + \frac{\theta_1(t)}{Z_1(t)} \left(\widehat{L}(t) - G_t(\theta_L)\right).$$

The residual risk equals

$$\begin{aligned} E\left(L(T) - G_T(\theta_L)\right)^2 &= L_0^2/EZ_1^2(T) \\ &\quad + E\int_0^T h^2(t)\frac{Z_1^2(T)}{Z_1^2(t)}dt. \end{aligned}$$

We refer to the calculations of Hipp [2] which are easily adapted to our more general situation. Existence and admissibility of θ_L are investigated in the recent paper by Rheinländer and Schweizer [6]. In cases for which the optimal hedging strategy for the constant 1 cannot be given, a possible alternative would be the locally optimal hedging strategy which is defined by

$$\theta_L^l(t) = g(t) + \frac{a(t)}{b^2(t)} \left(\widehat{L}(t) - G_t(\theta_L)\right).$$

The hedging strategy $\theta_L^l(t)$ minimizes the residual risk

$$\overline{E}\left(L(T) - G_T(\theta)\right)^2$$

under the probability measure $\overline{P}$ with a density proportional to

$$\exp\left(\int_0^T \frac{a^2(t)}{b^2(t)}dt\right).$$

Often, the performance of these alternative hedging strategies is quite poor.

If (2.1) holds, then $\theta_1(t)$ can be computed via a partial differential equation which is derived in Hipp [2]: let $h = h(x,t)$ be the solution of

$$h_t + \frac{\alpha^2}{\beta^2}h + \frac{1}{2}\beta^2 h_{xx} - \alpha h_x = 0, \ h(x,T) = 1. \qquad (2.2)$$

Then

$$\frac{\theta_1(t)}{Z_1(t)} = A(t) - \frac{h_x}{h}.$$

Example 2.2. *Let $X(t)$ be defined by*

$$dX(t) = aX(t)dt + X(t)^r dW(t), \ X(0) = x_0.$$

The Ansatz

$$h(x,t) = \exp\left(C(t)x^{2(1-r)} + D(t)\right)$$

yields

$$\begin{aligned} C(t) &= -\frac{a}{2(1-r)}\left(\tan(a(1-r)(T-t) - \frac{\pi}{4}) + 1\right) \\ D(t) &= (1-r)(1-2r)\int_t^T C(s)ds. \end{aligned}$$

If $r = 1$, then κ is constant. If $r \neq 1$, then we have a singularity and must restrict the range for T :

$$\begin{aligned} T &< \frac{3\pi}{4a(1-r)} \text{ if } a(1-r) > 0, \\ T &< -\frac{\pi}{4a(1-r)} \text{ if } a(1-r) < 0. \end{aligned}$$

For fixed liability $L(T) = 1$ we consider two hedging strategies:

$$\begin{aligned} &\theta_1(t) \\ \theta^l(t) &= A(t)(1 - G_t(\theta^l)) \end{aligned}$$

the globally optimal strategy $\theta_1(t)$ and the locally optimal hedging strategy $\theta^l(t)$. Under condition (2.1), which is true here, the residual risk for $\theta^l(t)$ equals

$$E\left(1 - G_T(\theta^l)\right)^2 = E\exp\left(-\int_0^T \kappa(t)dt\right).$$

The following figures deal with the case $a = 1, r = 1/2$. In Figure 2.1 we show the residual risks for both strategies: the functions

$$\exp(-1 - \tan(T/2 - \pi/4))$$

and

$$\exp\left(1 + \sqrt{3}\tanh\left(\frac{\sqrt{3}}{2}T - \frac{1}{2}\log(2-\sqrt{3})\right)\right)$$

the last one not going to zero if $T \to \infty$. Figure 2.2 shows simulations for the gains process for the globally optimal, Figure 2.3 for the locally optimal strategy.The hedging strategies $\theta_1(t)$ in these simulations are given in Figure 2.4. Surprisingly, there is little for the investment manager to do until day 130.

3. A pde for the continuous case

We shall now derive a pde for the value function of our stochastic control problem in the continuous case $\gamma = c = e = 0$. For $s, x, l \in \mathbb{R}$ let $U(s, x, l, t)$ be the residual risk of the optimal hedging strategy θ for the constant liability s in the interval (t, T), i.e. θ minimizes

$$E\left[\left(s - \int_t^T \theta(r)dX(r)\right)^2 \middle| X(t) = x,\ L(t) = l\right]$$

and the minimal value equals $U(s, x, l, t)$. Notice that

$$U(s, x, l, t) = s^2 U(1, x, l, t) =: s^2 A(x, l, t).$$

If A is smooth, the relation

$$U(s, x, l, t) = E\left[U(s - \int_t^u \theta(r)dX(r), X(u), L(u), u)\right]$$

together with Itô's lemma yields for $u = t + dt$

$$\begin{aligned} U(s, x, l, t) &= EU(s - \theta dX, x + dX, l + dL, t + dt) \\ &= U(s, x, l, t) + Rdt \end{aligned}$$

where

$$\begin{aligned} R &= U_t - \theta\alpha U_s + \frac{1}{2}\theta^2\beta^2 U_{ss} + \alpha U_x + \frac{1}{2}\beta^2 U_{xx} \\ &\quad + aU_l + \frac{1}{2}(b^2 + d^2)U_{ll} - \theta\beta^2 U_{sx} - \theta\beta b U_{sl} + \beta b U_{xl}. \end{aligned}$$

From $R = 0$ we obtain the following equation for $A(x, l, t)$:

$$\begin{aligned} 0 &= s^2 A_t - 2s\theta\alpha A + \theta^2\beta^2 A + \alpha s^2 A_x + \frac{1}{2}\beta^2 s^2 A_{xx} \\ &\quad + as^2 A_l + \frac{1}{2}(b^2 + d^2)s^2 A_{ll} \\ &\quad - 2\theta\beta^2 s A_x - 2\theta\beta b s A_l + \beta b s^2 A_{xl}. \end{aligned}$$

Here, $\theta = \theta(t)$ is the optimal choice for θ in the interval $(t, t + dt)$, which must be chosen to minimize the residual risk, which leads to

$$-2\alpha s A + 2\theta\beta^2 A - 2\beta^2 s A_x - 2\beta b s A_l = 0$$

or

$$\theta = \theta(t) = s\left[\frac{\alpha}{\beta^2} + \frac{A_x}{A} + \frac{\beta}{b}\frac{A_l}{A}\right].$$

Inserting this and omitting s^2 which is a common factor to all terms we obtain

$$\begin{aligned} 0 &= A_t - \alpha A_x - \frac{\alpha^2}{\beta^2}A + \left(a - 2\frac{\alpha b}{\beta}\right)A_l \\ &\quad -\frac{1}{A}(\beta A_x + bA_l)^2 + \frac{1}{2}\beta^2 A_{xx} + \frac{1}{2}(b^2+d^2)A_{ll} + \beta b A_{xl} \end{aligned}$$

For the function $F = 1/A$ we obtain the somewhat simpler pde

$$\begin{aligned} 0 &= -F_t + \alpha F_x - \frac{\alpha^2}{\beta^2}F - \left(a - 2\frac{\alpha b}{\beta}\right)F_l \\ &\quad +d^2F_l^2/F - \frac{1}{2}\beta^2 F_{xx} - \frac{1}{2}(b^2+d^2)F_{ll} - \beta b F_{xl} \end{aligned} \tag{3.1}$$

The boundary conditions are

$$A(x,l,T) = F(x,l,T) = 1.$$

We cannot give a general solution to (3.1); in special cases, however, we obtain the optimal hedging strategies which are known so far:

- α^2/β^2 deterministic, i.e. not depending on x or l, only on t: in this case F also does not depend on x, l, the same is true for A, and therefore

  $$\theta(t) = \frac{\alpha}{\beta^2}s,$$

  the solution given by Schweizer in [4].

- α^2/β^2 does not depend on l, but only on x and t : Then F (and A) also does not depend on l, and (3.1) reduces to

  $$0 = -F_t + \alpha F_x - \frac{\alpha^2}{\beta^2}F - \frac{1}{2}\beta^2 F_{xx},$$

  and the optimal θ is given by

  $$\theta(t) = s\left(\frac{\alpha}{\beta^2} - \frac{F_x}{F}\right).$$

  This is exactly the pde (2.2), and the optimal hedging strategy given above for this case. Notice that in this case we have a short Itô representation for Z.

- $d = 0$: In this case, we have a linear pde, and we again have a short Itô representation for Z. The resulting pde differs from (2.2), but the case $d = 0$ was not considered in Hipp [3].

4. Examples with jump processes

In the continuous case, the variance optimal martingale density was of major importance. This is no longer true in the pure jump case: Consider $N(t)$ a counting process, $N(0) = 0$, all jumps of size 1. The only random variable Z which admits a short Itô representaton of the form

$$Z = z + \int_0^T z(t)dN(t)$$

which is a martingale density for $N(t)$, equals - up to a norming constant -

$$Z = 1_{(N(T)=0)}.$$

We start with examples in which the mean variance tradeoff process is constant.

Example 4.1. *Let $X(t)$ be Poisson with intensity λ, and $L(t)$ be compound Poisson with the same jump times and jump sizes $Y_k, k = 1, 2, ...$ with mean μ_1 and second moment μ_2. With T_k the time of occurence of claim k, the optimal hedging strategy reads*

$$\begin{aligned} \theta(t) &= \mu_1,\ 0 \le t \le T_1, \\ &= Y_{k-1},\ T_{k-1} < t \le T_k, \end{aligned}$$

and the residual risk equals

$$E\left(L(T) - G_T(\theta)\right)^2 = (\mu_2 - \mu_1^2)(1 - \exp(-\lambda T)).$$

Example 4.2. *Now interchange the rôle of X and $L : X(t)$ is compound Poisson, and $L(t)$ is Poisson. Here, with $\alpha = \mu_1/\mu_2$ and $\beta = 1 - \mu_1\alpha$,*

$$\theta(t) = \alpha \sum_{j=1}^{L(t)} \prod_{i=j}^{L(t)} (1 - \alpha Y_i) + \alpha,$$

and the residual risk is

$$E\left(L(T) - G_T(\theta)\right)^2 = E \sum_{j=1}^{L(T)} \beta^j + 2E \sum_{j=1}^{L(T)-1} j\beta^{j+1}$$

which remains bounded if $T \to \infty$.

Example 4.3. *Let $X(t), L(t)$ be independent Poisson processes with intensities μ, λ. Here we have*

$$\widehat{L}(t) = \lambda(T-t) + L(t)$$

and

$$\theta(t) = \widehat{L}(t) - G_{t-}(\theta).$$

Let $S_1, ..., S_k$, $k = X(T)$, be the jump times for X. Since hedging stops at S_k, we have

$$\begin{aligned} L(T) - G_T(\theta) &= L(T) - G_{S_k-}(\theta) + \theta(S_k) \\ &= L(T) - L(S_k) - \lambda(T - S_k). \end{aligned}$$

So we obtain the following residual risk for θ :

$$\begin{aligned} E\left(L(T) - G_T(\theta)\right)^2 &= \lambda E\left(T - S_{X(T)}\right)^2 \\ &= \lambda E\left(S_1^2 \middle| S_1 \le T\right) \mathbb{P}\{\mathbb{X}(T) > 0\} + \lambda T \mathbb{P}\{X(T) = 0\} \\ &\to 2\lambda/\mu^2, \ T \to \infty. \end{aligned}$$

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