Schäfer, U.

A Feasibility Result for the Block Interval Gaussian Algorithm

A new class of block interval matrices is given for which the block interval Gaussian algorithm is feasible.

The interval Gaussian algorithm (I.G.A.) (see [3]) can be used to bound the solutions of linear systems of equations with matrices and right-hand sides both of which are varying within given intervals. It is applied to an interval matrix $[A]$ and an interval vector $[b]$ and it is feasible if and only if in each elimination step the zero is not contained in at least one coefficient of the regarded interval submatrix. Otherwise it breaks down. Given an interval matrix $[A]$ it has been shown in [5] that the nonsingularity of $[A]$ (i.e. each $A \in [A]$ is nonsingular) cannot guarantee the feasibility of the I.G.A. for $[A]$. So, one is interested in classes of interval matrices for which the I.G.A.'s feasibility is guaranteed.

In [1] a new class of interval matrices for which the I.G.A.'s feasibility is guaranteed has been presented. (This class includes the interval tridiagonal ([5]) and the interval arrowhead ([6]) matrices.) The proof concerning the feasibility uses the fact that no overestimation is produced by interval arithmetic during the elimination process.

In this note we want to extend this idea to the block case, although it has been stated in [1] that the only positive conceivable situation is when all diagonal blocks are themselves diagonal interval matrices. We derive a theorem where the diagonal blocks are 2×2 interval matrices and this result is not contained in the results given in [2] where the block interval Gaussian algorithm has been introduced.

In the sequel $\mathbf{IR}$ and $\mathbf{IR}^{m\times n}$ denote the set of real intervals and the set of real $m \times n$ interval matrices, respectively.

Theorem 1. *Let $[A]$ be a nonsingular block interval arrowhead matrix:*

$$[A] = \begin{pmatrix} [A_1] & O & \dots & O & [B_1] \\ O & [A_2] & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & O & \vdots \\ O & \dots & O & [A_p] & [B_p] \\ [C_1] & \dots & \dots & [C_p] & [a_s] \end{pmatrix}, \quad [A_i] = \begin{pmatrix} [a_{11}^{(i)}] & [a_{12}^{(i)}] \\ [a_{21}^{(i)}] & [a_{22}^{(i)}] \end{pmatrix} \in \mathbf{IR}^{2\times 2}, \quad [a_s] \in \mathbf{IR},$$

$$[C_i] \in \mathbf{IR}^{1\times 2},\ [B_i] \in \mathbf{IR}^{2\times 1}, \qquad i = 1, ..., p.$$

Furthermore, let $[A_i]$ be nonsingular, $0 \notin [a_{kl}^{(i)}], k, l = 1, 2$, $[B_i] = ([b_i]\, 0)^T$ or $[B_i] = (0\, [b_i])^T$, $[C_i] = ([c_i]\, 0)$ or $[C_i] = (0\, [c_i])$, where $[b_i]$, $[c_i] \in \mathbf{IR}$, $i = 1, ..., p$. Then using

$$[A_i]^{-1} := \begin{pmatrix} \dfrac{1}{[a_{11}^{(i)}] - \frac{[a_{12}^{(i)}][a_{21}^{(i)}]}{[a_{22}^{(i)}]}} & \dfrac{1}{[a_{21}^{(i)}] - \frac{[a_{22}^{(i)}][a_{11}^{(i)}]}{[a_{12}^{(i)}]}} \\ \dfrac{1}{[a_{12}^{(i)}] - \frac{[a_{22}^{(i)}][a_{11}^{(i)}]}{[a_{21}^{(i)}]}} & \dfrac{1}{[a_{22}^{(i)}] - \frac{[a_{12}^{(i)}][a_{21}^{(i)}]}{[a_{11}^{(i)}]}} \end{pmatrix}$$

the block interval Gaussian algorithm is feasible for $[A]$.

Proof. Due to the assumptions the first p block interval Gaussian steps are feasible and one gets

$$[a_s'] := [a_s] - \sum_{i=1}^{p} \left([C_i][A_i]^{-1}\right)[B_i].$$

Now, there are four cases:

$$\text{Case 1}: \left.\begin{array}{rcl} [C_i] & = & ([c_i]\, 0) \\ [B_i] & = & ([b_i]\, 0)^T \end{array}\right\} \Rightarrow \left([C_i][A_i]^{-1}\right)[B_i] = \frac{[c_i][b_i]}{[a_{11}^{(i)}] - \frac{[a_{12}^{(i)}][a_{21}^{(i)}]}{[a_{22}^{(i)}]}}.$$

$$\text{Case 2}: \quad \left.\begin{array}{rcl}[C_i] & = & (0\,[c_i]) \\ [B_i] & = & ([b_i]\,0)^T\end{array}\right\} \Rightarrow \Big([C_i][A_i]^{-1}\Big)[B_i] = \frac{[c_i][b_i]}{[a_{12}^{(i)}] - \frac{[a_{22}^{(i)}][a_{11}^{(i)}]}{[a_{21}^{(i)}]}}.$$

$$\text{Case 3}: \quad \left.\begin{array}{rcl}[C_i] & = & ([c_i]\,0) \\ [B_i] & = & (0\,[b_i])^T\end{array}\right\} \Rightarrow \Big([C_i][A_i]^{-1}\Big)[B_i] = \frac{[c_i][b_i]}{[a_{21}^{(i)}] - \frac{[a_{22}^{(i)}][a_{11}^{(i)}]}{[a_{12}^{(i)}]}}.$$

$$\text{Case 4}: \quad \left.\begin{array}{rcl}[C_i] & = & (0\,[c_i]) \\ [B_i] & = & (0\,[b_i])^T\end{array}\right\} \Rightarrow \Big([C_i][A_i]^{-1}\Big)[B_i] = \frac{[c_i][b_i]}{[a_{22}^{(i)}] - \frac{[a_{12}^{(i)}][a_{21}^{(i)}]}{[a_{11}^{(i)}]}}.$$

I.e. each interval entry of $[A]$ appears only once in the expression for $[a_s^{'}]$. Then according to [4]

$$[a_s^{'}] = \Big\{ a_s - \sum_{i=1}^{p} C_i A_i^{-1} B_i \,\Big|\, a_s \in [a_s],\, C_i \in [C_i],\, A_i \in [A_i],\, B_i \in [B_i],\, i = 1, ..., p \Big\}.$$

So, $0 \notin [a_s^{'}]$, since $[A]$ is nonsingular. Therefore, the block interval Gaussian algorithm is feasible.

Example 1. *We consider*

$$[A] = \begin{pmatrix} [2,5] & [2,3] & [2,3] \\ [-3,-1] & 2 & 0 \\ [-5,1] & 0 & [2,3] \end{pmatrix}, \qquad \begin{array}{ll} [A_1] := \begin{pmatrix} [2,5] & [2,3] \\ [-3,-1] & 2 \end{pmatrix}, & [B_1] := \begin{pmatrix} [2,3] \\ 0 \end{pmatrix}, \\ [C_1] := \Big([-5,1]\ 0 \Big), & [a_s] := [2,3]. \end{array}$$

It is easy to check that Theorem 1 can be applied to $[A]$. More important is the fact that the I.G.A. without pivoting applied to $[A]$ breaks down, since after two elimination steps the trailing submatrix is $\left[-\frac{651}{48}, \frac{639}{48}\right]$. In addition, one can conclude that $[A]$ cannot be an H-matrix. (Otherwise the I.G.A. without pivoting applied to $[A]$ would be feasible.) So, Theorem 3.1 in [2] cannot be applied to $[A]$. Finally, one can show that neither Theorem 3.3 nor Corollary 3.4 in [2] can be applied to $[A]$.

Example 2. *We consider a 5×5 interval tridiagonal matrix. If the third column is exchanged with the 5th column and if the third row is exchanged with the 5th row, then a block interval arrowhead matrix defined as in Theorem 1 arises:*

$$[A] = \begin{pmatrix} [a_1] & [b_1] & 0 & 0 & 0 \\ [c_1] & [a_2] & [b_2] & 0 & 0 \\ 0 & [c_2] & [a_3] & [b_3] & 0 \\ 0 & 0 & [c_3] & [a_4] & [b_4] \\ 0 & 0 & 0 & [c_4] & [a_5] \end{pmatrix} \rightsquigarrow [\tilde{A}] = \begin{pmatrix} \begin{pmatrix} [a_1] & [b_1] \\ [c_1] & [a_2] \end{pmatrix} & O & \begin{pmatrix} 0 \\ [b_2] \end{pmatrix} \\ O & \begin{pmatrix} [a_5] & [c_4] \\ [b_4] & [a_4] \end{pmatrix} & \begin{pmatrix} 0 \\ [c_3] \end{pmatrix} \\ \begin{pmatrix} 0 & [c_2] \end{pmatrix} & \begin{pmatrix} 0 & [b_3] \end{pmatrix} & [a_3] \end{pmatrix}.$$

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Dr. Uwe Schäfer, Institut für Angewandte Mathematik, Universität Karlsruhe, D-76128 Karlsruhe, Germany