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Dense sets of additive functions

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Abstract. We consider the topological vector space $\mathcal{A}$ of all additive functions from $\mathbb{R}$ to $\mathbb{R}$ with the Tychonoff topology induced by $\mathbb{R}^{\mathbb{R}}$ and we prove that the following subsets of $\mathcal{A}$ and their complements (with respect to $\mathcal{A}$) are dense: the set of all additive injections, surjections, bijections, involutions, additive functions with countable image, additive functions with big graph. We are using a lemma which characterizes the density of subsets of $\mathcal{A}$.

1. Introduction

Consider the real (locally convex) topological vector space $\mathbb{R}^{\mathbb{R}}$ of all functions from $\mathbb{R}$ to $\mathbb{R}$ with the usual Tychonoff topology and let

$$\mathcal{A} = \{a \in \mathbb{R}^{\mathbb{R}} : \ a \text{ is additive}\},$$

$$\mathcal{A}_{\mathrm{c}} = \{a \in \mathcal{A} : \ a \text{ is continuous}\}.$$

Then $\mathcal{A}, \mathcal{A}_{\mathrm{c}}$ are closed subspaces of $\mathbb{R}^{\mathbb{R}}$ and, in fact, $\mathcal{A}_{\mathrm{c}}$ is one–dimensional. We consider $\mathcal{A}$ with the topology induced by $\mathbb{R}^{\mathbb{R}}$, and by a Hamel basis of $\mathbb{R}$ we mean a basis of the vector space $\mathbb{R}$ over the field $\mathbb{Q}$.

Theorem 1. *For any Hamel basis H of $\mathbb{R}$ the operator $\Lambda : \mathcal{A} \to \mathbb{R}^H$ defined by $\Lambda a = a|_H$ is a linear homeomorphism.*

The proof will be given in section 2. Since $\mathbb{R}$ and H are equipotent, the topological vector spaces $\mathbb{R}^{\mathbb{R}}$ and $\mathbb{R}^H$ are isomorphic, and we have the following

Corollary. *The topological vector spaces $\mathcal{A}$ and $\mathbb{R}^{\mathbb{R}}$ are isomorphic.*

It is the aim of this paper to show that some well known sets of additive functions are dense subsets of $\mathcal{A}$. Namely we can prove

Theorem 2. *The following eight subsets of $\mathcal{A}$ and their complements* (*with respect to $\mathcal{A}$*) *are dense:*

$$\mathcal{A}_{\mathrm{inj}} = \{a \in \mathcal{A} : \ a \text{ is injective}\},$$

$$\mathcal{A}_{\mathrm{sur}} = \{a \in \mathcal{A} : \ a(\mathbb{R}) = \mathbb{R}\},$$

$$\mathcal{A}_{\mathrm{bij}} = \mathcal{A}_{\mathrm{inj}} \cap \mathcal{A}_{\mathrm{sur}},$$

$$\mathcal{A}_{\mathrm{inv}} = \{a \in \mathcal{A} : \ a \circ a = \mathrm{id}_{\mathbb{R}}\},$$

$$\mathcal{A}_{\mathbb{Q}} = \{a \in \mathcal{A} : \ a(\mathbb{R}) \subset \mathbb{Q}\},$$

$$\mathcal{A}_{\mathrm{small}} = \{a \in \mathcal{A} : \ a(\mathbb{R}) \text{ is countable}\},$$

$$\mathcal{A}_{\mathrm{inj}} \setminus \mathcal{A}_{\mathrm{sur}}, \quad \mathcal{A}_{\mathrm{sur}} \setminus \mathcal{A}_{\mathrm{inj}}.$$

For the proof the following lemma will be used:

Lemma. *A subset $\mathcal{D}$ of $\mathcal{A}$ is dense in $\mathcal{A}$ if and only if the following condition is satisfied:*

(P) *For $M \in \mathbb{N}$, for reals $h_1, \dots, h_M, h'_1, \dots, h'_M$ being linearly independent over $\mathbb{Q}$, and for $\varepsilon > 0$, there exists $a \in \mathcal{D}$ such that*

$$h'_m - \varepsilon < a(h_m) < h'_m + \varepsilon \qquad (m = 1, \dots, M). \tag{1}$$

In applications it is sometimes possible to verify (P) with (1) replaced by

$$a(h_m) = h'_m \qquad (m = 1, \dots, M), \tag{2}$$

the $\varepsilon > 0$ then being superfluous. The lemma will be proved in section 3 and the Theorem 2 in section 4.

The elements of $\mathcal{A}_{\text{small}}$ are the additive functions with small graph (cf. [1; p.287]). There are also additive functions with big graph (cf. [1; p.287]); the definition will be given in section 5, where also the following result will be proved:

Theorem 3. *The set $\mathcal{A}_{\text{big}}$ of additive functions with big graph is dense in $\mathcal{A}$, and also $\mathcal{A} \setminus \mathcal{A}_{\text{big}}$ is dense.*

Concerning $\mathcal{A}_{\text{big}}$ we can show that it has property (P) (with (2) instead of (1)), hence we can apply the lemma, a procedure which is similar to some steps in the proof of Theorem 2. The difference is that now we use transfinite induction, therefore some basic facts on ordinal numbers are required. For the lemma we need the following fact (the proof of which is left to the reader):

Remark. Given $M \in \mathbb{N}$, reals $y_1, \dots, y_M$, a finite set H_0 of reals linearly independent over $\mathbb{Q}$, and a positive ε, then there exist reals $h'_1, \dots, h'_M$ such that $H_0 \cup \{h'_1, \dots, h'_M\}$ is linearly independent over $\mathbb{Q}$ and

$$|h'_m - y_m| < \varepsilon \qquad (m = 1, \dots, M).$$

2. Proof of Theorem 1

Obviously Λ is a linear bijection. We shall show that it is also a homeomorphism. To see that Λ is continuous consider the cartesian product projections $\pi_x : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}$, $p_h : \mathbb{R}^H \to \mathbb{R}$, and note that

$$p_h \circ \Lambda = \pi_h|_{\mathcal{A}} \qquad (h \in H).$$

To get continuity of $\Lambda^{-1} : \mathbb{R}^H \to \mathbb{R}^{\mathbb{R}}$ observe that for finite sums

$$x = \sum_{h \in H} r_h h$$

with rationals r_h we have

$$\pi_x \circ \Lambda^{-1} = \sum_{h \in H} r_h p_h.$$

3. Proof of the lemma

Let $\mathcal{D}$ be a subset of $\mathcal{A}$. Assume (P) and let H be any Hamel basis of $\mathbb{R}$. Due to Theorem 1 it is enough to prove that $\Lambda(\mathcal{D})$ is dense in $\mathbb{R}^H$. Let $\mathcal{U}$ be an open and non–empty subset of $\mathbb{R}^H$. To show that $\mathcal{U} \cap \Lambda(\mathcal{D}) \neq \emptyset$ we may assume

$$\mathcal{U} = \bigtimes_{h \in H} U_h,$$

where the U_h are non–empty open subsets of $\mathbb{R}$ such that $U_h = \mathbb{R}$ for $h \in H \setminus \{h_1, \dots, h_M\}$ (the $h_1, \dots, h_M$ being different elements of H). When applying the remark from the end of the introduction we find $\varepsilon > 0$ and reals $h'_1, \dots, h'_M$ such that $h_1, \dots, h_M, h'_1, \dots, h'_M$ are linearly independent over $\mathbb{Q}$ and

$$(h'_m - \varepsilon, h'_m + \varepsilon) \subset U_{h_m} \qquad (m = 1, \dots, M).$$

By using (P) we get an element a of $\mathcal{D}$ satisfying (1). Then $\Lambda a \in \mathcal{U}$ and this ends the proof of the density of $\mathcal{D}$. The fact that any dense subset $\mathcal{D}$ of $\mathcal{A}$ satisfies (P) is obvious.

4. Proof of Theorem 2

We shall show the density of $\mathcal{A}_{\text{inv}}, \mathcal{A}_{\text{inj}} \setminus \mathcal{A}_{\text{sur}}, \mathcal{A}_{\text{sur}} \setminus \mathcal{A}_{\text{inj}}$ by applying the lemma. The density of $\mathcal{A}_{\mathbb{Q}}$ can be shown directly. Then the rest follows from the following inclusions:

$$\mathcal{A}_{\text{inv}} \subset \mathcal{A}_{\text{inj}} \cap \mathcal{A}_{\text{sur}} \cap (\mathcal{A} \setminus \mathcal{A}_{\mathbb{Q}}) \cap (\mathcal{A} \setminus \mathcal{A}_{\text{small}}),$$

$$\mathcal{A}_{\mathbb{Q}} \subset \mathcal{A}_{\text{small}} \cap (\mathcal{A} \setminus \mathcal{A}_{\text{inj}}) \cap (\mathcal{A} \setminus \mathcal{A}_{\text{sur}}) \cap (\mathcal{A} \setminus \mathcal{A}_{\text{bij}}) \cap (\mathcal{A} \setminus \mathcal{A}_{\text{inv}}),$$

$$\mathcal{A}_{\text{sur}} \subset \mathcal{A} \setminus (\mathcal{A}_{\text{inj}} \setminus \mathcal{A}_{\text{sur}}), \quad \mathcal{A}_{\text{inj}} \subset \mathcal{A} \setminus (\mathcal{A}_{\text{sur}} \setminus \mathcal{A}_{\text{inj}}).$$

For $M \in \mathbb{N}$ and for reals $h_1, \dots, h_M, h'_1, \dots, h'_M$ being linearly independent over $\mathbb{Q}$, let H be a Hamel basis of $\mathbb{R}$ containing them.

The function $a \in \mathcal{A}$ defined by putting

$$a(h_m) = h'_m, \quad a(h'_m) = h_m \qquad (m = 1, \dots, M)$$

and

$$a(h) = h \qquad \text{for } h \in H \setminus \{h_1, \dots, h_M, h'_1, \dots, h'_M\}$$

satisfies (2) and

$$(a \circ a)(h) = h \qquad (h \in H),$$

hence also

$$(a \circ a)(x) = x \qquad (x \in \mathbb{R}).$$

Therefore the density of $\mathcal{A}_{\text{inv}}$ follows from the lemma.

To apply the lemma to $\mathcal{A}_{\text{inj}} \setminus \mathcal{A}_{\text{sur}}$ it is enough to consider any $a \in \mathcal{A}$ satisfying (2) and such that $a|_H$ is an injection of H onto a proper subset of H.

In the case of $\mathcal{A}_{\text{sur}} \setminus \mathcal{A}_{\text{inj}}$ we choose any $a \in \mathcal{A}$ such that (2) holds and $a|_H$ maps H onto H but is not one–to–one.

The density of $\mathcal{A}_{\mathbb{Q}}$ can easily be given by using condition (P) in full (i.e. with the $\varepsilon > 0$ in it). Note that the density of $\mathcal{A}_{\mathbb{Q}}$ follows also from Theorem 1 since

$$\Lambda(\mathcal{A}_{\mathbb{Q}}) = \mathbb{Q}^H.$$

5. Big graph

Following [1; p.287] we say that an additive function $a : \mathbb{R} \to \mathbb{R}$ has a big graph if $B \cap$ Graph$(a) \neq \emptyset$ for every Borel subset B of $\mathbb{R}^2$ such that

$$\{x \in \mathbb{R} : (x, y) \in B\} \text{ has continuum cardinality.} \tag{3}$$

Such functions have a lot of interesting properties, we refer the reader to [1; pp.288–291, 297].

For the proof of Theorem 3 denote by γ the first ordinal such that the set of smaller ordinals has continuum cardinality. Let all Borel subsets B of $\mathbb{R}^2$ with property (3) be arranged in a transfinite sequence $(B_\alpha)_{\alpha<\gamma}$. To apply the lemma let also an $M \in \mathbb{N}$ be given as well as reals $h_1, \ldots, h_M, h'_1, \ldots, h'_M$ which are linearly independent over $\mathbb{Q}$. Now we can find transfinite sequences $(x_\alpha)_{\alpha<\gamma}$ and $(y_\alpha)_{\alpha<\gamma}$ of reals such that

$$(x_\alpha, y_\alpha) \in B_\alpha, \quad x_\alpha \notin \mathrm{Lin}(\{x_\beta : \beta < \alpha\} \cup \{h_1, \ldots, h_M\}) \qquad (\alpha < \gamma) \tag{4}$$

(where Lin denotes linear hull in the vector space $\mathbb{R}$ over $\mathbb{Q}$). Since the set $\{x_\alpha : \alpha < \gamma\} \cup \{h_1, \ldots, h_M\}$ is linearly independent over $\mathbb{Q}$, there exists $a \in \mathcal{A}$ such that (2) and

$$a(x_\alpha) = y_\alpha \qquad (\alpha < \gamma)$$

hold. It follows from the first part of (4) that a has a big graph.

The density of $\mathcal{A} \setminus \mathcal{A}_{\mathrm{big}}$ follows, e.g., from the density of its subset $\mathcal{A}_{\mathbb{Q}}$.

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Reference

[1] Marek Kuczma, *An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality.* Prace Naukowe Uniwersytetu Śląskiego w Katowicach nr 489, Państwowe Wydawnictwo Naukowe & Uniwersytet Śląski, Warszawa–Kraków–Katowice 1985.

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