Commodious Axiomatization of Quantifiers in Multiple-Valued Logic*

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Abstract

We provide a concise axiomatization of a broad class of generalized quantifiers in many-valued logic, so- called distribution quantifiers. Although sound and complete axiomatizations for such quantifiers exist, their size renders them virtually useless for practical purposes. We show that for certain lattice-based quantifiers relatively small axiomatizations can be obtained in a schematic way. This is achieved by providing an explicit link between skolemized signed formulas and filters/ideals in Boolean set lattices.

1 Introduction

In this paper we provide tools for a concise axiomatization of a broad class of generalized quantifiers in many-valued logic, so-called distribution quantifiers [6]. The task of axiomatizing such quantifiers has been solved satisfactorily in theory: sound and complete axiomatizations exist for Hilbert style as well as for Gentzen style calculi, see e.g. [7, 5, 10, 1]. For the purpose of automated theorem proving, however, this is not enough. There, one needs minimal axiomatizations which, moreover, one must be able to find in a reasonable way. In this paper we obtain axiomatizations in the form of sequent or tableau rules for a broad class of quantifiers. The key observation, due to Zabel [10], is that the bodies of quantified signed formulas, when a Skolem term or universal term is substituted for the quantified variable, can be used to characterize upsets and downsets of the set lattice over the set of truth values N. This leads to schematic and concise tableau rules for quantifiers which are defined as generalized meet and join in a lattice of truth values.

*Due to lack of space all proofs have been omitted, however, they were available to the referees. A full version of this paper will be published elsewhere and can be obtained from the author.

2 Basic Definitions

Definition 1 *A* first-order signature Σ *is a triple* $\langle \mathbf{P}_\Sigma, \mathbf{F}_\Sigma, \alpha_\Sigma \rangle$*, where* $\mathbf{P}_\Sigma$ *is a non-empty family of* predicate symbols*,* $\mathbf{F}_\Sigma$ *is a possibly empty family of function symbols disjoint from* $\mathbf{P}_\Sigma$*, and* α_Σ *assigns a non-negative arity to each member of* $\mathbf{P}_\Sigma \cup \mathbf{F}_\Sigma$*. Let* $Term_\Sigma$ *be the set of* Σ-terms *over* object variables $Var = \{x_0, x_1, \ldots\}$*, and let* $Term^0_\Sigma$ *be the set of ground terms in* $Term_\Sigma$*.* Atoms *are defined as* $At_\Sigma = \{p(t_1, \ldots, t_n) | p \in \mathbf{P}_\Sigma, \alpha_\Sigma(p) = n, t_i \in Term_\Sigma\}$*.*

Definition 2 *A* first-order language *is a triple* $\mathbf{L} = \langle \Theta, \Lambda, \alpha \rangle$*, where* Θ *is a finite or denumerable set of* logical connectives *and* α *defines the arity of each connective. Connectives with arity* 0 *are called* logical constants*.* Λ *is a finite or denumerable set of* quantifiers*. The set* L_Σ *of* $\mathbf{L}$*-formulas over* Σ *is inductively defined as the smallest set with the following properties:*

1. $At_\Sigma \subseteq L_\Sigma$*.*
2. *If* $\phi_1, \ldots, \phi_m \in L_\Sigma$*,* $\theta \in \Theta$*, and* $\alpha(\theta) = m$ *then* $\theta(\phi_1, \ldots, \phi_m) \in L_\Sigma$*.*
3. *If* $\lambda \in \Lambda$*,* $\phi \in L_\Sigma$*, and* $x \in Var$ *then* $(\lambda x)\phi \in L_\Sigma$*.*

Definition 3 ([6]) *The* set of truth values N *is an arbitrary finite set.* $|N|$ *denotes the cardinality of* N*. If* $\mathbf{L} = \langle \Theta, \Lambda, \alpha \rangle$ *is a first-order language then we call a triple* $\mathbf{A} = \langle N, A, Q \rangle$*, where* N *is a truth value set,* A *assigns to each* $\theta \in \Theta$ *a function* $A(\theta) : N^{\alpha(\theta)} \rightarrow N$*, and* Q *assigns to each* $\lambda \in \Lambda$ *a function* $Q(\lambda) : \mathcal{P}^+(N) \rightarrow N$ *a* first-order matrix *for* $\mathbf{L}$ *(we abbreviate* $2^N - \{\emptyset\}$ *by* $\mathcal{P}^+(N)$*).* $Q(\lambda)$ *is called the* distribution function *of the quantifier* λ*.*

Example 1 *Let* $N = \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$*. Then one can define generalizations of the classical quantifiers via* $Q(\forall) = \min$*,* $Q(\exists) = \max$*, where* min *and* max *are defined wrt the natural order of* N*. In particular, for* $|N| = 2$ *one has* $N = \{0, 1\}$ *and* $Q(\forall)(\{1\}) = 1$*,* $Q(\forall)(\{0\}) = Q(\forall)(\{0, 1\}) = 0$*.*

Definition 4 *A pair $\mathcal{L} = \langle \mathbf{L}, \mathbf{A} \rangle$ consisting of a first-order language $\mathbf{L}$ and a first-order matrix for $\mathbf{L}$ is called* $|N|$-valued first-order logic.

Definition 5 *Let $\mathcal{L}$ be a first-order logic. A* Herbrand structure[1] $\mathbf{M}$ *over Σ is an* interpretation $\mathbf{I}$ *that maps $p \in \mathbf{P}_\Sigma$ to functions $\mathbf{I}(p) : (Term^0_\Sigma)^{\alpha(p)} \rightarrow N$. W.l.o.g. Σ must contain at least one 0-ary function symbol. For a structure $\mathbf{M}$ over Σ and a closed first-order formula we define a* valuation function $v_\mathbf{M} : L_\Sigma \rightarrow N$ *via:*

1. *If $\phi = p(t_1, \ldots, t_n)$ then $v_\mathbf{M}(t) = \mathbf{I}(p)(t_1, \ldots, t_n)$.*
2. *If $\phi = \theta(\phi_1, \ldots, \phi_m)$ then $v_\mathbf{M}(\phi) = A(\theta)(v_\mathbf{M}(\phi_1), \ldots, v_\mathbf{M}(\phi_m))$.*
3. *Let the* **distribution** *of $\psi(x)$ be $d_\mathbf{M}(\psi(x)) = \{v_\mathbf{M}(\psi(t)) | t \in Term^0_\Sigma\}$.*
   If $\phi = (\lambda x)\psi(x)$ then $v_\mathbf{M}(\phi) = Q(\lambda)(d_\mathbf{M}(\psi(x)))$.

Definition 6 *Let $S \subseteq N$. A signed first-order Σ-formula $S\ \phi$ is said to be* (first-order) satisfiable *iff there is a structure $\mathbf{M}$ over Σ such that $v_\mathbf{M}(\phi) \in S$. In this case we say that $\mathbf{M}$ is a* model *of $S\,\phi$ and write $\mathbf{M} \models S\ \phi$. $S\ \phi$ is* valid, *in symbols $\models S\ \phi$, iff every Σ-structure is a model of ϕ.*

Definition 7 *A* lattice *L is an ordered set $\langle N, \leq \rangle$ such that any two elements of N have a unique supremum (called* join, *we use the symbol $\sqcup$) and a unique infimum (called* meet, *we use the symbol $\sqcap$) in N.*

A lattice can be alternatively defined with join and meet alone in which case it is explicitly stipulated that they are associative, commutative, idempotent and absorptive. With this in mind we write $\sqcup\{i_1, \ldots, i_n\}$ for $i_1 \sqcup (i_1 \sqcup (\cdots (i_{n-1} \sqcup i_n) \cdots))$ and similarly with $\sqcap$.

We are only working with *bounded lattices* that is we assume there is a (unique) minimal element $\perp$ and maximal element $\top$ in L. A lattice L is *distributive* iff for all $a, b, c \in L$: $a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$.

A special case are *Boolean set lattices* for N: $\mathbf{2^N} = \langle 2^N, \emptyset, N, \cap, \cup \rangle$, where $\cap$ is set intersection and $\cup$ is set union and N any set. In our present setting we associate the elements of $\mathbf{2^N}$ with distributions.

Definition 8 *Let L be a lattice and $i \in L$. Then $\uparrow i = \{x \in L | \ x \geq_L i\}$ and $\downarrow i = \{x \in L | \ x \leq_L i\}$ are called the* upset, *respectively, the* downset *of i. The* interval *between i and j is defined as $[i, j] = \{x \in L | \ i \leq_L x \leq_L j\}$.*

[1] We work with Herbrand structures to avoid technical complications. The result that any satisfiable Σ-formula is satisfiable already in a Herbrand structure over some $\overline{\Sigma} \supseteq \Sigma$ is proved exactly like the classical result, see [1].

We say an element $x \in L$ is covered *by $y \in L$ iff $x < y$ and for any z $x \leq z < y$ implies $x = z$. The elements that cover $\perp$ are called the* atoms *of L.*

Obviously, $[i, j] = (\uparrow i) \cap (\downarrow j)$ holds for all $i, j \in L$.

Definition 9 *Let L be a lattice and J, F sets of elements of L. If the properties*

1. *$a, b \in J$ $(a, b \in F)$ imply $a \sqcup b \in J$ $(a \sqcap b \in F)$*
2. *$a \in L$, $b \in J$ $(b \in F)$ and $a \leq b$ $(a \geq b)$ imply $a \in J$ $(a \in F)$*

hold then J is called an ideal *(F is called a* filter*) of L. For each $a \in L$, $\downarrow a$ is an ideal of L. This particular ideal is called the* principal ideal generated by a. Principal filters *are defined dually. A* maximal ideal *(a* maximal filter*) is an ideal (a filter) such that the only ideal (filter) properly containing it is L.*

In a finite lattice, every filter and ideal is principal.

Definition 10 *Let L be a lattice. An element $x \in L$ is called* meet-irreducible *if (i) $x \neq \top$ and (ii) $x = a \sqcap b$ implies $x = a$ or $x = b$ for all $a, b \in L$. $\mathcal{M}(L)$ denotes the meet-irreducible elements of L.*

3 Standard Axiomatization of Distribution Quantifiers

The problem one needs to solve in order to provide Gentzen or tableau rules[2] for signed quantified formulas is: given a quantified signed formula $S(\lambda x)\phi(x)$ we are looking for a formula of the form

$$\bigvee_{i \in I} \bigwedge_{j \in J_i} S_{ij}\ \phi(z_{ij}) \qquad (1)$$

where $S_{ij} \subseteq N$ and the z_{ij} are certain ground terms falling in two categories: either z_{ij} is a Skolem constant c, then it must be new relative to the proof in which the formula occurs or z_{ij} is a term t, then it stands for an arbitrary ground term. We stipulate for (1) that any structure $\mathbf{M}$ over Σ can be extended to a structure $\overline{\mathbf{M}}$ over $\overline{\Sigma} \supseteq \Sigma$ such that $S\,(\lambda x)\phi(x)$ is satisfiable in $\mathbf{M}$ iff (1) is satisfiable in $\overline{\mathbf{M}}$ for every possible substitution of ground terms for z_{ij} of universal type. This is nothing else than soundness and completeness of a tableau rule which is directly derived from (1):

[2] We use ground versions of tableau rules á la Smullyan [9] as opposed to free variable tableau rules [4] in order to avoid technical complications. It is completely straightforward to give free variable versions of the results in this paper.

$$
\begin{array}{c}
S\,(\lambda x)\phi(x) \\
\hline
\begin{array}{c|c|c}
S_{11}\,\phi(z_{11}) & \cdots & S_{r1}\,\phi(z_{r1}) \\
\vdots & & \vdots \\
S_{1s_1}\,\phi(z_{1s_1}) & \cdots & S_{rs_r}\,\phi(z_{rs_r})
\end{array}
\end{array}
\qquad (2)
$$

where $I = \{1, \ldots, r\}$, $J_i = \{1, \ldots, s_i\}$ and the z_{ij} are either c_{ij} or t according to their status in (1).

Such rules are given e.g. in [7, 1] for the special case where all signs S, $S_{11}, \ldots, S_{rs}$ are singletons. [5] deals with the general case which can easily be obtained from the following lemma:

Lemma 1 ([5]) *Let $(Q(\lambda))^{-1}(S) = \{\emptyset \neq I \subseteq N \mid Q(\lambda)(I) \in S\}$. Then a signed quantified formula $S\,(\lambda x)\phi(x)$ is satisfied iff there is an $I = \{i_1, \ldots, i_r\} \in (Q(\lambda))^{-1}(S)$ such that for each $i_k \in I$, there is a constant term c_k not occurring in $\phi(x)$ such that (i) all $\{i_k\}\,\phi(c_k)$ and (ii) for any ground term t, $I\,\phi(t)$ are simultaneously satisfiable.*

If $(Q(\lambda))^{-1}(S) = \{I_1, \ldots, I_r\}$, $I_i = \{k_{i1}, \ldots, k_{i|I_i|}\}$ then let $I = \{1, \ldots, r\}$, $s_i = |I_i| + 1$, $S_{ij} = \{k_{ij}\}$, $z_{ij} = c_{ij}$ for $j \leq |I_i|$ and $S_{i(|I_i|+1)} = I_i$, $z_{i(|I_i|+1)} = t$ in (1) to obtain a sound and complete tableau rule from the preceding lemma.

Example 2 *Consider $N = \{0, \frac{1}{2}, 1\}$ and $\forall$ as defined in Example 1. In $(Q(\forall))^{-1}(\{0, \frac{1}{2}\})$ are all subsets of N that contain at least one of 0 and $\frac{1}{2}$, all in all six sets. Hence the tableau rule according to Lemma 1 is as shown in Figure 1.*

The obvious problem with such rules is that they can become rather big. This is unavoidable in general: Zach [11] shows that for similar characterizations as (1) there exist combinations of quantifiers and signs such that $|I| = 2^{|N|-1}$. Hence, the branching factor can be exponential with respect to the cardinality of the set of truth values. On the other hand, Lemma 1 tends to give "fat" rules even when "slim" ones exist. For instance, the tableau rule (3) below is a sound and complete rule for the premiss from Example 2 and it is considerably smaller.

4 Skolemized Distribution Quantifiers and Boolean Set Lattices

In the following we show that in many cases concise representations of distribution quantifiers can be obtained. The starting point is the observation that signed formulas of the form $I\,\phi(c)$ and $J\,\phi(t)$ can be used to characterize certain sets of distributions. The important thing to note is that these sets of distributions play special rôles within the set lattice $\mathbf{2^N}$.

For $\emptyset \neq F \subseteq N$ we introduce the abbreviation $\mathcal{U}(F) = \{X \subseteq N \mid X \cap F \neq \emptyset\}$.

Lemma 2 *Assume $\mathcal{U}(F) = (Q(\lambda))^{-1}(S)$. Then any structure $\mathbf{M}$ over Σ can be extended to a structure $\overline{\mathbf{M}}$ over $\overline{\Sigma} \supseteq \Sigma$ such that $v_{\mathbf{M}}((\lambda x)\phi(x)) \in S$ iff $v_{\overline{\mathbf{M}}}(\phi(c)) \in F$, where c does not occur in Σ.*

As $F\,\phi(c)$ is considerably shorter than the set of extensions corresponding to the members of $\mathcal{U}(F)$ one obtains simplified tableau rules whenever $\mathcal{U}(F) = (Q(\lambda))^{-1}(S)$ holds for some $F \subseteq N$. For instance, as $\mathcal{U}(\{0, \frac{1}{2}\}) = (Q(\forall))^{-1}(\{0, \frac{1}{2}\})$ in Example 2, the rule shown in Figure 1 can be simplified to:

$$
\frac{\{0, \frac{1}{2}\}\,(\forall x)\phi(x)}{\{0, \frac{1}{2}\}\,\phi(c)} \qquad (3)
$$

The usefulness of Lemma 2 comes from the fact that the family of sets $\mathcal{U}(F)$ has an interesting structure:

Lemma 3 *For any $F \in \mathbf{2^N}$, $\mathcal{U}(F) = \bigcup_{i \in F} \uparrow\{i\}$, in particular, $\mathcal{U}(\{i\}) = \uparrow\{i\}$.*

A principal filter generated by an atom of $\mathbf{2^N}$ is a maximal filter of $\mathbf{2^N}$ (a maximal filter on N for short) and in the finite case all maximal filters on N are of this form.

Thus, whenever $(Q(\lambda))^{-1}(S)$ is a union of maximal filters one may choose its upset representation to obtain a single-extension rule. In particular, whenever $(Q(\lambda))^{-1}(S)$ is a maximal filter on N, say $\uparrow\{i\}$, then there is a single-extension rule with exactly one formula, namely $\{i\}\,\phi(c)$. It turns out that at least in the finite case this can be generalized:

Lemma 4 *For finite N the principal filter $\uparrow I$ of $\mathbf{2^N}$ is equal to $\bigcap_{i \in I} \uparrow\{i\}$.*

In other words, whenever $(Q(\lambda))^{-1}(S)$ is a filter of $\mathbf{2^N}$ there is a single extension rule containing the formulas $\{\{i\}\,\phi(c_i) \mid i \in I\}$ for some $I \subseteq N$. Consequently, whenever we have a representation of $(Q(\lambda))^{-1}(S)$ of the form $\bigcup_{k \in K} F_k$, where the F_k are filters of $\mathbf{2^N}$, then, by repeated application and disjunctive combination of Lemma 2, there is a tableau rule for $S\,(\lambda x)\phi(x)$ with $|K|$ extensions.

Similarly as signed formulas of the form $I\,\phi(c)$ characterize distribution quantifiers whose distributions correspond to filters of $\mathbf{2^N}$, signed formulas of the form $I\,\phi(t)$ characterize distribution quantifiers whose distributions correspond to certain ideals of $\mathbf{2^N}$. To make

$$
\begin{array}{c}
\{0,\frac{1}{2}\}\ (\forall x)\phi(x) \\
\hline
\begin{array}{c|c|c|c|c|c}
\{0\}\ \phi(c) & \{0\}\ \phi(c) & \{0\}\ \phi(c) & \{0\}\ \phi(c) & & \{\frac{1}{2}\}\ \phi(c) \\
 & \{\frac{1}{2}\}\ \phi(d) & \{\frac{1}{2}\}\ \phi(d) & & \{\frac{1}{2}\}\ \phi(d) & \\
 & & \{1\}\ \phi(e) & \{1\}\ \phi(e) & & \{1\}\ \phi(e) \\
\{0\}\ \phi(t_1) & \{0,\frac{1}{2}\}\ \phi(t_2) & N\ \phi(t_3) & \{0,1\}\ \phi(t_4) & \{\frac{1}{2}\}\ \phi(t_5) & \{\frac{1}{2},1\}\ \phi(t_6)
\end{array}
\end{array}
$$

Figure 1: Tableau rule for $\{0,\frac{1}{2}\}\ (\forall x)\phi(x)$, see Example 2.

this idea more precise, we define $\mathcal{P}(I) = \{X \subseteq N \mid \emptyset \neq X \subseteq I\}$.

Lemma 5 *Let $(\lambda x)\phi(x)$ be a Σ-formula, $\mathbf{M}$ a Σ-structure, assume $\mathcal{P}(I) = (Q(\lambda))^{-1}(S)$. Then for all ground terms t: $v_{\mathbf{M}}((\lambda x)\phi(x)) \in S$ iff $v_{\mathbf{M}}(\phi(t)) \in I$.*

Obviously, $\mathcal{P}(I) \cup \{\emptyset\}$ is the principal ideal of $\mathbf{2^N}$ generated by I, that is $\downarrow I$. In finite lattices every ideal is principal, hence, whenever $(Q(\lambda))^{-1}(S) \cup \{\emptyset\}$ is an ideal $\downarrow I$ of $\mathbf{2^N}$, then there is a single-extension rule with exactly one formula, namely $I\ \phi(t)$. So the characterization of distributions that are ideals of $\mathbf{2^N}$ is even slightly easier than that of filters of $\mathbf{2^N}$.

5 Distribution Quantifiers and $\mathbf{2^N}$

It could be argued that the case when $(Q(\lambda))^{-1}(S)$ can be straightforwardly repesented as a DNF combination of filters and ideals is relatively rare and therefore the results above are not particularly relevant. There are two objections to this argument: first, as Zabel [10, Section 1.3.3] points out, even when $(Q(\lambda))^{-1}(S)$ has no representation as a DNF combination of filters and ideals, it still can possibly be partitioned into $(Q(\lambda))^{-1}(S) = \mathcal{I} \cup \mathcal{R}$ such that $\mathcal{I}$ has a representation as a DNF combination of filters and ideals. In this case, at least the part of the standard rule (2) that corresponds to $\mathcal{I}$ can be simplified. Second, in the following we show that many "naturally" defined quantifiers in fact have a representation as a DNF combination of filters and ideals.

Example 3 *Consider the truth value set $\mathcal{FOUR} = \{\bot, f, t, \top\}$ with the ordering indicated in Figure 2(a) and a quantifier Π on $\mathcal{FOUR}$ defined as $Q(\Pi) = \sqcap$, where $\sqcap$ is the* meet *operator on the lattice $\mathcal{FOUR}$.*

Let us look at the set of distributions corresponding to Π and the sets of signs $\{\bot\}$, $\{f,\bot\}$, and $\{f\}$, respectively. One computes $(Q(\Pi))^{-1}(\{\bot\}) = 2^N - \{\{f,\top\},\{t,\top\},\{f\},\{t\},\{\top\},\{\}\}$, $(Q(\Pi))^{-1}(\{f\}) = \{\{f\},\{f,\top\}\}$, and $(Q(\Pi))^{-1}(\{\bot,f\})$ is the union of the latter. It turns out that all three sets of distributions can be simply characterized using filters and ideals of the set lattice of $\mathcal{FOUR}$, see Figure 2(b):

$$
\begin{array}{ll}
(Q(\Pi))^{-1}(\{\bot,f\}) & = \uparrow\{f\} \cup \uparrow\{\bot\} = \mathcal{U}(\{\bot,f\}) \\
(Q(\Pi))^{-1}(\{\bot\}) & = \uparrow\{f,t\} \cup \uparrow\{\bot\} \\
(Q(\Pi))^{-1}(\{f\}) & = \uparrow\{f\} \cap \downarrow\{f,\top\}
\end{array}
$$

This leads to the following tableau rules:

$$
\frac{\{\bot,f\}\ (\Pi x)\phi(x)}{\{\bot,f\}\ \phi(c)}
\qquad
\frac{\{\bot\}\ (\Pi x)\phi(x)}{\begin{array}{c|c} \{f\}\ \phi(c) & \{\bot\}\ \phi(e) \\ \{t\}\ \phi(d) & \end{array}}
\qquad
\frac{\{f\}\ (\Pi x)\phi(x)}{\begin{array}{c} \{f\}\ \phi(c) \\ \{f,\top\}\ \phi(t) \end{array}}
$$

This motivating example is generalized in the following theorem:

Theorem 6 *Assume $L = \langle N, \sqcap, \sqcup \rangle$ is a lattice*[3] *over a finite set of truth values N. We define distribution quantifiers Π and Σ via: $Q(\Pi) = \sqcap$, $Q(\Sigma) = \sqcup$.*

1. *If $i \in \mathcal{M}(L)$ then* $(Q^{-1}(\Pi))(\{i\}) = \mathcal{U}(\{i\}) \cap (\mathcal{P}(\Uparrow i) \cup \{\emptyset\})$
2. *If L is distributive and $i \in \mathcal{M}(L)$ then* $(Q^{-1}(\Pi))(\Downarrow i) = \mathcal{U}(\Downarrow i)$
3. *For all $i \in N$ and distributive L:* $(Q^{-1}(\Pi))(\{i\}) = \left(\bigcap_{m \in M_i} \mathcal{U}(\Downarrow m)\right) \cap (\mathcal{P}(\Uparrow i) \cup \{\emptyset\})$ *where M_i are the minimal elements of $\mathcal{M}(L) \cap \Uparrow i$.*
4. *For all $i \in N$, distributive L, and M_i is as above:* $(Q^{-1}(\Pi))(\Downarrow i) = \bigcap_{m \in M_i} \mathcal{U}(\Downarrow m)$
5. *For all $i \in N$: $(Q^{-1}(\Pi))(\Uparrow i) = \mathcal{P}(\Uparrow i)$*

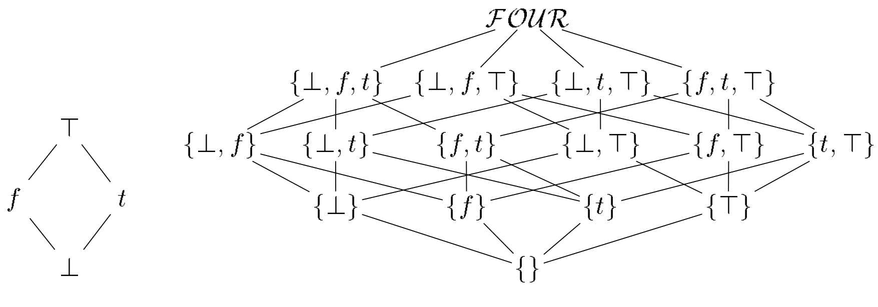
By duality of Π and Σ the previous theorem holds as well if Σ is substituted for Π, "join" for "meet", etc. This duality generalizes the well known duality between the classical quantifiers $\forall$ and $\exists$. Of course, duality extends to the associated tableau rules:

Corollary *Cases (1)–(5) in the previous theorem give rise to the following sound and complete tableau rules for distribution quantifiers:*

If i is meet-irreducible (1)	*L distributive* (2)	*For all i, L*[4] (5)
$\dfrac{\{i\}\ (\Pi x)\phi(x)}{\begin{array}{c}\{i\}\ \phi(c) \\ \Uparrow i\ \phi(t)\end{array}}$	$\dfrac{\Downarrow i\ (\Pi x)\phi(x)}{\Downarrow i\ \phi(c)}$	$\dfrac{\Uparrow i\ (\Pi x)\phi(x)}{\Uparrow i\ \phi(t)}$

[3] We use $\Uparrow i$, $\Downarrow i$ for the upset and downset of an element $i \in L$ to discern it from upsets and downsets in $\mathbf{2^N}$.

[4] It is straightforward to see that in the case of (5) even a lower semi-lattice and for its dual an upper semi-lattice is sufficient.



(a) $\mathcal{FOUR}$ (b) Set lattice of $\mathcal{FOUR}$

Figure 2: The truth value set $\mathcal{FOUR}$ with an ordering and the set lattice of $\mathcal{FOUR}$.

For all i, L distributive, $M_i{=}\{m_1,\ldots,m_r\}$ *as above:*

$$\frac{\{i\}\ (\Pi x)\phi(x)}{\begin{array}{c}\Downarrow m_1\ \phi(c_1)\\ \vdots\\ \Downarrow m_r\ \phi(c_r)\\ \Uparrow i\ \phi(t)\end{array}}\ (3) \qquad \frac{\Downarrow i\ (\Pi x)\phi(x)}{\begin{array}{c}\Downarrow m_1\ \phi(c_1)\\ \vdots\\ \Downarrow m_r\ \phi(c_r)\end{array}}\ (4)$$

Whenever i is the bottom (top) element of L, a signed formula of the form $\Uparrow i\ \phi$ ($\Downarrow i\ \phi$) is tautological and can be removed in the rules above.

Note that all signs occurring in the conclusions are of the form $\Uparrow i$, $\Downarrow i$ or $\{i\}$. As for all these signs and quantifiers based on distributive lattices rules exist we have a complete inference system.

Rule (2) can be considered to be a special case of rule (4): if i is meet-irreducible then $M = \{i\}$, thus (4) collapses to (2). Similarly, rule (3) collapses to a rule with $\Downarrow i\ \phi(c)$ and $\Uparrow i\ \phi(t)$ in the conclusion. A second application of this (3) in which t is chosen to be c, yields $\Uparrow i\ \phi(c)$. As $\{i\} = \Uparrow i \cap \Downarrow i$ the result is equivalent to the conclusion of (1).

Example 4 *Two of the rules given in Example 3 are instances of the corollary: as* $\mathcal{FOUR}$ *is distributive and f is meet-irreducible the left and the right rule in Example 3 can be obtained from schemata (2) and (1), respectively. For* $\{\bot\}$ *we have to invoke rule schema (3) to obtain* $\frac{\{\bot\}\ (\Pi x)\phi(x)}{\begin{array}{c}\{\bot,f\}\ \phi(c)\\ \{\bot,t\}\ \phi(d)\end{array}}$ *which in fact is simpler than the rule derived "by hand".*

As a further example we give the rule for $\Downarrow D$ *and* Π *in* $\mathcal{NINE}$ *(cf. Figure 3). A meet-irreducible representation of D is* $F \sqcap G$*, hence:* $\frac{\Downarrow D\ (\Pi x)\phi(x)}{\begin{array}{c}\Downarrow F\ \phi(c)\\ \Downarrow G\ \phi(d)\end{array}}$. *A rule constructed with Lemma 1 would have* several hundred extensions.

Hähnle [5, Section 5.4] lists rules for the special case when L is a finite chain and signs are either upsets, downsets or singletons. Those rules can be obtained from the corollary immediately as special cases, because a chain is distributive and all its elements are meet-irreducible.

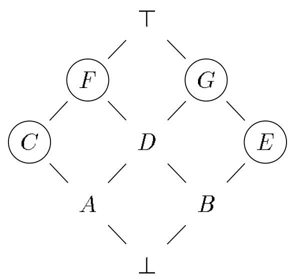


Figure 3: $\mathcal{M}(\mathcal{NINE})$, see Example 4.

We stress that propositional connectives do not occur at all in our results. This means that a many-valued logic needs not be lattice-based in general for our results to apply. It suffices that the *quantifiers* can be defined with the help of lattices. One may even use a different lattice for each quantifier given the lattices are *upset/downset-compatible*.[5]

6 Conclusion

Summary It has been pointed out by Carnielli [2, p. 488] that, even for singleton signs, it is a difficult problem to find minimal rules for distribution quantifiers automatically in a feasible way that is without enumerating all possible rules. Zabel [10] gave

[5] I.e. an upset wrt to one of the lattices is also an upset or downset wrt the other lattices. This is important for logics based on bilattices [3] where two different lattices occur naturally. As an upset (downset) in one of the lattices that constitute a bilattice is an upset (downset) in the other lattice as well, combination of the rules is no problem.

simplified singleton signs rules in the case when the distributions to be characterized form a sublattice of the Boolean set lattice. We generalized Zabel's results to include upsets and downsets as signs and provided a class of quantifiers that allow systematic construction of concise rules with at most one extension. As in the rule conclusions again only upsets, downsets and singletons occur one obtains complete calculi for those classes of logics. In particular, it is sufficient to consider merely $2|N|$ different signs, the upsets and downsets of N, because $\{i\} = \Uparrow i \cap \Downarrow i$. Together with the result that there exists a broad class of propositional many-valued logics with the same property (so-called *regular logics*, see [5]) this implies that there is a substantial class of many-valued *first-order* logics which is proof-theoretically not a lot more complex than classical logic.

Related Work Fitting [3] gives signed tableau rules for the case when the set of truth values forms a certain type of *bilattice*, however, as Fitting requires his rules to follow Smullyan's uniform notation [9], the class of logics for which they work is somewhat restricted.

Similar results as Lemmata 2 and 5, but restricted to singleton sets as signs, appear in [10] as Corollaries 1.3.3 and 1.3.4. Zabel also pointed out that the premisses of those lemmata are "les plus communs dans les logiques polyvalentes"; he did not try, however, to make this statement more precise.

Zach [11, Section 1.7] and Baaz & Fermüller [1, Example 4.20] gave single extension rules for singleton signs and quantifiers induced by certain connectives (the latter were shown in turn to be related to upper semi-lattices over the set of truth values).

Recently, Salzer [8] described a minimization algorithm for distribution quantifiers. He proves that optimal dual (CNF-based) rules are produced by his algorithm for arbitrary quantifiers and signs in finitely-valued logics. Salzer gives also general tableau rule schemata for quantifiers based on upper/lower semi-lattices and interval-shaped signs. This result is more general than the results presented here, but the number of signs occurring in Salzer's rules cannot be restricted *a priori*: all $2^{|N|} - 2$ different (non-trivial) signs may occur. Using the results of the present paper one can show, however, that in the case of distributive lattices only upsets and downsets need occur in Salzer's rule as well.

Future Work It is possible to generalize the present results to certain infinite lattices and thus infinitely-valued logics. As usual, one must work with prime ideals and prime filters instead of irreducible elements.

One can exploit that $\mathbf{2^N}$ is Boolean, hence complemented. This should extend the sets that can be characterized with skolemized formulas. The complement of $\Downarrow D$ in $\mathcal{NINE}$, for instance, is rather simple to represent, because it is the union of two upsets of join-irreducible elements.

As already mentioned above, in a many-valued logic several quantifiers and connectives may be present each of which is based on regular signs, but wrt underlying orders that are not upset/downset-compatible. Is it possible to give complete rule systems for such logics that still can exploit the results of this paper?

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