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Modelling turbulent dissipation rate for Rayleigh-Bénard convection

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Abstract – Both the two-point correlation technique with the assumption of local homogeneity and invariant theory were used to study the turbulence closure for the dissipation rate in turbulent natural convection. The structure of an analytically derived equation for the turbulent dissipation rate was analyzed by using direct numerical simulation data of turbulent natural convection in different fluids. It was found that the local homogeneity assumption holds for the two-point velocity correlations of third rank. However, the same conclusion does not seem to be applicable to the two-point temperature-velocity correlation. The time scale ratio $R = \tau_\theta/\tau$ should be included in modelling of the buoyant production term in the dissipation equation. The sink term in the dissipation rate equation was modeled using invariants of anisotropy tensor and the DNS data for Rayleigh-Bénard convection were shown in the anisotropy invariant map.

1. Introduction

Turbulence modelling plays an important role in computational fluid dynamics and computational heat and mass transfer, which are emerging as major tools for solving flow problems encountered in engineering applications. The most widely used approach for the prediction of turbulent flows is based on the equations for the moments of fluctuating flow quantities. The modelling of the dissipation rate correlation which is one of the unknown quantities in the second moment equations is very important for reliable flow prediction but is recognized to be quite difficult due to the lack of adequate data for the validation of the fundamental closure assumptions.

With the development of advanced numerical simulation techniques, it is possible to test various closure assumptions directly against a simulation database and make significant progress towards the development of turbulence closure. Based on the two-point correlation technique which was first introduced by Chou [1], and subsequently improved by Kolovandin & Vatutin [2], the structure of an analytically derived equation for the dissipation correlation was analyzed using direct numerical simulation (DNS) data of turbulent channel flow at low Reynolds number by Jovanović, Ye & Durst [3]. The detailed closure assumptions for dissipation rate were investigated by the authors against DNS data for a variety of turbulent shear flows [4, 5].

For some complex flow and heat transfer problems, e.g. flows dominated by buoyancy, it is recognized that turbulence modelling requires special care [6]. The influence of the molecular Prandtl number Pr, time scale ratio R and the turbulence Reynolds number Re_t has to be carefully considered in order to obtain accurate models [7, 8]. Thus, it is necessary to study the above mentioned closure assumptions for the turbulence dissipation rate in turbulent flows induced by thermal buoyancy.

In this paper, DNS data (Wörner & Grötzbach [8]) for turbulent natural convection, specially for Rayleigh-Bénard convection, in air and sodium are used to study the closure for the dissipation rate. First, the structure of the equation that governs the turbulence dissipation rate, including the buoyancy influence, is analyzed. The assumption of local homogeneity which is used to simplify the derived equation of dissipation rate is examined using the DNS data. Then the invariant theory introduced by Lumley & Newman [9] is used to derive the closure for the sink term in the dissipation rate equation. The buoyant production term should also be correctly modeled in turbulent natural convection with low Rayleigh number, although this term has a smaller contribution than other terms in the budget of the dissipation rate equation for forced convection, and usually can be neglected at high Reynolds number. In this paper, the influence of the Pr number and time scale ratio R on the closure of this term is considered.

2. Basic equations

The equations which describe the transport of the second-order moments $\overline{u_i u_j}$ can be derived from the Navier-Stokes equations. For incompressible buoyant flow under the Boussinesq assumption one can write:

$$\frac{\partial \overline{u_i u_j}}{\partial t} + U_k \frac{\partial \overline{u_i u_j}}{\partial x_k} + \overline{u_j u_k}\frac{\partial U_i}{\partial x_k} + \overline{u_i u_k}\frac{\partial U_j}{\partial x_k} + \beta g_j \overline{\theta u_i} + \beta g_i \overline{\theta u_j}$$
$$+ \frac{\partial \overline{u_i u_j u_k}}{\partial x_k} + \frac{1}{\rho}[\overline{u_j \frac{\partial p}{\partial x_i}} + \overline{u_i \frac{\partial p}{\partial x_j}}] + 2\nu \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} - \nu \Delta_x \overline{u_i u_j} = 0, \quad (1)$$

where Δ_x corresponds to the Laplace operator ($\Delta_x = \partial^2/\partial x_l \partial x_l$) with respect to the variable x. The dissipation correlations of $\overline{u_i u_j}$ are represented by the terms:

$$\epsilon_{ij} = \nu \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}. \quad (2)$$

2.1. Two-point correlation technique

The dissipation tensor (2) can be expressed as functions of two-point correlations by introducing a coordinate system relative to two arbitrary points A and B [1]:

$$\xi_k = (x_k)_B - (x_k)_A, \quad (3)$$
$$(x_k)_{AB} = \frac{1}{2}[(x_k)_A + (x_k)_B]. \quad (4)$$

From equations (3)-(4) various partial differential operators can be derived, which are functions of $(x_k)_{AB}$ and ξ_k, such as:

$$\left(\frac{\partial}{\partial x_k}\right)_A = \frac{1}{2}\left(\frac{\partial}{\partial x_k}\right)_{AB} - \frac{\partial}{\partial \xi_k}, \quad (5)$$
$$\left(\frac{\partial}{\partial x_k}\right)_B = \frac{1}{2}\left(\frac{\partial}{\partial x_k}\right)_{AB} + \frac{\partial}{\partial \xi_k}, \quad (6)$$
$$\left(\frac{\partial}{\partial x_k}\right)_A \left(\frac{\partial}{\partial x_k}\right)_B = \frac{1}{4}\left(\frac{\partial^2}{\partial x_k \partial x_k}\right)_{AB} - \frac{\partial^2}{\partial \xi_k \partial \xi_k}. \quad (7)$$

Applying the operator (7) to the velocity fluctuation product at two points, $(u_i)_A(u_j)_B$, taking the average and setting ξ_k equal to zero, we get:

$$\epsilon_{ij} = \nu \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} = \frac{1}{4}\nu \Delta_x \overline{u_i u_j} - \nu(\Delta_\xi \overline{u_i u_j'})_0, \quad (8)$$

its trace ϵ is:

$$\epsilon = \nu \overline{\frac{\partial u_s}{\partial x_k}\frac{\partial u_s}{\partial x_k}} = \underbrace{\frac{1}{4}\nu \Delta_x \overline{u_s u_s}}_{inhomogeneous} \underbrace{- \nu(\Delta_\xi \overline{u_s u_s'})_0}_{homogeneous}, \tag{9}$$

where the prime $'$ in Eq. (9) indicates a value for the two-point correlation function at point B ($\overline{(u_i)_A(u_j)_B} = \overline{u_i u_j'}$) and the subscript $_0$ represents the zero separation between the two points. The inhomogeneous parts of ϵ_{ij} and ϵ are especially important near the wall [3]. The homogeneous parts in the equations (8) and (9) can be obtained through the derivatives of the two point velocity correlations of second rank.

2.2. Dynamic equation for ϵ_h

The equations for $\nu(\Delta_\xi \overline{u_i u_j'})_0$ can be obtained by applying the operator $\nu\Delta_\xi$ to the dynamic equations for the two point velocity correlation $\overline{(u_i)_A(u_j)_B}$ and taking the limit $\xi_k \to 0$. Since the components of the dissipation tensor ϵ_{ij} can be analytically interpreted in terms of its trace ϵ and second order velocity correlation $\overline{u_i u_j}$ [1, 2], we are interested only in the contracted form of the equation for $\nu(\Delta_\xi \overline{u_i u_j'})_0$, which can be expressed as follows:

$$\begin{aligned}
&-\nu\frac{\partial}{\partial t}(\Delta_\xi \overline{u_s u_s'})_0 - \nu U_k \frac{\partial}{\partial x_k}(\Delta_\xi \overline{u_s u_s'})_0 = \nu[(\Delta_\xi \overline{u_k u_s'})_0 + (\Delta_\xi \overline{u_s u_k'})_0]\frac{\partial U_s}{\partial x_k} \\
&+\frac{\nu}{4}\left[2\overline{u_s u_k}\Delta_x \frac{\partial U_s}{\partial x_k} + (\Delta_x U_k)\frac{\partial}{\partial x_k}\overline{u_s u_s}\right] + \nu\left[\left(\frac{\partial}{\partial \xi_l}\overline{u_s u_k'}\right)_0 - \left(\frac{\partial}{\partial \xi_l}\overline{u_k u_s'}\right)_0\right]\frac{\partial^2 U_s}{\partial x_l \partial x_k} \\
&+2\nu\left(\frac{\partial^2}{\partial \xi_l \partial \xi_k}\overline{u_s u_s'}\right)_0 \frac{\partial U_k}{\partial x_l} + \nu\beta g_s[(\Delta_\xi \overline{\theta u_s'})_0 + (\Delta_\xi \overline{u_s \theta'})_0] + \frac{\nu}{2}\frac{\partial}{\partial x_k}[(\Delta_\xi \overline{u_s u_k u_s'})_0 + (\Delta_\xi \overline{u_s u_s' u_k'})_0] \\
&+\nu[\Delta_\xi \frac{\partial}{\partial \xi_k}(\overline{u_s u_s' u_k'} - \overline{u_s u_k u_s'})]_0 + \frac{\nu}{2\rho}\frac{\partial}{\partial x_s}[(\Delta_\xi \overline{p u_s'})_0 + (\Delta_\xi \overline{u_s p'})_0] \\
&-\frac{\nu}{\rho}[\Delta_\xi \frac{\partial}{\partial \xi_s}(\overline{p u_s'} - \overline{u_s p'})]_0 - \frac{1}{2}\nu^2 \Delta_x (\Delta_\xi \overline{u_s u_s'})_0 - 2\nu^2(\Delta_\xi \Delta_\xi \overline{u_s u_s'})_0.
\end{aligned} \tag{10}$$

Chou [1] has shown that this complicated equation can be simplified, if the assumption of local homogeneity for the small scale structure of turbulence is applied. This assumption permits the following properties valid for homogeneous turbulence to be used (see [10]):

$$\overline{u_s u_k'} = \overline{u_k u_s'}, \tag{11}$$

$$\overline{u_s u_s' u_k'} = -\overline{u_s u_k u_s'}, \tag{12}$$

$$\overline{p u_s'} = -\overline{u_s p'}. \tag{13}$$

Differentiating equations (11)-(13), setting $\xi = 0$ and substituting the corresponding results into Eq. (10), we get the approximate equation for the homogeneous part of the dissipation rate:

$$\begin{aligned}
&\frac{\partial \epsilon_h}{\partial t} + U_k \frac{\partial \epsilon_h}{\partial x_k} \simeq 2\nu(\Delta_\xi \overline{u_k u_s'})_0 \frac{\partial U_s}{\partial x_k} + \frac{\nu}{4}\left[2\overline{u_s u_k}\Delta_x \frac{\partial U_s}{\partial x_k} + (\Delta_x U_k)\frac{\partial}{\partial x_k}\overline{u_s u_s}\right] \\
&+\nu\beta g_s[(\Delta_\xi \overline{\theta u_s'})_0 + (\Delta_\xi \overline{u_s \theta'})_0] + 2\nu\left(\frac{\partial^2}{\partial \xi_l \partial \xi_k}\overline{u_s u_s'}\right)_0 \frac{\partial U_k}{\partial x_l} - 2\nu\left(\Delta_\xi \frac{\partial}{\partial \xi_k}\overline{u_s u_k u_s'}\right)_0 \\
&+\frac{1}{2}\nu\Delta_x \epsilon_h - 2\nu^2(\Delta_\xi \Delta_\xi \overline{u_s u_s'})_0,
\end{aligned} \tag{14}$$

where $\epsilon_h = -\nu(\Delta_\xi \overline{u_s u_s'})_0$.

One of the differences between the equation (14) and the approximate equation derived by Jovanović et al. [3] is that the second-order derivatives of the triple correlation $\frac{\nu}{2}\frac{\partial}{\partial x_k}[(\Delta_\xi \overline{u_s u_k u'_s})_0 + (\Delta_\xi \overline{u_s u'_s u'_k})_0]$ disappear because of the assumption of local homogeneous turbulence for small scales. Using the DNS data of channel flow they showed that away from the wall the properties of homogeneous turbulence (Eq. (11) and Eq. (13)) can be used. However they argued that, because of the lack of the appropriate correlations from the simulation database, it is difficult to justify the applicability of the assumption of local homogeneity for the second-order derivatives of the triple correlation in Eq. (10). Hence they kept the term $\frac{\nu}{2}\frac{\partial}{\partial x_k}[(\Delta_\xi \overline{u_s u_k u'_s})_0 + (\Delta_\xi \overline{u_s u'_s u'_k})_0]$ in the governing equation of ϵ_h. This point will be discussed in the next section.

The term $\nu\beta g_s[(\Delta_\xi \overline{\theta u'_s})_0 + (\Delta_\xi \overline{u_s \theta'})_0]$ in Eq. (14) accounts for the influence of buoyancy. It is interesting to notice that this term should vanish, provided that the assumption of local homogeneity for small scale of turbulence can be used, because similar to Eq. (13), for homogeneous turbulence the relation $\overline{\theta u'_s} = -\overline{u_s \theta'}$ is hold. The present DNS data for turbulent natural convection will be used to clarify this question.

3. Validation of the derived governing equation for ϵ_h

3.1. DNS data for turbulent Rayleigh-Bénard convection

Rayleigh-Bénard convection takes place in an infinite fluid layer which is bounded by two horizontal isothermal walls. The lower one, corresponding to $x_3 = 0$, is heated and the upper one ($x_3 = 1$) is cooled. The dimensionless numbers used for characterizing Rayleigh-Bénard convection are the Rayleigh number Ra, the Prandtl number Pr, the Grashof number $Gr = Ra/Pr$, the turbulence Reynolds number $Re_t = k^2/(\nu\epsilon_h)$ (where $k = \frac{1}{2}\overline{u_s u_s}$) and the turbulence Peclet number $Pe_t = Re_t Pr$, all given in Table 1. The simulations were performed with the TURBIT code (Grötzbach [11]). The parameters and grid data of the simulations, as well as the averaging procedure, are referred to [7, 8]. The DNS data used in this paper are available in [12].

The transport equation for ϵ can be simply expressed as [3]:

$$\frac{D}{Dt}\epsilon = P^1_\epsilon + P^2_\epsilon + P^3_\epsilon + P^4_\epsilon + P_{\epsilon b} + T_\epsilon + \Pi_\epsilon + D_\epsilon - \Upsilon, \tag{15}$$

where the terms P^1_ϵ - P^4_ϵ are production of ϵ. Other terms on the right hand of Eq.(15) represent buoyant production, turbulent diffusion ($T_\epsilon + \Pi_\epsilon$), viscous diffusion and viscous destruction of ϵ respectively. In the following treatment, all terms of the dissipation equation are normalized according to [8].

Pr	Ra	Gr	Re_t	Pe_t
0.7	630,000	8.9×10^5	154	107
0.7	381,000	5.4×10^5	109	76
0.006	24,000	4×10^6	2240	13
0.006	6,000	10^6	497	3

Table 1: Parameter of the simulation data. Values of Re_t and Pe_t are calculated at $x_3 = 0.5$.

3.2. Two-point velocity correlations of third rank

In order to determine the applicability of the local homogeneity for the derivatives of two-point velocity correlations of the third rank, i.e.

$$(\Delta_\xi \overline{u_s u_k u'_s})_0 + (\Delta_\xi \overline{u_s u'_s u'_k})_0 \simeq 0, \tag{16}$$

the following correlation is considered by means of the two-point correlation technique:

$$\nu \frac{\partial}{\partial x_k} \overline{\frac{\partial u_i}{\partial x_l} \frac{\partial u_k u_i}{\partial x_l}} = \frac{1}{4} \nu \frac{\partial}{\partial x_k} \Delta_x \overline{u_i u_k u_i} - \frac{1}{2} \nu \frac{\partial}{\partial x_k} [(\Delta_\xi \overline{u_i u_k u'_i})_0 + (\Delta_\xi \overline{u_i u'_k u'_i})_0]. \tag{17}$$

If the assumption of local homogeneity is fulfilled, the equation (17) can be approximated as follows:

$$\underbrace{\nu \frac{\partial}{\partial x_k} \overline{\frac{\partial u_i}{\partial x_l} \frac{\partial u_k u_i}{\partial x_l}}}_{Tl} \simeq \underbrace{\frac{1}{4} \nu \frac{\partial}{\partial x_k} \Delta_x \overline{u_i u_k u_i}}_{Tr}. \tag{18}$$

Figure 1 shows the comparison between the two terms Tl and Tr in Eq. (18). The results support the approximation given by Eq. (18) except for a few points in the vicinity of the wall. This means that the assumption of local homogeneity, Eq. (16), is satisfied in the flow field away from the wall.

3.3. Two-point temperature-velocity correlations

Using the two-point correlation technique, the buoyant production $P_{\epsilon b}$ of ϵ can be written as:

$$P_{\epsilon b} = -2\nu\beta g_i \overline{\frac{\partial \theta}{\partial x_l} \frac{\partial u_i}{\partial x_l}} = -\frac{1}{2}\nu\beta g_i \Delta_x \overline{\theta u_i} + \nu\beta g_i [(\Delta_\xi \overline{\theta u'_i})_0 + (\Delta_\xi \overline{u_i \theta'})_0]. \tag{19}$$

If the flow is locally homogeneous,

$$(\Delta_\xi \overline{\theta u'_i})_0 + (\Delta_\xi \overline{u_i \theta'})_0 \simeq 0, \tag{20}$$

then $P_{\epsilon b}$ can be approximated as

$$P_{\epsilon b} \simeq -\frac{1}{2}\nu\beta g_i \Delta_x \overline{\theta u_i}. \tag{21}$$

The DNS data of Rayleigh-Bénard convection with the largest turbulence Peclet number ($Pe_t = 107$, at $x = 0.5$ for air, $Ra = 630,000$) are chosen to test the equation (21). Figure 2 shows a comparison between the data for $P_{\epsilon b}$ obtained from the direct numerical simulation and the approximation Eq. (21). It is clear that the assumption of local homogeneity for the two-point temperature-velocity correlations is not applicable to the investigated turbulent natural convection. This is probably due to the low turbulence Peclet number. The turbulence Reynolds number for this data is, however, sufficiently large to ensure the statistical state of the small structure of the velocity field, at least far away from the wall, to be locally homogeneous.

3.4. Budget of the approximate equation for ϵ_h

Based on the above analysis, we will keep the derivatives of the two-point temperature-velocity correlation in Eq. (14). In contrast, the derivatives of the two-point velocity correlations of third rank, which are usually interpreted as the turbulent transport, are neglected in Eq. (14). Using the terms of the ϵ equation (15) from the simulation data, terms in Eq. (14) can be evaluated

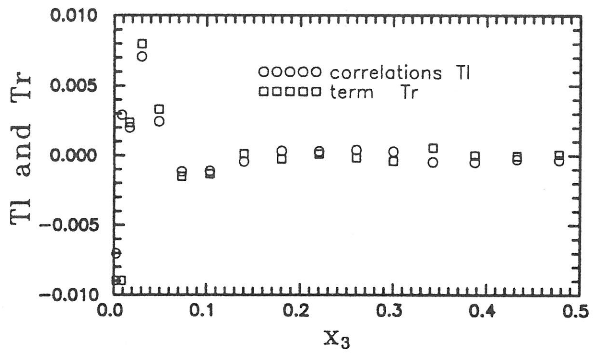

Figure 1: Distribution of correlation T_l (DNS data $Ra = 381,000$, $Pr = 0.71$) and T_r deduced from the DNS data.

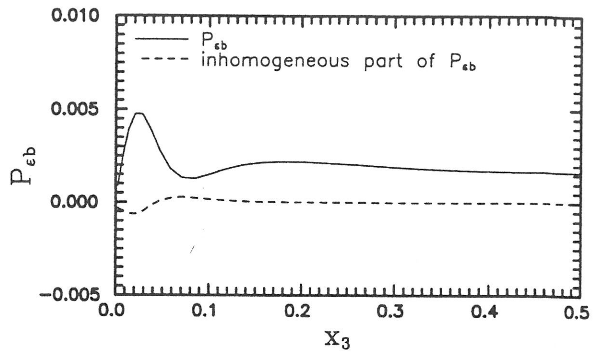

Figure 2: Distribution of $P_{\epsilon b}$ (DNS data $Ra = 630,000$, $Pr = 0.71$) and the approximation given by Eq. (21)

[3]. Figure 3 shows the budget of Eq. (14) computed from the DNS data of fully developed Rayleigh-Bénard convection for air ($Ra = 381,000$). The results given in Fig. 3 confirm that the derived equation (14) balances the data reasonably well across the flow field, especially away from the wall.

Neglecting the turbulent transport term in the current model for ϵ_h might be critical in isothermal turbulence shear flows. In the center of channel flow, e.g., this term is expected to balance the sink term in the ϵ equation, although in this region every term is small. Some further work is still required to clarify this point.

4. Modeling the approximate equation for ϵ_h

In Rayleigh-Bénard convection where no mean flow is present, the equations governing the dynamics of the turbulent dissipation rate for homogeneous part ϵ_h can be further simplified:

$$\frac{\partial \epsilon_h}{\partial t} \simeq \underbrace{\nu \beta g_s [(\Delta_\xi \overline{\theta u_s'})_0 + (\Delta_\xi \overline{u_s \theta'})_0]}_{T_b} \underbrace{-2\nu^2 (\Delta_\xi \Delta_\xi \overline{u_s u_s'})_0 - 2\nu \left(\Delta_\xi \frac{\partial}{\partial \xi_k} \overline{u_s u_k u_s'} \right)_0}_{T_s} + \frac{1}{2} \nu \Delta_x \epsilon_h . \tag{22}$$

The closure for the buoyant production term T_b and the sink term T_s in the equation (22) will be described below.

4.1. Model for the sink term T_s

The modeling of the sink term is always important, since it is one of the dominant terms in the dissipation rate equation. Using the invariant theory [9] and the DNS data of turbulent channel flow at low Reynolds number, Jovanović et al. [4] have investigated in detail the closure for the sink term T_s in Eq. (22). Some limiting values of T_s at different turbulent states were utilized to construct the model for T_s valid across the entire anisotropy invariant map (AIM), e.g. $(T_s)_{2C} = -0.0517\sqrt{20 Re_t} \epsilon_h^2/k$ for two-component turbulence, $(T_s)_{1C} = -1.4\epsilon_h^2/k$ for one-component turbulence, $(T_s)_{iso} = -7\frac{\sqrt{3}}{90} f_\epsilon \epsilon_h^2/k$ for isotropic turbulence and $(T_s)_{2C-iso} = -1.2\epsilon_h^2/k$ for two-component isotropic turbulence. Invariant functions were used to match the derived expressions for T_s. A similar method is used here.

Let us first locate the data of the Rayleigh-Bénard convection along the AIM (see Fig. 4). In order to compare the results, the data for forced convection of turbulent channel flow are also

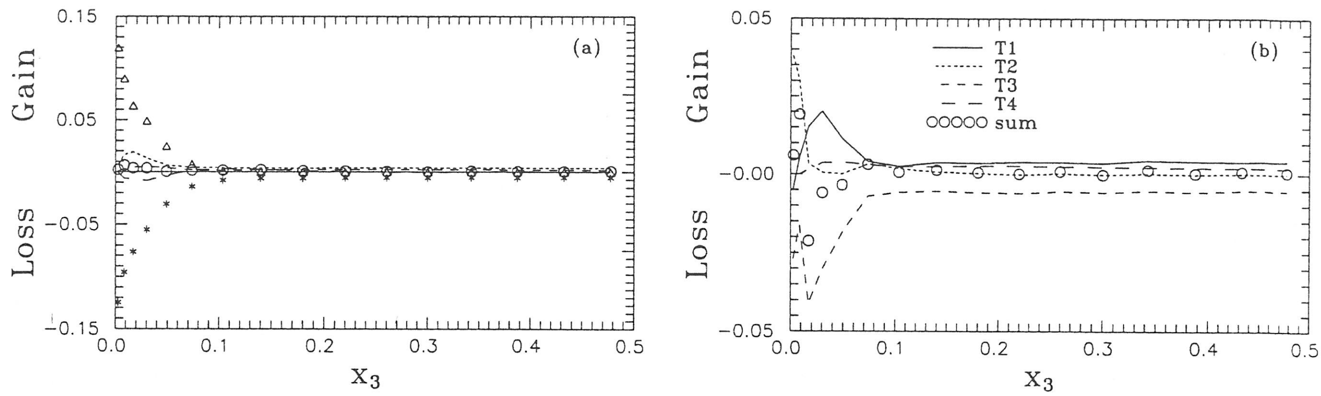

Figure 3: Terms in budget of Eq. (14) deduced from DNS data ($Ra = 381,000$, $Pr = 0.71$). (a) Budget of ϵ Eq. (15): ——, $P_\epsilon^1 + P_\epsilon^2 + P_\epsilon^3$; - - -, P_ϵ^4; — — —, $P_{\epsilon b}$; — — — —, $T_\epsilon + \Pi_\epsilon$; $\triangle$, D_ϵ; $*$, $-\Upsilon$; $\circ$, sum of all terms. (b) Budget of Eq. (14): ——, $T1 = -2\nu[\Delta_\xi(\partial/\partial\xi_k)(\overline{u_s u_k u_s'})]_0$; - - -, $T2 = 1/2\nu\Delta_x\epsilon_h$; — — —, $T3 = -2\nu^2(\Delta_\xi\Delta_\xi\overline{u_s u_s'})_0$; — — — —, $T4 = \nu\beta g_s[(\Delta_\xi\overline{\theta u_s'}) + (\Delta_\xi\overline{u_s\theta'})]$; $\circ$, sum of all terms.

plotted in Fig. 4. II and III denote the second and third invariants of the anisotropy tensor of the Reynolds stress, where $II = a_{ij}a_{ji}$, $III = a_{ij}a_{ik}a_{jk}$, $a_{ij} = 2\overline{u_i u_j}/k - 1/3\delta_{ij}$. For turbulent Rayleigh-Bénard convection ($a_{ij} = 0\,(i \neq j)$), only the diagonal components of the anisotropy tensor a_{ij} exist.

The level of anisotropy for Rayleigh-Bénard convection is quite smaller compared to the forced convection both in the near wall and in the buffer region. This is expected, since there is no mean strain in turbulent natural convection. Very close to the wall the invariants in the AIM for sodium reside on the boundary for two-component turbulence. The right hand boundary of the AIM is approched for $x_3 \to 0.5$. The invariants for air located however on the left hand boundary of the AIM. Considering the possible values of T_s for different turbulent states, the sink term may be written as:

$$T_s = -\psi\frac{\epsilon_h^2}{k}, \tag{23}$$

$$\psi = (1-F)\psi_{2C} + F\psi_{axi}, \tag{24}$$

$$\psi_{2C} = [0.02 + 0.03\exp(-Re_t)]\sqrt{20Re_t}, \tag{25}$$

$$\psi_{axi} = 1.4 + \{1 - 9[\frac{3}{4}(\frac{4}{3}III)^{2/3} - III]\}(\frac{7\sqrt{3}}{90}f_\epsilon - 1.4), \quad III > 0 \tag{26}$$

$$\psi_{axi} = 1.2 + \{1 - 9[\frac{3}{4}(\frac{4}{3}III)^{2/3} - III]\}(\frac{7\sqrt{3}}{90}f_\epsilon - 1.2), \quad III < 0 \tag{27}$$

$$f_\epsilon = \frac{54\sqrt{3}}{7}[1 - 0.222\exp(-0.336\sqrt{Re_t})], \tag{28}$$

where the decay function f_ϵ proposed by Coleman & Mansour [13] is used. The parameter F, which is defined as $F = J^2/[1 - 9[3/4(4/3 \mid III \mid)^{2/3} - III]$, where $J = 1 - 9(1/2II - III)$, equals unity when the stress field is isotropic but vanishes in two-component turbulence. It is found that the function F can match the stress fields for two-component turbulence and axisymmetric turbulence quite good for the present simulation data. Expressions (26) and (27) are identical to the model derived in [4]. In the equation for ψ_{2C} a modification factor $[0.02 + 0.03\exp(-Re_t)]$ deduced from the present DNS data, is added here to take account the

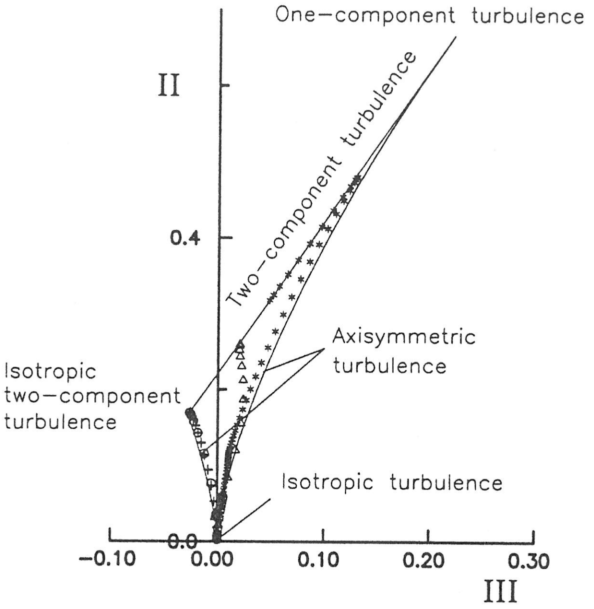

Figure 4: Anisotropy invariant map of Reynolds stress. $\triangle$, DNS $Ra = 6,000, Pr = 0.006$; $\circ$, $Ra = 381,000, Pr = 0.71$; $+$, $Ra = 630,000, Pr = 0.71$; $*$, DNS data (Kim, Moin and Moser, 1987)

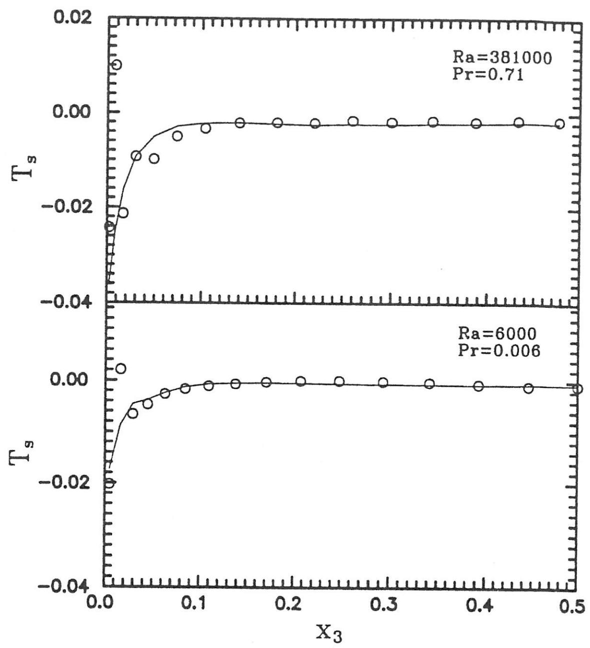

Figure 5: Distribution of sink term T_s. — —, Eq. (23); $\circ$, value evaluated from DNS data ($T_s = T1 + T3$, see Fig. 3).

influence of the turbulence Reynolds number.

The approximate form of T_s (Eq. 23) is tested by using the DNS data for Rayleigh-Bénard convection. The data presented by circle symbol in Fig. 5 are evaluated from the numerical simulation, where fourth order derivatives are required (see [3]). In the near wall region, where large gradients of the statistical quantities exist, the spatial discretization is not fine enough to calculate the high order derivatives with high accuracy. Therefore the circle data scatter somewhat. In general, the results in Fig. 5 show a good agreement between the evaluated and the predicted data.

4.2. Model for the buoyant production T_b

For the closure of T_b in Eq. (22) it is assumed that the derivatives with respect to ξ can be expressed in terms of a single-point second order correlation $\overline{\theta u_s}$. Using the scaling analysis proposed by Tennekes & Lumley the buoyant production in Eq. (22) can be approximated:

$$\nu\beta g_s[(\Delta_\xi\overline{\theta u'_s})_0 + (\Delta_\xi\overline{u_s\theta'})_0] \simeq C_{\epsilon 3}\frac{\epsilon_h}{k}G, \tag{29}$$

where $G = -\beta g_s\overline{u_s\theta}$. In the literature the coefficient $C_{\epsilon 3}$ is usually adopted as a constant or corrected by considering the influence of the flow form [14]. However the influence of Prandtl number should be also taken into account. It is appropriate to consider $C_{\epsilon 3}$ as a function of Prandtl number as well as the ratio of turbulent thermal and mechanical time scale $R = \tau_\theta/\tau = (\overline{\theta^2}/2\epsilon_\theta)/(k/\epsilon)$ (ϵ_θ is the dissipation of $\overline{\theta^2}$). The insight into this dependence can be clearly seen in the modelling of molecular destruction $\epsilon_{i\theta}$ ($\epsilon_{i\theta} = (\nu+\kappa)(\overline{\frac{\partial u_i}{\partial x_l}\frac{\partial \theta}{\partial x_l}})$) in the transport equation of heat fluxes $\overline{u_i\theta}$ (Shikazono & Kasagi[15]). The same derivatives $\overline{\frac{\partial u_i}{\partial x_l}\frac{\partial \theta}{\partial x_l}}$ appear in both terms $P_{\epsilon b}$ and $\epsilon_{i\theta}$.

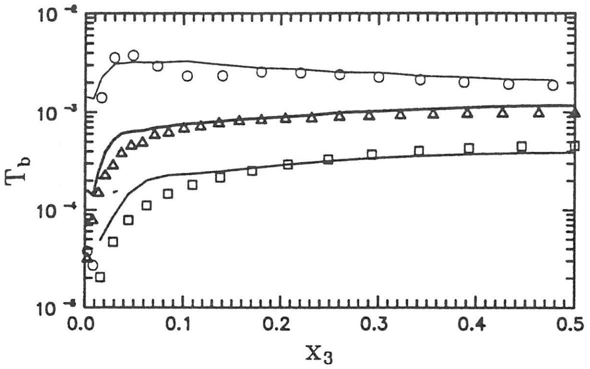

Figure 6: Distribution of the buoyant production T_b in Eq. (22).——, Eq. (23); ○, DNS data Ra = 381,000, Pr = 0.71; △, Ra = 24,000, Pr = 0.006; □, Ra = 6,000, Pr = 0.006.

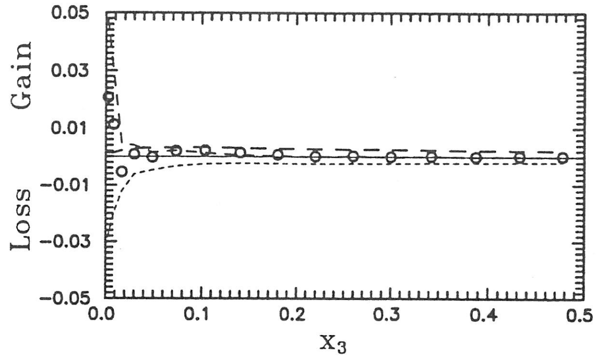

Figure 7: Terms in the budget Eq. (22) deduced from the DNS data Ra = 381,000, Pr = 0.71. - - -, T_s calculated from Eq. (23); — — —, $1/2\nu\Delta_x\epsilon_h$; ———, T_b calculated from Eq. (30); ○, sum of all terms.

The scale ratio R, as shown by Wörner and Grötzbach [8], is not an universal coefficient, but strongly depends on the Prandtl number and turbulence level. Introducing the heat flux anisotropy invariant $A_\theta = \overline{u_i\theta}\,\overline{u_i\theta}/\overline{\theta^2}\,\overline{u_iu_i}$ instead of R, one can avoid to solve a separate transport equation for ϵ_θ. This parameter (A_θ) was used, e.g. by Haroutunian & Launder [16] in the $\overline{u_i\theta}$ transport equation for free buoyant shear flows. However, it is difficult to find appropriate functions in terms of A_θ, which can account for the wall effects for the model of $P_{\epsilon b}$ or ϵ_θ. Based on the above considerations, the buoyant production of Eq. (22) can be proposed as follows:

$$T_b = \nu\beta g_s[(\Delta_\xi\overline{\theta u_s'})_0 + (\Delta_\xi\overline{u_s\theta'})_0] \simeq \left(\frac{Pr}{R}\right)^{0.7}\frac{\epsilon_h}{k}G, \tag{30}$$

where the term $(Pr/R)^{0.7}$ is obtained by fitting our DNS data. Above proposed form for T_b, as shown in Figure 6, agrees well with the DNS data except for the region near the wall. The small difference between the results from Eq. (30) and the DNS data in this region will not influence the budget of dissipation rate seriously, since the contribution of this term in Eq. (22) is smaller than that of the other two terms in the near wall region, which can be seen in Fig 7.

5. Conclusions

In this paper the closure of the equation for turbulent dissipation rate, which was developed earlier based on two-point correlation technique and invariant theory for thin shear turbulence, was investigated by means of the DNS data of turbulent Rayleigh-Bénard convection in air and sodium.

The assumption of local homogeneity for the small scale turbulence, which is used to simplify the analytically derived equation for the turbulent dissipation rate, was tested. It was found that for the derivatives of two-point velocity correlations of third rank the above mentioned assumption can be used, but not for the derivatives of the two-point velocity/temperature correlation of second rank.

It was found that the anisotropy of Reynolds stress for the turbulent natural convection is weaker than for the wall bounded turbulent shear flow. The derived closure for the sink term in ϵ_h equation shows a good agreement with the results from the DNS data. For turbulent natural convection the buoyant production term in the dissipation rate equation plays an important role. An appropriate form for this term was proposed by introducing the time scale ratio R, which can account for the influence of Prandtl number and turbulence level. The budget of the modeled equation for ϵ_h was analyzed. With the closure proposed in this study it is possible, except for a few points close to the wall, to balance the present DNS data to a reasonable degree of accuracy across the channel.

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