Necessary Conditions for Subdivision Surfaces

Hartmut Prautzsch   Ulrich Reif*

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Abstract

Subdivision surfaces are considered which consist of tri- or quadrilateral patches in a mostly regular arrangement with finitely many irregularities. A sharp estimate on the lowest possible degree of the patches is given. It depends on the smoothness and flexibility of the underlying subdivision scheme.

Keywords

Subdivision, piecewise polynomial surface, arbitrary topology, extraordinary point.

1 Introduction

This paper deals with subdivision algorithms acting on arbitrary 2D control nets as the schemes described in [Doo78, DS78, CC78, Loo87, Qu90, DGL90, Kob94, PR96]. Here we will generalize the degree estimate in [Rei94].

A surface $\mathcal{S}$ generated by one of the above mentioned algorithms consists of infinitely many either tri- or quadrilateral patches which are locally arranged as illustrated in Figure 1 schematically. Note that these patches may consist of smaller patches themselves as indicated by the broken lines.

For our analysis it suffices to consider such a local patch configuration. Note that it has one *extraordinary point* which does not belong to any of its surrounding patches. Further we assume that the patches are piecewise polynomial of degree d where d denotes the total or bidegree in case of tri- or quadrilateral patches, respectively.

If certain patches have simple C^k-joints and if the subdivision surface $\mathcal{S}$ viewed as the graph of a function has a Taylor polynomial at its extraordinary point of genuine degree r, then we are able to show that d is greater than or equal to $rk + r$, in general.

Moreover, in case of triangular patches and if the subdivision scheme can be described by finite masks, we can even show, that, in general d must be greater than or equal to $r[3k + 1]/2$ where $[x]$ denotes the largest integer not exceeding x.

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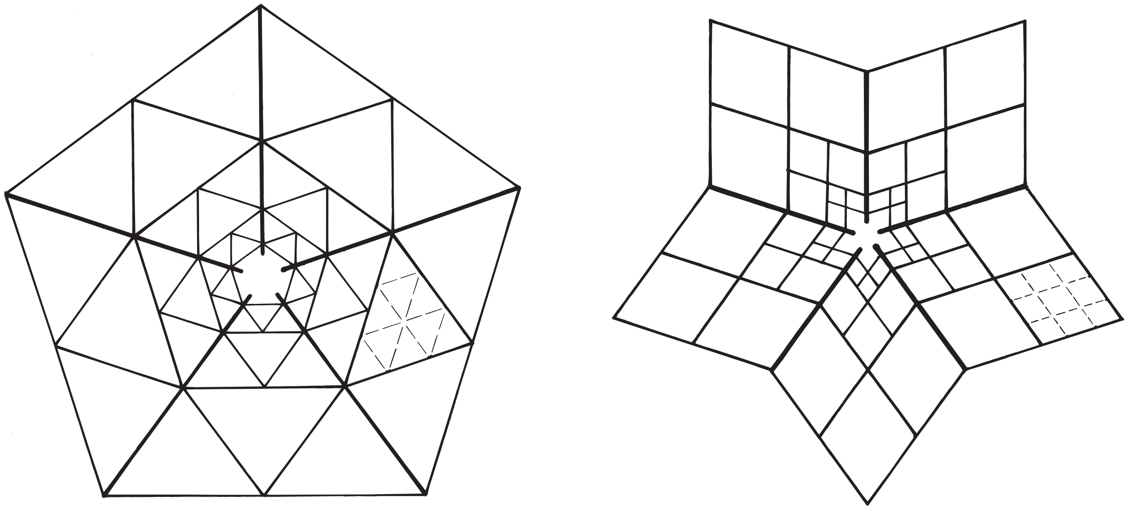

Figure 1: Subdivision surfaces with extraordinary point.

2 C^k-properties of the subdivision surface

To state the continuity properties of the surface $\mathcal{S}$ precisely, let $\mathbf{e}_1 := (1,0)$, $\mathbf{e} := (\cos \pi/3, \sin \pi/3)$, $\mathbf{e}_2 := (0,1)$, and

$$\Omega := \{a\mathbf{e}_1 + b\mathbf{e} | 1 \leq a+b \leq 2\}$$

or

$$\Omega := \{a\mathbf{e}_1 + b\mathbf{e}_2 | 1 \leq \max\{a,b\} \leq 2\}$$

in case of tri- or quadrilateral patches. These domains Ω are depicted in Figure 2.

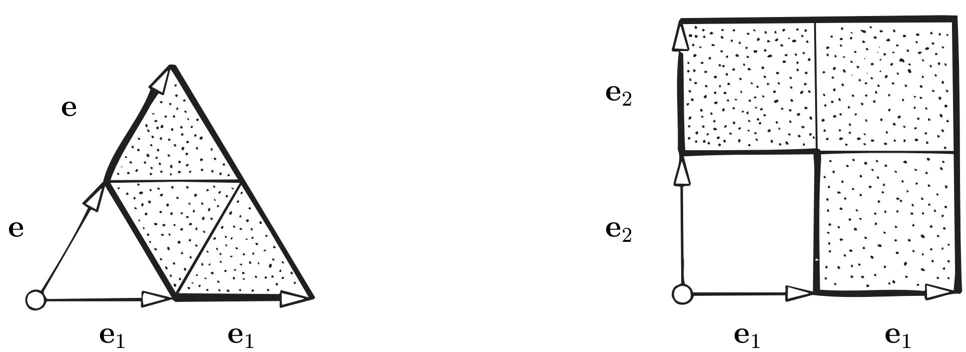

Figure 2: The domains Ω

Then we assume that the surface $\mathcal{S}$ can be parametrized by a map

$$\mathbf{s} : \mathrm{Z}_n \times \mathrm{N}_0 \times \Omega \to \mathrm{R}^3$$

consisting of piecewise polynomial patches $\mathbf{s}^i_m(\mathbf{u}) = \mathbf{s}(i, m, \mathbf{u})$ of degree $\leq d$.

Further let R be the rotation of R^2 around the origin with angle $\pi/3$ in the tri- and $\pi/2$ in the quadrilateral case, respectively. Then the double patches

$$\mathbf{s}^{i,i+1}_m(\mathbf{u}) := \begin{cases} \mathbf{s}^i_m(\mathbf{u}) & \text{for } \mathbf{u} \in \Omega \\ \mathbf{s}^{i+1}_m(R^{-1}\mathbf{u}) & \text{for } \mathbf{u} \in R\Omega \end{cases}$$

defined on $\Omega \cup R\Omega$, see Figure 3, are assumed to be k-times differentiable. Note that these double patches have overlaps according to

$$\mathbf{s}^{i,i+1}_m(R\mathbf{u}) = \mathbf{s}^{i-1,i}_m(\mathbf{u}) \ , \quad \mathbf{u} \in \Omega \ . \tag{2.1}$$

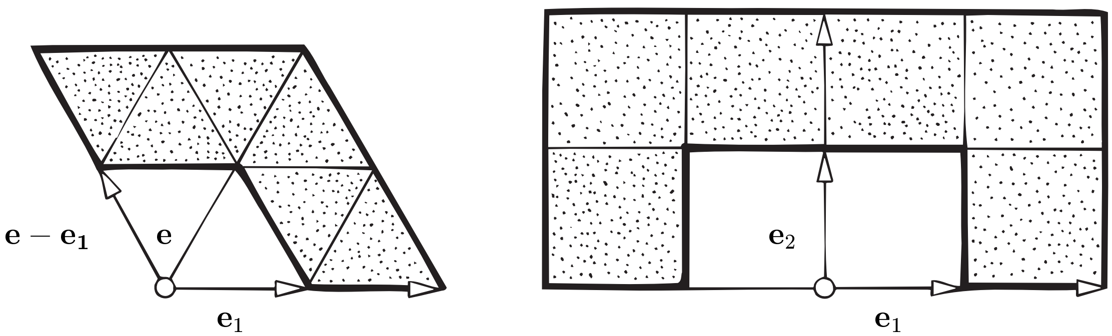

Figure 3: The domains of the double patches $\mathbf{s}^{i,i+1}_m$.

For fixed $i \in \mathrm{Z}_n$ the patches $\mathbf{s}^i_m$ form *segments*

$$\mathbf{s}^i : \cup_{m \in \mathrm{N}_0} 2^{-m}\Omega \to \mathrm{R}^3 \ , \quad \text{where } \mathbf{s}^i(\mathbf{u}) = \mathbf{s}^i_m(2^m\mathbf{u}) \quad \text{for } \mathbf{u} \in 2^{-m}\Omega \ .$$

Figure 4 shows the domain of these segments.

The map $\mathbf{s}$ is assumed to satisfy standard smoothness conditions, i.e. for all $i \in \mathrm{Z}_n$ the segments $\mathbf{s}^i$ and the double patches $\mathbf{s}^{i,i+1}_m$ are k-times continuously differentiable.

For any fixed m, the patches $\mathbf{s}^i_m$ form layers $\mathbf{s}_m(i, \mathbf{u}) = \mathbf{s}(i, m, \mathbf{u})$. Evidently the space of layers complying with the above smoothness requirements is linear and finite dimensional. For later use, we denote the underlying space of the coordinate functions by $\mathcal{L}_{d,k}$ and endow it with the maximum norm $\| \ \|_{d,k}$.

3 A necessary C^k-condition

In order to prepare us for the degree estimate let us assume, in this section that there is a linear map $\mathcal{U} : \mathcal{L}_{d,k} \to \mathcal{L}_{d,k}$ such that for all initial layers $\mathbf{s}_0$

$$\mathbf{s}_m = \mathcal{U}^m \mathbf{s}_0 \ ,$$

Figure 4: The domains of the segments $\mathbf{s}^i$.

where $\mathcal{U}^m$ is applied to each coordinate of $\mathbf{s}_0$ separately. Hence we assume that the underlying subdivision scheme acts *stationary* on the surface layers.

Now, let $1, \lambda, \mu, \ldots$ be the eigenvalues of $\mathcal{U}$ listed with all algebraic multiples and ordered by their modulus. For simplicity we suppose $\lambda = \mu \in (0, 1)$ and assume that there are two eigenvectors x and y in $\mathcal{L}_{d,k}$ associated with λ such that $\mathbf{x} = (x, y)$ is a regular parametrization of a surface in R^2 without self-intersections. Following [Rei95b] we call $\mathbf{x}$ the *characteristic map* of the underlying subdivision scheme. Figure 5 shows an example of such a characteristic map.

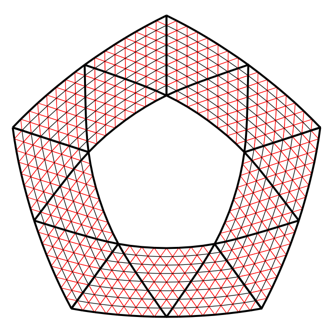

Figure 5: Characteristic map of the butterfly scheme for $n = 5$.

Applying the subdivision scheme to the characteristic map as initial surface ring we obtain the subdivision surface

$$\mathbf{x}(i, m, \mathbf{u}) : \mathrm{Z}_n \times \mathrm{N}_0 \times \Omega \to \mathrm{R}^2$$

where $\mathbf{x}(i, m, \mathbf{u}) = \lambda^m \mathbf{x}(i, 0, \mathbf{u})$. This surface can be used to reparametrize the given subdivision surface $\mathcal{S}$:

Let $\mathbf{r}(\mathbf{x}) : \mathrm{R}^2 \to \mathrm{R}^3$ be this new parametrization. Then we have

$$\mathbf{r}(\mathbf{x}(i, m, \mathbf{u})) = \mathbf{s}(i, m, \mathbf{u}) \ .$$

This parametrization $\mathbf{r}$ can easily be analyzed. Specializing the results in [Pra95] we obtain, e.g., the following theorem.

Theorem 3.1 *The map* $\mathbf{r}$ *is* k*-times differentiable if and only if all eigenvalues* ν *of* $\mathcal{U}$ *whose modulus lies in the closed interval* $[\lambda^k, \lambda]$

- *have equal geometric and algebraic multiplicites*
- *and there exists a* $j \in \{1, \ldots, k\}$ *such that* $\nu = \lambda^j$
- *and* $\mathcal{U}v = \nu v$ *implies* $v \in span\{x^\alpha y^\beta | \alpha + \beta = j\}$.

Remark The situation is similar for subdivision schemes acting stationary on control nets and if λ and μ are distinct or complex conjugate [Pra95, Rei95a].

For the following discussion we point out that the smoothness of the subdivision surfaces generated by $\mathcal{U}$ implies according to Theorem 3.1 a certain polynomial precision of the subdivision scheme $\mathcal{U}$. The situation is similar for non-stationary subdivision schemes. This polynomial precision then will lead to the anounced degree estimate.

4 General subdivision schemes

A subdivision scheme can be viewed very generally as an operator $\mathcal{S}$ mapping initial data $\mathbf{c}$ from some space $\mathcal{C}$ to subdivision surfaces $\mathbf{s} = \mathcal{S}(\mathbf{c})$. A subdivision scheme $\mathcal{S}$ is called

- *convergent*, if for all $\mathbf{s} := \mathcal{S}(\mathbf{c}), \mathbf{c} \in \mathcal{C}$, there exists a unique *limit point*

$$\mathbf{p} := \lim_{\mathbf{u} \to (0,0)} \mathbf{s}^i(\mathbf{u}) \ , \quad i \in \mathrm{Z}_n \ .$$

- r*-flexible*, if for any choice of Taylor coefficients $a_{p,q}$ there exists a $\mathbf{c}$ such that the limit point is the origin, and the coordinate functions x, y, z of $\mathbf{s} = \mathcal{S}(\mathbf{c})$ satisfy

$$z = \sum_{2 \le p+q \le r} a_{p,q} x^p y^q + o\left((|x| + |y|)^r\right) \quad \text{as} \quad (x, y) \to (0, 0) \ . \qquad (4.1)$$

Let us briefly discuss the concept of r-flexibility. First, note that the particular form (4.1) causes no loss of generality. If $\mathcal{S}$ is a C^1-scheme, then $\mathbf{s}$ can always be parametrized explicitly over the tangent plane at $\mathbf{p}$, locally. Choosing coordinates so that $\mathbf{p}$ is the origin and the xy-plane is the tangent plane there yields (4.1). Flexibility is a measure for the variety of shapes which can be modeled by a subdivision scheme. For instance, it is well known that the Catmull-Clark algorithm can be modified to be a C^2-scheme. Then, however, $\mathbf{p}$ is enforced to be a flat spot, i.e. $a_{p,q} = 0$ for $p+q = 2$, and the scheme is not 2-flexible. Or think of a C^3-scheme which is only 2-flexible. If $a_{2,0}a_{0,2} - a_{1,1}/4 \neq 0$ then the surface is locally either elliptic or hyperbolic, and the third order coefficients will not change the shape of the surface essentially. However, the lack of 3-flexibility will become apparent if a monkey saddle like $z = x(x^2 - y^2)$ is to be modeled, since its shape is substantially distinct from any quadratically parametrized surface.

5 Characteristic maps for general schemes

Definition 5.1 *Let* $\mathbf{s}$ *be a regular surface generated by an r-flexible subdivision scheme such that the limit point is the origin and the xy-plane is the tangent plane, there. With* $\sigma_m := \|x_m^2 + y_m^2\|_{2d,k}$ *define the sequence*

$$\mathbf{x}_m := \left(x_m/\sqrt{\sigma_m}, y_m/\sqrt{\sigma_m}\right) \ .$$

Then $\mathbf{x}$ *is a* characteristic map *of* $\mathbf{s}$ *if there exists a subsequence* $\mathbf{x}_{\kappa(m)}$ *such that*

$$\mathbf{x} = \lim_{m\to\infty} \mathbf{x}_{\kappa(m)} \ .$$

Note that each subdivision surface $\mathbf{s}$ has at least one characteristic map $\mathbf{x} \in (L_{d,k})^2$ since the coordinates of $\mathbf{x}_m$ are bounded in $L_{d,k}$, which is finite dimensional and closed with respect to $\|\cdot\|_{d,k}$. Further, $\mathbf{x} \not\equiv 0$ since the sum of the squares of its components has norm 1.

Lemma 5.1 *Let* $\mathbf{s}$ *be a regular surface, then* $\mathbf{x}$ *is* ν*-periodic, i.e.*

$$\mathbf{x}(\cdot, i) \equiv \mathbf{x}(\cdot, i+\nu) \text{ for all } i \in \mathrm{Z}_n \ ,$$

if and only if $\nu = 0 \mod n$.

Proof Let $\mathbf{x}$ be ν-periodic. Since $\mathbf{x}$ is n-periodic by definition, it is also μ-periodic with μ the greatest common divisor of n and μ. Let τ_m and $\tau := \lim_{m\to\infty} \tau_{\kappa(m)}$ be the outer boundaries of $\mathbf{x}_m$ and $\mathbf{x}$, respectively. Then τ is μ-periodic and consequently its winding number is $\#\tau = n/\mu$. Hence, $\#\tau_{\kappa(m)} = n/\mu$ for m large enough by continuity.

Now assume that $\nu \neq 0 \mod n$, then $\mu < n$ and $\#\tau_{\kappa(m)} \geq 2$. This implies that $\tau_{\kappa(m)}$ has self-intersections, i.e. there exist sequences $s^1_{\kappa(m)} \neq s^2_{\kappa(m)}$ such that

$$\tau_{\kappa(m)}(s^1_{\kappa(m)}) = \tau_{\kappa(m)}(s^2_{\kappa(m)})$$

and consequently

$$\begin{aligned} x_{\kappa(m)}(s^1_{\kappa(m)}) &= x_{\kappa(m)}(s^2_{\kappa(m)}) \\ y_{\kappa(m)}(s^1_{\kappa(m)}) &= y_{\kappa(m)}(s^2_{\kappa(m)}) \ . \end{aligned}$$

This implies that the projection of $\mathbf{s}$ to the tangent plane is not injective near the origin contradicting the regularity assumption on $\mathbf{s}$. □

The following theorem relates the minimal possible degree and the smoothness order of a characteristic map.

Theorem 5.1 *Let* $\mathbf{x}$ *be a characteristic map of a regular subdivision surface* $\mathbf{s}$ *and* d' *be the degree of* $\mathbf{x}$*. Then* $d' \geq (k+1)$ *if* $n \notin \{3,6\}$ *in the triangular or* $n \neq 4$ *in the quadrilateral case.*

Proof Assume that the degree d' is less than the specified bounds. Then, evidently the double patches $\mathbf{x}^{i,i+1}$ of $\mathbf{x}$ are not piecewise polynomial but in fact simply polynomial.

Identify the double patches $\mathbf{x}^{i,i+1}$ with their polynomial extensions to R^2. Then iterating (2.1) yields

$$\mathbf{x}^{i,i+1} \circ R^\nu = \mathbf{x}^{i-\nu,i-\nu+1} \ , \quad \nu \in \mathrm{N} \ .$$

In the triangular case, $R^6 = \mathrm{Id}$. Hence, $\mathbf{x}$ is 6-periodic contradicting Lemma 5.1 for $n \notin \{3,6\}$. In the quadrilateral case, $R^4 = \mathrm{Id}$. Hence, $\mathbf{x}$ is 4-periodic contradicting Lemma 5.1 for $n \neq 4$. □

In the triangular case we can improve the estimate under further conditions which are met by all subdivision schemes described by finite masks.

Let Δ be the triangle $\mathbf{0}\mathbf{e}_1\mathbf{e}$ and D_1, D_2, D_3 be the directional derivatives with respect to $\mathbf{e}_1$, $\mathbf{e} - \mathbf{e}_1$, and $\mathbf{e}$, respectively. Then consider the space B^d_k of all C^k-functions with compact support in R^2 which are polynomial of degree d over each triangle $\mathbf{a}_1\mathbf{e}_1 + \mathbf{a}_2\mathbf{e}_2 + \Delta$ and $\mathbf{a}_1\mathbf{e}_1 + \mathbf{a}_2\mathbf{e}_2 + R\Delta$, where $\mathbf{a}_1, \mathbf{a}_2 \in \mathrm{Z}^2$.

Since B^d_k has no constants besides the zero function the composed operators

$$D_1 D_2 : B^d_k \to B^{d-2}_{k-2} \quad \text{and} \quad D_1 D_2 D_3 : B^d_k \to B^{d-3}_{k-2}$$

are one-to-one. As a consequence of this and the fact $B^0_0 = \{0\}$ we obtain the following lemma

Lemma 5.2 *For $d \le (3k+1)/2$ one has $B_k^d = \{0\}$.*

Hence Theorem 5.1 can be extended as follows:

Theorem 5.2 *Let $\mathbf{x}$ be a characteristic map of a regular subdivision surface with triangular patches. If each patch of $\mathbf{x}$ lies on a function in $(B_k^d)^2$, then*

$$\text{degree } \mathbf{x} \ge 3k/2 + 1 \ .$$

6 The degree estimate

Now we are prepared for the announced degree estimate.

Theorem 6.1 *If $\mathcal{S}$ is a convergent r-flexible subdivision scheme and r is even, then*

i) $d \ge r(3[k]/2+1)$ if all surfaces $\mathcal{S}(\mathbf{c})$ consist only of patches in $(B_k^d)^3$ and $n \notin \{3,6\}$,

ii) $d \ge r(k+1)$ if $n \notin \{3,6\}$ in the triangular or $n \ne 4$ in the quadrilateral case.

Proof Let the initial data $\mathbf{c}$ be chosen such that $\mathbf{s} := S(\mathbf{c})$ satisfies

$$z = (x^r + y^r) + o\left((|x| + |y|)^r\right) \ .$$

With σ_m as in (5.1) let

$$\begin{aligned} \tilde{x}_m &:= x_m/\sqrt{\sigma_m} \\ \tilde{y}_m &:= y_m/\sqrt{\sigma_m} \\ \tilde{z}_m &:= z_m/\sigma_m^{r/2} \ . \end{aligned}$$

Then

$$\tilde{z}_m = (\tilde{x}_m^r + \tilde{y}_m^r) + o\,(1) \quad \text{as} \quad \sigma_m \to 0 \ .$$

Since $\lim_{m\to\infty} \sigma_m = 0$, the latter equation is equivalent to

$$\tilde{z}_m = (\tilde{x}_m^r + \tilde{y}_m^r) + o\,(1) \quad \text{as} \quad m \to \infty \ .$$

Choose a convergent subsequence

$$\hat{x} := \lim_{m\to\infty} \tilde{x}_{\kappa(m)} \ , \quad \hat{y} := \lim_{m\to\infty} \tilde{y}_{\kappa(m)}$$

then $\mathbf{x} = (\hat{x}, \hat{y})$ is a characteristic map of $\mathbf{s}$. By continuity, we obtain from (6)

$$\hat{z} := \lim_{m\to\infty} \tilde{z}_{\kappa(m)} = (\hat{x}^r + \hat{y}^r) \ .$$

Denote the degree of $\hat{x}, \hat{y}, \hat{z}$ by d_x, d_y, d_z, respectively, then

$$d \ge d_z = r \max(d_x, d_y) \ ,$$

and a comparison with Theorem 5.1 completes the proof. □

Clearly, Theorem 6.1 gives also an estimate if r is odd since r-flexibility implies $(r-1)$-flexibility. A sharper estimate for odd r can be obtained for subdivision schemes with an r-flexible z-coordinate, i.e. schemes which can for all Taylor coefficients generate surfaces satisfying (4.1) with a common characteristic map.

Theorem 6.2 *If $\mathcal{S}$ is a convergent subdivision scheme with an r-flexible z-coordinate, then*

i) $d \geq r(3[k]/2+1)$ *if all surfaces $\mathcal{S}(\mathbf{c})$ consist only of patches in $(B_k^d)^3$ and $n \notin \{3,6\}$,*

ii) $d \geq r(k+1)$ *if $n \notin \{3,6\}$ in the triangular or $n \neq 4$ in the quadrilateral case.*

Proof Let $x, y, \hat{x}$ and $\hat{y}$ be as in the proof of Theorem 6.1. Without loss of generality we assume that $\hat{x}$ is of equal or higher degree than $\hat{y}$. (In the triangular case we consider the total degree and in the quadrilateral case the degree in one variable.) Now we choose $z = x^r + o((|x|+|y|)^r)$ and define

$$\tilde{z}_m := z_m/\sigma_m^{r/2}$$
$$\tilde{z}_m := \lim_{m\to\infty} \tilde{z}_{\kappa(m)} = \hat{x}^r$$

as before. Then the proof can be completed as for Theorem 6.1 □

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