Multivariate Splines with Convex B-Patch Control Nets are Convex

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Abstract: In this paper results from a forthcoming paper are presented concerning the convexity of multivariate spline functions built from B-patches. Conditions are given under which it is possible to define a control net for such spline functions. The control net is understood as a piecewise linear function. If it is convex, then so is the underlying spline.

Keywords: multivariate splines, B-patches, convexity, control nets, Greville-abscissae

1 Introduction

For the Bézier representation of a bivariate polynomial over some triangle $\Delta$ it is well-known that the convexity of the Bézier net implies the convexity of the polynomial over the triangle $\Delta$. This fact was first proved by Chang and Davis [1984] and later generalized to multivariate polynomials and their Bézier representations over a simplex [DM88, Bes89, Pra95]. Here it is shown that this property is, more generally, even shared by multivariate polynomials and their B-patch representations. Moreover it is also possible to extend the proof to multivariate spline functions and their B-patch control nets.

2 Multivariate B-splines

This paper is based on the B-splines constructed by Dahmen, Micchelli and Seidel [1992] from B-patches. To begin with let us recall the relevant properties and thereby introduce the notation used in this paper:

For any set of knots $u_0, \ldots, u_k \in \mathbb{R}^s$ or an $s \times k+1$ matrix $[u_0 \ldots u_k]$ the simplex spline $M(x|u_0 \ldots u_k)$ is defined as the solution of the functional equation

$$\int_{\mathbb{R}^s} f(x) M(x|u_0 \ldots u_k) dx = k! \int_\sigma f([u_0 \ldots u_k]t) dt$$

for all continuous functions $f(x)$ where

$$\sigma = \{ t \in \mathbb{R}^{k+1} | 0 \leq t, |t| = 1 \} , \quad |t| = \text{sum of all coordinates of } t$$

denotes the standard $k$-simplex.

Thus the above normalization implies that

$$\int_{\mathbb{R}^s} M(x|u_0 \ldots u_k) dx = 1.$$
Now for any \( s + 1 \) knot clusters \( u^\alpha \), \( \alpha = 0, \ldots, s, \beta = 0, \ldots, n \), consider the simplices \( \sigma_i \) with vertices \( u^0_{i_0}, \ldots, u^s_{i_s} \) where \( i = (i_0, \ldots, i_s) \in \mathbb{N}_{0}^{s+1} \) and \( |i| = n \). Then the corresponding splines

\[
B_i(x) = \frac{\text{vol}_{s} \sigma_i}{(n+s)!} M(x|u^0_{i_0} \ldots u^0_{i_s} \ldots u^s_{i_s} \ldots u^s_{i_s}),
\]

are the \textbf{multivariate B-splines} which were introduced in [DMS92] with the name B-weights.

Throughout the paper we will assume that bold indices \( i, j, \ldots \) are in \( \mathbb{N}_0^{s+1} \) and that the intersection \( \Omega \) of all simplices \( \sigma_i, |i| \leq n \) is non-empty. Then one has the following crucial property:

**Theorem 2.1** Let \( p(x) \) be any \( s \)-variate polynomial of total degree \( n \) and let \( p(x_1 \ldots x_n) \) be the unique symmetric multiaffine polynomial with the diagonal property \( p(x \ldots x) = p(x) \). Then for all \( x \in \Omega \) one has

\[
p(x) = \sum_{|i| = n} p[u^0_{i_0} \ldots u^0_{i_{n-1}} \ldots u^0_{i_{n-1}} \ldots u^s_{i_{n-1}}]B_i(x).
\]

For the proof one can use the properties of the so-called polar form \( p(x_1 \ldots x_n) \) and the recurrence relation of simplex splines to evaluate the left and respectively the right hand side of the equation recursively. A comparison then reveals the identity above.

A dimension count further shows that the \( \binom{n+s}{s} \) B-splines \( B_i \) are linearly independent \( \text{over} \ \Omega \).

**Remark 2.2** Theorem 2.1 also shows that for \( s = 1 \) the \( B_i(x) \) are the common \textit{univariate} B-splines. Further if \( u^0_0 = \cdots = u^0_n \) for all \( \alpha \), then the \( B_i(x) \) are the truncated Bernstein polynomials over \( \Omega \).

### 3 Control nets

In order to describe the control net of a polynomial

\[
p(x) = \sum c_i B_i(x), \quad x \in \Omega,
\]

we need the B-spline representation of \( x \). From Theorem 2.1 we obtain

\[
x = \sum x_i B_i(x), \quad \text{where } x_i = \frac{1}{n} \sum_{\alpha=0}^{s} \sum_{\beta=0}^{i_{\alpha}-1} u^\alpha_{\beta}.
\]

In particular, if \( s = 1 \), then the \( x_i \) are the so-called \textbf{Greville abscissa} and if

\[
u^\alpha_{\beta} = u^\alpha \quad \text{for all } \alpha \text{ and } \beta
\]

then the \( x_i \) lie on a regular grid, i.e.

\[
x_i = (i_0 u^0 + \cdots + i_s u^s)/n.
\]

Next we will construct a triangulation whose vertices are the abscissae \( x_i \) and define the control net of \( p \) as the piecewise linear function \( c(x) \) which is linear over each simplex of this triangulation and which interpolates the \( c_i \) at the \( x_i \).
If the $x_i$ are not too far away from the vertices of a regular grid, then we can obtain a triangulation from a triangulation of the regular grid. Therefore we will first describe a triangulation for the case $u_\beta^\alpha = u^\alpha$. Then we change the triangulation by moving the $u_\beta^\alpha$ independently from each into general positions and present conditions under which the triangulation remains a triangulation with disjoint simplices.

For the construction of a Bézier net Dahmen and Micchelli [1988] used a triangulation due to Allgower and Georg:

Let $\pi$ be the simplex $u^0 \ldots u^s$ and $\rho$ the subsimplex whose vertices have the barycentric coordinates

$$
\frac{1}{n} \begin{bmatrix} n \\ 0 \\ 0 \\ \vdots \\ 0 \\ n \\ 0 \\ \vdots \\ 0 \\ n \end{bmatrix}, \quad \frac{1}{n-1} \begin{bmatrix} n-1 \\ 1 \\ \vdots \\ 1 \\ n \end{bmatrix}, \quad \frac{1}{n-1} \begin{bmatrix} n-1 \\ 0 \\ \vdots \\ 0 \\ n \end{bmatrix}
$$

with respect to $\pi$. Let $a_0, \ldots, a_s$ be these vertices in any arbitrarily fixed order. Counting indices modulo $s+1$ the vertex $a_0$ and the ordered sequence of vectors

$$v_0 = a_1 - a_0, \ldots, v_s = a_{s+1} - a_s$$

describe a simple closed path through all vertices of $\rho$. Note that $a_i$ and $v_i, \ldots, v_{i+s}$ describe the same path. Now if any two successive vectors, say $v_i$ and $v_{i+1}$ are interchanged, then $a_i; v_{i+1}, \ldots, v_{i+s+1}$ describes a path around a simplex $\rho'$ which shares an $s-1$-dimensional face with $\rho$. By further transpositions of successive vectors one gets paths around successively adjacent simplices. All the simplices obtained in this way form a triangulation of the entire space $\mathbb{R}^s$. This triangulation is also formed by all hyperplanes spanned by the knot $u_0$ and any $s-1$ vectors out of $\{v_0, \ldots, v_s\}$ and translates of these hyperplanes by integer multiples of the $v_i$. Thus this triangulation respects the simplex $\pi$ and can be restricted to $\pi$.

**Remark 3.1** If the $a_i$ denote the vertices of $\sigma$ in a different order, then the construction above results in a different triangulation.

### 4 Conditions on the knot clusters

Assume that all knots in every cluster coincide, i.e. $u_\beta^\alpha = u^\alpha$ for all $\alpha$ and $\beta$. Then the above triangulation has the following property:

**Lemma 4.1** The union of all simplices with vertex $a_0$ forms the set of all points

$$x = a_0 + \mu_0 v_0 + \cdots + \mu_s v_s , \quad \text{where } \mu_i \in [0, 1] .$$

**Proof**

Let $\mu_0 \geq \cdots \geq \mu_s$. Then since $v_0 + \cdots + v_s = 0$, we can write $x = a_0 + \mu_0 v_0 + \cdots + \mu_s v_s$ as

$$x = (1 - \mu_0 + \mu_s)(a_0 + v_0 + \cdots + v_{s-1} + v_s)$$

$$+ (\mu_{s-1} - \mu_s)(a_0 + v_0 + \cdots + v_{s-1})$$

$$\vdots$$

$$+ (\mu_0 - \mu_1)(a_0 + v_0)$$
which is a convex combination of the vertices of the simplex given by the loop $a_0, v_0 \ldots v_s$. Similarly any ordering of the $u_i$ corresponds to a loop $a_0, w_0 \ldots w_s$ where $(w_0, \ldots, w_s)$ is a permutation of $(v_0, \ldots, v_s)$ and vice versa. This completes the proof since all these loops describe all the simplices with vertex $a_i$. ■

Now we move the $u^\alpha \beta$ independently from each other into general positions. This will also change the positions of the $x_i$ and the shape and positions of the simplices of the triangulation given in Section 3. The new triangulation is still feasible under the following mild restrictions on the knot positions:

**Theorem 4.2** If for all $\alpha = 0, \ldots, s$ and $\beta = 0, \ldots, n$

$$u^\alpha \beta \in u^\alpha + [v_0 \ldots v_s][0, 1/2)^{s+1}$$

then any two simplices of the new triangulation have disjoint interiors.

We omit the full proof here and derive only the crucial property on which the proof is based:

$$x_i = \frac{1}{n} \sum_{\alpha=0}^{s} \sum_{\beta=0}^{i=0} u^\alpha \beta \in \frac{1}{n} \sum_{\alpha=0}^{s} \sum_{\beta=0}^{i=0} u^\alpha + [v_0 \ldots v_s][0, 1/2)^{s+1}$$

Thus different $x_i$ lie in disjoint convex regions.

5 B-patches with convex control nets

Consider the control net of a single B-patch. It is a piecewise linear function defined over some triangulation with the vertices $x_i$. In general, this triangulation does not form a convex domain for the control net. Therefore we need to explain what is meant by a convex net:

First let $q(x) = [x q(x)]$ be the graph of a quadratic polynomial $q(x)$ and let $c_i \in \mathbb{R}^{s+1}, \ i = 2$, be its B-spline control points with respect to the knots $u^\alpha \beta$, $\alpha = 0, \ldots, s; \beta = 0, 1, 2$, and further let $b_i$ be the Bézier points of $q(x)$ over the simplex $u^0 \ldots u^0$. Then it follows from Theorem 2.1 that

$$c_i = b_i \quad \text{for all } i \leq (1, \ldots, 1)$$

and furthermore that the points $b_i$ and the points $c_i$, for $i = e_i + e_j$, $i$ fixed, $j = 0, \ldots, s$, span the same plane. Thus we have the following property:

**Lemma 5.1** The Bézier and the B-spline control nets of the quadratic polynomial $q(x)$ above are identical over the intersection of their domains.

Hence we say that the B-spline control net of the quadratic polynomial $p(x)$ is convex if the associated Bézier net of $p(x)$ is convex.
Next consider again a polynomial of degree $n$
\[
p(x) = \sum_{i=0}^{n} c_i B_i(x)
\]
given by its B-spline representation over the knot clusters $u^0_{\alpha}, \alpha = 0, \ldots, s; \beta = 0, \ldots, n$. Let $p[x_1 \ldots x_n]$ be the polar form of $p(x)$. Then the quadratic polynomials
\[
p_i(x) = p[x \ x \ u^0_{\alpha} \ldots u^0_{\alpha-1} \ldots u^0_{\beta} \ldots u^0_{\beta-1}], \quad |i| = n - 2,
\]
have the B-spline representations
\[
p_i(x) = \sum_{j=2}^{n} c_{i+j} B_{j}(x)
\]
over the knots $u^0_{\alpha+\beta}, \alpha = 0, \ldots, s; \beta = 0, 1, 2$. Now we can state the main result of this section.

**Theorem 5.2** If the control nets of all quadratic polynomials $p_i(x), |i| = n - 2$, are convex, then $p(x)$ is convex over the intersection $\Omega$ of all simplices $u^0_{\alpha} \ldots u^0_{\beta}, |i| \leq n$.

Let us sketch the proof: Let $D^2_{\nu}f(x)$ be the second derivative of the function $f$ with respect to the direction $\nu$. Then one can use, e.g., the multidimensional analog of Proposition 8.2 in [Ram87] to derive
\[
D^2_{\nu}p(x) = \frac{n(n-1)}{2} \sum_{|i|=n-2} (D^2_{\nu}p_i)B_i(x).
\]
Since the $p_i$ have a convex Bézier net, they are convex functions, see e.g., [DM88]. Hence the second directional derivatives $D^2_{\nu}p_i$ are non-negative which implies that $D^2_{\nu}p(x)$ is non-negative and thus the convexity of $p(x)$ over $\Omega$.

6 Splines with convex control nets

The results above for a single B-patch can be extended to splines:
Let $u^a, a \in Z$, be the vertices of some triangulation $T$ covering the entire space $\mathbb{R}^s$. Here we think of $T$ as a subset of $Z^{s+1}$ such that the simplices $u^{a_0} \ldots u^{a_s}, a = (a_0, \ldots, a_s) \in T$ form the triangulation. In the following we will always assume that $T$ contains each simplex only once, i.e., for any $a \in T$ there is no other permutation of $a$ in $T$. Further let $u^0_{\beta}, \beta = 0, \ldots, n$ be associated knot clusters and assume that the intersections $\Omega_a$ of all simplices $u^{a_0}_0 \ldots u^{a_s}_{i_s}, |i| \leq n$, are non-empty for all $a \in T$. Then consider the spline
\[
s(x) = \sum_{a \in T} \sum_{|i|=n} c_i^a B_i^a(x)
\]
where $B_i^a$ is the B-spline over the knots $u^0_{\beta}, \alpha = a_0, \ldots, a_s; \beta = 0, \ldots, i_s$. In order to define the control net of $s(x)$ as a piecewise linear function we need the abscissae
\[
x_i^a = \frac{1}{n} \sum_{a=a_0, \ldots, a_s} \sum_{\beta=0}^{i_s} u^0_{\beta}.
\]
Then for each \( a \in T \) we construct a triangulation having the abscissae \( x_i^a \) as vertices as described in Section 4 using the loops

\[
\nu_0^a = u^{a_1} - u^{a_2}, \ldots, \nu_{s}^a = u_{a_0} - u_s^a.
\]

In order to obtain a correct triangulation of all \( x_i^a, a \in T \), we need to restrict the positions of the knots. Such a condition is given by the following extension of Theorem 4.2:

**Theorem 6.1** Let \( \Omega_\alpha \) be the intersections

\[
\Omega_\alpha = \cap \{ [\nu_0^a \ldots \nu_{s}^a] | 0,1/2)^{s+1} \mid a \in T, \ \alpha \text{ is a coordinate of } a \}
\]

and for all \( \alpha \in \mathcal{L} \) and \( \beta = 0, \ldots, n \) let \( u^\beta_\alpha \in u^\alpha + \Omega_\alpha \). Then any two simplices of the triangulation of the \( x_i^a, a \in T, |i| = n \), have disjoint interiors.

Theorem 6.1 enables us to define the control net of \( s(x) \) as the piecewise linear function which is composed of the control nets of the patches

\[
s_a(x) = \sum_{|i|=n} c_i^a B_i^a(x), \quad x \in \Omega_a.
\]

Note that the control nets of the patches over the sets \( \Omega_a \) are always continuous, but the entire control net of \( s(x) \) is continuous only if \( c_i^a = c_i^b \) whenever \( x_i^a = x_i^b \). Now, for this control net of \( s(x) \) we can state the main result presented in this paper:

**Theorem 6.2** Let the control net of \( s(x) \) be continuous and such that the subnets for all patches \( s_a(x), x \in \Omega_a \), satisfy the conditions of Theorem 5.2. Then the spline function \( s(x) \) is convex for all \( x \in \mathbb{R} \).
References


