Freeform splines[1]

Hartmut Prautzsch[2]

Universität Karlsruhe, IBDS, D-76128 Karlsruhe, Germany

Abstract

In this paper we show how one can construct a regularly parametrized G^2-spline surface of arbitrary topology from one control net. The surface is piecewise bisextic around extraordinary points and bicubic elsewhere. Furthermore, the bisextic representation of the surface allows for subdivision algorithms. The underlying ideas can also be used to construct subdividable G^k-splines of bidegree $2k+2$.

Keywords: G^k-splines, n-sided patches, n-sided holes, subdivision.

1 Introduction

Constructing C^k-spline surfaces of arbitrary shape by existing methods is quite cumbersome. Moreover, the degree of such surfaces is relatively high, namely $O(k^2)$ [Mögerle '92, Hahn '89], and it is not clear how to present such surfaces by a single spline control net as one can do it with tensor product or box spline surfaces.

The purpose of this paper is to introduce a new methodology enabling us to construct G^k-surfaces with **regular** parametrizations of low degree from one single spline control net. In order to make the underlying ideas more easily accessible, I restrict myself here, exemplary, to the currently most interesting case of G^2-splines. The general case plus further results and details will be described in a second paper.

2 The control net

To start with consider a control net in $\mathbb{R}^3$ with quadrilateral meshes such that extraordinary vertices $\bullet$ are sufficiently isolated by regular vertices $\circ$. We will view each regular subnet of at least 4×4 points as the control net of a uniform bicubic C^2 tensor product spline surface. Hence, the total control net defines a C^2-surface where each extraordinary vertex corresponds to a

[1]Presented at Oberwolfach, June 5, 1995

[2]e-mail: prau@ira.uka.de

non-four sided hole as illustrated in Figure 1. Note that an n-sided hole is surrounded by a ring $\mathbf{r}$ of $3n$ patches $\mathbf{r}_1, \ldots, \mathbf{r}_{3n}$, where $\mathbf{r}_i(u,v) : [0,1]^2 \to \mathbb{R}^3$.

Our idea is to represent these bicubic rings as bisextic splines and to change the inner boundary such that these rings can be filled smoothly with a surface of total degree 2.

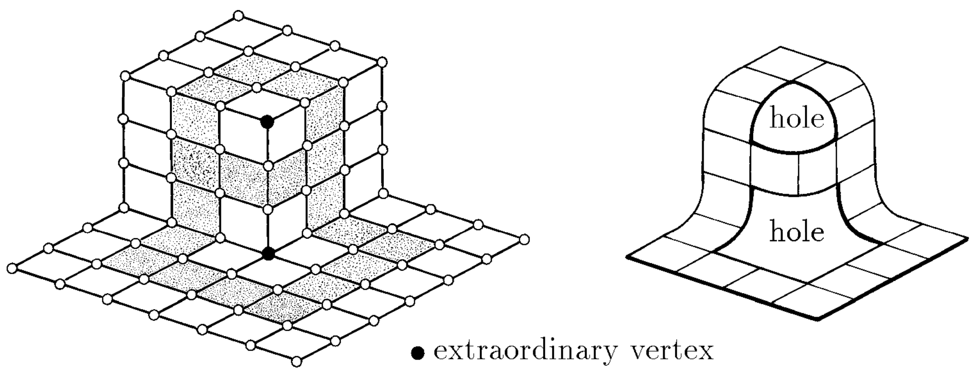

Figure 1: Control net with corresponding piecewise bicubic tensor product spline.

3 The construction

Let $\mathbf{r}$ be a ring surrounding an n-sided hole as considered above and let $\mathbf{x}$ be a similar, but planar surface ring having as many patches $\mathbf{x}_i(u,v) : [0,1]^2 \to \mathbb{R}^2$ with the same connectivity as the spatial ring $\mathbf{r}$. Figure 2 shows the bicubic B-spline control and the Bézier-net of one possible, planar ring $\mathbf{x}$, for $n = 5$, cf. also Remark 3.

The n-sided hole surrounded by $\mathbf{x}$ can be filled by a planar surface $\mathbf{y}$ consisting of n bicubic patches having C^2-joints with $\mathbf{x}$. Figure 2, right, shows the Bézier nets of some pair $\mathbf{x}$ and $\mathbf{y}$. The Bézier points of $\mathbf{y}$ marked by $\circ$ are determined by the C^2-conditions while the inner Bèzier point $\bullet$ is determined by some suitable averaging.

It should be mentioned that the n patches of $\mathbf{y}$ have C^2-joints only if all Bézier points marked by $\circ$ happen to coincide with the inner point $\bullet$. But, since it is planar, $\mathbf{y}$ is a G^∞-surface, in any case.

Now, let $\mathbf{q} : \mathbb{R}^2 \to \mathbb{R}^3$ be any suitable polynomial of total degree ≤ 2. Then $\mathbf{q} \circ \mathbf{x}$ and $\mathbf{q} \circ \mathbf{y}$ together form a bisextic spline surface with C^2- and G^∞-joints, respectively.

We return to the bicubic surface ring $\mathbf{r}$ from above and represent it also as a bisextic C^2-spline. Then $\mathbf{r}$ is described by 7 rings of spline control points.

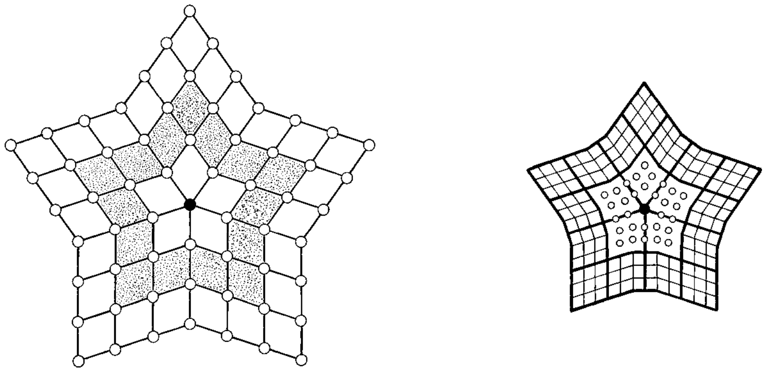

Figure 2: Control nets of a (here piecewise translational) planar ring $\mathbf{x}$.

Hence, if we substitute the three inner control rings of $\mathbf{r}$ by those of $\mathbf{q} \circ \mathbf{x}$, we obtain a C^2-surface whose hole associated with $\mathbf{r}$ can be filled smoothly by $\mathbf{q} \circ \mathbf{y}$.

4 Concluding remarks

1 Least squares fit: Let $\mathbf{x} = [x, y]$ and $\mathbf{m} = [1, x, y, x^2, xy, y^2] : \mathbb{R}^2 \to \mathbb{R}^6$. Then the bisextic surface ring $\mathbf{m} \circ \mathbf{x}$ has $84n$ spline control points in $\mathbb{R}^6$ forming a $84n \times 6$ spline control matrix M_x. Similarly, $\mathbf{r}$ has a $84n \times 3$ spline control matrix R. Further, we can represent $\mathbf{q}$ in monomial form by a 6×3 matrix Q such that $\mathbf{q} = \mathbf{m}Q$. Hence, $\mathbf{q} \circ \mathbf{x}$ has the $84n \times 3$ spline control matrix $M_x Q$. Using a least square fit to minimize (the square sum of the entries) in $M_x Q - R$ we obtain

$$Q = UR, \quad \text{where } U = (M_x^{\mathrm{t}} M_x)^{-1} M_x^{\mathrm{t}} \ .$$

The piecewise bisextic surface ring $\mathbf{m} \circ \mathbf{y}$ is described by $42n{+}1$ Bézier points in $\mathbb{R}^6$. They form a $42n{+}1 \times 6$ Bézier control matrix M_{yB}. Thus the Bézier points of $\mathbf{q} \circ \mathbf{y}$ are given by the rows of the $42n{+}1 \times 3$ matrix $M_{yB} Q$.

2 Smaller masks: One can use less than $84n$ control points to determine a projection $\mathbf{q}$. For example, let M_4 be the $36n \times 6$ matrix whose rows present the $36n$ spline control points of the four inner control rings of $\mathbf{m} \circ \mathbf{x}$. Similarly let R_4 be the 36×3 spline control matrix determined by the four inner control rings of the bisextic representation of $\mathbf{r}$. Then one obtains a projection $\mathbf{q}$ by

$$Q = U_4 R_4 \ , \quad U_4 = (M_4^{\mathrm{t}} M_4)^{-1} M_4^{\mathrm{t}} \ ,$$

i.e. $\mathbf{q} \circ \mathbf{x}$ has the spline control matrix $M_x Q$ and $\mathbf{q} \circ \mathbf{y}$ has the Bézier control matrix $M_{yB} Q$.

3 Good parametrizations: First implementations showed that it is best to use a planar surface ring $\mathbf{x} = [x, y]$ such that x and y are two independent eigenvectors of the Catmull/Clark scheme [1978] associated with the second largest eigenvalue $\lambda \in (0, 1)$.

4 Subdivision: Let $\mathbf{x}$ and λ be as in Remark 3. Then $\mathbf{x}$ fits smoothly around the scaled surface ring $\lambda\mathbf{x}$ and both rings together form a regular C^2-spline without self-intersections, cf. [Prautzsch '95]. Hence, together the surfaces $\mathbf{q} \circ \lambda^i \mathbf{x}$, $i \in \mathbb{N}$, form the surface $\mathbf{q} \circ \mathbf{y}$. Note that if the monomial representation of $\mathbf{q}(\mathbf{x})$ is given by the 6×3 matrix Q, then the monomial representation of $\mathbf{q}(\lambda^i \mathbf{x})$ is given by the matrix DQ where $D = \text{diag}[1, \lambda, \lambda, \lambda^2, \lambda^2, \lambda^2]$. Thus if $\mathbf{q}$ is computed as in Remark 2, then the four inner control rings of $\mathbf{q} \circ \lambda^i \mathbf{x}$ are given by $M_4 D^i U_4 R_4$. Since U_4 is the pseudoinverse of M_4, one has that $U_4 M_4$ is the 6×6 identity matrix and moreover $M_4 D^i U_4 = (M_4 D U_4)^i$.

Together with the subdivision algorithm for bisextic C^2-splines the matrices $S = M_4 D U_4$, for $n = 3, 5, 6, \ldots$, define a stationary G^2-subdivision algorithm as considered in [Prautzsch '95]: One subdivides the bisextic B-spline representation of $\mathbf{r}$ and takes the three inner control rings of the refined representation as the three outer control rings of a bisextic surface ring $\mathbf{r}^1$, i.e. $\mathbf{r}$ and $\mathbf{r}^1$ have a C^2-joint. The four inner control rings of $\mathbf{r}^1$ are then determined by SR_4 or, which is the same, by $\mathbf{q} \circ \lambda\mathbf{x}$. Similarly, one can compute a next surface ring $\mathbf{r}_2$ from $\mathbf{r}_1$ and successively further surface rings $\mathbf{r}^i$, $i \geq 3$. The four inner control rings of $\mathbf{r}^i$, $i \geq 1$, are given by $S^i R_4$ or equivalently by $\mathbf{q} \circ \lambda^i \mathbf{x}$. Since the three outer control rings of $\mathbf{r}_i$ come from $\mathbf{r}^{i-1}$ or $\mathbf{q} \circ \lambda^{i-1} \mathbf{x}$, one has $\mathbf{r}^i = \mathbf{q} \circ \lambda^i \mathbf{x}$. Thus the surface rings $\mathbf{r}, \mathbf{r}^1, \mathbf{r}^2, \ldots$ form a G^2-surface.

5 Unique nets: Rather than changing the control points of $\mathbf{r}$, we may change the three outmost control rings of $\mathbf{q} \circ \lambda\mathbf{x}$ such that $\mathbf{q} \circ \lambda\mathbf{x}$ and $\mathbf{q} \circ \lambda\mathbf{y}$ fill the hole of the original bicubic surface smoothly. Then each control point of the original "bicubic net" corresponds to a basis function for the surface after the holes are filled.

6 Convex hull property: The matrix $M_x U$ has negative entries in general. However, if we worked only with constant polynomials $\mathbf{q}$, we would have obtained a positive matrix P such that PR represents the control net of the least squares fit $\mathbf{q} \circ \mathbf{x}$. Hence, for any sufficiently small positive α the matrix $A = \alpha M_x U + (1 - \alpha) P$ is positive and the control net AR lies in the convex hull of the net R.

7 Projective invariance: A freeform spline in $\mathbb{R}^4$ represents a rational freeform spline in projective 3-space. The control and weight points on the edges of a tree spanning the initial control net form a projective invariant representation of the entire surface.

8 Singular C^k-parametrization: Independently Ulrich Reif [1995] constructed singularly parametrized C^k-splines with quasi control nets. In our set up we obtain these special splines by choosing $\mathbf{y}$ (and then $\mathbf{x}$ so as to fit smoothly around $\mathbf{y}$) such that all interior Bézier points, $\circ$, of $\mathbf{y}$ coincide.

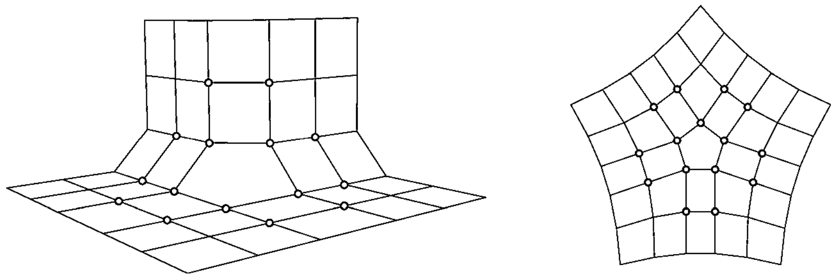

Figure 3: The spline control net of a freeform spline (left) and of the parametrization $\mathbf{x}$ (right).

9 Example: The method described above can also be used to obtain G^k-freeform splines for any k. Figure 3, left side, shows the control net of a biquartic G^1-freeform spline. The right side shows the spline control net of the parametrizing surface ring $\mathbf{x} = [x, y]$. Here, x and y are the eigen vectors of Doo's algorithm, see [Catmull et al. '78]. Figure 4, right side, shows the surface ring $\mathbf{r}$ together with the surface $\mathbf{q} \circ \mathbf{y}$. Here, only the 15 control points marked by hollow dots in Figure 3 were used to determine the least

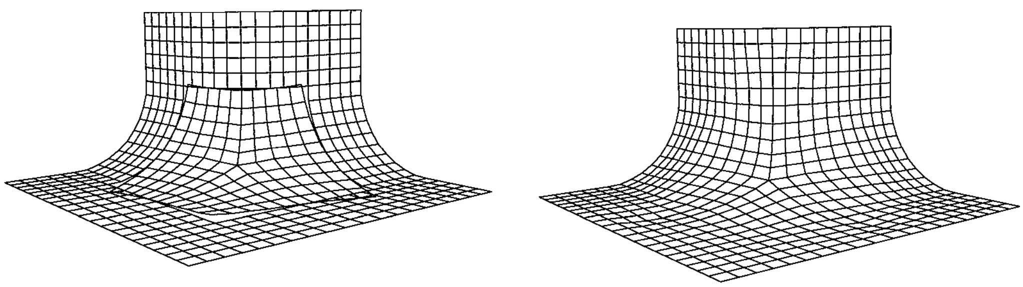

Figure 4: The surface ring $\mathbf{r}$ with least squares fit $\mathbf{q} \circ \mathbf{y}$ (left) and the corresponding C^1-freeform spline (right).

squares fit **q**, cf. Remark 2. On the right side of Figure 4 one sees the resulting G^1-freeform spline. It was determined in a slightly different way than described above. Namely, only the 15 control points of **r** marked in Figure 3 were replaced by the corresponding control points of $\mathbf{q} \circ \mathbf{x}$. The two outer control rings of $\mathbf{q} \circ \mathbf{y}$ were then changed such that the changed surfaces **r** and $\mathbf{q} \circ \mathbf{y}$ have C^1-joint along their common boundary.

Acknowledgement: I would like to thank Stefan Keppler who implemented the method with several modifications and produced Figure 4 and together with Georg Umlauf Figure 3.

References

E. Catmull and J. Clark (1978). Recursively generated B-spline surfaces on arbitrary topological meshes. *Computer-Aided Design*, 10(6):350–355.

J. Hahn (1989). Filling polygonal holes with rectangular patches. In W. Strasser and H.-P. Seidel, editors, *Theory and Practice of Geometric Modelling*, pages 81–91. Springer, Berlin.

H. Mögerle (1992). *G-Splines höherer Ordnung.* PhD thesis, Universität Stuttgart.

H. Prautzsch (1995). C^k-analysis of subdivision surfaces at extraordinary points. http://i33www.ira.uka.de, submitted.

U. Reif (1995). TURBS – Topologically Unrestricted Rational B-splines. preprint 95-16, 1995. institut@mathematik.uni-stuttgart.de.