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The magnetic response of a proximity-coupled superconductor-normal metal sandwich is studied within the framework of the quasiclassical theory. The magnetization is evaluated for finite values of the applied magnetic field (linear and nonlinear response) at arbitrary temperatures and is used to fit recent experimental low-temperature data. The hysteretic behavior predicted from a Ginzburg-Landau approach and observed in experiments is obtained within the quasiclassical theory and shown to exist also outside the Ginzburg-Landau region.

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I. INTRODUCTION

A superconductor in electric contact with a normal metal induces superconducting correlations on the normal side. This *proximity effect* has been studied extensively, both theoretically and experimentally (see Ref. 1 and references therein). The superconducting properties of the normal metal show up, e.g., in the conductance or the magnetization (Meissner effect). The vanishing of the resistance of a normal wire in contact to superconducting islands has been observed. In recent experiments (typically with wires consisting of a superconducting core covered by a normal metal) by Oda and Nagano², Mota *et al.*^{3,4}, and Bergmann *et al.*⁵ the magnetization was measured. In each of these experiments, the magnetic field was expelled from the normal-metal part (N) of the sample by superconducting screening currents induced by the presence of the superconductor (S).

In the present work, we study theoretically the proximity and Meissner effect in an NS sandwich using the quasiclassical Eilenberger formalism^{6,7}. In materials with a high concentration of impurities (dirty limit), the quasiclassical theory can be reformulated leading to the Usadel equation⁸. In both approaches the induced pair amplitude in the normal metal, the density of states, the critical current, etc., have been calculated (see, e.g., Refs. 9–11). In general, for a realistic geometry the solution of the Eilenberger or Usadel equation together with the Maxwell equations can be performed only numerically. We have solved the combined system of equations in a wide range of parameters (temperature, external magnetic field, layer thickness). In addition we have paid special attention to the non-linear response that shows interesting hysteretic behavior at low temperatures.

In Section II we introduce the quasiclassical Green's function formulation for the clean and dirty limit. The geometry and the characteristic length scales are also defined there. In Section III we evaluate the Meissner screening current and present some results on the space dependence of the various quantities. Finally, in Section IV results for the susceptibility in a wide range of temperatures and values of the magnetic field are presented and discussed. Throughout the paper we use units with $\hbar = k_B = c = 1$.

II. THE MODEL: CLEAN AND DIRTY CASE

Our theoretical description is based on the quasiclassical Green's function technique^{6,7}. We will study two limiting cases, the clean limit with complete absence of impurities and the dirty limit with high concentration of impurities such that the elastic scattering rate $1/\tau_{el}$ is large compared to the superconducting order parameter Δ and the temperature T . We introduce two different coherence lengths for both the superconducting and the normal side of the double-layer structure. In a superconducting material we have in the clean limit

$$\xi_c^S = \frac{v_F}{2\Delta},$$

and in the dirty case

$$\xi_d^S = \sqrt{\frac{D}{2\Delta}}.$$

Here v_F is the Fermi velocity, Δ the pair potential and $D = \frac{1}{3}v_F l_{el} = \frac{1}{3}v_F^2 \tau_{el}$ the diffusion constant. These lengths are temperature-independent for $T \ll T_c$. In a normal metal we define the temperature-dependent coherence lengths in the clean limit

$$\xi_c^N(T) = \frac{v_F}{2\pi T},$$

and in the dirty limit

$$\xi_d^N(T) = \sqrt{\frac{D}{2\pi T}}.$$

For definiteness we first consider a one-dimensional geometry with a bulk superconductor in the region $x < 0$ and a normal metal layer in the region $0 \leq x \leq d$ in perfect electric contact with the superconductor. (Later we will also present some results for the cylinder geometry of the experiments.) Since we assume $d \gg \xi_c^S$, we can neglect the spatial dependence of the pair potential in the superconductor. This has also been confirmed numerically¹². Therefore, we assume the following form for the pair potential

$$\Delta(x) = \Delta \Theta(-x), \quad \text{Im}\Delta = 0. \quad (1)$$

For the vector potential we use the Coulomb gauge. Together with the boundary conditions it is written as

$$\mathbf{A} = (0, A(x), 0), \quad A(0) = 0, \quad \left. \frac{dA}{dx} \right|_{x=d} = H, \quad (2)$$

where H is the external applied field. We have neglected the magnetic field on the superconducting side, i.e., we assume that the penetration depth on the superconducting side is much smaller than the other relevant length scales.

We first consider the clean limit defined by

$$\xi_c^S, \xi_c^N(T) \ll l_{el}.$$

The Eilenberger equations for this system read

$$\left[(\omega + ie v_y A(x)) \hat{\tau}_3 - \Delta \hat{\tau}_1 + v_x \frac{\partial}{\partial x}, \hat{g}_\omega(v_x, v_y, x) \right] = 0, \quad (3)$$

which is to be combined with the normalization condition for the Green's functions

$$\hat{g}_\omega^2(v_x, v_y, x) = 1. \quad (4)$$

The Matsubara frequencies are $\omega = \pi T(2n + 1)$, $\hat{\tau}_i$ are the Pauli matrices and v_x, v_y are the x - and y -components of the Fermi velocity. We assume ideal transmission, i.e. there are no surface potentials at the N-S-boundary and the Fermi velocities are equal on both sides. The boundary condition reads $\hat{g}_\omega(v_x, v_y, 0-) = \hat{g}_\omega(v_x, v_y, 0+)$. At the interface to the vacuum we assume specular reflection, $\hat{g}_\omega(v_x, v_y, d) = \hat{g}_\omega(-v_x, v_y, d)$.

The current density in the y -direction is given by

$$j(x) = -ie2\pi N(0)T \sum_\omega \langle v_y \text{Tr} \hat{\tau}_3 \hat{g}_\omega(v_x, v_y, x) \rangle, \quad (5)$$

where $N(0)$ is the density of states at the Fermi level and $\langle \dots \rangle$ denotes averaging over the Fermi surfaces that are assumed to be spherical.

In systems with high concentration of nonmagnetic impurities, such that

$$\xi_d^S, \xi_d^N(T) \gg l_{el},$$

the Eilenberger equation reduces to the Usadel equation for the isotropic part of the Green's function, $\hat{g}_\omega(x) = \langle \hat{g}_\omega(v_x, v_y, x) \rangle$. For our geometry it can be written as

$$D \frac{d}{dx} \hat{g}_\omega(x) \frac{d}{dx} \hat{g}_\omega(x) = [\omega \hat{\tau}_3 - \Delta(x) \hat{\tau}_1 + D e^2 A(x)^2 \hat{\tau}_3 \hat{g}_\omega(x) \hat{\tau}_3, \hat{g}_\omega(x)] \quad (6)$$

We take the same boundary conditions as in the clean case, i. e. we neglect the magnetic field on the superconducting side. In the dirty limit the current density is given by the London-like expression

$$j(x) = \pi e^2 N(0) D T \sum_\omega \text{Tr} \hat{\tau}_3 \hat{g}_\omega(x) [\hat{\tau}_3, \hat{g}_\omega(x)] A(x), \quad (7)$$

which together with the Maxwell equation

$$\frac{d^2}{dx^2} A(x) = 4\pi j(x) = \frac{1}{\lambda^2(x)} A(x), \quad (8)$$

defines a local penetration depth $\lambda(x)$.

III. MEISSNER EFFECT

A. Clean Limit

In the clean limit an analytic solution of Eq. (3) for the normal metal layer has been found by Zaikin¹⁰. It turns out that the $\hat{\tau}_3$ component of the Green's function in the normal metal layer is spatially constant. Consequently the current density (5) in the normal metal layer is spatially constant as well and can be expressed as

$$j_n = \frac{K_c}{4\pi e \xi_c^N(T_c)^3} \frac{T}{T_c} \sum_{\omega > 0} \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} d\varphi \sin^2 \theta \cos \varphi \sin 2\phi \times \left[\left(\sqrt{1 + \left(\frac{\omega}{\Delta}\right)^2 \sinh \Phi} + \frac{\omega}{\Delta} \cosh \Phi \right)^2 + \cos^2 \phi \right]^{-1}. \quad (9)$$

Here we have introduced a dimensionless constant

$$K_c = 32e^2 N(0) v_F^2 \xi_c^N(T_c)^2 = \frac{24}{\pi} \left(\frac{\xi_c^N(T_c)}{\lambda_N} \right)^2, \quad (10)$$

where $\lambda_N = (4\pi e^2 n_e / m)^{-1/2}$ is the normal-metal penetration depth, defined analogously to the London penetration depth with the normal electron density n_e replacing the superfluid density. Furthermore,

$$\Phi = \frac{2\omega d}{v_F \cos \theta} \quad (11)$$

is the length of a classical trajectory divided by the thermal coherence length $\xi_c^N(T)$ and

$$\phi = 2e \tan \theta \cos \varphi \int_0^d A(x) dx \quad (12)$$

is the Aharonov-Bohm phase connected with this trajectory. The last equation shows, that the relationship between current and vector potential is completely non-local. This phase factor leads to a shift in the energies of the Andreev levels in normal metal layer⁹. The fact that these bound states are extended over the whole thickness of the N-layer and that the density of Andreev levels is spatially constant, leads to a constant current density. We can add, that in cylindrical geometries if the normal layer is not thin compared to the radius, this second condition is not satisfied. Hence, as was pointed out by Nazarov¹³, the current density is not constant in space.

To evaluate the Meissner effect, we have to solve the Maxwell equations with the boundary condition given in Eq. (2). The solution in the N-layer is

$$B(x) = H - 4\pi j_n(d - x). \quad (13)$$

For the susceptibility of the N-layer we find

$$\chi = -\frac{j_n d}{2H}. \quad (14)$$

Since the solution of the problem may lead to $4\pi j_n d > H$ the magnetic field in the region $0 \leq x \leq d - H/4\pi j_n$ may change its sign relative to the applied field. This over-screening effect was first found by Zaikin¹⁰. In Fig. 1 we have plotted the current density and the magnetic field in the N-layer for different temperatures in the limit of small magnetic fields. Below $T \sim 0.1T_c$ we find the over-screening. Below this temperature the screening quickly reaches its saturation value (solid curve) and a susceptibility of 3/4 as compared to complete screening. Thus, even at $T = 0$ the screening is incomplete.

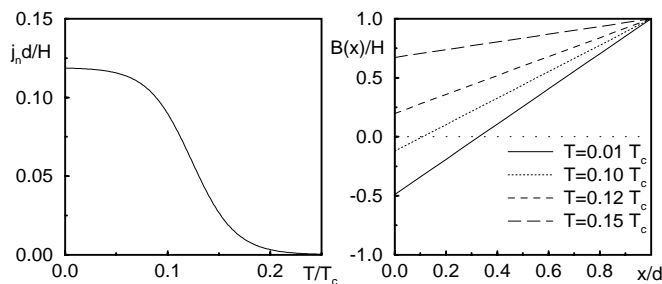


FIG. 1. Current density and magnetic field in the N-layer (clean limit). Layer thickness $d = 20\xi_c^N(T_c)$, dimensionless material constant $K_c = 10$.

Next we study the nonlinear response. For this purpose we write the self-consistency equation for the integrated dimensionless vector potential $a = e \int_0^d A(x) dx$

$$a = e \frac{1}{2} H d^2 - \frac{4\pi}{3} d^3 e j_n(a). \quad (15)$$

Solving Eq. (15) for the applied field H we get, together with Eq. (14), the magnetization curve $\chi(H)$

$$\chi(a) = -\frac{j_n(a) d}{2H(a)} \quad (16)$$

$$H(a) = \frac{2a}{ed^2} + 4\pi j_n(a) \frac{2d}{3}$$

parameterized by the integrated vector potential.

B. Dirty Limit

Here Eq. (6) cannot be solved analytically anymore. To proceed numerically we reduce it to the scalar equation

$$2\omega F_\omega(x) - 2\Delta G_\omega + 4De^2 A^2(x) F_\omega(x) G_\omega(x) =$$

$$D \left[G_\omega(x) \frac{d^2}{dx^2} F_\omega(x) - F_\omega(x) \frac{d^2}{dx^2} G_\omega(x) \right], \quad (17)$$

where $F_\omega(x) = \frac{1}{2} \text{Tr} \hat{\tau}_1 \hat{g}_\omega(x)$ is the off-diagonal part of the Green's function and $G_\omega(x) = (1 - F_\omega(x)^2)^{1/2}$ is fixed by the normalization (4). The Maxwell equation is given by Eq. (8) with the boundary conditions given in (2). We write for the local penetration depth

$$\frac{1}{\lambda^2(x)} = \frac{K_d}{\xi_d^N(T_c)^2} \frac{T}{T_c} \sum_{\omega > 0} F_\omega^2(x), \quad (18)$$

where we have defined an dimensionless constant $K_d = 4(l_{el}/\pi\lambda_N)^2/3$.

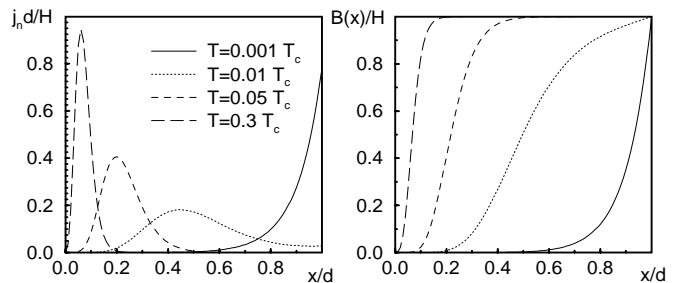


FIG. 2. Current density and magnetic field in the N-layer (dirty limit). Layer thickness $d = 50\xi_d^N(T_c)$ and dimensionless material constant $K_d = 100$.

We have solved this system of equations numerically for a range of parameters. Some results are shown in Fig. 2 for a layer thickness of $50\xi_d^N(T_c)$ and $K_d = 100$ in a weak magnetic field, where we neglect the term $\sim A^2$ in the Usadel equation. At the highest temperature, $T = 0.3T_c$, the field penetrates through the whole N-layer. Near the N-S-boundary, where the induced superconductivity is strong, the field decays rapidly to zero. At lower temperatures the field-free region increases, because the induced superconductivity extends to a region of linear size $\sim \xi_d^N(T)$. At the lowest temperature shown here, $T = 0.001T_c$, the field expulsion is London like and the field as well as the current density fall off exponentially. The reason is that at this temperature the local penetration depth near the N-Vacuum-boundary is practically constant. In the intermediate temperature regime the current flows in a well defined region inside the N-layer, and consequently the screening takes place in this region.

To calculate the nonlinear response we have to solve the complete Usadel equation Eq. (6) together with the Maxwell equations including the local penetration depth (18) numerically. From the solution of the Maxwell equations, we find the susceptibility of the normal-metal part

$$\chi = -\frac{1}{4\pi} \left(\frac{A(d)}{Hd} - 1 \right). \quad (19)$$

Results for the clean and the dirty case are discussed in the next section.

IV. SUSCEPTIBILITY

A. Linear Response

First we study the case of linear response. In the clean limit we linearize Eq. (9) with respect to the Aharonov-Bohm phase ϕ . The resulting expression has been analyzed analytically by Zaikin in some limits¹⁰ and numerically by Higashitani and Nagai¹². We have performed the remaining integration over θ numerically for all temperatures. In Fig. 3 some results for different layer thicknesses are shown. The temperature at which screening sets in, as well as the slope of the temperature dependence, changes drastically when the layer thickness increases. At $T = 0$ the integrals can be solved analytically. For $d \gg \lambda_N$, the saturation value is equal to $3/4$ of a perfect diamagnet, independent of the thickness¹⁰.

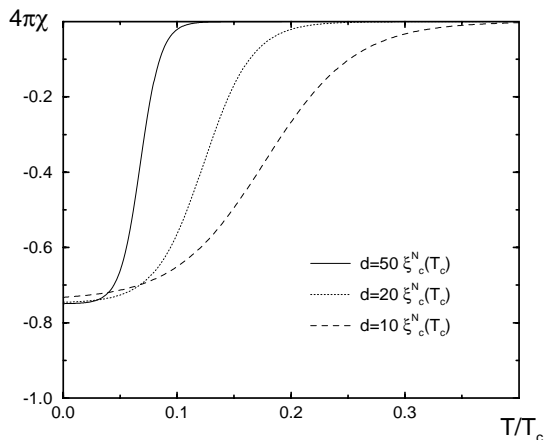


FIG. 3. Linear susceptibility (clean case) for $K_c = 10$.

Here we should mention that the temperature dependence of the susceptibility cannot be fitted to a power law $\chi \propto T^{-\alpha}$ in any temperature range, in contrast to earlier theoretical predictions¹⁴ which suggest powers in the range $1/2 \leq \alpha \leq 1$ depending on the impurity concentration. Rather it satisfies an exponential law. This explains qualitatively some experimental results³⁻⁵, where exponents α with values up to 2 had been found.

In the dirty limit, we neglect the term $\sim A^2$ in Eq. (6) and solve the resulting differential equation numerically. In Fig. 4 some results of our calculation are shown. In comparison with the clean case, the screening sets in at higher temperatures and the saturation values can be larger (depending on K_d). The temperature dependence is well described by

$$\chi \sim (T^{-1/2} - \text{const}) \quad (20)$$

in the intermediate temperature regime well below T_c and above the saturation temperature. Our results agree with previous works in the applicable limits. In earlier work¹⁴, a generalized Ginzburg-Landau approach was used. Narikiyo and Fukuyama¹⁵ linearized Eq. (6) with respect to F and solved the system of equations for an infinite system numerically. However, our calculation is free of the limitations of the Ginzburg-Landau theory and is valid at any temperature. Furthermore, we have taken into account the nonlinearity of the Usadel equation and finite-size effects.

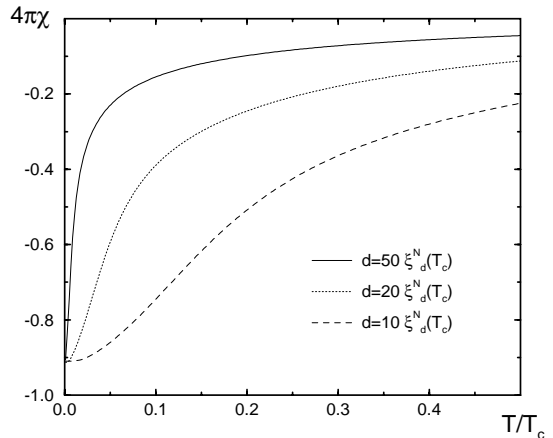


FIG. 4. Linear susceptibility (dirty limit) for $K_d = 100$.

The behavior shown in Fig. 4 has also been observed experimentally in very dirty samples^{2,5}. In these experiments, a temperature dependence described by $\chi \sim (T^{-1/2} - \text{const})$ was found and saturation values of the susceptibility of 90 – 95% of a perfect diamagnet.

In Fig. 5 we show a comparison between theory and experiment. The experimental data¹⁶ are measured on a cylindrical geometry with a superconducting core surrounded by a normal metal layer of thickness d . The theory has been generalized to this kind of geometry and also been solved numerically. The parameters for the theory were calculated from experimental values for d and l_{el} and from theoretical values for v_F and λ_N . No fitting parameter was used. The agreement is quite remarkable at low temperatures. At higher temperatures ($T \gtrsim 0.1T_c$) the dirty limit theory cannot be applied anymore and this explains the disagreement in this temperature regime.

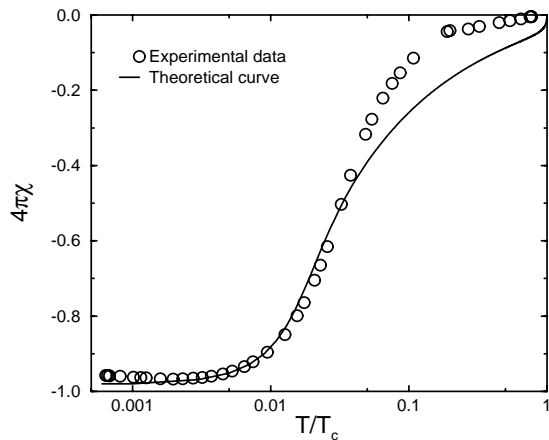


FIG. 5. Comparison between experiment (AgNb)¹⁶ and dirty limit theory. Experimental parameters: Radius of the S-core = 35 μm, $d = 14.5 \mu\text{m}$, and $l_{el} = 2.0 \mu\text{m}$. In the theory we have used $d = 41.5 \xi_d^N(T_c)$ and $K_d = 10000$.

B. Nonlinear Response

We will now turn to the nonlinear response. In Fig. 6 some numerical results for the clean case are shown. At the highest temperature shown here, the susceptibility drops to zero around $h \approx 0.02$. At lower temperatures the solution is not unique anymore. There is one solution with the maximal field expulsion, the value of which depends on temperature (lower thick lines), and one solution, for which the field completely penetrates in the N-layer and there is no screening (upper thick lines). The third solution with the negative derivative is unstable (thin lines). The region in which the solution is non-unique grows, as the temperature is lowered, but the lower boundary of this region stays at the same value of the field. Since only one solution can be stable, there must be a jump in the susceptibility at a certain point. In principle we could determine this point by comparing the free energies of the N-layer for the two different solutions. Unfortunately the free-energy functional proposed by Eilenberger⁶ cannot be applied for $\Delta = 0$ (normal region).

In experiments the situation is often quite different. On lowering or raising the field, the jump in the susceptibility would not occur at the value predicted theoretically, but effects like “superheating” or “supercooling” are likely to occur. These effects were observed in very clean samples⁴. When the field is increased, the susceptibility will remain on the lower branch up to a certain field value and then will jump to the upper branch. On the other hand, when the field is decreased, the system will stay on the upper branch with complete field penetration. At a certain value of the field, the field will suddenly be expelled and the system will jump to the lower branch. These jump fields do not necessarily coincide with the

boundaries of the instability zone.

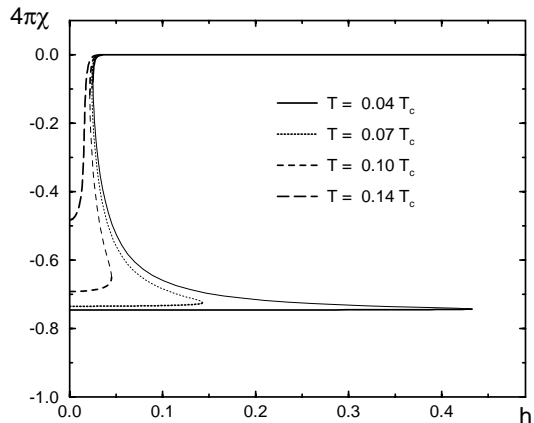


FIG. 6. Nonlinear susceptibility (clean case) of an N-layer of thickness $20 \xi_c^N(T_c)$ and $K_c = 10$ as a function of the applied field for various temperatures. The dimensionless field is defined by $h = H \pi \xi_c^N(T_c)^2 / \Phi_0$.

The boundaries of the hysteretic regions are shown in Fig. 7 for different layer thicknesses. The superheating field depends exponentially on temperature, whereas the supercooling field depends not very strongly on temperature. At low temperatures the superheating field tends to saturate at a certain value, which is independent of the layer thickness. Further calculations show that the saturation value is proportional to the material constant K_c .

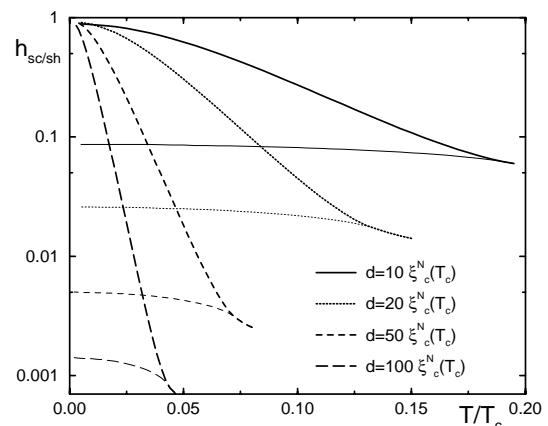


FIG. 7. Limits of the instability region: supercooling field h_{sc} (thin lines) and superheating field h_{sh} (thick lines) for $K_c = 10$. h is defined as in the previous figure.

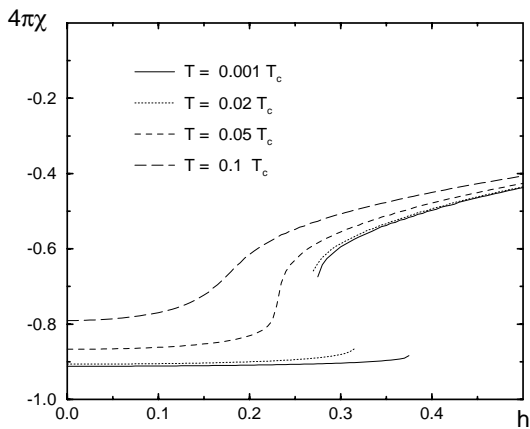


FIG. 8. Nonlinear susceptibility (dirty case) of an N-layer of thickness $10\xi_d^N(T_c)$ and $K_d = 100$ as a function of the applied field for various temperatures. The dimensionless field is defined by $h = H\pi\xi_d^N(T_c)^2/\Phi_0$.

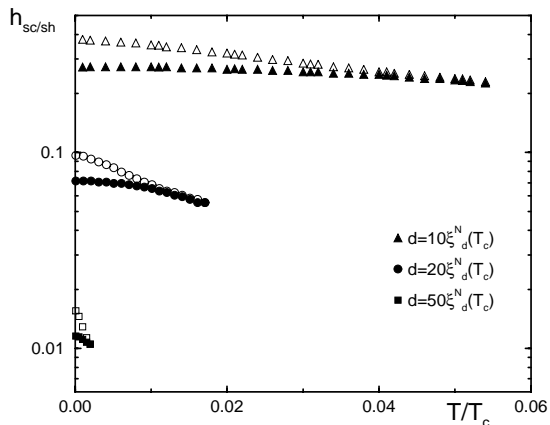


FIG. 9. Limits of the instability region (dirty case): supercooling field h_{sc} (open symbols) and superheating field h_{sh} (filled symbols) for $K_d = 10$. h is defined as in the previous figure.

In the dirty limit there is also a temperature-dependent hysteretic effect, as shown in Fig. 8. Above temperatures of about $0.03T_c$ the magnetization curve is unique. Below these temperatures there is a region with constant lower boundary and increasing upper boundary as in the clean case. The absolute values of the susceptibility are drastically different. The saturation value at low fields and low temperatures can reach complete field expulsion for appropriately chosen values of K_d . On the other hand the jump in the susceptibility is less than in the clean case. On increasing the field the susceptibility is reduced only by $\approx 50\%$ at the jump and reaches slowly zero, if the field is increased further. On lowering the field, the jump is a slightly smaller. This hysteretic behavior was

observed⁵.

In Fig. 9 the limits of the region with hysteretic behavior for different layer thicknesses in the dirty limit are shown. As in the clean case the superheating field depends exponentially on the temperature, but not as strongly. Also, the supercooling field tends to become a constant at low temperatures and does not very strongly depend on temperature. In contrast to the clean case the saturation values of both fields at low temperatures depend strongly on the layer thickness.

Finally, we would like to comment on some of our assumptions. We have assumed ideal interfaces between normal metal and superconductor, specular reflection at the interface between normal metal and vacuum, and spherical Fermi surfaces in both materials. The formalism presented in this paper can be modified to describe more general physical situations. Including e.g. non-ideal interfaces will weaken the proximity effect and diminish the diamagnetic response of the normal metal. In this paper, we have concentrated on presenting a model calculation in which the main aspects of the diamagnetic response of NS sandwiches can be studied. Despite our simplifying assumptions, the calculation was shown to be in reasonable agreement with experiment, see Fig. 5. Further calculations will be necessary to describe the high-temperature behaviour correctly and to study arbitrary concentrations of impurities, i.e., cases that are neither clean nor dirty.

In conclusion, we have applied the quasiclassical (Eilenberger or Usadel) theory to calculate the Meissner effect in a proximity sandwich. We have evaluated the full non-linear magnetic response at all temperatures and for various thicknesses of the normal layer. We have shown that both in the clean and in the dirty limit a hysteretic behavior of the magnetization of a normal metal layer in proximity with a superconductor is possible, as has been observed experimentally in both limits.

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