Charge – Vortex Duality

in Double-Layered Josephson Junction Arrays

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Abstract

A system of two parallel Josephson junction arrays coupled by interlayer capacitances is considered in the situation where one layer is in the vortex-dominated and the other in the charge-dominated regime. This system shows a symmetry (duality) of the relevant degrees of freedom, i.e. the vortices in one layer and the charges in the other. In contrast to single-layer arrays both contribute to the kinetic energy. The charges feel the magnetic field created by vortices, and, vice versa, the vortices feel a gauge field created by charges. For long-range interaction of the charges the system exhibits two Berezinskii-Kosterlitz-Thouless transitions, one for vortices and another one for charges. The interlayer capacitance suppresses both transition temperatures. The charge-unbinding transition is suppressed already for relatively weak coupling, while the vortex-unbinding transition is more robust. The shift of the transition temperature for vortices is calculated in the quasi-classical approximation for arbitrary relations between the capacitances (both weak and strong coupling).

Two-dimensional (2D) Josephson junction arrays have attracted much attention because of the experimental and technological progress and the rich underlying physics (see Ref. [1] for review). Classical 2D Josephson junction arrays, where the Josephson coupling energy E_J between the superconducting islands dominates, is a standard example of a system exhibiting the Berezinskii-Kosterlitz-Thouless (BKT) transition – the unbinding of vortex-antivortex pairs at a certain temperature T_J [2,3]. The transition separates a superconducting phase at $T < T_J \propto E_J$, where vortices are bound, from a resistive phase. It was realized later (see e.g. Refs. [4–7]) that charging effects, associated with the capacitances of the islands to the ground C_0 and of the junctions C, lead to quantum fluctuations of the phase and suppress the BKT transition temperature. Beyond a critical value of the charging energy $E_C = \min\{e^2/2C, e^2/2C_0\}$ the transition temperature vanishes, and the superconducting phase ceases to exist. The next step [8–11] was to understand that in the "extreme" quantum limit $E_C \gg E_J$, where the quantum fluctuations of the phases are very strong, the vortices are ill-defined objects. In this regime the charges on the islands become the relevant variables. If, furthermore, $C_0 \ll C$, the interaction between the charges is (nearly) the same as that of the vortices in the quasi-classical array. In particular, the charges can be considered as a 2D Coulomb gas [13], and they undergo a BKT transition at temperature $T_C \propto E_C$. The phase below the transition is insulating. A finite value of the Josephson coupling between the islands suppresses this transition. As another example we mention the influence of dissipation (e.g. Ohmic dissipation) on the phase transition in the array, which was first noted in Ref. [12]. We are not going to review these theoretical results, however it is necessary to stress that the theory of 2D Josephson junction arrays is far from being settled.

Below we describe another, more complicated system - two parallel 2D Josephson junction arrays with capacitive coupling between them (no Josephson coupling[1]). Probably the most interesting situation arises when one array is in the quasi-classical (vortex) regime while another one is in the quantum (charge) regime. Then the vortices in one layer and the charges in the other one are well-defined *dynamical* variables. Another important feature of the present system is that the strength of interaction between charges and vortices is controlled by the interlayer coupling C_x and consequently this interaction may be weak or strong, whereas in usual 2D Josephson junction arrays the strength of charge-vortex interaction is of the same order as either the charge-charge or the vortex-vortex interaction. We also show that the physical realization of this interaction is rather different from that in one array. Hence, at least for weak interlayer coupling, one should expect two BKT transitions, the first for charges in one layer, and the second for vortices in the other one. In this article we provide the theoretical description of the coupled system and calculate the shifts of the transition temperatures due to the interlayer interaction.

We consider two parallel Josephson junction arrays, i.e. (square) lattices of superconducting islands connected by Josephson links. As usual, we suppose that the magnitude of the order parameter in the islands is constant while its phase fluctuates from island to island. The partition function of the system may be expressed conveniently in terms of these phases $\phi_{i\mu}$ (the indices i label the islands in each array and $\mu = 1, 2$ refers to the number of array)

$$Z = \prod_i \int_0^{2\pi} d\phi_{i1}^{(0)} d\phi_{i2}^{(0)} \sum_{\{m_{i1}\},\{m_{i2}\}} \int D\phi_{i1}(\tau) D\phi_{i2}(\tau) \exp(-S\{\phi\}) \, . \tag{1}$$

Here the path integration over phases is carried out with the boundary conditions

$$\phi_{i\mu}(0) = \phi_{i\mu}^{(0)}; \quad \phi_{i\mu}(\beta) = \phi_{i\mu}^{(0)} + 2\pi m_{i\mu},$$

[1]Multi-layered systems *with* Josephson coupling between layers have been discussed in the literature (see e.g. [14,15]). The analog of the BKT transition is in this case the disruption of vortex rings. In the limit of weak Josephson couplings this system is reduced to the 2D XY-model, while in the opposite case of strong coupling it is essentially the 3D XY-model. This situation, however, is absolutely different from the one we describe below.

where β is the inverse temperature, and $\hbar = 1$. The Euclidean effective action $S\{\phi\}$ is

$$S\{\phi\} = \int_0^\beta d\tau \left\{ \frac{C_{01}}{8e^2}\sum_i(\dot{\phi}_{i1})^2 + \frac{C_1}{8e^2}\sum_{\langle ij\rangle}(\dot{\phi}_{i1}-\dot{\phi}_{j1})^2 + \frac{C_{02}}{8e^2}\sum_i(\dot{\phi}_{i2})^2 + \frac{C_2}{8e^2}\sum_{\langle ij\rangle}(\dot{\phi}_{i2}-\dot{\phi}_{j2})^2 + \right.$$
$$\left. + \frac{C_x}{8e^2}\sum_i(\dot{\phi}_{i1}-\dot{\phi}_{i2})^2 + E_{J1}\sum_{\langle ij\rangle}[1-\cos(\phi_{i1}-\phi_{j1})] + E_{J2}\sum_{\langle ij\rangle}[1-\cos(\phi_{i2}-\phi_{j2})] \right\} . \tag{2}$$

Here $C_{0\mu}$ are the capacitances of the islands in the array μ relative to the ground, C_μ are the capacitances of the junctions in the array μ, and C_x are the interlayer capacitances. Furthermore, $E_{J\mu}$ are the Josephson coupling constants in the layers. Here and below we use the symbol $\sum_{\langle ij\rangle}$ to denote the summation over nearest neighbors only, and each pair is counted once; the symbol $\sum_{ij}$ stands for the summation for all values i and j (in particular, each pair except $\langle ii\rangle$ is counted twice).

From now on we choose parameters such that the array 1 is in the charge (quantum) regime while the array 2 is in the quasi-classical (vortex) regime. In terms of the phase variables this means that in the array 1 the phases on each grain are strongly fluctuating in time, while in array 2 they are nearly time-independent. This regime is described by the conditions

$$E_{J1} \ll e^2/\tilde{C}_1, \quad E_{J2} \gg e^2/\tilde{C}_2,$$

with $\tilde{C}_\mu = \max\{C_{0\mu}, C_\mu, C_x\}$. Below we first calculate the shift of the BKT transition temperature for vortices in the array 2. This does not require the introduction of charges and vortices and may be done in the phase representation. Then, we turn to the BKT transition for charges in the array 1. For this purpose we move from a description in terms of phases to one in terms of charges and vortices, and use the duality of the resulting action to investigate the transition. At the same time, we will show that charges and vortices in this system can be considered as two-dimensional dynamical particles with masses. The charge-charge and vortex-vortex interaction are essentially those of 2D Coulomb particles, while the charge-vortex interaction is more peculiar.

BKT transition for vortices

The shift of the BKT transition temperature for vortices in the array 2 due to the coupling to the array 1 can be calculated easily if we set the small parameter E_{J1} to zero[2]. Then the action for the phases ϕ_{i1} becomes Gaussian and the latter may be integrated out. After that the shift of the BKT temperature for vortices may be obtained by means of the quasi-classical expansion [7,16].

The first step requires a comment. The path integration over the phases of the islands $\phi_{i1}(\tau)$ in the array 1 is, as usual, performed by a linear shift of variables in order to eliminate

[2]This means, in particular, that the results obtained below are valid also in the case when the array 1 is in the normal state. Because of the e-periodicity in this case the boundary conditions in the array 1 should read $\phi_{i1}(\beta) - \phi_{i1}(0) = 4\pi m_{i1}$. As we will show, this does not change the final result.

terms linear in ϕ_{i1}. However, the new (shifted) variables do not generally satisfy the boundary conditions, and consequently the integration is not possible. If, nevertheless, the array 2 is in the quasi-classical regime, the contributions of all non-zero winding numbers m_{i2} to the partition function are exponentially small in comparison with the contribution of $m_{i2} = 0$ (see, e.g. [16]). If we neglect these small contributions, the phases ϕ_{i2} become periodic, and the boundary conditions in the array 1 are met automatically. After integration over the phases ϕ_{i1} we find

$$Z = \prod_i \int_0^{2\pi} d\phi_{i2}^{(0)} \int D\phi_{i2}(\tau) \exp(-\tilde{S}\{\phi_{i2}\}) \tag{3}$$

with the effective action

$$\tilde{S}\{\phi\} = \int_0^\beta d\tau \left\{ \frac{1}{8e^2} \sum_{ij} \dot{\phi}_{i2}[\Lambda_2]_{ij}\dot{\phi}_{j2} + E_{J2} \sum_{\langle ij \rangle}[1 - \cos(\phi_{i2} - \phi_{j2})] \right\}. \tag{4}$$

Here $[\Lambda_2]_{ij}$ is the effective capacitance matrix for the layer 2 (see also below)

$$[\Lambda_2]_{ij} = -\frac{C_x^2}{C_1}[\hat{Q}_1^{-1}]_{ij} + C_2[\hat{Q}_2]_{ij} \tag{5}$$

and the matrices $\hat{Q}_\mu$ $(\mu = 1, 2)$ have a form

$$[\hat{Q}_\mu]_{ij} = \left[\begin{array}{ll} 4 + \frac{C_{0\mu}+C_x}{C_\mu} & i = j \\ -1 & i \text{ and } j \text{ are nearest neighbors} \\ 0 & \text{otherwise} \end{array} \right. .$$

Since the array 2 is supposed to be in the quasi-classical regime, only weakly time-dependent periodic paths $\phi_{i2}(\tau)$ are important. Hence we may write the phases in the form $\phi_{i2}(\tau) = \phi_{i2}^{(0)} + f_i(\tau)$, where

$$f_i(\tau) = \beta^{-1} \sum_{n=1}^{\infty} [f_i(\omega_n) \exp(-i\omega_n\tau) + f_i^*(\omega_n) \exp(i\omega_n\tau)].$$

is expressed as a sum over Matsubara frequencies $\omega_n = 2\pi n\beta^{-1}$. Now the action may be expanded to quadratic terms in $f_i(\tau)$, yielding

$$\begin{aligned} \tilde{S}\{\phi_{i2}^{(0)}, f_i(\tau)\} = \beta E_{J2} \sum_{\langle ij \rangle}[1 - \cos(\phi_{i2}^{(0)} - \phi_{j2}^{(0)})] + \\ + \int_0^\beta d\tau \left\{ \frac{1}{8e^2} \sum_{ij} \dot{f}_i[\Lambda_2]_{ij}\dot{f}_j + \frac{E_{J2}}{2} \sum_{\langle ij \rangle}[f_i(\tau) - f_j(\tau)]^2 \cos(\phi_{i2}^{(0)} - \phi_{j2}^{(0)}) \right\}. \end{aligned} \tag{6}$$

Note that the first term is the classical action of 2D Coulomb gas [3]. Finally, one performs the cumulant expansion [7,16] in the last term in brackets in Eq. (6). As a result the action has exactly the same form as the classical one, but with the renormalized temperature

$$\beta \to \beta^{-1} - \frac{1}{2} \int_0^\beta d\tau \left\langle (f_i(\tau) - f_j(\tau))^2 \right\rangle. \tag{7}$$

Here the angular brackets denote the averaging with the effective action

$$S_{eff} = (8e^2)^{-1} \int_0^\beta d\tau \sum_{ij} \dot{f}_i(\tau)[\Lambda_2]_{ij}\dot{f}_j(\tau).$$

From this we obtain the shift of the transition temperature due to the interlayer coupling

$$T_J = T_{J0} - e^2 A/3, \tag{8}$$

where

$$A = \mathrm{Re}([\Lambda_2^{-1}]_{ii} - [\Lambda_2^{-1}]_{\langle ij\rangle}),$$

and the second term is the matrix element taken for neighboring islands. (We have assumed that the matrix Λ_2 depends on the distance between the islands only). Here T_{J0} is the transition temperature for a classical 2D Josephson junction array [3] (to be of order E_{J2}). Eq. (8) is the result we were aiming at, however in order to obtain some analytical expressions we evaluate the quantity A in some approximations. In Fourier representation the matrices $Q_\mu(\mathbf{k})$ have the form

$$Q_\mu(\mathbf{k}) = (ka)^2 + (C_\mu)^{-1}(C_x + C_{0\mu}), \quad ka \le 1,$$

with a being the lattice parameter. Consequently the matrix Λ_2^{-1} is

$$\Lambda_2^{-1}(k) = \frac{C_1(ka)^2 + C_x + C_{01}}{(C_1(ka)^2 + C_x + C_{01})(C_2(ka)^2 + C_x + C_{02}) - C_x^2}. \tag{9}$$

If we replace the first Brillouin zone by a circle cut-off at $k < a^{-1}$, the integration over the angular variable can be performed easily, and we obtain

$$A = \frac{a^2}{2\pi}\int_0^{1/a} kdk\Lambda_2^{-1}(k)[1 - J_0(ka)].$$

In the range of integration the Bessel function J_0 can be approximated by its expansion

$$J_0(x) \approx 1 - x^2/4.$$

Finally, in the case $C_{0\mu} \ll C_x$ (this situation is the most interesting) we obtain

$$T_{J0} - T_J = \frac{e^2}{48\pi C_{eff}}, \quad \frac{1}{C_{eff}} = \frac{1}{C_2} - \frac{C_x}{C_2^2}\ln\frac{C_xC_1 + C_xC_2}{C_xC_1 + C_xC_2 + C_1C_2}. \tag{10}$$

It is seen that the effect of layer 1 is merely the renormalization of the effective capacitance. As a result, the BKT transition for vortices in the layer 2 is suppressed. We should emphasize that the result (10) is valid for arbitrary capacitances C_x, C_1 and C_2. The only restriction is the validity of the quasi-classical approximation. The shift of the transition temperature should be small, or, in other words, $e^2/C_{eff} \ll E_{J2}$. In particular, for $C_x \ll C_2$ (weak coupling between the layers) one obtains $C_{eff} \approx C_2$ irrespectively of C_1 — vortices in layer 2 do not feel the presence of the layer 1. In the case $C_1 = C_2 \ll C_x$ the effective capacitance is $C_{eff} = 2C_2$, while for $C_1 = C_x \gg C_2$ one has $C_{eff} = 2C_x/3$. It is seen that in the latter case the temperature begins to feel the presence of the first layer, however the absolute value of the shift becomes now small.

Charge – vortex duality and BKT transition for charges

Before we turn to the description of the BKT transition for charges, it is necessary to stress the following. As shown by the effective action (2), the interlayer capacitance C_x not only couples the layers, but also renormalizes the capacitances C_{01} and C_{02} of the islands to the ground. Hence the logarithmic interaction between the charges in each layer has a finite range for any non-zero C_x due to the screening, and the BKT transition, is, strictly speaking, absent. One should realize, however, that in the situation $C_{01} \ll C_x \ll C_1$ the screening length $\xi_1 \sim a(C_1/C_x)^{1/2}$ can be very large. Below we assume that these inequalities are satisfied and the range of interaction ξ_1 is large enough to make it meaningful to speak about the charge-unbinding transition. (Note that like any phase transition in a finite system this transition is smeared; in other words, the resistance grows exponentially, and, strictly speaking, for finite ξ it is impossible to distinguish between insulating and resistive phases). On the other hand, already for relatively weak coupling $C_x \sim C_{cr} \ll C_1$ this description becomes meaningless and the insulating phase is absent. As we show below, in the small range $C_x \ll C_{cr}$ the transition temperature for charges in layer 1 does not feel the presence of the layer 2 and hence is essentially the charge-unbinding temperature for one Josephson junction array [10]. Nevertheless, the charge-vortex description required to obtain this result gives rise to an interesting physical model to be described below.

Now we move from the phase description (1),(2) to a charge-vortex description. First we introduce the large capacitance matrix

$$\hat{C} = \begin{pmatrix} \hat{C}_1 & -\hat{C}_x \\ -\hat{C}_x & \hat{C}_2 \end{pmatrix}. \tag{11}$$

Here $\hat{C}_\mu$ is the capacitance matrix in the array μ while $\hat{C}_x = C_x\delta_{ij}$. The inverse matrix in the Fourier representation reads as

$$C^{-1}(\mathrm{k}) = \frac{1}{(C_1(ka)^2 + C_x + C_{01})(C_2(ka)^2 + C_x + C_{02}) - C_x^2} \times \tag{12}$$
$$\times \begin{pmatrix} C_2(ka)^2 + C_x + C_{02} & C_x \\ C_x & C_1(ka)^2 + C_x + C_{01} \end{pmatrix} \equiv \begin{pmatrix} \hat{\Lambda}_1^{-1}(k) & \hat{\Lambda}_x^{-1}(k) \\ \hat{\Lambda}_x^{-1}(k) & \hat{\Lambda}_2^{-1}(k) \end{pmatrix} \equiv [\hat{C}^{-1}]^{\mu\nu}(k).$$

Indices $\mu, \nu = 1, 2$ again label the array. The matrix $\hat{C}^{-1}$ describes the interaction of charges. We have also introduced for later convenience the matrices Λ_μ^{-1}, describing the interaction of charges within layer μ, as well as Λ_x^{-1} referring to charges in different layers (cf. Eq. (9). Then the effective action (2) can be rewritten in terms of integer charges $q_{i\mu}$ of each island and phases $\phi_{i\mu}$

$$S\{q,\phi\} = \int_0^\beta d\tau \left\{ 2e^2 \sum_{ij} \sum_{\mu,\nu} q_{i\mu}(\tau)[\hat{C}^{-1}]_{ij}^{\mu\nu} q_{j\nu}(\tau) + \sum_i \left[q_{i1}(\tau)\dot{\phi}_{i1}(\tau) + q_{i2}(\tau)\dot{\phi}_{i2}(\tau) \right] + \right.$$
$$\left. + E_{J1} \sum_{\langle ij \rangle} [1 - \cos(\phi_{i1} - \phi_{j1})] + E_{J2} \sum_{\langle ij \rangle} [1 - \cos(\phi_{i2} - \phi_{j2})] \right\}. \tag{13}$$

Now it is possible to introduce vortex degrees of freedom by means of the Villain transformation [17] (see also [18]). It is important that this procedure deals only with the phase variables and does not affect the charge interaction (the first term in Eq. (13)). The phase terms (the second and the third one in Eq. (13)), however, have exactly the same form in the problem of two arrays as for a single Josephson junction layer. The procedure for a single-layer array array is discussed in details in Refs. [10,11], the generalization to the double-layer system is straightforward. The partition function becomes

$$Z = \prod_i \sum_{\{q_{i1},q_{i2}\}} \sum_{\{v_{i1},v_{i2}\}} \exp(-S\{q,v\}), \tag{14}$$

where the effective action for integer charges $q_{i\mu}$ and vorticities $v_{i\mu}$ is

$$\begin{aligned} S\{q,v\} = \int_0^\beta d\tau \Bigg\{ & 2e^2 \sum_{ij}\sum_{\mu,\nu} q_{i\mu}(\tau)[\hat{C}^{-1}]_{ij}^{\mu\nu} q_{j\nu}(\tau) + \frac{1}{4\pi E_{J1} F(\epsilon_1 E_{J1})} \sum_{ij} \dot{q}_{i1}(\tau) G_{ij}^{(1)} \dot{q}_{j1}(\tau) + \\ & + \frac{1}{4\pi E_{J2}} \sum_{ij} \dot{q}_{i2}(\tau) G_{ij}^{(2)} \dot{q}_{j2}(\tau) + \pi E_{J1} F(\epsilon_1 E_{J1}) \sum_{ij} v_{i1} G_{ij}^{(1)} v_{j1} + \pi E_{J2} \sum_{ij} v_{i2} G_{ij}^{(2)} v_{j2} + \\ & + i \sum_{ij} \dot{q}_{i1}(\tau) \Theta_{ij} v_{j1}(\tau) + i \sum_{ij} \dot{q}_{i2}(\tau) \Theta_{ij} v_{j2}(\tau) \Bigg\}. \end{aligned} \tag{15}$$

Here we introduced the discrete time variable; the time lattice spacing in the array μ is of order $\epsilon_\mu \sim (8E_{J\mu}E_{C\mu})^{-1/2}$, $E_{C\mu} \equiv e^2/2C_\mu$. The time integration and derivatives are continuous notations for a summation over time lattice and for a discrete derivative

$$\dot{f}(\tau) = \epsilon_\mu^{-1}[f(\tau+\epsilon_\mu) - f(\tau)],$$

respectively. The function

$$F(x) = \frac{1}{2x \ln(J_0(x)/J_1(x))} \to \frac{1}{2x\ln(4/x)}, \quad x \ll 1,$$

is introduced to "correct" the Villain transformation for small E_J [17]. As we see, its entire effect is to renormalize (to increase) the Josephson coupling in the layer 1; the renormalized coupling $\tilde{E}_{J1}$ reads as

$$\tilde{E}_{J1} \sim (8E_{J1}E_{C1})^{1/2} (\ln(E_{C1}/E_{J1}))^{-1}. \tag{16}$$

Note that for $E_{J1} \ll E_{C1}$ (charge regime) one obtains $\tilde{E}_{J1} \ll E_{C1}$.

The kernel

$$\Theta_{ij} = \arctan(\frac{y_i - y_j}{x_i - x_j})$$

describes the phase configuration at site i around a vortex at site j. Finally, the kernel $G_{ij}^{(\mu)}$ is the lattice Green's function, i.e. the Fourier transform of k^{-2}. At large distances between the sites i and j it depends only on the distance r between the sites and has a form (see e.g. [13])

$$G_{ij}^{(\mu)} = \ln(\xi_\mu/r), \quad a \ll r \ll \xi_\mu, \quad \xi_\mu = a(C_\mu/C_x)^{1/2}. \tag{17}$$

Later on, we assume that the linear size of each array is much less that the range of interaction ξ_μ. This means, in particular, that we assume $C_x \ll C_2$.

The action (15) depends on the charges and vorticities in both layers. However, in our situation, when the layers 1 and 2 are in the charge and vortex regimes, respectively, the vortices in the layer 1 and the charges in the layer 2 may be integrated out [10]. To do this we suppose the latter variables to be continuous (strongly fluctuating), and neglect the kinetic term for charges in the layer 2 ($\dot{q}_2 G^{(2)} \dot{q}_2$). Then after performing the Gaussian integration we obtain the effective action for charges q_{i1} in the layer 1 and vorticities v_{i2} in the layer 2 (to be referred below as q_i and v_i)

$$S\{q,v\} = \int_0^\beta d\tau \left\{ \frac{2E_{C1}}{\pi} \sum_{ij} q_i(\tau) G_{ij}^{(1)} q_j(\tau) + \frac{1}{4\pi \tilde{E}_{J1}} \sum_{ij} \dot{q}_i(\tau) G_{ij}^{(1)} \dot{q}_j(\tau) + \pi E_{J2} \sum_{ij} v_i G_{ij}^{(2)} v_j + \right.$$

$$\left. + \frac{\pi}{8E_{C2}} \sum_{ij} \dot{v}_i(\tau) \left[G_{ij}^{(2)} - \frac{C_x^2}{4\pi^2 C_1 C_2} \sum_{kl} \Theta_{ik} G_{kl}^{(1)} \Theta_{lj} \right] \dot{v}_j(\tau) + \frac{iC_x}{2\pi C_1} \sum_{ijk} \dot{v}_i(\tau) \Theta_{ik} G_{kj}^{(1)} q_j(\tau) \right\}. \quad (18)$$

To derive Eq. (18) we have taken into account the explicit expression for the large capacitance matrix (12).

The action (18) is the central result of this section. It looks rather similar to the effective charge-vortex action in one Josephson junction, but the most important difference is that while in one layer either charges or vortices are well-defined, Eq. (18) describes the system of *well-defined* dynamic variables on each site — charges in the layer 1 and vortices in the layer 2. We postpone the discussion of physics in this system until the next section, however it is clear that the action shows a duality between charges and vortices. The second term in the square brackets is small if $C_x \ll C_1, C_2$. Both kinetic terms for charges and vortices violate the duality due to the numerical coefficients. However close enough to the transitions these terms produce only small renormalization of the transition temperature, and are not important. Another interesting feature of this action is that the last term, describing the interaction between charges and vortices, is also small, while in a single-layer array the interaction is always of the same order of magnitude as another terms.

It is obvious that for long-range interaction of the charges in the layer 1 they also exhibit the BKT transition, and under the conditions where the action (18) was obtained the transition temperature does not feel the presence of the layer 2:

$$T_{C0} - T_C = \frac{\tilde{E}_{J1}}{24\pi}.$$

Charge and vortex motion

To understand the physics described by the action (18) it is instructive to map this model onto the 2D Coulomb gas. For this purpose we move from the space-time lattice to the continuous medium and introduce the coordinates of the vortex centers and charges

$$q_i(\tau) \to \sum_m q_m \delta(\mathbf{r} - \mathbf{r}_m(\tau))$$
$$v_i(\tau) \to \sum_n v_n \delta(\mathbf{r} - \mathbf{R}_n(\tau)). \quad (19)$$

Here $q_m = \pm 1$ and $v_n = \pm 1$ represent charges and vortices respectively; $\mathbf{r}_m(\tau)$ and $\mathbf{R}_n(\tau)$ are the corresponding coordinates of the charges and of the vortex centers. Now the partition function reads

$$Z = \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} \int D\mathbf{r}_1(\tau) \dots D\mathbf{r}_{2M}(\tau) D\mathbf{R}_1(\tau) \dots D\mathbf{R}_{2N}(\tau) \exp(-S\{\mathbf{r}, \mathbf{R}\}), \tag{20}$$

and we are going to deal with the effective action $S\{\mathbf{r}, \mathbf{R}\}$, describing the behavior of the system of $2M$ charges (of which M are positive, $q = 1$, and the other M are negative, $q = -1$), and of N positive ($v = 1$) and N negative ($v = -1$) vortices.

The first and third terms of the action (18) can be easily transformed by means of decomposition (19). The first one produces the potential energy of charge interaction,

$$S_{int}^{(q)} = \frac{2E_{C1}}{\pi} \int_0^\beta d\tau \sum_{m,n=1}^{2M} q_m q_n G^{(1)}(\mathbf{r}_m(\tau) - \mathbf{r}_n(\tau)). \tag{21}$$

In principle, the summation includes the terms with $m = n$; these, however, may be excluded from this sum, giving rise to the chemical potential for charges. The third term in Eq. (18) yields the interaction of vortices

$$S_{int}^{(v)} = \pi E_{J2} \int_0^\beta d\tau \sum_{m,n=1}^{2N} v_m v_n G^{(1)}(\mathbf{R}_m(\tau) - \mathbf{R}_n(\tau)). \tag{22}$$

Here again the term with $m = n$ gives rise to the chemical potential for vortices. The terms (21) and (22) are essentially the action for (classical) Coulomb gases of charges and vortices, respectively [3].

If we neglect the small correction proportional to the C_x^2/C_1C_2 in the fourth term in Eq.(18) then the second and fourth terms can be transformed to the kinetic energy of charges and vortices respectively [10]. The second term gives

$$S_{kin}^{(q)} = \frac{1}{2\pi \tilde{E}_{J1}} \int_0^\beta d\tau \sum_{m,n=1}^{2M} q_m q_n \dot{r}_m^\gamma M_{\gamma\delta}(\mathbf{r}_m - \mathbf{r}_n) \dot{r}_n^\delta. \tag{23}$$

We have introduced the mass tensor [19]

$$M_{\gamma\delta}(\mathbf{r}) = -\nabla_\gamma \nabla_\delta G^{(\mu)}(\mathbf{r}). \tag{24}$$

It decreases proportional to r^{-2} for $r \gg a$, and consequently may be approximated by a local function

$$M_{\gamma\delta}(\mathbf{r}) = M\delta_{\gamma\delta}\delta(\mathbf{r}), \quad M = \frac{\pi}{a^2}.$$

Then the kinetic term for charges takes a simple form

$$S_{kin}^{(q)} = \frac{1}{2a^2 \tilde{E}_{J1}} \int_0^\beta d\tau \sum_{m=1}^{2M} \dot{r}_m^2(\tau). \tag{25}$$

Similarly, the fourth term in Eq.(18) produces the kinetic term for vortices

$$S_{kin}^{(v)} = \frac{\pi^2}{8a^2 E_{C2}} \int_0^\beta d\tau \sum_{m=1}^{2N} \dot{R}_m^2(\tau). \tag{26}$$

Finally, the last term in Eq.(18)) is responsible for the interaction between charges and vortices. The corresponding term in S{**r**,**R**} is

$$S_{qv} = \frac{iC_x}{2\pi C_1 a^2} \int_0^\beta d\tau \sum_{mn} v_m q_n \int d\mathbf{r}' \nabla_{\mathbf{R}_m} \Theta(\mathbf{R}_m - \mathbf{r}') G^{(1)}(\mathbf{r}' - \mathbf{r}_n) \dot{\mathbf{R}}_m(\tau). \tag{27}$$

The integral over **r** can be calculated explicitly, yielding

$$S_{qv} = -\int_0^\beta d\tau \sum_m i v_m \dot{\mathbf{R}}_m(\tau) \mathbf{A}_m(\mathbf{R}_m) \tag{28}$$

with

$$\mathbf{A}_m(\mathbf{R}_m) = \sum_n q_n \mathbf{a}(\mathbf{R}_m(\tau) - \mathbf{r}_n(\tau)),$$

$$\mathbf{a}(\mathbf{r}) = -\frac{1}{4a^2} \frac{C_x}{C_1} (1 + 2\ln(\xi_1/r))[\hat{z} \times \mathbf{r}].$$

The resulting action is

$$S\{\mathbf{r}, \mathbf{R}\} = S_{int}^{(q)} + S_{int}^{(v)} + S_{kin}^{(q)} + S_{kin}^{(v)} + S_{qv}. \tag{29}$$

The action (29) is essentially that of two 2D Coulomb systems. The charges and the vortices can be viewed as particles with masses

$$M_q = \frac{1}{a^2 \tilde{E}_{J1}} \quad \text{and} \quad M_v = \frac{\pi^2}{4a^2 E_{C2}} \tag{30}$$

respectively. Charges interact via the effective capacitance, vortices via the usual logarithmic interaction with strength E_{J2}. Furthermore, the vortices produce the vector potential **a** for the charges[3]; the magnetic field associated with this vector potential is

$$B = \pm \frac{1}{ea^2} \frac{C_x}{C_1} \ln \frac{\xi_1}{r}, \quad a \ll r \ll \xi_1. \tag{31}$$

Its sign depends of the signs of the corresponding vortex and charge. Apart from its quite peculiar functional form, another important feature of this field is the small factor C_x/C_1.

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[3]This seeming asymmetry is rather artificial. In Eq.(18) one can rewrite after a partial integration the charge-vortex interaction term in order to obtain the vector potential for vortices, created by charges, as well. However, this vector potential contains always the small factor C_x/C_1.

Summary

We have investigated the system of two 2D Josephson junction arrays coupled by capacitances C_x, in the situation when the arrays 1 and 2 are in the charge and vortex regime, respectively. In the case of weak coupling $C_x \ll C_1, C_2$ the system shows an (approximate) duality between dynamical charges in one layer and dynamical vortices in the other one. In contrast to a single layer array, both variables are well-defined. The system is equivalent to two 2D Coulomb gases of massful particles. The charges feel the magnetic field created by vortices, and, vice versa, the vortices feel the gauge field created by charges. In this respect the system resembles the composite fermion model of the fractional quantum Hall effect, however the magnetic field is now small and has another functional form, so one may expect different physics. In this regime the system shows two BKT transitions, one for charges and another for vortices, and the coupling between the layers suppresses both transitions. Although one could expect the suppression of one transition and the enhancement of another one, the suppression of both transitions is rather natural, since the capacitance C_x also renormalizes the capacitances of the islands to the ground. The BKT transition for charges vanishes even for very small values of C_x, however, the BKT transition for vortices survives under condition $e^2/C_{eff} \ll E_{J2}$ irrespective of the relations between the capacitances C_1, C_2 and C_x. The shift of this temperature due to the capacitance effects is calculated within the phase representation for both cases of weak and strong coupling. The effect of the layer 1 is to renormalize the capacitance matrix in the layer 2. For weak coupling $C_x \ll C_2$ (irrespective of C_1) the vortices do not feel the presence of another layer, and the temperature remains the same as for one layer. However, for $C_x \gg C_2$ different situations are possible.

In summary, we would like to emphasize that the system of two coupled Josephson junction arrays may exhibit quite rich and interesting physics. We have investigated some limiting cases, however, the further rich behavior of this system can be expected in other cases. In particular, the magnetic field created by vortices seems to be rather unusual and interesting. We hope that experimental studies of this system will be performed in the near future.

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