How smooth are subdividable surfaces at extraordinary points ?

Hartmut Prautzsch
Universität Karlsruhe, IBDS
D-76128 Karlsruhe
Germany

e-mail: prau@ira.uka.de

November 3, 1997

Abstract

A stationary subdivision scheme as Catmull and Clark's is described by a matrix iteration around an extraordinary point. We show how higher order smoothness of a limiting surface depends on the spectral properties of the matrix and give necessary and sufficient conditions.

The results are also useful to construct subdivision algorithms for surfaces of any smoothness order at extraordinary points.

Keywords

Subdivision, extraordinary points, regular G^k-surfaces, matrix iteration.

1 Introduction

Fifteen years after the first efforts to analyze the behaviour of Catmull and Clark's subdivision algorithm at extraordinary points Ulrich Reif ['93] showed that all arguments given earlier by Doo & Sabin ['78], Loop ['87], Ball & Storry ['88], and others to verify tangent plane continuity are incomplete.

One year later in 1994 Reif derived a degree estimate explaining why previous attempts to find subdividable curvature continuous surfaces failed.

However, despite all efforts since 1978 two questions remained:

(1) Is there a rigorous method to show how smooth a subdividable surface is at extraordinary points?

(2) And, are there subdivision algorithms for arbitrary control nets generating G^k-surfaces?

Both questions were answered positively in 1995. Developing and building on some ideas of Reif ['94] I resolved the first, while the second has been solved simultaneously by Reif ['97] and myself ['97].

The purpose of this paper is to publish the smoothness characterization presented in 1995 [Prautzsch '95]. In the meantime we have applied these results successfully to (1) improve Catmull and Clark's, Qu's, Loop's and the butterfly algorithm [Prautzsch & Umlauf '97a, '97b], and (2) to extend the degree estimate by Reif ['94] to subdivision surfaces of arbitrary high smoothness [Prautzsch & Reif '97].

To introduce the notation and the problem I first present a simple class of subdivision schemes derived from uniform tensor product spline subdivision. It serves as an example for the following general smoothness analysis, which applies to all stationary subdivision schemes. In particular, it also applies to subdivision algorithms on triangular nets.

The concepts used in the analysis are new in that a parametrization is used under which the subdivision surfaces are self-similar under scaling.

2 The midpoint scheme

The subdivision algorithm for uniform tensor product splines [Lane & Riesenfeld '80] has a simple generalization. We describe it by two operators.

The **refinement operator** R maps any control net C of arbitrary topology to the net RC that connects the midpoints of all edges in C with both their endpoints and the centroids of both abutting meshes.

The **averaging operator** A maps any net C to the net AC that connects the centroids of all meshes sharing a common edge.

As an illustration, Figure 1 shows some net C (light edges), the net RC (light edges and broken edges) and the net ARC (heavy edges).

Figure 1: Refining and averaging a net.

The net operator $M_n = A^{n-1}R$, which refines a net and averages it $(n-1)$-times successively is called the **midpoint subdivision operator**. We say that the sequence of nets $M_n^i C, i \in \mathbb{N}$, is obtained from C under the **midpoint scheme** M_n.

In particular, M_2 and M_3 represent specific instances of the Doo-Sabin and the Catmull-Clark algorithm, respectively. A sequence of nets obtained under M_3 is shown in Figure 2.

Note that if n is odd, then the nets $M_n^i C$ have no **extraordinary meshes**, that means non-quadrilateral meshes, and if n is even, then the nets have no

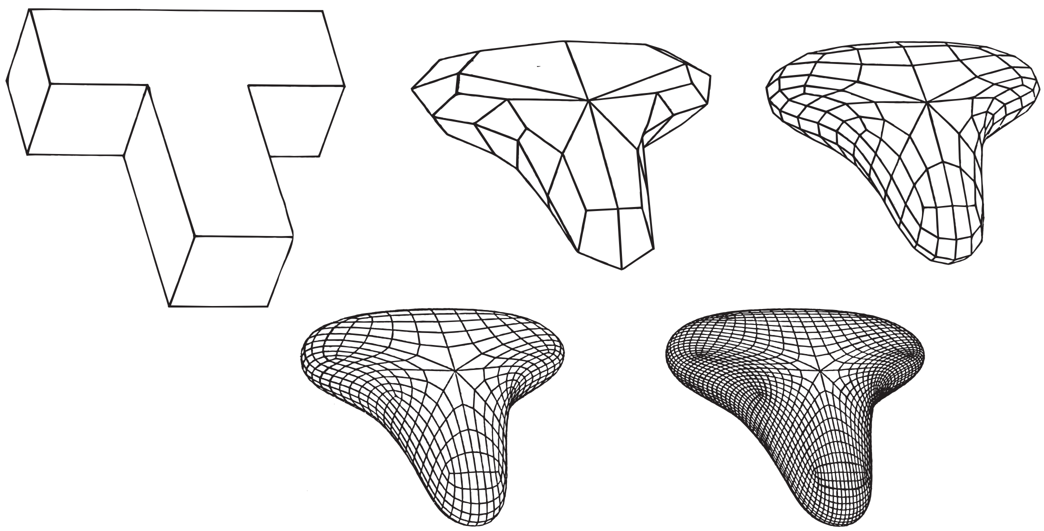

Figure 2: Subdivision under the Catmull-Clark algorithm.

extraordinary vertices, that means interior vertices with valence $\neq 4$.

Furthermore, each extraordinarity of any net $M_n^i C$ is obtained by affine combinations of vertices around an extraordinarity of the preceding net $M_n^{i-1} C$ called the **corresponding extraordinarity**.

3 The limiting surface

On a regular quadrilateral net the midpoint scheme does nothing else than the subdivision algorithm for uniform tensor product spline surfaces of bidegree n. Consequently any net sequence obtained under the midpoint scheme converges to a piecewise polynomial surface $\mathbf{s}$, where the sequences of corresponding extraordinarities converge to the so-called **extraordinary points** of $\mathbf{s}$.

In the sequel we study the limiting surface at an extraordinary point. Therefore, without loss of generality, we assume from now on, that the nets $M^i C$ have just one m-sided mesh or vertex of valence m surrounded by several rings of quadrilateral meshes as illustrated in Figure 3.

Each regular subnet of $n \times n$ quadrilateral meshes of a net $M^i C$ forms the

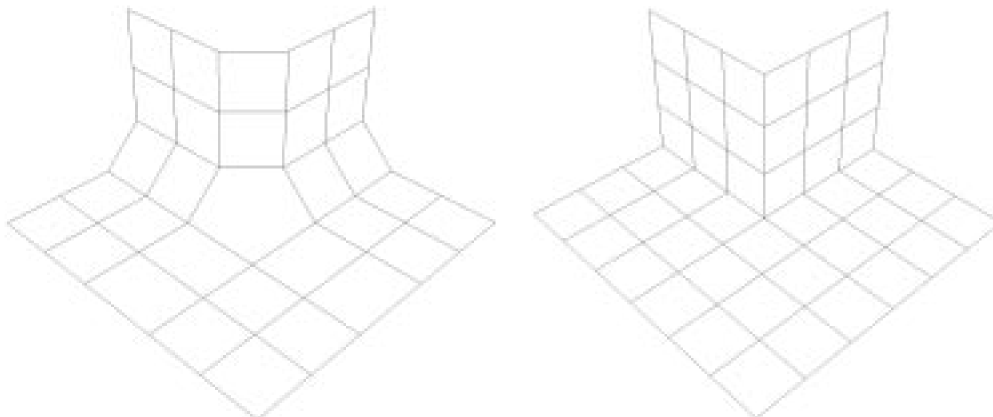

Figure 3: Control nets with one extraordinarity.

B-spline control net of some polynomial patch of $\mathbf{s}$. The patches defined by all these subnets of M^iC form an $(n-1)$-times differentiable surface $\mathbf{s}_i$ which is part of the limiting surface $\mathbf{s}$. Moreover the surface $\mathbf{s}_i$ without the surface $\mathbf{s}_{i-1}$ forms a surface ring $\mathbf{r}_i$ consisting of $3m\rho_n^2$ patches, where

$$\rho_n = \begin{cases} n-1, & \text{if } n \text{ is} \quad \text{even} \\ n-2, & \phantom{\text{if } n \text{ is}} \quad \text{odd} \end{cases} .$$

Together the rings $\mathbf{r}_i$ form the limiting surface $\mathbf{s}$. We can partition the surface rings $\mathbf{r}_i$ into $3m$ macro patches $\mathbf{r}_i^1, \ldots, \mathbf{r}_i^{3m}$, where each patch consists of $\rho_n \times \rho_n$ patches. This is illustrated schematically in Figure 4, where $m = 5$. The dashed lines show the patches of $\mathbf{r}_1^5$ for $\rho_n = 3$.

4 The standard parametrization

Every macro patch can be parametrized over $[0,1]^2$ and an entire ring $\mathbf{r}_i$ over $3m$ copies of $[0,1]^2$. Therefore we can parametrize $\mathbf{r}_i$ over $\Omega = \{1, \ldots, 3m\} \times [0,1]^2$, which means

$$\mathbf{r}_i : \Omega \to \mathbb{R}^3 \quad , \quad (j|u,v) \mapsto \mathbf{r}_i(j|u,v) = \mathbf{r}_i^j(u,v) \ .$$

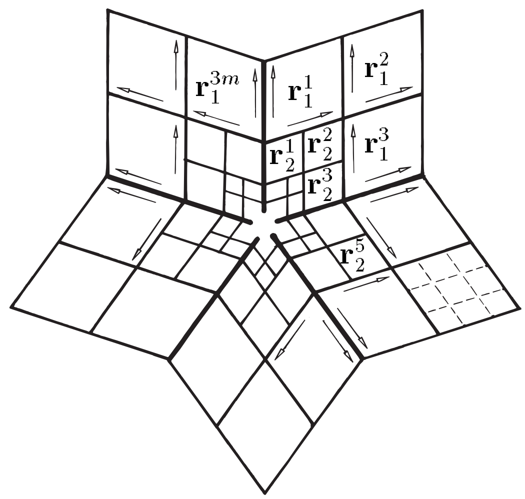

Figure 4: The adjacency of the patches $\mathbf{r}_i^j$ for $m = 5$ and $\rho = 1, 3$.

Together, all rings form a C^{n-1}-surface. This property is captured by the following smoothness conditions:

$$
(4.1) \quad \begin{array}{llll}
\frac{\partial^k}{\partial v^k}\mathbf{r}_{i+1}^j(u,1) = \frac{\partial^k}{\partial v^k}\mathbf{r}_i^j(\frac{u}{2},0) & & & \\
\frac{\partial^k}{\partial v^k}\mathbf{r}_{i+1}^{j+1}(u,1) = \frac{\partial^k}{\partial v^k}\mathbf{r}_i^j(\frac{1}{2}+\frac{u}{2},0) & \frac{\partial^k}{\partial u^k}\mathbf{r}_i^j(1,v) & = & \frac{\partial^k}{\partial u^k}\mathbf{r}_i^{j+1}(0,v) \\
\frac{\partial^k}{\partial u^k}\mathbf{r}_{i+1}^{j+1}(1,v) = \frac{\partial^k}{\partial u^k}\mathbf{r}_i^{j+2}(0,\frac{1}{2}+\frac{v}{2}) & \frac{\partial^k}{\partial v^k}\mathbf{r}_i^{j+1}(u,0) & = & \frac{\partial^k}{\partial v^k}\mathbf{r}_i^{j+2}(u,1) \\
\frac{\partial^k}{\partial u^k}\mathbf{r}_{i+1}^{j+2}(1,v) = \frac{\partial^k}{\partial u^k}\mathbf{r}_i^{j+2}(1,\frac{v}{2}) & \frac{\partial^k}{\partial v^k}\mathbf{r}_i^{j+2}(u,0) & = & -\frac{\partial^k}{\partial u^k}\mathbf{r}_i^{j+3}(0,u)
\end{array}
$$

for all $u, v \in [0,1]$, $i \in \mathbb{N}$, $k = 0, 1, \ldots, n-1$, and $j = 1, 4, 7, \ldots, 3n-2$, where $\mathbf{r}_i^{3n+1} = \mathbf{r}_i^1$.

Moreover, if $\mathbf{c}_1, \ldots, \mathbf{c}_p$ denote the control points of $\mathbf{r}_i$, then $\mathbf{r}_i$ can also be written as

$$\mathbf{r}_i(j|u,v) = \sum_{l=1}^{p} \mathbf{c}_l B_l(j|u,v) \ ,$$

where $B_l(j|u,v)$ for each fixed j is some segment of a tensor product B-spline in u and v.

Example: Using the numbering given in Figures 3 and 4 we obtain for the midpoint scheme $M_2, j = 2$ and $u, v \in [0,1]$

$$\begin{bmatrix} B_1 & B_2 & B_3 \\ B_4 & B_5 & B_6 \\ B_7 & B_8 & B_9 \end{bmatrix} = \begin{bmatrix} N_0(u)N_{-2}(v) & \cdots & N_0(u)N_0(v) \\ \vdots & & \vdots \\ N_{-2}(u)N_{-2}(v) & \cdots & N_{-2}(u)N_0(v) \end{bmatrix}$$

and for $j = 3$ and $u, v \in [0,1]$

$$\begin{bmatrix} B_4 & B_5 & B_6 \\ B_7 & B_8 & B_9 \\ B_{10} & B_{11} & B_{12} \end{bmatrix} = \begin{bmatrix} N_0(u)N_{-2}(v) & \cdots & N_0(u)N_0(v) \\ \vdots & & \vdots \\ N_{-2}(u)N_{-2}(v) & \cdots & N_{-2}(u)N_0(v) \end{bmatrix} ,$$

where N_l denotes the uniform B-spline of degree 2 over the knots $l, l+1, l+2, l+3$.

5 Matrix iteration

Any surface ring $\mathbf{r}_i : \Omega \to \mathbb{R}^3$ has three coordinate functions and all these coordinate functions form the linear space

$$\mathcal{R} := \{\sum_{l=1}^{p} c_l B_l(j|u,v) | c_l \in \mathbb{R}\} \ .$$

Since the midpoint scheme is given by affine combinations, there is a linear operator ϕ on $\mathcal{R}$ such that

$$\phi(\mathbf{r}_i) = \mathbf{r}_{i+1} \ ,$$

where ϕ is to be applied to each coordinate of $\mathbf{r}_i$ separately.

The subdivision matrix representing ϕ with respect to the basis functions B_l is stochastic (i.e., it has non-negative entries with row sums one.) Therefore the leading eigenvalue of ϕ is one with the constant functions as associated eigenvectors.

One can check that some power of the subdivision matrix has a strictly positive row, which implies that one is a single eigenvalue of ϕ, see [Micchelli & Prautzsch '89, Thm. 2.1].

Consequently for any $\mathbf{r} \in \mathcal{R}^3$ the surface rings $\phi^i(\mathbf{r})$ converge to a constant map over Ω, which represents the extraordinary point of the subdivision surface formed by all these rings.

6 Characteristic maps

From now on let $\mathcal{R}$ be an arbitrary real linear space of functions $r : \Omega \to \mathbb{R}$, and let ϕ be an arbitrary linear map on $\mathcal{R}$ such that all surface rings $\mathbf{r}$ and $\phi(\mathbf{r})$ in $\mathcal{R}^3$ satisfy the smoothness conditions (4.1).

We suppose that any sequence $\phi^i(\mathbf{r}), \mathbf{r} \in \mathcal{R}^3$, converges to a point $\mathbf{r}_\infty$. This property is equivalent to the fact that the dominant eigenvalue of ϕ equals one and is simple, see e.g. [Micchelli & Prautzsch '89, p. 844].

Further, let $1, \lambda, \mu, \ldots$ be the eigenvalues of ϕ listed with all their algebraic multiples and ordered by their modulus. If there are two real eigenvectors x and y associated with λ and μ, respectively, the map $\mathbf{x} = (x, y) : \Omega \to \mathbb{R}^2$ is called a **characteristic map** of ϕ, see [Reif '95].

The Doo-Sabin, the Catmull-Clark and the midpoint algorithm have a characteristic map, see [Reif '95, Umlauf '96].

7 Triangular nets

The set up described in Section 6 is general enough to cover also subdivision schemes for triangular nets such as the butterfly or Loop's algorithm.

For example, the surfaces obtained by Loop's algorithm consist of triangular polynomial patches. Around an extraordinary point these triangular patches are arranged as illustrated in Figure 5 for $m = 5$. The dotted patches form a surface ring $\mathbf{r}_i$ which can be partitioned into $3m$ quadrilateral patches $\mathbf{r}_i^j$ as in Figure 4. Repartitioning the remaining surface similarly we obtain the tessellation of Figure 4. The broken lines indicate again that the triangular patches may be macro patches consisting of smaller patches.

8 Smoothness at extraordinary points

Under the assumptions made in Section 6 we are able to show the following theorem.

Theorem 8.1 *(For $k = 1$ see also [Reif '95])*
For almost all surface rings $\mathbf{r}_0 \in \mathcal{R}^3$ the iterates $\mathbf{r}_i = \phi^i(\mathbf{r}_0)$ form a regular

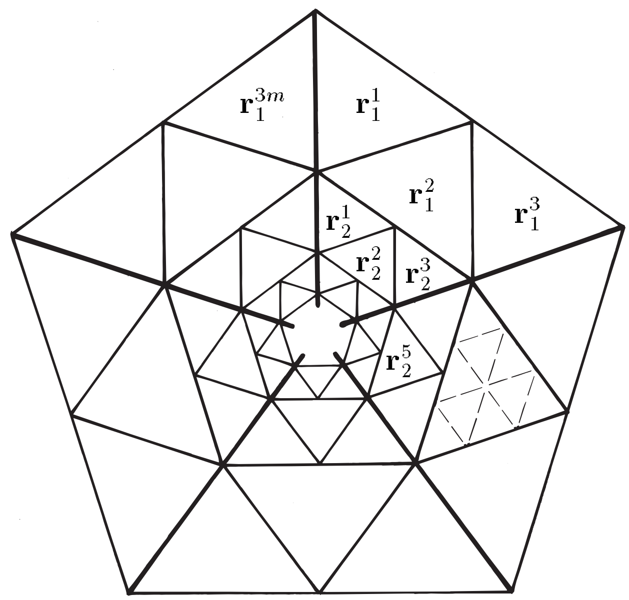

Figure 5: The patch configuration around an extraordinary point in case of triangular patches.

G^k-surface in a neighborhood of the extraordinary point, where $1 \leq k < n$, if

- *ϕ has a regular invertible characteristic map $\mathbf{x} = (x, y)$,*
- *all eigenvalues ν of ϕ, with $|\lambda| \geq |\nu| \geq |\mu|^k$ have equal algebraic and geometric muliplicities, and the associated eigenspace is a subspace of* span $\{x^\alpha y^\beta | \lambda^\alpha \mu^\beta = \nu,\ \alpha \text{ and } \beta \in \mathbb{N}_0\}$.

Remark *The condition on the eigenvalues implies that an eigenvalue of ϕ is either of the form $\lambda^\alpha \mu^\beta$ or that its modulus is smaller than $|\mu|^k$.*

Proof

Since we suppose that $\mathbf{x}$ is regular and invertible, the scaled versions $\mathbf{x}_i = \phi^i(\mathbf{x}) = (\lambda^i x, \mu^i y)$ are also regular and invertible. Furthermore, $\mathbf{x}_i$ lies closer to the origin than $\mathbf{x}_{i-1}$. Hence all planar rings $\mathbf{x}_i$, $i \in \mathbb{N}$, form a parametrization of some neighborhood U of the origin $\mathbf{o}$ that does not contain the origin.

We parametrize the subdivision surface $\mathbf{s}$ consisting of the rings $\mathbf{r}_i$ and the extraordinary point $\mathbf{r}_\infty$ over $U \cup \{\mathbf{o}\}$ in the following way

$$(8.2) \qquad \mathbf{s}(\xi,\eta) = \begin{cases} \mathbf{r}_i \circ \mathbf{x}_i^{-1}(\xi,\eta) & \text{if} \quad (\xi,\eta) \in \mathbf{x}_i(\Omega) \\ \mathbf{r}_\infty & \text{if} \quad (\xi,\eta) = \mathbf{o} \end{cases} .$$

Now we show that any coordinate s of $\mathbf{s} : U \cup \{\mathbf{o}\} \to \mathbb{R}^3$ is differentiable: Each coordinate r_0 of $\mathbf{r}_0$ can be written as

$$r_0 = \sum_\gamma a_\gamma + b \ ,$$

where the a_γ are eigenvectors of ϕ of the form

$$a_\gamma = \sum \rho_{\alpha\beta} x^\alpha y^\beta \ , \quad \lambda^\alpha \mu^\beta = const. \ ,$$

with associated eigenvalue $\nu = \lambda^\alpha \mu^\beta$, and where

$$\sup_\Omega |\phi^i(b)| = o(|\mu|^{ik}) \ .$$

Since the iterates $\phi^i(a_\gamma) = \lambda^{i\alpha}\mu^{i\beta} a_\gamma$ are of the form

$$\phi^i(a_\gamma) = \sum \rho_{\alpha\beta} (\lambda^i x)^\alpha (\mu^i y)^\beta \ ,$$

they form a polynomial component of s under the parametrization (8.2), namely

$$(\phi^i(a_\gamma) \circ \mathbf{x}_i^{-1})(\xi,\eta) = \sum \rho_{\alpha\beta} \xi^\alpha \eta^\beta \ .$$

Further, the contraction rate of the sequence $\phi^i(b)$ is not slowed down under a fixed reparametrization and a differentiation since these are linear operations. Thus we have for all $\alpha, \beta \in \mathbb{N}_0$

$$\sup_{\mathbf{x}(\Omega)} \left| \frac{\partial^\alpha}{\partial \xi^\alpha} \frac{\partial^\beta}{\partial \eta^\beta} (\phi^i(b) \circ \mathbf{x}^{-1}) \right| = o(|\mu|^{ik}) \ .$$

Reparametrizing by $\mathbf{x}_i^{-1}$ instead of $\mathbf{x}^{-1}$ means to substitute $(\lambda^{-i}\xi, \mu^{-i}\eta)$, where $(\xi,\eta) \in \mathbf{x}_i(\Omega)$, for $(\xi,\eta) \in \mathbf{x}_1(\Omega)$. Thus taking the inner derivatives into account we obtain

$$\begin{aligned} \sup_{\mathbf{x}_i(\Omega)} \left| \frac{\partial^\alpha}{\partial \xi^\alpha} \frac{\partial^\beta}{\partial \eta^\beta} (\phi^i(b) \circ \mathbf{x}_i^{-1}) \right| &= o(|\lambda|^{-i\alpha} |\mu|^{i(k-\beta)}) \\ &= o(|\mu|^{i(k-\alpha-\beta)}) \\ &= o(||(\xi,\eta)||^{k-\alpha-\beta}) \ . \end{aligned}$$

Hence the iterates $\phi^i(b)$ form a component of s whose derivatives up to order k converge to zero as $(\xi, \eta) \to \mathbf{o}$.

To prove that $\mathbf{s}$ is regular around $\mathbf{r}_\infty$ we write

$$\mathbf{r}_0 = \mathbf{a} + \mathbf{b}x + \mathbf{c}y + \mathbf{z} \ ,$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ and $\mathbf{z} \in \mathcal{R}^3$ with $\sup_\Omega ||\phi^i(\mathbf{z})|| = o(|\mu|^i)$. Then it follows as above that

$$\mathbf{s}(\xi, \eta) = \mathbf{a} + \mathbf{b}\xi + \mathbf{c}\eta + \mathbf{r}(\xi, \eta) \ ,$$

where $||\mathbf{r}|| = o(||(\xi, \eta)||)$. This map is regular for small (ξ, η) provided that $\mathbf{b}$ and $\mathbf{c}$ are linearly independent. This completes the proof. ■

9 Remarks

The linear maps ϕ describing the Catmull-Clark algorithm around extraordinary points have all a regular invertible characteristic map, see [Peters & Reif '97]. Hence, Theorem 8.1 can be applied. This shows that the Catmull-Clark algorithm generates tangent plane continuous but in general no curvature continuous surfaces, see [Umlauf '96]. The same is true for Loop's algorithm, see [Umlauf '97].

Moreover, one can use Theorem 8.1 to design better subdivision schemes. For example one can diagonalize the linear maps ϕ of the Catmull-Clark scheme, change non-suitable eigenvalues and keep the same eigenvectors. This then results in subdivision schemes generating G^2-surfaces with zero curvature at extraordinary points, see [Prautzsch & Umlauf '97a]. Similarly one can improve the smoothness order of Qu's algorithm from G^1 to G^3 [Umlauf '96], of Loop's algorithm from G^1 to G^2 and of other algorithms [Umlauf 96, Prautzsch & Umlauf '97b].

It is also possible to build subdivision schemes for G^k-surfaces with non-zero curvature at extraordinary points: In [Prautzsch '97, Reif '97] the linear map ϕ has the eigenvalues $1, \lambda, \lambda, \lambda^2, \lambda^2, \lambda^2, 0, \dots, 0$ which means that ϕ is a projection onto the space $\text{span}\{x^\alpha y^\beta | \alpha + \beta = 0, 1, 2\}$. The surfaces generated are piecewise bisextic G^2-surfaces. The characteristic map $\mathbf{x} = (x, y)$, however, is piecewise bicubic.

Similarly one can build subdivision schemes for piecewise polynomial G^k-surfaces of bidegree $2k + 1$.

If the characteristic map in Theorem 8.1 exists, is regular and invertible, then the converse of Theorem 8.1 is also true, i.e., the differentiability of the parametrization (8.2) implies the spectral properties listed in Theorem 8.1. This follows directly from the Taylor expansion of $\mathbf{x}$ around $\mathbf{o}$, see [Prautzsch '95].

The degree estimates for piecewise polynomial subdivision surfaces in [Prautzsch & Reif '97] are based on the spectral properties given in Theorem 8.1. There are subdivision schemes that are derived from linear dependent "basis" functions, for example the box splines over the four direction mesh. Then it can happen that the rings $\mathbf{r}_i$ around an extraordinary point are not related under a linear map ϕ although their control nets are.

Nevertheless, it is always possible to group consecutive rings to larger rings that are related under a linear map [Reif '96].

10 Complex eigenvalues

So far we have assumed that the subdivision map ϕ has real subdominant eigenvalues λ and μ. In the sequel we assume that λ and μ are complex conjugate eigenvalues of ϕ and that x and $\overline{x} = y$ are associated complex conjugate eigen vectors.

The **characteristic map** of ϕ is then given by the map $\mathbf{x} = (\mathrm{Re}\ x, \mathrm{Im}\ x) : \Omega \to \mathbb{R}^2$. This extends the definition in Section 6.

Theorem 8.1 remains valid also if the subdominant eigenvalues λ and μ are complex conjugate, i.e., $\lambda = \overline{\mu}$. We can repeat the theorem, where we only change the definition of the characteristic map $\mathbf{x}$ and write $\overline{\lambda}$ rather than μ. The assumptions are still the same as in Section 6.

Theorem 10.1 *(For $k = 1$ see also [Reif '95])*
For almost all surface rings $\mathbf{r}_0 \in \mathcal{R}^3$ the iterates $\mathbf{r}_i = \phi^i(\mathbf{r}_0)$ form a regular G^k-surface in a neighborhood of the extraordinary point, where $1 \leq k < n$, if

- *ϕ has a regular invertible characteristic map $\mathbf{x} = (Re\ x, Im\ y)$,*
- *all eigenvalues ν of ϕ, with $|\lambda| \geq |\nu| \geq |\lambda|^k$ have equal algebraic and geometric muliplicities, and the associated eigenspace is a subspace of span $\{x^\alpha \overline{x}^\beta | \lambda^\alpha \overline{\lambda}^\beta = \nu,\ \alpha \text{ and } \beta \in \mathbb{N}_0\}$.*

Proof

We can use the proof of Theorem 8.1 with slight modifications:

Since we suppose that $\mathbf{x}$ is regular and invertible, the scaled and rotated versions $\mathbf{x}_i = \phi^i(\mathbf{x}) = (\text{Re } \lambda^i x, \text{Im } \lambda^i x)$ are also regular and invertible. Furthermore, $\mathbf{x}_i$ lies closer to the origin than $\mathbf{x}_{i-1}$. Hence all planar rings $\mathbf{x}_i, i \in \mathbb{N}$, form a parametrization of some neighborhood U of the origin $\mathbf{o}$ that does not contain the origin.

We parametrize the subdivision surface $\mathbf{s}$ consisting of the rings $\mathbf{r}_i$ and the extraordinary point $\mathbf{r}_\infty$ over $U \cup \{\mathbf{o}\}$ in the following way

$$(10.2) \qquad \mathbf{s}(\xi, \eta) = \begin{cases} \mathbf{r}_i \circ \mathbf{x}_i^{-1}(\xi, \eta) & \text{if} \quad (\xi, \eta) \in \mathbf{x}_i(\Omega) \\ \mathbf{r}_\infty & \text{if} \quad (\xi, \eta) = \mathbf{o} \end{cases} .$$

Now we show that any coordinate s of $\mathbf{s} : U \cup \{\mathbf{o}\} \to \mathbb{R}^3$ is differentiable:

Each coordinate r_0 of $\mathbf{r}_0$ can be written as

$$r_0 = \sum_\gamma (a_\gamma + \overline{a}_\gamma) + b \ ,$$

where the a_γ are eigenvectors of ϕ of the form

$$a_\gamma = \sum \rho_{\alpha\beta} x^\alpha \overline{x}^\beta \ , \quad \lambda^\alpha \overline{\lambda}^\beta = const. \ ,$$

with associated eigenvalue $\nu = \lambda^\alpha \overline{\lambda}^\beta$, and where

$$\sup_\Omega |\phi^i(b)| = o(|\lambda|^{ik}) \ .$$

Since the iterates $\phi^i(a_\gamma) = \lambda^{i\alpha} \overline{\lambda}^{i\beta} a_\gamma$ are of the form

$$\phi^i(a_\gamma) = \sum \rho_{\alpha\beta} (\lambda^i x)^\alpha (\overline{\lambda}^i \overline{x})^\beta \ ,$$

they form complex conjugate polynomial components of s under the parametrization (10.2), namely

$$(\phi^i(a_\gamma) \circ \mathbf{x}_i^{-1})(\xi, \eta) = \sum \rho_{\alpha\beta} (\xi + i\eta)^\alpha (\xi + i\eta)^\beta \ .$$

Following the proof of Theorem 8.1 exactly word for word we can show that the iterates $\phi^i(b)$ form a component of s whose derivatives up to order k converge to zero as $(\xi, \eta) \to \mathbf{o}$.

To prove that $\mathbf{s}$ is regular around $\mathbf{r}_\infty$ we write

$$\mathbf{r}_0 = \mathbf{a} + \mathbf{b}x + \overline{\mathbf{b}}\overline{x} + \mathbf{z} \ ,$$

where $\mathbf{a} \in \mathbb{R}^3, \mathbf{b} \in \mathbb{C}^3$, and $\mathbf{z} \in \mathcal{R}^3$ with $\sup_\Omega ||\phi^i \mathbf{z}|| = o(|\mu|^i)$. Then it follows as above that

$$\mathbf{s}(\xi,\eta) = \mathbf{a} + \mathbf{b}\xi + \mathbf{c}\eta + \mathbf{r}(\xi,\eta) \ ,$$

where $||\mathbf{r}|| = o(||(\xi,\eta)||)$. This map is regular for small (ξ,η) provided that $\mathbf{b}$ and $\overline{\mathbf{b}}$ are linearly independent. This completes the proof. ■

If the parametrization of $\mathbf{s}$ given in (10.2) is k-times differentiable for all initial rings $\mathbf{r}_0$, then the converse of Theorem 10.1 also holds, i.e., ϕ has the spectral properties required in the Theorem. See [Prautzsch '95] for a proof.

References

A.A. Ball and D.J.T. Storry (1988). Conditions for tangent plane continuity over recursively generated B-spline surfaces. *ACM Transactions on Graphics*, 7(2):83–102.

E. Catmull and J. Clark (1978). Recursively generated B-spline surfaces on arbitrary topological meshes. *Computer Aided Design*, 10(6):350–355.

D.W.H. Doo and M. Sabin (1978). Behaviour of recursive division surfaces near extraordinary points. *Computer Aided Design*, 10(6):356–360.

D.W.H. Doo (1978). A subdivision algorithm for smoothing down irregular shaped polyhedrons. In *Proceedings: Interactive Techniques in Computer Aided Design*, pages 157–165, Bologna, 1978. IEEE.

N. Dyn, J. Gregory, and D. Levin (1990). A butterfly subdivision scheme for surface interpolation with tension control. *ACM Transactions on Graphics*, 9:160–169.

J.M. Lane and R.F. Riesenfeld (1980). A theoretical development for the computer generation and display of piecewise polynomial surfaces. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 2(1):35–46.

C.T. Loop (1987). Smooth Subdivision Surfaces Based on Triangles. Master's thesis, University of Utah, 1987.

C.A. Micchelli and H. Prautzsch (1989). Uniform refinement of curves. *Linear Algebra and its Applications*, 114:841–870.

J. Peters and U.Reif (1997). Analysis of algorithms generalizing B-spline subdivision. preprint aviable at http://www.cs.perdue.edu/homes/jorg/ftp/readme.html, 1997.

H. Prautzsch and U. Reif (1997). Necessary conditions for subdivision surfaces. Report 97/04, Universität Stuttgart SFB 404, 1997. aviable at http://i33www.ira.uka.de.

H. Prautzsch and G. Umlauf (1997a). A G^2-subdivision algorithm. preprint, 1997. http://i33www.ira.uka.de.

H. Prautzsch and G. Umlauf (1997b). Improved triangular subdivision schemes. preprint, 1997. http://i33www.ira.uka.de.

H. Prautzsch (1995). C^k-analysis of subdivision surfaces at extraordinary points. technical report 1995, avaible at http://i33www.ira.uka.de, 1995.

H. Prautzsch (1997). Freeform Splines. *CAGD*, 14:201–206.

U. Reif (1993). *Neue Aspekte in der Theorie der Freiformflächen beliebiger Topologie*. PhD thesis, Mathematisches Institut A der Universität Stuttgart, 1993.

U. Reif (1995). Some new results on subdivision algorithms for meshes of arbitrary topology, in Wavelets and Multilevel Approximation. In C.K. Chui and L.L. Schumaker, editors (1995), *Series in Approximations and Decompositions*, volume 2, pages 367–374. World Scientific, 1995.

U. Reif (1996a). private communication, July 1996.

U. Reif (1996b). A Degree Estimate for Subdivision Surfaces of Higher Regularity. *Proc. of the AMS*, 124(7):2167–2174.

U. Reif (1997). TURBS – Topologically Unrestricted Rational B-splines. preprint 95-16, 1997. institut@mathematik.uni-stuttgart.de, to appear in Constructive Approximation.

G. Umlauf (1996). Verbesserung der Glattheitsordnung von Unterteilungsalgorithmen für Flächen beliebiger Topologie. Master's thesis, Universität Karlsruhe, Fakultät für Informatik, 1996.

G. Umlauf (1997). The characteristic map of triangular subdivision schemes. preprint, 1997.