Lower and upper bounds for (sums of) binomial coefficients

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Abstract

Apparently there is no closed form for the partial sum of a row of PASCAL's triangle. In this paper lower and upper bounds for binomial coefficients and their sums are deduced. In the case of single coefficients these bounds differ only by a constant factor which is arbitrarily close to 1 for sufficiently large n. In the case of sums the gap between lower and upper bound is larger but still small enough to be useful in some applications. The upper bound obtained for sums is somewhat better than that resulting from a Chernoff bound.

1 Introduction

We are interested in expressions of the form $\sum_{i=0}^{m} \binom{n}{i}$ or more generally

$$S_n(k,m) := \sum_{i=k}^{m} \binom{n}{i},$$

where $0 \le k \le m \le n$ all are natural numbers. Hence we can use the identity

$$\binom{n}{i} = \frac{n!}{i!\,(n-i)!}. \tag{1.1}$$

The only two cases which are immediately clear are

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n \qquad \text{and} \qquad \sum_{i=0}^{m} \binom{2m+1}{i} = \frac{1}{2} 2^{2m+1}. \tag{1.2}$$

But according to Graham/Knuth/Patashnik [GKP89, page 165] we have to face the fact that in general *"there is no closed form for the partial sum of a row of Pascal's triangle"*. So the only hope is to find closed forms for *approximations* of the $S_n(k,m)$.

Because of identity 1.1 we first take a short look at STIRLING's formula[1]

$$n! \sim \sqrt{2\pi n}\,\frac{n^n}{e^n} \tag{1.3}$$

More precisely

$$n! = \sqrt{2\pi n}\,\frac{n^n}{e^n}\,(1+h(n)) \tag{1.4}$$

$$\text{where} \qquad h(n) = \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{5140n^3} + \cdots \in O\left(\frac{1}{n}\right) \tag{1.5}$$

In other words we have[2]

1.1 Lemma. $\forall\varepsilon \;\; \exists n_0 > 0 \;\; \forall n > n_0 :$

$$\frac{1}{1+\varepsilon}\sqrt{2\pi n}\,\frac{n^n}{e^n} < n! < (1+\varepsilon)\sqrt{2\pi n}\,\frac{n^n}{e^n} \tag{1.6}$$

2 Approximations for binomial coefficients

We now simply put the inequations 1.6 into identity 1.1 for the binomial coefficients and using the abbreviations

$$B(x,y) := \frac{1}{\sqrt{2\pi}}\sqrt{\frac{x}{y(x-y)}}\,\left(\frac{x}{y}\right)^y \left(\frac{x}{x-y}\right)^{x-y} \tag{2.1}$$

we get

2.1 Lemma. $\forall\varepsilon \;\; \exists n_0 > 0 \;\; \forall n > 2n_0 \;\; \forall i[n_0 < i < n - n_0]$:

$$\frac{1}{(1+\varepsilon)}B(n,i) < \binom{n}{i} < (1+\varepsilon)B(n,i) \tag{2.2}$$

2.2 Proof: In order to obtain an upper bound for $\binom{n}{i}$ we want use the upper bound for $n!$ and the lower bounds for $i!$ and $(n-i)!$ from 1.6. Thus the condition $n > n_0 \wedge i > n_0 \wedge n - i > n_0$ must be satisfied which is equivalent to $n > 2n_0 \wedge n_0 < i < n - n_0$ (and to the symmetrical condition $n > 2n_0 \wedge n_0 < n - i < n - n_0$). One then gets:

$$\begin{aligned}\binom{n}{i} &< \frac{(1+\varepsilon')\sqrt{2\pi n}n^n}{e^n}\,\frac{(1+\varepsilon')e^i}{\sqrt{2\pi i}\;i^i}\,\frac{(1+\varepsilon')e^{n-i}}{\sqrt{2\pi(n-i)}(n-i)^{n-i}}\\ &= (1+\varepsilon')^3\frac{1}{\sqrt{2\pi}}\sqrt{\frac{n}{i(n-i)}}\,\left(\frac{n}{i}\right)^i \left(\frac{n}{n-i}\right)^{n-i}\end{aligned}$$

[1] We write $f(n) \sim g(n)$ iff $\lim_{n\to\infty}\frac{f(n)}{g(n)} = 1$.

[2] We prefer to write $\forall x[P(x)]\,Q(x)$ or even simpler $\forall P(x)\,Q(x)$ instead of $\forall x\,(P(x) \Rightarrow Q(x))$. Unless otherwise noticed, the range of quantified variables is always a set of numbers. The domain of variables denoted by greek symbols or letters from the end of the alphabet (δ, ε, x, y, z) are the positive real numbers, and the domain of variables denoted by other letters (like i, k, m, n, n_0, …) are the nonnegative integers.

Choosing ε' such that $3\varepsilon' + 3\varepsilon'^2 + \varepsilon'^3 \le \varepsilon$ gives the desired result.

The lower bound is obtained analogously. ∎

Above all we are interested in the asymptotic behavior of $\binom{n}{i}$ (and their sums) for large n. But what about i? If i is *fixed* while n is increasing, writing

$$\binom{n}{i} = \frac{n(n-1)\cdots(n-i+1)}{1\cdot 2\cdots i} = \frac{1}{i!}n^i + O(n^{i-1}) = \frac{1}{i!}n^i(1+O(\frac{1}{n}))$$

makes it immediately clear that $\binom{n}{i}$ grows polynomially. Let us now consider the case where i is not necessarily a constant, but a function of n.

As a simple example consider an even $n = 2m$ and $i = m$. Then we get the following well known formula from 2.2:

$$\binom{2m}{m} \sim \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{m}}\,2^m\,2^m = \sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{n}}\,2^n \tag{2.3}$$

Without loss of generality we will now always write i in the form $i(n) = \frac{n}{a(n)}$.[3] It obviously suffices to consider functions satisfying $2 \le a(n)$. And since the case of constant i — that is $a(n) = \frac{n}{i}$ — has already been taken care of, in what follows we will always assume that $a(n) \in o(n)$ (that is $\lim_{n\to\infty} \frac{a(n)}{n} = 0$) without explicitly mentioning it.

Furthermore, to keep notation a little bit more readable from now on **we will always write only a instead of $a(n)$, even if it is *not* a constant!**

Using lemma 2.1 we may – under certain circumstances – compute as follows:

$$\begin{aligned}\binom{n}{\frac{n}{a}} &< (1+\varepsilon)\frac{1}{\sqrt{2\pi}}\sqrt{\frac{n}{\frac{n}{a}(n-\frac{n}{a})}}A(n,\frac{n}{a}) \\ &= (1+\varepsilon)\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{n}}\sqrt{\frac{a^2}{a-1}}a^{\frac{n}{a}}\left(\frac{a}{a-1}\right)^{\frac{a-1}{a}n} \\ &= (1+\varepsilon)\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{n}}\sqrt{\frac{a^2}{a-1}}\left(a^{\frac{1}{a}}\left(\frac{a}{a-1}\right)^{\frac{a-1}{a}}\right)^n\end{aligned} \tag{2.4}$$

Therefore let us first of all define

$$C(x) := x^{\frac{1}{x}}\left(\frac{x}{x-1}\right)^{\frac{x-1}{x}} \tag{2.5}$$

Simple computations show that $C(2) = 2$, $\lim_{x\to\infty} C(x) = 1$ and $\frac{d}{dx}C(x) < 0$ for $x > 2$. A plot of $C(x)$ is shown in figure 1. Please note that $\lim_{x\to 2+0}\frac{d}{dx}C(x) = 0$ (and the point of inclination lies between 3.23235 and 3.23236).

Furthermore straightforward computations show, that for $x > 1$ the second factor has a maximum value of $e^{1/e} < 1.45$ at $x = \frac{e}{e-1}$.

[3]In what follows we write $\frac{n}{a(n)}$ even if it is no integer. In this case it is to be understood as $\lfloor\frac{n}{a(n)}\rfloor$.

2.3 Lemma. *If $e_0 = e^{1/e}$ then $\forall x > 2$:*

$$x^{\frac{1}{x}} < C(x) < e_0\, x^{\frac{1}{x}}. \tag{2.6}$$

If x is sufficiently large, the bounds can be improved:
$\forall \varepsilon > 0 \quad \exists x_0 \quad \forall x > x_0$:

$$1 < C(x) < 1 + \varepsilon \tag{2.7}$$

The simple transformation

$$C(x) = x^{\frac{1}{x}} \left(\frac{x}{x-1}\right)^{\frac{x-1}{x}} = \left(x\left(\frac{x-1+1}{x-1}\right)^{x-1}\right)^{\frac{1}{x}} = \left(\left(1+\frac{1}{x-1}\right)^{x-1} x\right)^{\frac{1}{x}}$$

shows, that the inner exponential expression converges monotonically towards e (from below) as $2 \le x \to \infty$. Since for $y > 2$ the function $g(y) = (xy)^{1/x}$ also grows monotonically, we get

2.4 Lemma. $\forall x \ge 2$

$$(2\,x)^{\frac{1}{x}} \le C(x) < (e\,x)^{\frac{1}{x}} \tag{2.8}$$

If x is sufficiently large, the lower bound can be improved:
$\forall \delta[1 > \delta > 0] \quad \exists x_0 > 2 \quad \forall x > x_0$:

$$((e-\delta)\,x)^{\frac{1}{x}} < C(x) < (e\,x)^{\frac{1}{x}} \tag{2.9}$$

Using the notation $C(x)$ we can now write:

2.5 Lemma. $\forall \varepsilon \quad \exists n_0 \quad \forall n[n > n_0 a]$:

$$\frac{1}{(1+\varepsilon)}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{n}}\sqrt{\frac{a^2}{a-1}}C(a)^n < \binom{n}{\frac{n}{a}} < (1+\varepsilon)\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{n}}\sqrt{\frac{a^2}{a-1}}C(a)^n. \tag{2.10}$$

2.6 Proof: According to lemma 2.1 the upper bound in equation 2.4 is correct for given ε and the accompanying n_0, if $n > 2n_0$ and $n_0 < \frac{n}{a} < n - n_0$. The latter is equivalent to $n_0 a < n$ and $n_0 \frac{a}{a-1} < n$. Because of $a \ge 2$, the strictest conditions of all is $n_0 a < n$. The lower bound is computed analogously. ∎

2.7 Corollary. *If $a \ge 2$ is a constant, then $\binom{n}{\frac{n}{a}}$ grows exponentially*[4].

We now turn to the case, where a is not a constant but a monotonically increasing and unbounded function. Remember that nevertheless we'll continue to write only a instead of $a(n)$!

Using the lemmata 2.4 and 2.5 one gets:

[4] We say that a function f grows exponentially if $f(x)$ can be bounded from above and from below by functions c^x for some constants $c > 1$. Note that this is not the same as growing more than polynomially.

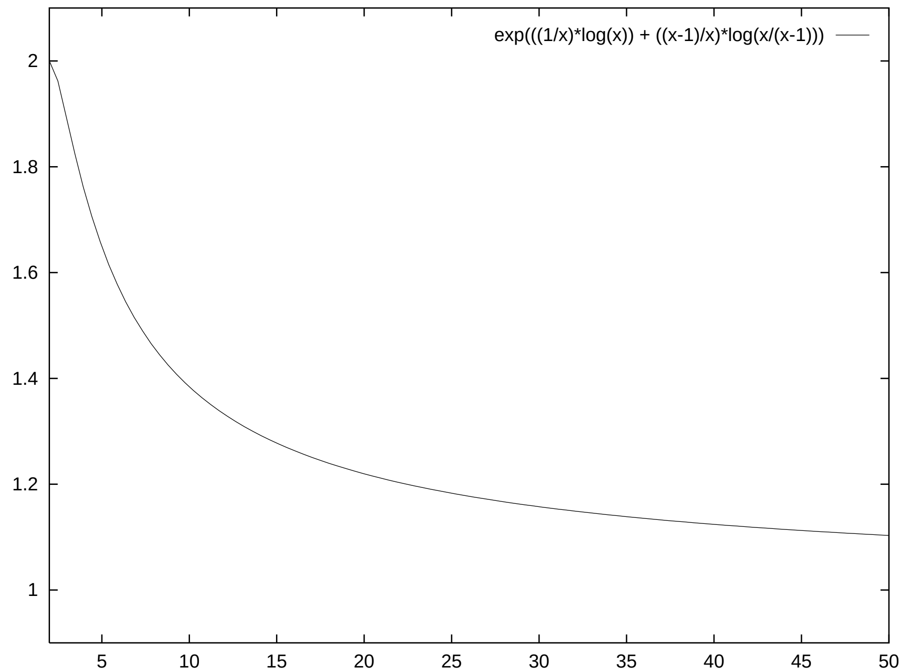

Figure 1: A plot of $C(x)$ for $2 \leq x \leq 50$.

2.8 Lemma. *If a is a monotonically increasing and unbounded function, then:* $\forall\delta[1 > \delta > 0]\quad \forall\varepsilon \quad \exists n_0 \quad \forall n[n > n_0 a]$*:*

$$\frac{1}{(1+\varepsilon)}\frac{1}{\sqrt{2\pi}}\sqrt{\frac{a}{n}}\;((e-\delta)\;a)^{\frac{n}{a}} < \binom{n}{\frac{n}{a}} < (1+\varepsilon)\frac{1}{\sqrt{2\pi}}\sqrt{\frac{a}{n}}\;(e\;a)^{\frac{n}{a}} \tag{2.11}$$

2.9 Proof: For sufficiently large n a good lower bound for $\sqrt{\frac{a^2}{a-1}}$ is $\sqrt{a}$ and a good upper bound is $\sqrt{\frac{a^2}{a-\varepsilon a}} = \sqrt{\frac{1}{1-\varepsilon'}}\sqrt{a}$ in inequation 2.10. The rest follows from lemma 2.4. ∎

2.10 Corollary. *If a is a monotonically increasing and unbounded function, then $\binom{n}{\frac{n}{a}}$ does not grow exponentially.*

2.11 Proof: The interesting part of the upper bound in inequation 2.11 can be written as

$$(e\;a)^{\frac{n}{a}} = e^{\frac{n}{a}}\;2^{\frac{n\,\mathrm{ld}(a)}{a}}$$

Since a grows beyond all bounds, both factors grow slower than c^n for every $c > 1$. ∎

3 Approximations for sums of binomial coefficients

We use the trivial observation that for $0 \leq k \leq m \leq \frac{n}{2}$ holds:

$$\binom{n}{m} \leq \sum_{i=k}^{m} \binom{n}{i} \leq (m-k+1)\binom{n}{m} \tag{3.1}$$

Setting $k = \frac{n}{b}$ and $m = \frac{n}{a}$ and using lemma 2.5 we immediately get the following result:

3.1 Lemma. *Let $2 < a < b \in o(n)$. Then* $\quad \forall \varepsilon \quad \exists n_0 \quad \forall n [n > n_0 a]$:

$$\frac{1}{(1+\varepsilon)} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{n}} \sqrt{\frac{a^2}{a-1}} C(a)^n \leq S_n\left(\frac{n}{b}, \frac{n}{a}\right)$$
$$\leq (1+\varepsilon)\frac{1}{\sqrt{2\pi}}\left(\sqrt{n}(b-a) + \frac{ab}{\sqrt{n}}\right)\sqrt{\frac{1}{b^2(a-1)}} C(a)^n.$$

3.2 Proof:

$$\left(\frac{n}{a} - \frac{n}{b} + 1\right)\frac{1}{\sqrt{n}}\sqrt{\frac{a^2}{a-1}} = \left(\frac{n(b-a)+ab}{ab}\right)\frac{1}{\sqrt{n}}\sqrt{\frac{a^2}{a-1}}$$
$$= \left(\sqrt{n}(b-a) + \frac{ab}{\sqrt{n}}\right)\sqrt{\frac{1}{b^2(a-1)}}$$

■

Since estimation 3.1 is a simple one, we are no longer in a situation where the lower and upper bound are arbitrarily close to each other if n is sufficiently large. Usually there will be a gap in the order of at least $\frac{n}{a}$. Because the formulas are looking a little bit simpler, let us consider the special case of sums starting at 0 from now on. Then we get

3.3 Lemma. $\forall \varepsilon \quad \exists n_0 \quad \forall n [n > n_0 a]$:

$$\frac{1}{(1+\varepsilon)} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{n}} \sqrt{\frac{a^2}{a-1}} C(a)^n \leq S_n\left(0, \frac{n}{a}\right) \leq (1+\varepsilon)\frac{1}{\sqrt{2\pi}}\sqrt{n}\sqrt{\frac{1}{(a-1)}} C(a)^n.$$

3.4 Proof: Since $a \in o(n)$ one can approximate $(\frac{n}{a}+1)$ by $(1+\varepsilon')\frac{n}{a}$ for sufficiently large n:

$$\left(\frac{n}{a}+1\right)\frac{1}{\sqrt{n}}\sqrt{\frac{a^2}{a-1}} \leq (1+\varepsilon')\frac{n}{a}\frac{1}{\sqrt{n}}\sqrt{\frac{a^2}{a-1}} = (1+\varepsilon')\sqrt{n}\sqrt{\frac{1}{a-1}}$$

■

Again we have

3.5 Corollary. *If $a \geq 2$ is a constant, then $S_n(0, \frac{n}{a})$ grows exponentially.*
$\forall \delta[1 > \delta > 0]\ \exists c_1\ \exists c_2\ \exists n_0\ \forall n[n > n_0 a]$:

$$c_1(C(a) - \delta)^n \leq S_n\left(0, \frac{n}{a}\right) \leq c_2(C(a) + \delta)^n \tag{3.2}$$

In the case of a monotonically increasing and unbounded a one can use the approximation for $C(a)$ from 2.4 once again.

3.6 Lemma. *If a is a monotonically increasing and unbounded function, then:*
$\forall \delta[1 > \delta > 0]\ \forall \varepsilon\ \exists n_0\ \forall n[n > n_0 a]$:

$$\begin{aligned}
\frac{1}{(1+\varepsilon)} \frac{1}{\sqrt{2\pi}} \sqrt{\frac{a}{n}}\, (e-\delta)^{\frac{n}{a}}\, 2^{\frac{\mathrm{ld}(a)}{a} n} \leq S_n\left(0, \frac{n}{a}\right) &\leq (1+\varepsilon) \frac{1}{\sqrt{2\pi}} \sqrt{\frac{n}{a}}\, e^{\frac{n}{a}}\, 2^{\frac{\mathrm{ld}(a)}{a} n} \\
\frac{1}{(1+\varepsilon)} \frac{1}{\sqrt{2\pi}} \sqrt{\frac{a}{n}}\, ((e-\delta)a)^{\frac{n}{a}} \leq S_n\left(0, \frac{n}{a}\right) &\leq (1+\varepsilon) \frac{1}{\sqrt{2\pi}} \sqrt{\frac{n}{a}}\, (ea)^{\frac{n}{a}} \\
\frac{1}{(1+\varepsilon)} \frac{1}{\sqrt{\pi n}}\, ((e-\delta)a)^{\frac{n}{a}} \leq S_n\left(0, \frac{n}{a}\right) &\leq (1+\varepsilon) \sqrt{\frac{n}{4\pi}}\, (ea)^{\frac{n}{a}}
\end{aligned}$$

3.7 Proof: The first line results from straightforward substitution of the results from lemma 2.8 into 3.1 and in each of the following lines we have simply rewritten or relaxed the bounds from the preceeding line. ∎

3.8 Corollary. *If a is a monotonically increasing and unbounded function, then $S_n(0, \frac{n}{a})$ does not grow exponentially.*

4 Comparison with Chernoff bound

Another possibility to obtain bounds on sums of binomial coefficients is to use approximations for the binomial distribution function. Specifically we'll now compare the results from the previous section with an often used so called Chernoff bound [Che52].

Let $b(k; n, p) = \binom{n}{k} p^i q^{n-i}$ and

$$B(k; n, p) = \sum_{i=k}^{n} b(i; n, p) = \sum_{i=k}^{n} \binom{n}{i} p^k q^{n-k} \tag{4.1}$$

the "tail" of the binomial distribution function. Setting $p = q = \frac{1}{2}$ one gets

$$B(k; n, \frac{1}{2}) = 2^{-n} \sum_{i=k}^{n} \binom{n}{i} = 2^{-n} \sum_{i=0}^{n-k} \binom{n}{i} = 2^{-n} S_n(0, n-k) \tag{4.2}$$

$$S_n(0, m) = 2^n B(n-m; n, \frac{1}{2}) \tag{4.3}$$

Hence estimations for $B(k; n, \frac{1}{2})$ are as helpful as estimations for $S_n(0, n-k)$.

4.1 Lemma. (Chernoff bound) *For $k > np$ holds:*

$$B(k; n, p) \leq \left(\frac{np}{k}\right)^k e^{k-np} \tag{4.4}$$

If we write $k = \frac{n}{b}$ similar to the $\frac{n}{a}$ before[5], we get for $p = \frac{1}{2}$:

$$B(\frac{n}{b}; n, \frac{1}{2}) \quad \leq \left(\tfrac{b}{2}\, e^{1-\frac{b}{2}}\right)^{\frac{n}{b}} \tag{4.5}$$

Together with equation 4.3 we get:

4.2 Corollary. *Let $b = \frac{a}{a-1}$. Then:*

$$S_n(0, \frac{n}{a}) \leq 2^n \left(\frac{b}{2}\, e^{1-\frac{b}{2}}\right)^{\frac{n}{b}}$$

4.3 Proof:

$$S_n(0, \frac{n}{a}) = 2^n B(n - \frac{n}{a}; n, \frac{1}{2}) = 2^n B(\frac{an-n}{a}; n, \frac{1}{2}) = 2^n B(\frac{n}{b}; n, \frac{1}{2})$$

■

We will now compare the upper bound $U(a,n)$ from lemma 3.3 with the upper bound $V(a,n)$ from corollary 4.2:

$$U(a,n) = \frac{1}{(1+\varepsilon)} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{n}} \sqrt{\frac{a^2}{a-1}}\, C(a)^n \quad \text{where} \quad C(a) = a^{\frac{1}{a}} \left(\frac{a}{a-1}\right)^{\frac{a-1}{a}}$$

$$V(a,n) = D(a)^n \quad \text{where} \quad D(a) = 2\left(\frac{a}{2(a-1)}\, e^{1-\frac{a}{2(a-1)}}\right)^{\frac{a-1}{a}}$$

Here, $D(a)$ results from the above corollary by substituting $\frac{a}{a-1}$ for b. First, let us compare the bases $C(a)$ and $D(a)$ of the exponential functions occuring in the approximations:

$$\frac{C(a)}{D(a)} = \frac{a^{\frac{1}{a}} \left(\frac{a}{a-1}\right)^{\frac{a-1}{a}}}{2\left(\frac{a}{2(a-1)}\, e^{1-\frac{a}{2(a-1)}}\right)^{\frac{a-1}{a}}} = \frac{2^{\frac{a-1}{a}}\, a^{\frac{1}{a}}}{2\, e^{\frac{a-1}{a}-\frac{1}{2}}} = \frac{2^{-\frac{1}{a}}\, a^{\frac{1}{a}}}{\sqrt{e}\, e^{-\frac{1}{a}}} = \frac{1}{\sqrt{e}} \left(\frac{e\, a}{2}\right)^{\frac{1}{a}} \tag{4.6}$$

As a function of a the fraction $\frac{C(a)}{D(a)}$ monotonically decreases from 1 to $\frac{1}{\sqrt{e}}$ as a increases from 2 to infinity. Now, the case of a constant $a = 2$ is trivial (see equation 1.2). Hence in all interesting cases (i.e. whenever $a > 2$) the upper bound U from the preceeding section is strictly better than the Chernoff bound V by a factor of almost $\left(\frac{D(a)}{C(a)}\right)^n$ which can come close to $e^{n/2}$ depending an a. The "almost" comes from the factors in front of $C(a)^n$, but they can be bounded by a function in $o(\sqrt{n})$ and hence only have secondary relevance.

[5] Again, b may be a *function* depending on n.

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