Towards the Curvature Cuff of a Canal Surface

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Abstract

By means of a special stereographic mapping, a sphere corresponds to a *point* in 4-space. A canal surface then corresponds to a *curve* in 4-space. The poster gives insights into this mapping and presents a construction of an osculating Dupin cyclide, which provides the principal curvatures for each generating circle of a curvature continuous canal surface.

Basic idea

A canal surface is the envelope of spheres, whose centers move on a so-called *spine curve*. These spheres will be mapped onto points in four-dimensional space, which we will call $\mathbb{M}$-space. Investigating a number of spheres takes over to investigating a number of points in $\mathbb{M}$. Hence, a canal surface corresponds to a curve in $\mathbb{M}$. A construction is given, how to receive an osculating conic in $\mathbb{M}$, which corresponds to an osculating Dupin cyclide in three-dimensional space. By means of this cyclide, it is easy to calculate the curvature values for a point of the canal surface.

$\mathbb{M}$-space

Let $\mathbb{M}$ be a four-dimensional Euclidean space, where points will be denoted by small hollow letters in homogeneous coordinates, eg. $\mathbb{y} = [y_1\, y_2\, y_3\, y_4\, y_0]$. The unit hypersphere, which is given by

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 = y_0^2,$$

will be called Möbius hypersphere and is denoted by Ψ. Other characteristic elements in $\mathbb{M}$ are the following:

- the north pole of Ψ: $\mathbb{n} = [0\,0\,0\,1\,1]$

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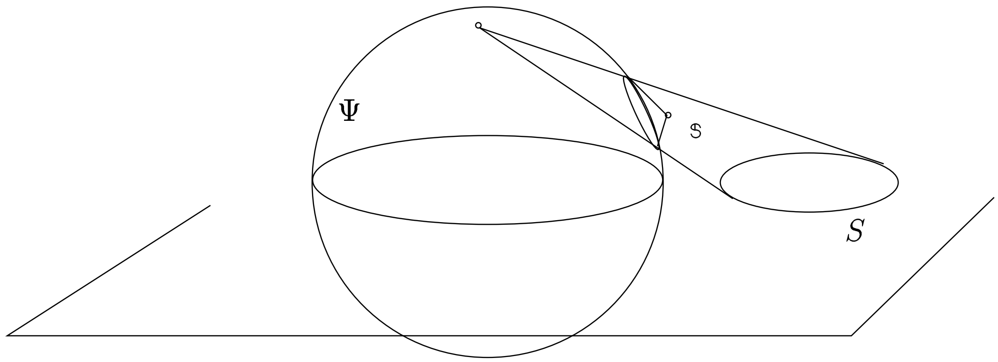

Figure 1: The sphere S is mapped onto the point $\mathbb{s}$

- the north hyperplane $\mathcal{N}$: $y_0 = y_4$
- the scalar product: $\mathbb{x}\mathbb{y} := x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 - x_0y_0$

A simple observation shows, that a point $\mathbb{y}$ lies outside (on / inside) the Möbius hypersphere Ψ iff $\mathbb{y}\mathbb{y}$ is greater than (equal / lower than) zero. Further, given a point $\mathbb{y}$, then all the points $\mathbb{x}$ with $\mathbb{x}\mathbb{y} = 0$ lie in the polar hyperplane of $\mathbb{y}$, denoted by $\mathcal{Y}$. The points $\mathbb{y}$ and $\mathbb{x}$ are a conjugate pair with respect to Ψ.

Note: $\mathcal{N}$ is the polar hyperplane of the north pole $\mathbb{n}$.

The mapping

Combining a standard stereographic projection with the polarity induced by $\mathbb{x}\mathbb{y} = 0$ leads to a mapping from spheres and planes of ordinary three-dimensional space to points in $\mathbb{M}$. First, the stereographic projection maps a sphere or plane of three-dimensional space to a sphere on the Möbius hypersphere. The polarity then maps its points to their unique pole.

More precisely, a sphere S with center $\mathbf{m} = [a\,b\,c]$ and radius r is mapped onto the point

$$\mathbb{s} = \begin{bmatrix} 2a \\ 2b \\ 2c \\ a^2+b^2+c^2-r^2-1 \\ a^2+b^2+c^2-r^2+1 \end{bmatrix} = \begin{bmatrix} 2\mathbf{m} \\ \mathbf{m}^t\mathbf{m} - r^2 - 1 \\ \mathbf{m}^t\mathbf{m} - r^2 + 1 \end{bmatrix} \in \mathbb{M}.$$

In particular, a sphere of radius ∞, i.e. a plane U, given by $\mathbf{u}^t\mathbf{x} - u_0 = u_1x + u_2y + u_3z - u_0 = 0$ is mapped onto the point

$$\mathbb{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} \mathbf{u} \\ u_0 \\ u_0 \end{bmatrix} \in \mathbb{M}.$$

Conversely, the preimage of the point $\mathbb{y} = [y_1\, y_2\, y_3\, y_4\, y_0]$, where $y_0 \neq y_4$, is the sphere defined by

$$\mathbf{m} = \frac{1}{y_0 - y_4} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad r = \frac{1}{y_0 - y_4} \cdot \sqrt{\mathbb{y}\mathbb{y}}.$$

In the case $y_0 = y_4$, $\mathbb{y}$ corresponds to the plane with the equation

$$y_1 x + y_2 y + y_3 z - y_0 = 0.$$

Some useful observations

The mapping induces a correspondence of figures in three-dimensional space to figures in $\mathbb{M}$. The most obvious ones are the following:

Property (P) Two spheres span a linear family of spheres, which will be called a *pencil*. The intersection of all spheres of the pencil is a circle.	Two points span a linear family of points. The join of all points of the above family is a line.
Property (N) Three spheres span a linear family of spheres, which we will call a *net*. The intersection of all spheres of the net is a pair of points.	Three points span a linear family of points. The join of all points of the above family is a plane.
Property (T) Two spheres are *tangent*, if they have exactly one point in common, in other words, if their span intersects in exactly one circle of zero radius.	Two points lie on a Ψ-tangent, if their span contains exactly one point of Ψ.
Property (O) Two spheres are *orthogonal*, if the cosine of their intersection angles equals zero.	Two points are conjugate, if the following equation holds: $\cos\varphi = \frac{(\mathbb{x}\mathbb{y})^2}{(\mathbb{x}\mathbb{x})(\mathbb{y}\mathbb{y})} = 0$

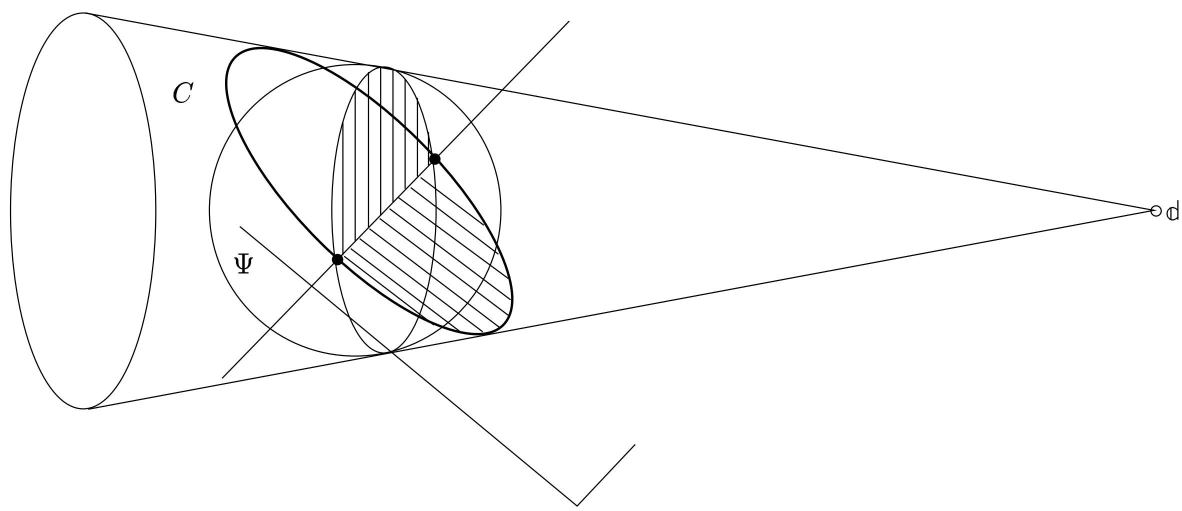

Figure 2: The Dupin conic has double contact with Ψ.

Dupin cyclides

It follows directly, that curves in $\mathbb{M}$ correspond to canal surfaces in three-dimensional space. We will now have a look at Dupin cyclides as special canal surfaces.

A Dupin cyclide is a canal surface, whose spine curve is of second order. Hence, the spine curve is planar and its spheres belong to a net. But even more: Dupin cyclides are simultanously the enveloppe of two families of spheres. Each sphere of one family is tangent to *all* spheres of the other family.

In terms of $\mathbb{M}$-space, this reads as follows: A Dupin cyclide corresponds simultanously to two curves of second degree, which will be called *Dupin conics*, denoted by C and D . Picking out a point $\mathbb{d}$ of conic D, we can trace lines from $\mathbb{d}$ to each point of the other conic C. All theses lines are tangent to the Möbius hypersphere Ψ (recall property (T)) . This makes $\mathbb{d}$ the vertex of a tangent hypercone of Ψ. The intersection of this hypercone with a suitable plane gives the conic C.

Applying property (N), we find that the suitable plane is the image of the net, to which the Dupin cyclide's spheres belong to. The plane intersects the Möbius hypersphere in a circle (of real or imaginary radius). As the conic lies on a tangent Ψ hypercone, C has double contact with the circle in two points (with real or imaginary coordinates).

Property (DC) A Dupin cyclide is a canal surface with a conic spine curve where one sphere exists, which is tangent to all spheres of the canal spheres.	A Dupin conic is a conic with two double contacts at points of Ψ.

Constructing a Dupin Conic

We will now construct a conic C, which is lying in the plane spanned by three given points $\mathbb{p}_1, \mathbb{p}_2, \mathbb{p}_3$. In particular, the conic C will satisfy the following conditions: C passes through $\mathbb{p}_1$, is tangent to the line $\mathbb{p}_1 + t\mathbb{p}_2$ and has two double contacts with the Möbius hypersphere Ψ. Hence, C is a Dupin conic.

In the given plane, we will use a local homogeneous coordinate system, taking $\mathbb{p}_1, \mathbb{p}_2, \mathbb{p}_3$ as base points: $\mathbb{y} = \mathbb{p}_1 x_1 + \mathbb{p}_2 x_2 + \mathbb{p}_3 x_3$.

Intersecting this plane with the Möbius hypersphere, we get a circle Γ,

$$\mathbb{y}\mathbb{y} = \sum_{i,k} \mathbb{p}_i \mathbb{p}_k x_i x_k = 0.$$

The two contact points lie on a line Δ with the equation

$$\mathbf{ux} = u_1 x_1 + u_2 x_2 + u_3 x_3 = 0,$$

where $\mathbf{x} = [x_1, x_2, x_3]^t$ are local coordinates. Γ and Δ carry a pencil of conics, which all are double tangent to the contact points, namely

$$\Gamma + \lambda \Delta^2 = \sum_{i,k} (\mathbb{p}_i \mathbb{p}_k - u_i u_k) x_i x_k = \sum_{i,k} c_{ik} x_i x_k = 0.$$

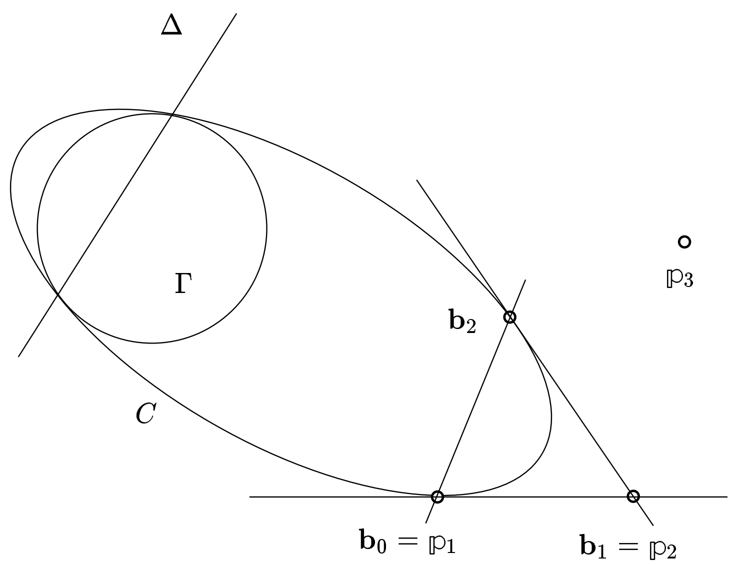

Figure 3: Finding the Dupin conic

Under these conics, we have to find one, that fulfills the above stated properties. Passing through $[1, 0, 0]^t$ with the tangent through $[0, 1, 0]^t$ leads to

$$c_{11} = c_{12} = 0.$$

Note that $c_{13} c_{22}^2 \neq 0$.

If we look at $\mathbf{b}_0 = [1,0,0]^t$ and $\mathbf{b}_1 = [0,1,0]^t$ as two Bézier points of the conic C, we have to find a third point $\mathbf{b}_2$ on C, whose tangent passes through $\mathbf{b}_1$. It lies on the polar of $\mathbf{b}_1$, which intersects C in the point

$$\mathbf{b}_2 = \begin{bmatrix} c_{22}c_{33} - c_{23}^2 \\ 2c_{13}c_{23} \\ -2c_{13}c_{22} \end{bmatrix}$$

To get the final Bézier representation of C,

$$\mathbf{b}_0 + \mathbf{b}_1 \cdot 2\beta t + \mathbf{b}_2 \cdot t^2,$$

it is easy to check that $\mathbf{b}_1$ must get the weight $\beta = \pm c_{13}$.

The Osculating Dupin Conic ...

Let a canal surface be given by its image curve $\mathbb{m}(t)$ in $\mathbb{M}$. The first and second order derivative with respect to t will be denoted by $\dot{\mathbb{m}}(t)$ and $\ddot{\mathbb{m}}(t)$.

The Taylor expansion at t_0 up to the second order gives

$$\mathbb{y}(t) = \mathbb{m} + \dot{\mathbb{m}} \cdot (t_0 + \Delta t) + \frac{1}{2}\ddot{\mathbb{m}} \cdot (t_0 + \Delta t)^2.$$

Comparing this with the equation of a conic, which is tangent to $\mathbb{p}_1$ and $\mathbb{p}_3$ from above, given by

$$\mathbb{y}(\tau) = \mathbb{p}_1 + \mathbb{p}_2 \cdot \tau + \mathbb{p}_3 \cdot \tau^2,$$

leads to

$$\mathbb{p}_1 = \mathbb{m}, \qquad \mathbb{p}_2 = \dot{\mathbb{m}}, \qquad \mathbb{p}_3 = \frac{1}{2}\ddot{\mathbb{m}}.$$

Now, we can proceed as above.

... is a Curvature Cuff

Using a theorem of Aurel Voss, we arrive at the osculating cyclide of a canal surface. Voss stated in 1919 :

The principal curvatures of the points along a generating circle do not depend on the torsion of the spin.

This leads at once to the theorem, which we exploited in $\mathbb{M}$:

Three consecutive spheres of a canal surface determine the osculating Dupin cyclide.	*Three consecutive points of an $\mathbb{M}$-curve determine an osculating Dupin conic.*

Résumé

Given a curvature continous canal surface, the poster presented the construction of its osculating Dupin cyclide. This was done by transfering the problem into a problem of four-dimensional space. The curves in this space correspond to canal surfaces. It turned out, that an osculating conic to the four-dimensional curve corresponds to the osculating Dupin cyclide in three-dimensional space, the curvature cuff.

References

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[2] A. Voss. Zur Theorie der Kanalflächen. *Sitzungsberichte der mathematisch-physikalischen Klasse der Bayerischen Akademie der Wissenschaften zu München*, 49(III):353–368, 1919.