Quantization of the Reduced Phase Space of Two-Dimensional Dilaton Gravity

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We study some two-dimensional dilaton gravity models using the formal theory of partial differential equations. This allows us to prove that the reduced phase space is two-dimensional without an explicit construction. By using a convenient (static) gauge we reduce the theory to coupled ordinary differential equations and we are able to derive for some potentials of interest closed-form solutions. We use an effective (particle) Lagrangian for the reduced field equations in order to quantize the system in a finite-dimensional setting leading to an exact partial differential Wheeler-DeWitt equation instead of a functional one. A WKB approximation for some quantum states is computed and compared with the classical Hamilton-Jacobi theory. The effect of minimally coupled matter is examined.

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I. INTRODUCTION

A tensor formulation of physical theories makes no overt reference to any particular frame of reference. To interpret these theories it is often necessary to extract from them coordinate independent information. In particular in theories of classical gravitation, coordinate freedom is either explicitly removed by working in a particular coordinate system or regarded as generating constraints for the subsequent analysis. For the classical theory it may simply be a matter of taste as to which procedure is adopted. However the discussion of quantization is often acutely sensitive to the choice adopted.

Following Witten's observation [35] that models of two dimensional dilaton gravity offer a means of studying the Hawking effect with back reaction there has been an enormous interest in such models. They arise naturally from certain truncations of low energy string effective actions [3] and symmetric configurations in higher dimensions [34]. Such models have been rendered completely integrable at the classical level by exploiting the local conformal flatness of all two dimensional manifolds and their quantization discussed from several alternative viewpoints [26].

In this paper we reexamine the conditions that are responsible for this remarkable integrability and offer an alternative quantization. The basic observation is that a particular conformal gauge reduces the classical integrability to the problem of solving a system of *ordinary* differential equations. Methods from the formal theory of partial differential equations [24,27] allows us to compute the dimension of the reduced phase space without explicitly constructing it. This technique should also prove useful in more complicated theories where explicit reduction is not possible.

Using methods from the Hamilton-Jacobi theory for systems with constraints [6,14] we construct local expressions for the dynamical degrees of freedom for dilaton gravity on the line. This is in marked difference to other approaches to the quantization of the reduced theory [18,22]. The quantum amplitudes are shown to satisfy a simple hyperbolic wave equation which is exactly soluble for appropriate boundary conditions. Similar quantum theories were already obtained by different authors [21,23] in an approximate minisuperspace approach. But here it is not necessary to make such an approximation because of the finite-dimensional reduced phase space.

A straightforward semi-classical analysis of the exact quantum description yields a WKB phase that encodes all the classical dilaton gravity solutions. We explicitly demonstrate that the integral curves that annihilate the gradient of the WKB phase form a family of exact classical vacuum solutions. This suggests that such a quantization of dilaton gravity deserves further scrutiny.

The article is organized as follows: After a brief discussion of the classical action and its field equations, we use in Section III a formal analysis to derive indirectly the dimension of the reduced phase space. In Section IV we explicitly reduce the field equations by a gauge fixing to a system of ordinary differential equations and construct its general solution. After considering some explicitly solvable models we proceed in Section VI to the Hamilton-Jacobi analysis

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of the system. Its results are used in Section VII for the quantization. Section VIII discusses the effect of minimally coupling a matter field. Finally, some conclusions are given.

II. CLASSICAL ACTION AND FIELD EQUATIONS

In two dimensions a general coordinate invariant Lagrangian density containing the metric $g_{\mu\nu}$, a scalar dilaton field Φ and their derivatives up to second order is given by

$$\mathcal{L}[g_{\mu\nu},\Phi] = \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + V(\Phi) + D(\Phi)R\right) \tag{1}$$

where R denotes the curvature scalar associated with the metric and D is a scalar function.

Using field redefinitions one can, however, considerably simplify this action. The kinetic term for Φ can be eliminated by a Weyl rescaling of the metric [18]

$$\bar{g}_{\mu\nu} = \Omega^2(\Phi)g_{\mu\nu}\,, \tag{2}$$

if Ω satisfies the differential equation

$$4D'(\Phi)\frac{\mathrm{d}\ln\Omega}{\mathrm{d}\Phi} = 1\,. \tag{3}$$

If we additionally redefine the dilaton field $\bar{\Phi} = D(\Phi)$, we obtain the action

$$\mathcal{L}[\bar{g}_{\mu\nu},\bar{\Phi}] = \sqrt{-\bar{g}}\left(\bar{V}(\bar{\Phi}) + \bar{\Phi}\bar{R}\right) \tag{4}$$

where the new potential $\bar{V}(\bar{\Phi})$ is given by

$$\bar{V}(\bar{\Phi}) = \frac{V(\Phi(\bar{\Phi}))}{\Omega^2(\Phi(\bar{\Phi}))}\,. \tag{5}$$

(One must be careful here, if D has critical points [1]).

Henceforth we will restrict our attention to this action and drop the bar over the fields. Variation with respect to the metric yields the "Einstein Equations"

$$\nabla_\mu\nabla_\nu\Phi - g_{\mu\nu}\left[\nabla^\rho\nabla_\rho\Phi + V(\Phi)\right] = 0\,, \tag{6}$$

whereas variation with respect to Φ leads to the additional equation

$$R + V'(\Phi) = 0 \tag{7}$$

determining the curvature scalar.

Before we start a detailed analysis of these field equations, we study briefly the relation between the potentials appearing in (1) and (4) for the most often considered case $D(\Phi) = a\Phi^n$ for some constants a, n. If $n \neq 2$ a solution of (3) is given by

$$\Omega(\Phi) = e^{-\Phi^{2-n}/4an(n-2)}\,; \tag{8}$$

while for $n = 2$

$$\Omega(\Phi) = \Phi^{1/8a}\,. \tag{9}$$

For $n = 1$ we obtain thus from (5)

$$\bar{V}(\bar{\Phi}) = V(\bar{\Phi}/a)e^{-\bar{\Phi}/2a^2}\,. \tag{10}$$

This implies especially that for an exponential potential $V(\Phi) \sim e^{\nu\Phi}$ the potential remains an exponential after the transformation but with a modified coefficient $\bar{\nu} = (2a\nu - 1)/2a^2$. Note that this results also holds for $\nu = 0$, i.e. if the potential consists just of a cosmological constant. Conversely, the potential becomes constant, if $a = 1/2\nu$.

For $n=2$ the transformation reads

$$\bar{V}(\bar{\Phi}) = V(\pm\sqrt{\bar{\Phi}/a})(\bar{\Phi}/a)^{-1/8a}\,. \tag{11}$$

Thus Lagrangian densities of the form $\bar{\mathcal{L}} = \bar{\Phi}R + \Lambda\bar{\Phi}^n$ as they are e.g. considered in Ref. [20] can be derived from a model in the form (1) with $D(\Phi) = -\Phi^2/8n$ and a "cosmological constant" as potential $V(\Phi) = \Lambda/(-8n)^n$.

A class of models which appeared first in effective string actions and which has found considerable interest due to the existence of black hole solutions [3,35] is described by the action

$$\mathcal{L}[g_{\mu\nu},\varphi] = \frac{1}{8}\sqrt{-g}e^{-2\varphi}\left(R + 4(\nabla\varphi)^2 + c\right) \tag{12}$$

where c is a constant. Using field redefinitions one can transform it to [25]

$$\bar{\mathcal{L}}[\bar{g}_{\mu\nu},\phi] = \sqrt{-\bar{g}}\left(\frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}q\phi\bar{R} + \frac{1}{8}ce^{\phi/q}\right) \tag{13}$$

with an arbitrary constant q. Elimination of the kinetic term leads then to a modification of the exponential. Note that this simple Liouville form of the transformed action is due to the factor 4 in (12). A different factor γ leads to a modified potential of the form $\mu\phi^{1-\gamma/4}e^{\phi/q}$.

III. FORMAL ANALYSIS

The first step in a formal analysis is always to complete the given system of partial differential equations to an involutive one [24,27]. This completion is closely related to the Dirac formalism for systems with constraints. Actually, one can interpret the Dirac algorithm as a completion procedure for the Hamilton-Dirac equations of the system [30].

In our case the involution analysis is rather simple, as it is straightforward to show that the combined field equations (6,7) are already in involution. An interesting fact hereby is that (6) entails (7), if we exclude the trivial case that Φ is constant. The integrability conditions of (6) require that either (7) holds or Φ must be constant. Similar effects are known from other theories coupled to gravity.

The arbitrariness of the general solution of a system of q-th order partial differential equations in n independent variables can be determined from its Cartan characters $\alpha_q^{(k)}$, $k=1,\dots,n$ [28]. A simple calculation for our system yields

$$\alpha_2^{(2)} = 2\,, \quad \alpha_2^{(1)} = 6\,. \tag{14}$$

By a comparison with a Taylor expansion of the general solution these characters can be interpreted in terms of numbers of arbitrary functions of different numbers of arguments. Here we obtain that the general solution of our field equations can be written as an algebraic expression containing two arbitrary functions of two arguments and two arbitrary functions of one argument.

Another way to represent the arbitrariness of the general solution is given by the Hilbert polynomial $H(r)$ of the field equations. It denotes the number of Taylor coefficients of order r which can be chosen arbitrarily. From (14) we obtain [27,28] (note the slightly different notation used there)

$$H(r) = 2r + 4\,. \tag{15}$$

It is important to note that $H(r)$ yields the correct values only for $r \geq 2$, as we are dealing with second-order equations. On the other hand the number of arbitrary Taylor coefficients of order less than or equal to 2 is determined by the dimension of the submanifold described by the field equations in the appropriate second-order jet bundle; thus in our case it is 20.

We must, however, adjust for the covariance under coordinate transformations. Especially the two functions of two variables stem obviously from this gauge covariance. We have recently shown how such a correction can be performed as soon as the gauge group is known [28,29]. The key is the introduction of a gauge corrected Hilbert polynomial which in turn leads to gauge corrected Cartan characters.

In our case we must subtract the invariance under the transformation

$$g_{\mu\nu} = \frac{\partial\bar{x}^\rho}{\partial x^\mu}\frac{\partial\bar{x}^\sigma}{\partial x^\nu}\bar{g}_{\rho\sigma}\,, \tag{16a}$$

$$\Phi = \bar{\Phi}\,. \tag{16b}$$

The transformation depends on two gauge functions $\bar{x}^\rho$ through their first derivatives. Thus if we expand again in a power series, we can give $G(r)$ coefficients of order r arbitrary values through gauge transformation where $G(r)$ is given by

$$G(r) = 2\binom{r+2}{r+1} = 2r + 4\,. \tag{17}$$

By comparison with the Hilbert polynomial we see that all the arbitrariness for $r \geq 2$ stems from this gauge covariance. Hence the gauge corrected Cartan characters vanish and the reduced phase space of this theory is finite-dimensional. Usually $G(r)$ yields the correct values only from a certain value of r on. In our case, however, one can easily see by writing out the first terms of the expansion that it is correct for all $r \geq 0$. Thus we can further conclude that 18 Taylor coefficients of order up to two can be given arbitrary values by gauge transformation. Since the general solution of our field equations contains only 20 arbitrary coefficients at these orders we obtain that the dimension of the reduced solution space is two. This fact was also proven in Refs. [1,18] using an explicit reduction.

Actually, in this simple case it is not necessary to use the Cartan characters to prove the finiteness of the reduced phase space. The easier concept of the strength of a differential equation introduced by Einstein [7,32] suffices here. A straightforward computation shows that the field equations are absolutely compatible and have a vanishing strength, if one takes the gauge symmetry into account. But since we are dealing with a two-dimensional space-time, this implies immediately that the gauge reduced solution space is finite-dimensional. However, the exact dimension can be computed only with the refined analysis used above.

We can understand this finiteness by considering the metric as an external field. (6) represents then a finite type system for the dilaton field Φ, as each second order derivative of Φ is determined by an equation. Thus the general solution of this system depends only on a finite number of parameters. All arbitrary functions stem therefore from the metric as solution of (7).

IV. REDUCTION TO ORDINARY DIFFERENTIAL EQUATIONS

The solution of every system of finite type can be obtained by solving systems of ordinary differential equations [4,17]. The reduction is based on the theory of complete systems and can be performed in a purely algorithmic way. However, in our case it will not be necessary to follow this procedure which would lead to a fairly complicated system of ordinary differential equations [27]. By choosing an appropriate gauge the reduction can be obtained directly.

We first exploit the well-known fact that every two-dimensional metric is (locally) conformally flat [8] and set

$$g_{\mu\nu} = e^{\lambda(x,t)}\eta_{\mu\nu} \tag{18}$$

where $\eta_{\mu\nu} = \mathrm{diag}\,(-1, 1)$ is the Minkowski metric. The curvature scalar of such a metric is given by

$$R = (\lambda_{tt} - \lambda_{xx})e^{-\lambda}\,. \tag{19}$$

Thus after some trivial manipulations the combined field equations can be written in the following form:

$$\Phi_{tt} - \tfrac{1}{2}(\Phi_t\lambda_t + \Phi_x\lambda_x) + \tfrac{1}{2}e^\lambda V(\Phi) = 0\,, \tag{20a}$$
$$\Phi_{xx} - \tfrac{1}{2}(\Phi_t\lambda_t + \Phi_x\lambda_x) - \tfrac{1}{2}e^\lambda V(\Phi) = 0\,, \tag{20b}$$
$$\Phi_{xt} - \tfrac{1}{2}(\Phi_x\lambda_t + \Phi_t\lambda_x) = 0\,, \tag{20c}$$
$$\lambda_{tt} - \lambda_{xx} + e^\lambda V'(\Phi) = 0\,. \tag{20d}$$

Banks and O'Loughlin [1] showed that the field equations imply the existence of a Killing vector

$$k_\mu = \epsilon_{\mu\nu}\nabla^\nu\Phi \tag{21}$$

orthogonal to the gradient of Φ. Thus we can always choose the gauge $\Phi_t = 0$. Then (20c) leads to $\lambda_t = 0$, if we discard the uninteresting case $\Phi = const$. This means that it suffices to study static metrics.

We will assume from now on that we are in a coordinate system where $\lambda_t = \Phi_t = 0$. The first two equations of (20) yield $\Phi_{xx} - \Phi_x\lambda_x = 0$. This can be integrated once and yields

$$\Phi_x = Ae^\lambda \tag{22}$$

with an integration constant A. Note that this implies that the sign of Φ_x never changes and that it is fixed by the sign of A. Substituting this in (20b) leads to

$$A\lambda_x = V(\Phi)\,. \tag{23}$$

Differentiating (22) allows one to eliminate λ and arrive finally at the simple equation

$$A\Phi_{xx} - V(\Phi)\Phi_x = 0\,. \tag{24}$$

There is no need to consider (20d), as it is an integrability condition and thus automatically satisfied.

Rewriting the potential as a derivative, $V(\Phi) = W'(\Phi)$, one can easily obtain an implicit solution of (24). Integrating once yields the first integral

$$A\Phi_x - W(\Phi) = B \tag{25}$$

for some constant B. Separation of variables leads to

$$x(\Phi) + C = \int_0^{\Phi} \frac{A\, d\varphi}{B + W(\varphi)}\,. \tag{26}$$

Once this expression is inverted to obtain Φ in explicit form, λ can be derived algebraically from (22)

$$\lambda = \ln\left(\frac{B + W(\Phi)}{A^2}\right)\,. \tag{27}$$

We have thus found a three-dimensional solution space. This is no surprise, as the field equations together with the used gauge conditions describe a three-dimensional manifold in the second-order jet bundle. A similar construction in light cone coordinates was presented in Ref. [19].

To conclude this section we briefly discuss the three occurring integration constants. C can obviously be set to any value by changing the origin of the coordinate system. Thus we can set it to zero without loss of generality. Similarly, A can be adjusted to any value by a coordinate scaling $x \to x/A$, $t \to t/A$, as under such a transformation $\lambda \to \lambda + \ln A^2$.

By contrast B has an invariant meaning. Since $A = \Phi_x e^{-\lambda}$, we obtain from (25)

$$B = e^{-\lambda}\Phi_x^2 - W(\Phi)\,. \tag{28}$$

This expression can be expressed covariantly as

$$B = g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi - W(\Phi)\,. \tag{29}$$

One can show that B corresponds to the ADM energy of the system [11].

Thus only one of the three integration constants parameterizing the general solution of the field equations has an invariant meaning. The other two can be absorbed in coordinate transformation. This effect is extensively discussed in Ref. [18].

This is exactly the result one would expect in ordinary gravity from the Birkhoff theorem: Up to coordinate transformations the static vacuum solutions form a one-parameter family. For this reason some authors speak of the generalized Birkhoff theorem of dilaton gravity [19].

V. SOME SOLVABLE MODELS

We start by considering a linear potential of the form $V(\Phi) = k\Phi + m$, i.e. the so-called Jackiw-Teitelboim or Liouville gravity [15,33] with a cosmological constant $k \neq 0$. In this case the field equations decouple and we obtain for the conformal factor the equation

$$\lambda_{xx} - ke^{\lambda} = 0 \tag{30}$$

which can be considered either as a special case of the Poisson-Boltzmann equation or as describing stationary solutions of the Liouville equation. Its general solution is given by [15]

$$\lambda(x) = -\ln\left\{\frac{k}{2\beta^2}\sinh^2\left[\beta(x-x_0)\right]\right\} \tag{31}$$

with two integration constants β, x_0. Obviously, x_0 is without physical significance and can be set zero. The curvature is constant

$$R = k\,. \tag{32}$$

From (23) we obtain immediately

$$\Phi(x) = -\frac{A}{k}\left[2\beta\coth\left(\beta(x-x_0)\right) - \mu\right]\,. \tag{33}$$

Next we consider potentials of the form $V(\Phi) = \alpha e^{\beta\Phi}$ as they occur in the effective string actions.[1] Here it is simpler to go back to the equations (22,23) and to introduce new dependent variables ψ, μ by $\psi = V(\Phi)$ and $\mu = Ae^{\lambda}$. This transformation yields the system

$$\psi_x = \beta\psi\mu = \beta A\mu_x\,. \tag{34}$$

Thus these new variables are related through

$$\psi(x) = \beta A[\mu(x) + D] \tag{35}$$

with an integration constant D. Eliminating ψ leads to a simple Bernoulli equation for μ which can be solved by separation of variables. We must distinguish two cases: If $D = 0$, we obtain

$$\mu(x) = \frac{1}{C - \beta x} \tag{36}$$

and for the curvature scalar

$$R = \frac{A\beta^2}{\beta x - C} \tag{37}$$

with a further integration constant C. Otherwise we find

$$\mu(x) = \frac{D}{Ce^{-\beta Dx} - 1} \tag{38}$$

and the curvature scalar

$$R = \frac{ADC\beta^2}{e^{\beta Dx} - C}\,. \tag{39}$$

By setting $C = 0$ in (37) and $C = 1$ in (39), respectively, we can move the singularity of the curvature to $x = 0$.

The third important model is provided by spherically symmetric gravity in 3+1 dimensions [34]. It can be reduced to a dilaton gravity action in two dimensions of the form (4) where the potential is given by $V(\Phi) = 1/\sqrt{2\Phi}$. As above we must distinguish two cases in the integral in (26). If $B = 0$, the solution can be given in explicit form

$$\Phi(x) = \frac{1}{A^2}(x + C)^2\,. \tag{40}$$

Otherwise an inversion is not possible. The implicit solution is

$$x + C = A\left[\sqrt{2\Phi} - B\ln\left(1 + \sqrt{2\Phi/B^2}\right)\right]\,. \tag{41}$$

In any case the curvature scalar is given by

$$R = \frac{1}{4}\Phi^{-3/2}\,. \tag{42}$$

[1] As already mentioned in Section II more generally one obtains a potential of the form $\alpha\Phi^{\gamma}e^{\beta\Phi}$. These models can still be solved exactly [16]; however, many case distinctions arise.

VI. HAMILTON-JACOBI THEORY

After the gauge reduction we obtained in Section IV the following system of one first-order and two second-order ordinary differential equations

$$\Phi_{xx} - e^{\lambda} V(\Phi) = 0\,, \tag{43a}$$

$$\lambda_{xx} - e^{\lambda} V'(\Phi) = 0\,, \tag{43b}$$

$$\Phi_x \lambda_x - e^{\lambda} V(\Phi) = 0\,. \tag{43c}$$

Note that the first-order equation produces together with any of the second-order ones the other second-order equation as an integrability condition. The two second-order equations, however, form a normal system and thus cannot generate the first-order one.

We now try to find an effective Lagrangian for the gauged equations of motion (43). A reasonable starting point is obtained by applying our gauge conditions to the full Lagrangian density (4) and integrating once by parts

$$\mathcal{L}_g[\Phi, \lambda] = \Phi_x \lambda_x + e^{\lambda} V(\Phi)\,. \tag{44}$$

The corresponding Euler-Lagrange equations are the two second-order equations in (43). Thus this action yields a too general dynamics, as it "looses" one condition! Performing a Legendre transformation on (44) shows that the missing equation demands the vanishing of the Hamiltonian of the system ("zero energy condition")

$$\mathcal{H}_g = \pi_\Phi \pi_\lambda - e^{\lambda} V(\Phi) = 0 \tag{45}$$

where the canonically conjugate momenta are given by $\pi_\Phi = \lambda_x$ and $\pi_\lambda = \Phi_x$, respectively.

If we denote Hamilton's principal function as usual by S, the Hamilton-Jacobi equation of the *unconstrained* system described by the Lagrangian $\mathcal{L}_g$ is

$$\frac{\partial S}{\partial x} + \frac{\partial S}{\partial \Phi}\frac{\partial S}{\partial \lambda} - e^{\lambda} V(\Phi) = 0\,. \tag{46}$$

Imposing the constraint (45) leads to a second equation for S, namely [6,14]

$$\frac{\partial S}{\partial \Phi}\frac{\partial S}{\partial \lambda} - e^{\lambda} V(\Phi) = 0\,. \tag{47}$$

Obviously, we can now discard (46) by simply looking for a principal function independent of x.

Ideally, one would like to find a complete integral $S(x, \Phi, \lambda, p_1, p_2)$ of (46) such that it satisfies the constraint (47) for $p_2 = 0$. Such a complete integral generates a canonical transformation to new coordinates (q_1, q_2, p_1, p_2) via

$$\pi_\Phi = \frac{\partial S}{\partial \Phi}\,, \quad \pi_\lambda = \frac{\partial S}{\partial \lambda}\,, \tag{48a}$$

$$q^1 = \frac{\partial S}{\partial p_1}\,, \quad q^2 = \frac{\partial S}{\partial p_2}\,. \tag{48b}$$

In these coordinates the system decouples [12] into an unconstrained one depending only on the canonical pair (q^1, p_1) plus a trivial one containing the gauge degree of freedom (q^2, p_2). p_2 is constrained to zero and q^2 remains completely arbitrary.

Unfortunately, we have not been able to construct such a complete integral. However, we found an incomplete integral [14] satisfying the full system (46,47)

$$S^{(0)}(\Phi, \lambda, p_1) = p_1 e^{\lambda} + \frac{W(\Phi)}{p_1} \tag{49}$$

where again $W'(\Phi) = V(\Phi)$. $S^{(0)}$ can be extended to a complete integral by making the ansatz

$$S(x, \Phi, \lambda, p_1, p_2) = S^{(0)}(\Phi, \lambda, p_1) + p_2\left[\Delta(\Phi, \lambda, p_1, p_2) - x\right]\,. \tag{50}$$

It is not difficult to show that such a function Δ always exists. The special form of (50) allows us to evaluate the canonical transformation (48) on the constraint surface $p_2 = 0$. There we obtain

$$\pi_\Phi = V(\Phi)/p_1 , \quad \pi_\lambda = p_1 e^\lambda , \tag{51a}$$
$$q^1 = e^\lambda - W(\Phi)/(p_1)^2 . \tag{51b}$$

We cannot compute q^2, but this does not matter, as it is purely gauge.

The new coordinates (q^1, p_1) are gauge independent observables, as one can easily check that their Poisson brackets with the Hamiltonian vanish (modulo the constraint). Furthermore we can relate them with the integration constants A, B used in Section IV

$$A = p_1 , \quad B = (p_1)^2 q^1 . \tag{52}$$

VII. QUANTIZATION

Since we have reduced dilaton gravity to the zero energy sector of a finite-dimensional dynamical system, we can quantize it in a simple way obtaining a standard Wheeler-DeWitt equation instead of a functional equation. We choose the usual representation of the momenta in terms of partial derivatives. The vanishing of the classical Hamiltonian (45) yields the following hyperbolic equation for the wave function $\mathbf{\Psi}(\mathbf{\Phi}, \lambda)$

$$\hbar^2 \frac{\partial^2 \mathbf{\Psi}}{\partial \Phi \partial \lambda} + e^\lambda V(\Phi) \mathbf{\Psi} = 0 . \tag{53}$$

The simple field redefinition $\mu = e^\lambda$, $\rho = W(\Phi)$ where again $W'(\Phi) = V(\Phi)$ transforms it into the Klein-Gordon equation (in characteristic coordinates)

$$\hbar^2 \frac{\partial^2 \mathbf{\Psi}}{\partial \rho \partial \mu} + \mathbf{\Psi} = 0 . \tag{54}$$

In order to validate our quantization procedure we compute the semi-classical limit of this theory using the WKB approach. Thus we make the following ansatz for $\mathbf{\Psi}$ depending on two real fields $S, \mathcal{A}$

$$\mathbf{\Psi}(\rho, \mu) = \mathcal{A}(\rho, \mu) e^{\frac{i}{\hbar} S(\rho,\mu)} . \tag{55}$$

(54) yields the following differential equation

$$\hbar^2 \mathcal{A}_{\rho\mu} + i\hbar(\mathcal{A} S_{\rho\mu} + \mathcal{A}_\mu S_\rho + \mathcal{A}_\rho S_\mu) - \mathcal{A} S_\rho S_\mu + \mathcal{A} = 0 . \tag{56}$$

Now we expand both functions in power series in $\hbar$: $\mathcal{A} = \mathcal{A}^{(0)} + \hbar \mathcal{A}^{(1)} + \ldots$ and $S = S^{(0)} + \hbar S^{(1)} + \ldots$ In the classical limit, i.e. for $\hbar \to 0$, this leads to

$$S^{(0)}_\rho S^{(0)}_\mu = 1 . \tag{57}$$

This is exactly the Hamilton-Jacobi equation (47) we obtained in the last section (transformed to the new coordinates ρ, μ) and we can reuse the incomplete integral (49). In the new coordinates ρ, μ the λ-Φ relation (27) derived in Section IV reads

$$A\mu = \frac{\rho + B}{A} . \tag{58}$$

Identifying p_1 with A one can easily that these classical trajectories are orthogonal with respect to the Minkowski metric to the curves described by $S^{(0)} = const$. Thus we obtain the correct classical limit.

For the next terms in the WKB approximation we obtain the following differential equations

$$S^{(0)}_\mu S^{(1)}_\rho + S^{(0)}_\rho S^{(1)}_\mu = 0 , \tag{59a}$$
$$S^{(0)}_{\rho\mu} \mathcal{A}^{(0)} + S^{(0)}_\rho \mathcal{A}^{(0)}_\mu + S^{(0)}_\mu \mathcal{A}^{(0)}_\rho = 0 , \tag{59b}$$
$$S^{(0)}_{\rho\mu} \mathcal{A}^{(1)} + S^{(1)}_{\rho\mu} \mathcal{A}^{(0)} + S^{(0)}_\rho \mathcal{A}^{(1)}_\mu + S^{(1)}_\rho \mathcal{A}^{(0)}_\mu + S^{(0)}_\mu \mathcal{A}^{(1)}_\rho + S^{(1)}_\mu \mathcal{A}^{(0)}_\rho = 0 . \tag{59c}$$

They can be solved easily by introducing the new variables $2\sigma^\pm = A\rho \pm \mu/A$

$$S^{(0)}(\sigma^+,\sigma^-) = \sigma^+ + C\,, \tag{59d}$$

$$S^{(1)}(\sigma^+,\sigma^-) = F(\sigma^-)\,, \tag{59e}$$

$$\mathcal{A}^{(0)}(\sigma^+,\sigma^-) = G(\sigma^-)\,, \tag{59f}$$

$$\mathcal{A}^{(1)}(\sigma^+,\sigma^-) = H(\sigma^-) + \Big[G(\sigma^-)F''(\sigma^-) + G'(\sigma^-)F'(\sigma^-)\Big]\sigma^+ \tag{59g}$$

with an arbitrary constant C and three arbitrary functions F, G, H.

VIII. MINIMALLY COUPLED MATTER

We now couple minimally a matter field ψ by adding

$$\mathcal{L}_M = \kappa\sqrt{-g}(\nabla\psi)^2 \tag{60}$$

with a coupling constant κ to the action (4). Its energy-momentum tensor is given in the conformal gauge (18) by

$$T^{00} = T^{11} = \frac{\kappa}{2}e^{-2\lambda}(\psi_t^2 + \psi_x^2)\,, \tag{61a}$$

$$T^{01} = -\kappa e^{-2\lambda}\psi_t\psi_x\,. \tag{61b}$$

Adding again the gauge fixing condition $\lambda_t = 0$ it is easy to show that we obtain exactly in the same way as before that $\Phi_t = 0$ and additionally that $\psi_t = 0$. Thus we can still reduce the field equations to ordinary differential equations. Note that this stems from the fact that there is no coupling between the dilaton field and the matter.

The reduced field equations have now the form

$$\Phi_{xx} - e^\lambda V(\Phi) = 0\,, \tag{62a}$$

$$\lambda_{xx} - e^\lambda V'(\Phi) = 0\,, \tag{62b}$$

$$\psi_{xx} = 0\,, \tag{62c}$$

$$\Phi_x\lambda_x - e^\lambda V(\Phi) + \kappa\psi_x^2 = 0\,. \tag{62d}$$

Again we can identify the last equation with a zero energy condition for the unconstrained system defined by the Lagrangian

$$\mathcal{L}_g[\Phi,\lambda,\psi] = \Phi_x\lambda_x + \kappa\psi_x^2 + e^\lambda V(\Phi)\,. \tag{63}$$

Quantizing the Hamiltonian constraint we obtain again a hyperbolic wave equation as Wheeler-DeWitt equation

$$\hbar^2\frac{\partial^2\mathbf{\Psi}}{\partial\Phi\partial\lambda} + \frac{\hbar^2}{4\kappa}\frac{\partial^2\mathbf{\Psi}}{\partial\psi^2} + e^\lambda V(\Phi)\mathbf{\Psi} = 0\,. \tag{64}$$

In the absence of matter Φ and λ entered the equation on equal footing. There was no way to decide whether $\Phi+\lambda$ or $\Phi-\lambda$ should be a timelike coordinate in the superspace. Now the sign of κ induces a (2+1)-split of the superspace. However, in general it is not clear which part of the split is timelike and which spacelike.

IX. CONCLUSION

A similar reduction to ordinary differential equations was used by Banks and O'Loughlin [1]. We would like to point out some differences in the obtained quantum theories. They do not consider whether their quantum theory yields the correct classical limit. Actually, it is easy to see that they would not obtain their classical model. The latter one depends on three fields, whereas their quantum theory knows only two degrees of freedom. The field g used in their parameterization of the metric simply disappears.

There exists an alternative way to endow the gauge reduced equations of motion with a Hamiltonian structure. In Section VI we started with the second-order system (43). Alternatively one can use the first-order formulation obtained in Section IV after one integration

$$\Phi_x = Ae^{\lambda}\,, \tag{65a}$$
$$A\lambda_x = V(\Phi)\,. \tag{65b}$$

These are the Euler-Lagrange equations for the first-order Lagrangian

$$\mathcal{L}_1[\Phi,\lambda] = \Phi\lambda_x + Ae^{\lambda} - \frac{W(\Phi)}{A}\,. \tag{66}$$

It is well-known that such a Lagrangian leads directly to generalized Poisson brackets [31] which can also be considered as Dirac brackets [13]. Applying this formalism to $\mathcal{L}_1$ yields

$$\{\lambda,\Phi\} = 1\,. \tag{67}$$

In this description we can thus interpret the dilaton and the conformal factor as canonically conjugate coordinates! However, we believe that (66) represents a dubious starting point for a quantization, as A is treated as a parameter. But we saw in Section VI that it can be identified with a dynamical variable!

Since we have not been able to find a complete integral of the Hamilton-Jacobi equation (46) we could not pursue this argument until the end. We have not constructed the full canonical transformation which leads to the decoupling of the Hamiltonian. Otherwise we could have used its regular, gauge-independent part for the quantization and thus quantize the fully reduced phase space.

Instead we have used a finite-dimensional classical system and imposed from the outside a gauge symmetry by considering only its zero energy sector. This symmetry corresponds to the residual gauge freedom left after fixing the gauge with the condition $\lambda_t = 0$. Then we proceed in the usual way following Dirac [5] by requiring that the wave function is annihilated by an operator version of the (first-class) constraint.

It appears natural to ask for the relationship between the quantum theory obtained this way and the one obtained by following the above mentioned Hamilton-Jacobi procedure. One can expect that they are not equivalent. This situation is very similar to the quantization of the free relativistic particle. The approach we took here corresponds to the covariant quantization. No gauge fixing is performed and we get a covariant wave function (the Klein-Gordon equation (54) is obtained in characteristic coordinates!).

One should probably study in more detail the relation between the residual gauge symmetry in the reduced field equations (43) and the symmetry generated by the constraint $\mathcal{H}_g = 0$. As mentioned in Section IV the integration constant A can be changed by a rescaling of x. In the context of the field equations we consider this as a gauge transformation. For the system described by the particle Lagrangian $\mathcal{L}_g$ this corresponds to a reparametrization of the evolution parameter x and is not contained in the gauge transformations generated by $\mathcal{H}_g$. Under these transformations $A = p_1$ remains invariant.

This connection can be made more transparent by using a reparametrization invariant action. To this end one introduces a new evolution parameter γ and sets $x = X(\gamma)$. This leads to the action (the dot denotes derivatives with respect to γ)

$$\mathcal{L}'_g[\lambda,\Phi,X] = \frac{\dot\Phi\dot\lambda}{\dot X} + \dot X e^{\lambda}V(\Phi)\,. \tag{68}$$

The original equations of motion are recovered, if one imposes the gauge fixing condition $X-\gamma = 0$. Since this condition depends explicitly on the evolution parameter, the gauge fixed Hamiltonian acquires a correction term [9,10]. Once this is taken into account, one obtains exactly the same quantum theory as we did in Section VII.

Finally, we would like to stress again that applying methods from the formal theory of partial differential equations allows us to compute the dimension of the fully reduced phase space without constructing it. This indicates that these techniques should also be useful for more complicated models where this construction cannot be performed explicitly.

This holds especially for systems where one can show that for a full gauge reduction one must pose in addition initial and/or boundary conditions. For instance in the case of standard four-dimensional general relativity it is easy to see that the gauge corrected Cartan characters cannot be obtained from any system of differential equations, as they do not satisfy all properties of Cartan characters. This implies that it is not possible to fix the gauge completely by imposing gauge conditions in the form of differential (or algebraic) equations. Nevertheless, one can determine the arbitrariness of the fully reduced phase space [27,29].

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