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Residence time distribution of laminar flow in microchannels of various cross-sectional shape

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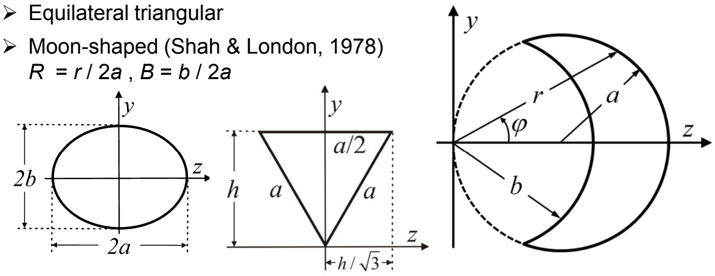
1. Introduction and motivation

- Liquid flows in microchannels are often associated with low Reynolds number (laminar flow) and high Schmidt numbers (small diffusion coefficients)
- Under these conditions the residence time distribution (RTD) is dominated by the non-uniform laminar velocity profile
- The diffusion-free RTD of fully developed laminar flow is known only for very few channel shapes (circular pipe, parallel plates)
- Microchannels have various shapes (rectangular, triangular, trapezoidal, elliptical, irregular curved, …)
- Goal: investigate influence of cross-sectional shape on diffusion-free RTD of fully developed laminar flow in straight channels

2. Considered channel shapes

- Elliptical (arbitrary aspect ratio)
- Equilateral triangular
- Moon-shaped (Shah & London, 1978)
  $R = r/2a$, $B = b/2a$



3. Evaluation of the RTD

- In the absence of diffusion the cumulative RTD is given by

$$F(\theta)=\frac{Q(\theta)}{Q_{\mathrm{total}}},\qquad \theta=\frac{t}{\bar{t}}=\frac{L/\lambda U_{\max}}{L/U_{\mathrm{mean}}}=\frac{1}{\lambda}\frac{U_{\mathrm{mean}}}{U_{\max}}=\frac{\theta_{\mathrm{F}}}{\lambda}$$

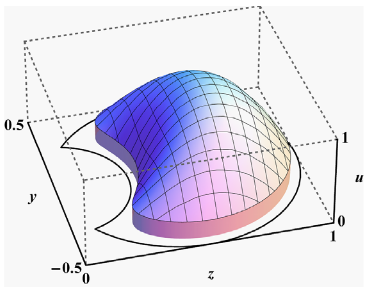
- The volumetric flow rate associated with a value $0 < \lambda \leq 1$ and the total volumetric flow rate are

$$Q(\theta)=\iint_{A_\lambda} u\,\mathrm{d}A$$

$$Q_{\mathrm{total}}=AU_{\mathrm{mean}}$$

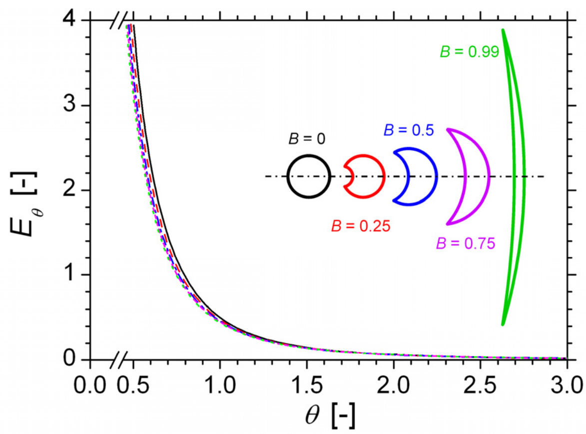
- Thus

$$F(\theta)=\frac{1}{A\theta_{\mathrm{F}}}\iint_{A_\lambda}\frac{u}{U_{\max}}\,\mathrm{d}A$$

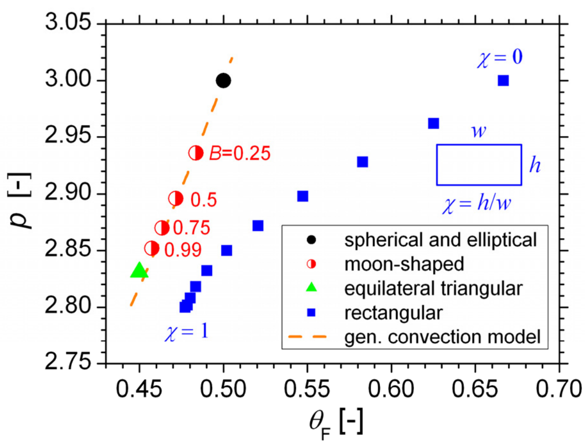


where A_λ is the area enclosed by the velocity isolines $u=\lambda U_{max}$

- While the channel shapes arising from various values of B are very different, the difference in θ_F and the RTD is rather small



- Value of fitting parameter p for different channel types



4. RTD of moon-shaped channels

- First appearance time

$$\theta_{\mathrm{F}}=\frac{1}{8}\,\frac{B(1+14B^2)\sqrt{1-B^2}-\left[8B^2(1+B^2)-1\right]\cos^{-1}B}{(R^2_{U,\max}-B^2)(R^{-1}_{U,\max}-1)\left[B\sqrt{1-B^2}+\left(1-2B^2\right)\cos^{-1}B\right]}$$

- Cumulative RTD (the outer integral is evaluated numerically)

$$F(\theta)=\frac{8a^2}{A\theta_{\mathrm{F}}}\int_{R_{\min,\lambda}}^{R_{\max,\lambda}}\int_{0}^{\varphi_\lambda}\frac{R^2-B^2}{R^2_{U,\max}-B^2}\,\frac{1-R^{-1}\cos\varphi}{1-R^{-1}_{U,\max}}\,R\,\mathrm{d}\varphi\,\mathrm{d}R$$

$$=\frac{8a^2}{A\theta_{\mathrm{F}}}\int_{R_{\min,\lambda}}^{R_{\max,\lambda}}\frac{R^2-B^2}{R^2_{U,\max}-B^2}\,\frac{R\varphi_\lambda-\sin\varphi_\lambda}{1-R^{-1}_{U,\max}}\,\mathrm{d}R$$

- Fitting model for $\theta \geq \theta_F$ with p as a free parameter (Wörner, 2010)

$$E_\theta=\frac{\Gamma\left(1+(p-2)\theta_{\mathrm{F}}^{-1}\right)}{\Gamma(p-1)\Gamma\left((p-2)(\theta_{\mathrm{F}}^{-1}-1)\right)}\,\frac{\theta_{\mathrm{F}}^{p-1}}{\theta^p}\left(1-\frac{\theta_{\mathrm{F}}}{\theta}\right)^{(p-2)(\theta_{\mathrm{F}}^{-1}-1)-1}$$

$$p=2+(\theta_{\mathrm{F}}^{-1}-1)^{-1}\quad\rightarrow\quad E_\theta=\frac{1}{1-\theta_{\mathrm{F}}}\,\frac{1}{\theta}\left(\frac{\theta_{\mathrm{F}}}{\theta}\right)^{\frac{1}{1-\theta_{\mathrm{F}}}}$$

Generalized convection model (Levenspiel, 1989)

5. Conclusions

- The RTD of spherical, elliptical, equilateral triangular and moon-shaped channels are all well described by the generalized convection model
- The RTD of rectangular channels shows a completely different behavior