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THE APPROXIMATE INVERSE IN ACTION WITH AN APPLICATION TO COMPUTERIZED TOMOGRAPHY*

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Abstract. The approximate inverse is a scheme to obtain stable numerical inversion formulæ for linear operator equations of the first kind. Yet, in some applications the computation of a crucial ingredient, the reconstruction kernel, is time-consuming and instable. It may even happen that the kernel does not exist for a particular semidiscrete system. To cure this dilemma we propose and analyze a technique that is based on a singular value decomposition of the underlying operator. The results are applied to the reconstruction problem in 2D-computerized tomography where they enable the design of reconstruction filters and lead to a novel error analysis of the filtered backprojection algorithm.

Key words. approximate inverse, mollification, Radon transform, computerized tomography, filtered backprojection

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1. Introduction.

The approximate inverse is a regularization scheme which applies especially to underdetermined (semidiscrete) systems. Yet in some applications the numerical computation of the necessary reconstruction kernel v_{discrete} is time-consuming and instable. It may even happen that v_{discrete} does not exist for a particular semidiscrete system. However, the reconstruction kernel v of the underlying infinite dimensional (continuous) problem may be at hand. In this paper we propose a procedure to find a substitute for v_{discrete} from v and we show that this procedure is sound.

In the following we recall the concept of the approximate inverse which belongs to the class of mollifier methods as considered, for instance, by Murio [19]. In a systematic way the approximate inverse generalizes a technique used by Grünbaum [5] and Davison and Grünbaum [3] for tomographic inversion.

Let $A : X \to Y$ be a continuous and *injective* operator between the real or complex infinite dimensional Hilbert spaces X and Y. We want to find a $f \in X$ such that

$$A_n f = g_n, \tag{1.1}$$

where $A_n : X \to \mathbb{C}^n$ and $g_n \in \mathbb{C}^n$ are defined via a mapping $\Psi_n : Y \to \mathbb{C}^n$ by $A_n = \Psi_n A$ and $g_n = \Psi_n g$ with $g \in \mathsf{R}(A)$, the range of A. Let us assume—for the time being—that A_n is continuous. The above setting describes most practical situations where the data can be recorded only in finitely many observation points.

Problem (1.1) is underdetermined and we can only search for its minimum norm solution $f_n^\dagger$, that is,

$$A_n^* A_n f_n^\dagger = A_n^* g_n \quad \text{and } f_n^\dagger \in \mathsf{N}(A_n)^\perp. \tag{1.2}$$

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Here $\mathsf{N}(A_n)^\perp$ is the orthogonal complement of the null space of A_n. If the range of A is nonclosed in Y, that is, the generalized inverse of A is unbounded, instabilities appear very likely in computing $f_n^\dagger$ directly from (1.2) under erroneous data g_n.

This reasoning led Louis and Maass [13] to the approximate inverse where one tries to reconstruct moments of f: $\langle f, e_n^i\rangle_X$, $i = 1,\dots,m$, with suitable *mollifiers* e_n^i. In case $X = L^2(\Omega)$, Ω a domain in $\mathbb{R}^d$, one can think of the e_n^i's as smooth approximations to δ-distributions located at points $x_i \in \Omega$.

The computations of the moments is achieved by approximating e_n^i in the range of A_n^*. To any e_n^i we associate a *reconstruction kernel* $\upsilon_n^i \in \mathbb{C}^n$ by minimizing the defect $\|A_n^*\upsilon_n^i - e_n^i\|_X$, that is, υ_n^i solves the normal equation

$$A_n A_n^* \upsilon_n^i \;=\; A_n e_n^i. \tag{1.3}$$

The above equation for υ_n^i is independent of the data g_n, therefore free of noise from measurement errors. We call (e_n^i, υ_n^i) a *mollifier/reconstruction kernel pair* for A_n.

The operator $S_n : \mathbb{C}^n \to \mathbb{C}^m$,

$$(S_n h)_i \;=\; \langle h, \upsilon_n^i\rangle_{\mathbb{C}^n}, \quad i = 1,\dots,m, \tag{1.4}$$

is called *approximate inverse* of A_n. Hence, $S_n g_n$ is an approximate solution of (1.1).

LEMMA 1.1. *If $g_n = A_n f$ then*

$$(S_n g_n)_i \;=\; \langle f_n^\dagger, e_n^i\rangle_X, \quad i = 1,\dots,m. \tag{1.5}$$

Proof. The reconstruction kernels satisfy $A_n^*\upsilon_n^i = \mathcal{P}_n e_n^i$ where $\mathcal{P}_n : X \to X$ is the orthogonal projector onto $\mathsf{R}(A_n^*) = \mathsf{N}(A_n)^\perp$. Hence,

$$(S_n g_n)_i \;=\; \langle f, A_n^*\upsilon_n^i\rangle_X \;=\; \langle \mathcal{P}_n f, e_n^i\rangle_X.$$

Since $\mathcal{P}_n f = f_n^\dagger$ (see (1.2)), we are finished with the proof. □

An interpretation of the approximate inverse as regularization scheme and further details are given by Louis [12]. He also shows how invariances of A improve the efficiency; see Remark 5.2 below.

For several reasons we wish to avoid solving (1.3): $A_n A_n^*$ may be densely populated and ill-conditioned, increasing n calls for a complete new computation of the kernels; invariances of A do not show in $A_n A_n^*$ in general.

We propose the following technique to approximate υ_n (we will drop the superscript i whenever considering a single pair (e_n, υ_n)). Suppose (e, υ) is a mollifier/reconstruction kernel pair for A, i.e., $A^*\upsilon = e$ (A is injective!). Then we expect $\Psi_n\upsilon$ to be an approximate solution of (1.3) where e_n is equal or close to e. In section 3.1 we show convergence of $\Psi_n\upsilon$ to a solution of (1.3). We also analyze the situation when the mollifier e is *not* in $\mathsf{R}(A^*)$ (section 3.2). Here we approximate υ_n by $\Psi_n\upsilon$ where $A^*\upsilon$ is close to e_n. We further discuss a technique to construct υ from e which can be implemented.

In some applications, for instance, if A is the Radon transform, $A_n : \mathsf{D}(A_n) \subset X \to \mathbb{C}^n$ is unbounded and A_n^* does not exist; see section 5. Consequently, the concept of approximate inverse cannot be applied to (1.1). Louis and Schuster [16] replaced A by a truncated singular value decomposition, thus circumventing the problem. We favor another cure which is closely related to our findings for a bounded A_n (section 4).

In section 5 we apply the results from the previous sections to the reconstruction problem in 2D-computerized tomography, mainly to illustrate our rather abstract results by a concrete application. As a byproduct we achieve a novel error estimate for the filtered backprojection algorithm as well as an alternative to design reconstruction filters.

To start this paper we introduce our technical set-up in the next section. Especially the operator A_n is defined precisely. In the appendix we prove an auxiliary mapping property of the Radon transform.

Hegland and Anderssen [6] investigated a mollification method being akin to our approximate inverse approach. However, the details are completely different and they require stronger conditions on A; for instance, A^{-1} has to be densely defined. Further, an implementation of their method requires an explicit knowledge of the preimages (under A) of the chosen basis functions. On the other hand, Hegland and Anderssen relate the regularization parameter (support width of the mollifier) to the discretization step size to bound the noise amplification error. This is an issue we do not address here.

2. Preliminaries. We specify our technical assumptions *that are required to hold throughout the paper* if not indicated otherwise.

The operator A is supposed to have the mapping property (2.1). Let there be Banach spaces X_1 and Y_1 such that the embeddings $X_1 \hookrightarrow X$ as well as $Y_1 \hookrightarrow Y$ are continuous, injective, and dense. Moreover,

$$A : X_1 \to Y_1 \text{ is continuous.} \tag{2.1}$$

Let Y_1' be the dual to Y_1. One may consider the spaces X_1 and Y_1 as abstract smoothness classes in X and Y, respectively.

We are now able to define the observation operator $\Psi_n : Y_1 \to \mathbb{C}^n$ precisely: given n functionals $\psi_{n,k} \in Y_1'$, $k = 1, \ldots, n$, let

$$(\Psi_n v)_k := \langle \psi_{n,k}, v \rangle_{Y_1' \times Y_1}, \quad k = 1, \ldots, n, \tag{2.2}$$

where $\langle \cdot, \cdot \rangle_{Y_1' \times Y_1}$ is the duality pairing on $Y_1' \times Y_1$.

In applications we have in mind, typically, Y_1 will be a Sobolev space of sufficient order such that point evaluations are continuous.

It will prove useful to transform equation (1.1) into an equivalent equation where $\mathbb{C}^n$ is replaced by a suitable subspace of Y; see (2.6) below. To this end we introduce a family $\{V_n\}_{n \in \mathbb{N}}$ of finite dimensional subspaces of Y being nested: $V_n \subset V_{n+1}$. Furthermore, each V_n is spanned by basis elements $\varphi_{n,k}$, $k = 1, \ldots, n$, which build a *Riesz system* with respect to Y, that is,

$$\sum_{k=1}^{n} |a_k|^2 \preceq \Big\| \sum_{k=1}^{n} a_k \, \varphi_{n,k} \Big\|_Y^2 \preceq \sum_{k=1}^{n} |a_k|^2 \quad \text{for all } n \in \mathbb{N}. \tag{2.3}$$

Our notation $A \preceq B$ indicates the existence of a generic constant $c > 0$ such that $A \le c\,B$. The constant c will not depend on the arguments of A and B. This means that the constants involved in (2.3) do not depend on n.

The spaces $\mathbb{C}^n$ and V_n are related one-to-one by the operator $Q_n : \mathbb{C}^n \to V_n$, $Q_n a := \sum_{k=1}^{n} a_k \, \varphi_{n,k}$. The composition of Ψ_n and Q_n creates a new operator $\Pi_n : Y_1 \to V_n$ as follows:

$$\Pi_n v := Q_n \Psi_n v = \sum_{k=1}^{n} \langle \psi_{n,k}, v \rangle_{Y_1' \times Y_1} \, \varphi_{n,k}.$$

The operator Π_n relates the observation operator Ψ_n to V_n. Considered as an operator mapping Y_1 into Y, Π_n is assumed to be uniformly bounded in n:

$$\|\Pi_n\|_{Y_1\to Y} \preceq 1 \quad \text{as } n\to\infty. \tag{2.4}$$

Our last ingredient is the *approximation property* (2.5): let there be a sequence $\{\rho_n\}\subset[0,1]$ converging monotonically to zero such that

$$\|v-\Pi_n v\|_Y \preceq \rho_n\,\|v\|_{Y_1} \quad \text{for all } v\in Y_1 \text{ as } n\to\infty. \tag{2.5}$$

We understand $\{\rho_n\}$ as optimal, that is, $\{\rho_n\}$ is the fastest converging admissible sequence in (2.5).

Now we apply Q_n from the left to both sides of (1.1) yielding

$$\widetilde{A}_n\, f = \widetilde{g}_n, \tag{2.6}$$

where $\widetilde{A}_n = Q_n A_n : X_1 \to V_n$ and $\widetilde{g}_n = Q_n g_n$.

For the solution of (1.1) and (2.6), respectively, by the approximate inverse we distinguish two scenarios.

First, we assume that $A_n : \mathsf{D}(A_n)\subset X\to\mathbb{C}^n$ is *bounded* where $\mathsf{D}(A_n) := X_1$ is the domain of definition of A_n. Thus, $A_n\in\mathcal{L}(X,\mathbb{C}^n)$. Typical examples are integral operators which are sufficiently smoothing.

Example 2.1. Let $A : L^2(0,1)\to L^2(0,1)$, $Af(x) := \int_0^1 \mathsf{k}(x,y)\,f(y)\,dy$, where the kernel k is such that $A : L^2(0,1)\to H^{1/2+\varepsilon}(0,1)$ is bounded for an $\varepsilon>0$. On the Sobolev space $H^{1/2+\varepsilon}$ point evaluations are continuous functionals, so $\Psi_n g = n^{-1/2}(g(x_1),\dots,g(x_n))^t$, $x_i\in\,]0,1[$, is the right choice if we are able to observe Af at x_i. Thus, $A_n^* w(y) = n^{-1/2}\sum_i \overline{\mathsf{k}(x_i,y)}\,w_i$ and $(A_nA_n^*)_{i,j} = n^{-1}\int_0^1 \overline{\mathsf{k}(x_i,y)}\,\mathsf{k}(x_j,y)\,dy$.

Second, we consider $A_n : \mathsf{D}(A_n)\subset X\to\mathbb{C}^n$ *unbounded*. Hence, the Hilbert space adjoint of A_n cannot be defined on all of $\mathbb{C}^n$ (otherwise A_n would have been continuous already). Here the worst case is $\mathsf{D}(A_n^*)=\{0\}$, so that the approximate inverse is *not* defined meaningful for (1.1). This happens for the Radon transform; see section 5.

3. Bounded semidiscrete operators A_n: Approximating the discrete reconstruction kernel. Let (2.1) hold true with $X_1 = X$ (topologically):

$$A : X\to Y_1 \ \text{ is continuous,} \tag{3.1}$$

that is, $A_n\in\mathcal{L}(X,\mathbb{C}^n)$ and $\widetilde{A}_n\in\mathcal{L}(X,Y)$. In what follows we will denote the adjoint of $A : X\to Y$ by A^*.

Now we study convergence of the minimum norm solution $f_n^\dagger$ of (1.1) as $n\to\infty$. From this we derive a kind of pointwise convergence of the approximate inverse S_n.

LEMMA 3.1. *If* (3.1) *then* $\|A-\widetilde{A}_n\|_{X\to Y}\le\rho_n\,\|A\|_{X\to Y_1}$.

Proof. Since $\widetilde{A}_n x = \Pi_n Ax$ for $x\in X$ one needs to apply (3.1) only. ∎

THEOREM 3.2. *Let* $f_n^\dagger$ (1.2) *be the minimum norm solution of* (1.1) *with* $g_n = A_n f$ *for* $f\in X$. *Then*

$$\lim_{n\to\infty}\left\|f-f_n^\dagger\right\|_X = 0.$$

Moreover, if the sequence of mollifiers $\{e_n^i\}_{n\in\mathbb{N}}$ *converges to* $e^i\in X$, $i=1,\dots,m$, *we have that*

$$\lim_{n\to\infty} S_nA_nf = Ef, \tag{3.2}$$

where

$$E : X \to \mathbb{C}^m \quad \textit{is defined by} \quad (Ef)_i := \langle f, e^i \rangle_X, \quad i = 1, \ldots, m. \tag{3.3}$$

Proof. Recall that $f_n^\dagger = \mathcal{P}_n f$ where $\mathcal{P}_n : X \to X$ is the orthogonal projection onto $\mathsf{N}(A_n)^\perp = \mathsf{N}(\widetilde{A}_n)^\perp$. Due to Lemma 3.1 and the injectivity of A we have that $\bigcap_{n\in\mathbb{N}} \mathsf{N}(\widetilde{A}_n) = \{0\}$. This yields the pointwise convergence of $\mathcal{P}_n$ to the identity operator in X as $n \to \infty$, thereby proving the first assertion. The second assertion follows readily from (1.5). □

Choosing special mollifiers e_n^i we will show below that $\|S_n A_n f - Ef\|_\infty \preceq \rho_n \|f\|_X$ as $n \to \infty$; see Corollary 3.8.

For an $e \in X$ we have either $e \in \mathsf{R}(A^*)$ or $e \in \partial\mathsf{R}(A^*)$ due to the injectivity of A ($\partial\mathsf{R}(A^*)$ is the topological boundary of $\mathsf{R}(A^*)$). The first situation leads to reconstruction kernels υ satisfying $A^*\upsilon = e$. In section 3.1 below we shall show that $\Psi_n \upsilon$ is an approximate solution of (1.3) for suitable e_n.

If we cannot find a mollifier e in the range of A^*, the equation $A^*y = e$ has no least squares solution. Thus, no reconstruction kernel is associated with e. We investigate the latter situation in section 3.2.

3.1. The special case $e \in \mathsf{R}(A^*)$. In Lemma 3.3 we derive a relation between the reconstruction kernels for A_n and $\widetilde{A}_n$.

LEMMA 3.3. *Let $(e, \widetilde{\upsilon}_n)$ be a mollifier/reconstruction kernel pair for $\widetilde{A}_n$ where $e \in X$ is arbitrary. Then, $(e, Q_n^* \widetilde{\upsilon}_n)$ is a mollifier/reconstruction kernel pair for A_n.*

Proof. The assertion follows from $A_n^* Q_n^* \widetilde{\upsilon}_n = \widetilde{A}_n^* \widetilde{\upsilon}_n = \mathcal{P}_n e$ where $\mathcal{P}_n$ is as in the proof of Theorem 3.2. □

Below we will need the *Gramian matrix* $G_n \in \mathbb{C}^{n\times n}$ relative to $\{\varphi_{n,1}, \ldots, \varphi_{n,n}\}$. This matrix has entries $(G_n)_{i,j} = \langle \varphi_{n,i}, \varphi_{n,j} \rangle_Y$. A quick calculation validates the equality $G_n \Psi_n z = Q_n^* \Pi_n z$ for all $z \in Y_1$.

THEOREM 3.4. *Adopt all assumptions specified in section* 2 *and assume* (3.1). *Let $(e_n, \widetilde{\upsilon}_n)$ be a mollifier/reconstruction kernel pair for $\widetilde{A}_n$ where $e_n = \widetilde{A}_n^* \upsilon$, $\upsilon \in Y_1$, and $\widetilde{\upsilon}_n \in \mathsf{N}(\widetilde{A}_n^*)^\perp$. Then,*

$$\|G_n \Psi_n \upsilon - Q_n^* \widetilde{\upsilon}_n\|_{\mathbb{C}^n} \;\preceq\; \rho_n \, \|\upsilon\|_{Y_1} + \inf_{y \in \mathsf{R}(\widetilde{A}_n)} \|\upsilon - y\|_Y \tag{3.4}$$

as $n \to \infty$. Note that $(e_n, Q_n^* \widetilde{\upsilon}_n)$ is a mollifier/reconstruction kernel pair for A_n.

Proof. Since $\|Q_n^*\|_{Y\to\mathbb{C}^n} \preceq 1$ by (2.3) we may estimate

$$\begin{aligned}
\|G_n \Psi_n \upsilon - Q_n^* \widetilde{\upsilon}_n\|_{\mathbb{C}^n} &\le \|Q_n^* \Pi_n \upsilon - Q_n^* \upsilon\|_{\mathbb{C}^n} + \|Q_n^* \upsilon - Q_n^* \widetilde{\upsilon}_n\|_{\mathbb{C}^n} \\
&\preceq \|\Pi_n \upsilon - \upsilon\|_Y + \|\upsilon - \widetilde{\upsilon}_n\|_Y \\
&\preceq \rho_n \, \|\upsilon\|_{Y_1} + \|\upsilon - \widetilde{\upsilon}_n\|_Y,
\end{aligned}$$

where we used (2.5) in the final step. The assertion will be proved if we bound $\|\upsilon - \widetilde{\upsilon}_n\|_Y$ by a multiple of $\inf\{\, \|\upsilon - y\|_Y \mid y \in \mathsf{R}(\widetilde{A}_n) \,\}$.

Recall that $\widetilde{\upsilon}_n$ is the *unique* solution in $\mathsf{N}(\widetilde{A}_n^*)^\perp$ of the normal equation

$$\widetilde{A}_n \widetilde{A}_n^* \widetilde{\upsilon}_n = \widetilde{A}_n e_n = \widetilde{A}_n \widetilde{A}_n^* \upsilon. \tag{3.5}$$

Let $P_n : Y \to Y$ be the orthogonal projector onto $\mathsf{N}(\widetilde{A}_n^*)^\perp$. Since $P_n \upsilon$ solves (3.5) as well, we obtain $\widetilde{\upsilon}_n = P_n \upsilon$. As $\mathsf{N}(\widetilde{A}_n^*)^\perp = \mathsf{R}(\widetilde{A}_n)$ we proceed with

$$\|\upsilon - \widetilde{\upsilon}_n\|_Y = \|\upsilon - P_n \upsilon\|_Y = \inf_{y \in \mathsf{R}(\widetilde{A}_n)} \|\upsilon - y\|_Y$$

which completes the proof. ∎

COROLLARY 3.5. *The assumptions are those from Theorem* 3.4. *If either* $\upsilon \in \mathsf{R}(A)$ *or all* A_n*'s are onto then*

$$\|G_n \Psi_n \upsilon - Q_n^* \widetilde{\upsilon}_n\|_{\mathbb{C}^n} \preceq \rho_n \|\upsilon\|_{Y_1} \quad \text{as } n \to \infty.$$

Proof. First we consider $\upsilon \in \mathsf{R}(A)$. Let $\upsilon = Az$ for $z \in X$. Now

$$\inf_{y \in \mathsf{R}(\widetilde{A}_n)} \|\upsilon - y\|_Y \le \|Az - \Pi_n Az\|_Y \preceq \rho_n \|Az\|_{Y_1}$$

by the approximation property (2.5). Second, if $A_n : X \to \mathbb{C}^n$ is onto we have that $\mathsf{R}(\widetilde{A}_n) = V_n$ which gives

$$\inf_{y \in \mathsf{R}(\widetilde{A}_n)} \|\upsilon - y\|_Y = \inf_{y \in V_n} \|\upsilon - y\|_Y \le \|\upsilon - \Pi_n \upsilon\|_Y \preceq \rho_n \|\upsilon\|_{Y_1}.$$

In both cases the assertion follows from (3.4). ∎

Even so $e_n = \widetilde{A}_n^* \upsilon$ converges to $e = A^* \upsilon$ due to Lemma 3.1, e_n may be an unsuitable mollifier for fixed (possibly small) n. It seems natural to work with e in the semidiscrete setting also. This more general situation is considered in the following lemma where we, however, allow a weighted norm in $\mathbb{C}^n$. Under the assumptions of Lemma 3.6 below, $\|A_n A_n^* \cdot \|_{\mathbb{C}^n}$ is a norm on $\mathbb{C}^n$ being, in general, weaker than the Euclidean norm in the following sense. There exist positive constants γ_n and Γ such that $\gamma_n \|z\|_{\mathbb{C}^n} \le \|A_n A_n^* z\|_{\mathbb{C}^n} \le \Gamma \|z\|_{\mathbb{C}^n}$ for all $z \in \mathbb{C}^n$ where Γ does not depend on n and where γ_n tends to zero as n grows.

LEMMA 3.6. *Let* $e = A^* \upsilon$ *for* $\upsilon \in Y_1$. *Further, let* $(e, \overline{\upsilon}_n)$ *be a mollifier/reconstruction kernel pair for* $\widetilde{A}_n$ *where* $\overline{\upsilon}_n \in \mathsf{N}(\widetilde{A}_n^*)^\perp$. *Under the assumptions of Theorem* 3.4 *and provided all* A_n*'s are onto we have that*

$$\|A_n A_n^* (G_n \Psi_n \upsilon - Q_n^* \overline{\upsilon}_n)\|_{\mathbb{C}^n} \preceq \rho_n \|\upsilon\|_{Y_1} \quad \text{as } n \to \infty.$$

Proof. Let $\widetilde{\upsilon}_n$ be as in Theorem 3.4. Hence,

$$\begin{aligned}\|A_n A_n^* (G_n \Psi_n \upsilon - Q_n^* \overline{\upsilon}_n)\|_{\mathbb{C}^n} &\le \|A\|^2_{X \to Y_1} \|G_n \Psi_n \upsilon - Q_n^* \widetilde{\upsilon}_n\|_{\mathbb{C}^n} \\ &\qquad + \|A_n \widetilde{A}_n^* \widetilde{\upsilon}_n - A_n \widetilde{A}_n^* \overline{\upsilon}_n\|_{\mathbb{C}^n} \\ &\preceq \rho_n \|\upsilon\|_{Y_1} + \|\widetilde{A}_n \widetilde{A}_n^* \widetilde{\upsilon}_n - \widetilde{A}_n \widetilde{A}_n^* \overline{\upsilon}_n\|_Y,\end{aligned}$$

where we used Corollary 3.5, (2.3), and the estimate

$$\|A_n A_n^*\|_{\mathbb{C}^n \to \mathbb{C}^n} = \|A_n\|^2_{X \to \mathbb{C}^n} \preceq \|\Pi_n A\|^2_{X \to Y} \preceq \|A\|^2_{X \to Y_1} \tag{3.6}$$

by (2.3) and (2.4). Since $\widetilde{A}_n \widetilde{A}_n^* \widetilde{\upsilon}_n = \widetilde{A}_n \widetilde{A}_n^* \upsilon$ and $\widetilde{A}_n \widetilde{A}_n^* \overline{\upsilon}_n = \widetilde{A}_n A^* \upsilon$ we obtain that

$$\|\widetilde{A}_n \widetilde{A}_n^* \widetilde{\upsilon}_n - \widetilde{A}_n \widetilde{A}_n^* \overline{\upsilon}_n\|_Y \preceq \|\widetilde{A}_n^* - A^*\|_{Y \to X} \|\upsilon\|_Y.$$

The assertion of Lemma 3.6 is now due to Lemma 3.1. ∎

We discuss the implications of Corollary 3.5 on the approximate inverse S_n of A_n (1.4). Here one has m mollifier/reconstruction kernel pairs (e_n^i, υ_n^i), $i = 1, \dots, m$; see (1.3). Now let $e_n^i = \widetilde{A}_n^* \upsilon^i$ where $\upsilon^i \in Y_1$, $i = 1, \dots, m$. Our investigations from above suggest to replace the (unknown) approximate inverse S_n by the (computable) operator Σ_n defined by

$$(\Sigma_n b)_i = \langle b, G_n \Psi_n \upsilon^i \rangle_{\mathbb{C}^n}, \quad i = 1, \dots, m. \tag{3.7}$$

As a direct consequence of Corollary 3.5, we can show that Σ_n is a reasonable substitute for S_n.

THEOREM 3.7. *The assumptions are those from Theorem* 3.4. *Further, let* (e^i_n, υ^i_n), $i = 1, \dots, m$, *be mollifier/reconstruction kernel pairs for* A_n *where* $e^i_n = \widetilde{A}^*_n \upsilon^i$. *Assume that all* υ^i*'s are in* Y_1. *If either all* υ^i*'s are in* $\mathsf{R}(A)$ *or all* A_n*'s are onto, then*

$$\|S_n A_n f - \Sigma_n A_n f\|_\infty \preceq \rho_n \max_{1\le i\le m} \|\upsilon^i\|_{Y_1} \, \|f\|_X \quad \text{as } n \to \infty. \tag{3.8}$$

Proof. Let $(e^i_n, \widetilde{\upsilon}^i_n)$ be the mollifier/reconstruction kernel pair for $\widetilde{A}_n$ where $\widetilde{\upsilon}^i_n \in \mathsf{N}(\widetilde{A}^*_n)^\perp$. From Lemma 3.3 we know that $(e^i_n, Q^*_n\widetilde{\upsilon}^i_n)$ is a mollifier/reconstruction kernel pair for A_n. Note that $Q^*_n\widetilde{\upsilon}^i_n$ may be different from the kernel υ^i_n used in S_n; however, $A^*_n\upsilon^i_n = A^*_n Q^*_n\widetilde{\upsilon}^i_n$. Thus,

$$(S_n A_n f)_i = \langle f, A^*_n \upsilon^i_n\rangle_X = \langle f, A^*_n Q^*_n \widetilde{\upsilon}^i_n\rangle_X = \langle A_n f, Q^*_n \widetilde{\upsilon}^i_n\rangle_{\mathbb{C}^n}$$

which implies that

$$|(S_n A_n f)_i - (\Sigma_n A_n f)_i| = |\langle A_n f, Q^*_n\widetilde{\upsilon}^i_n - G_n\Psi_n\upsilon^i\rangle_{\mathbb{C}^n}| \le \|A_n f\|_{\mathbb{C}^n} \, \|Q^*_n\widetilde{\upsilon}^i_n - G_n\Psi_n\upsilon^i\|_{\mathbb{C}^n}.$$

The estimate (3.8) follows now from (3.6) and from Corollary 3.5. ∎

The following fact on the convergence speed of the approximate inverse is worthwhile to mention; compare (3.2).

COROLLARY 3.8. *We have that*

$$\|S_n A_n f - Ef\|_\infty \preceq \rho_n \, \|f\|_X \max_{1\le i\le m} \|\upsilon^i\|_{Y_1} \quad \text{as } n \to \infty.$$

Proof. By the triangle inequality and by (3.8) it suffices to show that $\|\Sigma_n A_n f - Ef\|_\infty \preceq \rho_n \, \|f\|_X \, \max_{1\le i\le m} \|\upsilon^i\|_{Y_1}$. This is obtained from

$$\begin{aligned} \big|\big(\Sigma_n A_n f\big)_i - \langle f, e^i\rangle_X\big| &= \big|\langle \Psi_n A f, G_n \Psi_n \upsilon^i\rangle_{\mathbb{C}^n} - \langle f, A^*\upsilon^i\rangle_X\big| \\ &= \big|\langle \Pi_n g, \Pi_n \upsilon^i\rangle_Y - \langle g, \upsilon^i\rangle_Y\big|, \end{aligned}$$

where $g = Af$. The difference on the right-hand side may now be estimated as follows:

$$\begin{aligned} \big|\langle \Pi_n g, \Pi_n\upsilon^i\rangle_Y - \langle g, \upsilon^i\rangle_Y\big| &\le \|\Pi_n g - g\|_Y \, \|\Pi_n \upsilon^i\|_Y + \|\Pi_n\upsilon^i - \upsilon^i\|_Y \, \|g\|_Y \\ &\preceq \rho_n \, \|g\|_{Y_1} \, \|\upsilon^i\|_{Y_1} \preceq \rho_n \, \|f\|_X \, \|\upsilon^i\|_{Y_1}, \end{aligned}$$

where we used the uniform boundedness (2.4), the approximation property (2.5), and the continuity (3.1). ∎

3.2. The general case $e \in X$. The range of A^* is dense in X due to the injectivity of A. Therefore, we will assume only that the mollifier can be approximated arbitrarily close by an element in $\mathsf{R}(A^*)$.

Let $e^i \in X$ be mollifiers for $i = 1, \dots, m$. To any $\varepsilon_i > 0$ we can find a $\upsilon^i \in Y_1$ so that

$$\|e^i - A^*\upsilon^i\|_X \le \varepsilon_i, \quad i = 1, \dots, m. \tag{3.9}$$

Below we will demonstrate how to get υ^i from e^i knowing a singular value decomposition of A.

Since, in general, no reconstruction kernel is associated with e^i there will be no counterparts of Theorems 3.4 and 3.7, respectively. Instead, we are directly heading towards an estimate of $\Sigma_n A_n f - Ef$. Based on the e^i's and the υ^i's from above the operators E (3.3) and Σ_n (3.7) are well defined.

THEOREM 3.9. *Adopt the assumptions specified in section* 2 *and assume* (3.1). *Let the operators E and Σ_n be defined as in* (3.3) *and* (3.7), *respectively, where $e^i \in X$ and $\upsilon^i \in Y_1$ are related by* (3.9). *Then*

$$\big\|\Sigma_n A_n f \,-\, Ef\big\|_\infty \;\preceq\; \Big(\rho_n \max_{1\le i\le m} \|\upsilon^i\|_{Y_1} \,+\, \max_{1\le i\le m} \varepsilon_i\Big)\, \|f\|_X \qquad \text{as } n\to\infty. \tag{3.10}$$

Proof. By the triangle inequality and by (3.9) we get

$$\big|\big(\Sigma_n A_n f\big)_i - \langle f, e^i\rangle_X\big| \;\le\; \big|\langle \Psi_n A f, G_n \Psi_n \upsilon^i\rangle_{\mathbb{C}^n} - \langle f, A^*\upsilon^i\rangle_X\big| \,+\, \|f\|_X\, \varepsilon_i.$$

We may now proceed as in the proof of Corollary 3.8. ∎

We will now discuss the vital issue of constructing $\upsilon^i \in Y_1$ from $e^i \in X$ which satisfy (3.9) for ε_i arbitrarily small. For convenience let us suppress the superscript i.

The tool we employ is a singular value decomposition (SVD) of the operator A. In medical imaging SVDs are explicitly known; see, e.g., [9, 10, 15, 17, 18, 21].

Let $A : X \to Y$ be a compact operator and let $\{\mathrm{v}_k, u_k; \sigma_k \,|\, k \in \mathbb{N}_0\}$ be its *singular system*, that is,

$$Ax \;=\; \sum_{k=0}^{\infty} \sigma_k\, \langle x, \mathrm{v}_k\rangle_X\; u_k.$$

The sets of *singular functions* $\{\mathrm{v}_k\}$ and $\{u_k\}$ are orthonormal bases in X (A is injective) and $\overline{\mathsf{R}(A)}$, respectively. The positive numbers σ_k are the *singular values* of A satisfying $\lim_{k\to\infty} \sigma_k = 0$ (monotonically). The singular functions and the singular values are related via

$$A\mathrm{v}_k \;=\; \sigma_k\, u_k \quad \text{ and } \quad A^* u_k \;=\; \sigma_k\, \mathrm{v}_k.$$

We assume that all u_k's are in Y_1. For an arbitrary $e \in X$ we follow the approach of Dietz [4] and define

$$\upsilon_M \;:=\; \sum_{k=0}^{M-1} \sigma_k^{-1}\, \langle e, \mathrm{v}_k\rangle_X\; u_k \tag{3.11}$$

which is an element of Y_1. Dietz [4] implemented (3.11) to solve the cone beam reconstruction problem in 3D utilizing the formula of Grangeat.

Obviously,

$$\|e - A^*\upsilon_M\|_X^2 \;=\; \sum_{k=M}^{\infty} |\langle e, \mathrm{v}_k\rangle_X|^2 \;\to\; 0 \quad \text{as } M \to \infty. \tag{3.12}$$

Incorporating an abstract smoothness assumption on e, we are able to give convergence rates of $\|e - A^*\upsilon_M\|_X$ as $M \to \infty$.

LEMMA 3.10. *Suppose that $e \in \mathsf{R}\big((A^*A)^\alpha\big) = \mathsf{D}\big((A^*A)^{-\alpha}\big)$ for a $\alpha \ge 0$. Then*

$$\lim_{M\to\infty} \sigma_M^{-\alpha}\, \|e - A^*\upsilon_M\|_X \;=\; 0.$$

Moreover, the following error estimate holds:

$$\|e - A^*\upsilon_M\|_X \;<\; \sigma_M^\alpha \;\sqrt{\|e\|_X \;\|(A^*A)^{-\alpha}e\|_X}\,.$$

Proof. We have that

$$\begin{aligned}\|e - A^*\upsilon_M\|_X^2 \;&=\; \sum_{k=M}^{\infty} \sigma_k^{-2\alpha}\,|\langle e, \mathrm{v}_k\rangle_X|\;\sigma_k^{2\alpha}\,|\langle e, \mathrm{v}_k\rangle_X| \\ &\le\; \left(\sum_{k=M}^{\infty} \sigma_k^{-4\alpha}\,|\langle e, \mathrm{v}_k\rangle_X|^2\right)^{1/2} \;\sigma_M^{2\alpha}\;\left(\sum_{k=M}^{\infty} |\langle e, \mathrm{v}_k\rangle_X|^2\right)^{1/2}\end{aligned}$$

and both assertions follow readily. ∎

In view of (3.10) we realize that controlling the ε_i's tells only half of the story. To learn the whole story we look at $\|\upsilon_M\|_{Y_1}$.

LEMMA 3.11. *Suppose that $e \in \mathsf{R}\big((A^*A)^\alpha\big) = \mathsf{D}\big((A^*A)^{-\alpha}\big)$ for a $\alpha \ge 0$. Further, let there exist a $\beta \ge 0$ such that $\|u_k\|_{Y_1} \preceq \sigma_k^{-\beta}$ for all k. Then*

$$\|\upsilon_M\|_{Y_1} \;\preceq\; \|(A^*A)^{-\alpha}e\|_X \;\left(\sum_{k=0}^{M-1} \sigma_k^{4\alpha-2(1+\beta)}\right)^{1/2}.$$

Proof. The straightforward estimates

$$\begin{aligned}\|\upsilon_M\|_{Y_1} \;&\preceq\; \sum_{k=0}^{M-1} \sigma_k^{-2\alpha}\;|\langle e, \mathrm{v}_k\rangle_X|\;\sigma_k^{2\alpha-(1+\beta)} \\ &\preceq\; \left(\sum_{k=0}^{M-1} \sigma_k^{-4\alpha}\;|\langle e, \mathrm{v}_k\rangle_X|^2\right)^{1/2} \left(\sum_{k=0}^{M-1} \sigma_k^{4\alpha-2(1+\beta)}\right)^{1/2}\end{aligned}$$

verify the claim. ∎

THEOREM 3.12. *Let $A : X \to Y$ be compact with singular system $\{\mathrm{v}_k, u_k; \sigma_k \,|\, k \in \mathbb{N}_0\}$. Assume that $\sigma_k \asymp (k+1)^{-\gamma}$ for a $\gamma > 0$ as $k \to \infty$ ($a \asymp b$ abbreviates $a \preceq b \preceq a$) and that $\|u_k\|_{Y_1} \preceq \sigma_k^{-\beta}$ for $\beta \ge 0$.*

Assume the hypotheses of Theorem 3.9*; in particular, let the operators E and Σ_n be defined as in* (3.3) *and* (3.7)*, respectively, where $e^i \in \mathsf{D}\big((A^*A)^{-\alpha}\big)$ and $\upsilon_{M_i}^i$ are related by* (3.11).

If $\alpha > (1+\beta)/2 + 1/(4\gamma)$ and $M_i = M_i(n) \succeq \rho_n^{-1/(\alpha\gamma)}$ as $n \to \infty$ (ρ_n from (2.5)*), then*

$$\big\|\Sigma_n A_n f \;-\; Ef\big\|_\infty \;\preceq\; \rho_n\;\|f\|_X \;\max_{1\le i\le m} \|(A^*A)^{-\alpha}e^i\|_X \qquad \text{as } n \to \infty. \tag{3.13}$$

Proof. Since $\|e^i\|_X \preceq \|(A^*A)^{-\alpha}e^i\|_X$ we have that

$$\begin{aligned}\varepsilon_i \;=\; \|e^i - A^*\upsilon_{M_i}^i\|_X \;&\preceq\; \sigma_{M_i}^\alpha\;\|(A^*A)^{-\alpha}e^i\|_X \\ &\preceq\; (M_i+1)^{-\alpha\gamma}\;\|(A^*A)^{-\alpha}e^i\|_X \;\preceq\; \rho_n\;\|(A^*A)^{-\alpha}e^i\|_X\end{aligned}$$

by Lemma 3.10 and our assumption on $M_i = M_i(n)$ as $n \to \infty$. Further, by Lemma 3.11,

$$\|\upsilon_{M_i}^i\|_{Y_1} \;\preceq\; \|(A^*A)^{-\alpha}e^i\|_X \;\left(\sum_{k=0}^{\infty}(k+1)^{-\gamma\left(4\alpha-2(1+\beta)\right)}\right)^{1/2},$$

where the series converges due to $\gamma\big(4\alpha - 2(1+\beta)\big) > 1$. Recalling Theorem 3.9 we are finished with the proof of (3.13). $\square$

4. Unbounded semidiscrete operators A_n. Here we consider (2.1) where X_1 is a proper subspace of X with a stronger topology.

As we will see in the next section it may happen that $A_n : X_1 \subset X \to \mathbb{C}^n$ is unbounded. In the extremest case we even have to deal with $\mathsf{D}(A_n^*) = \{0\}$, that is, the approximate inverse with respect to the topology in X is not defined for (1.1).

Basically, this leaves us with the situation already investigated in section 3.2. Indeed, if $(e^i, v^i) \in X \times Y_1$, $i = 1, \dots, m$, are mollifier/reconstruction kernel pairs satisfying (3.9) then E (3.3) as well as Σ_n (3.7) are well defined. Even for unbounded operators A_n both Theorems 3.9 and 3.12 remain valid with a slight modification: we have to assume that $f \in X_1$. In (3.10) as well as in (3.13) we have to replace $\|f\|_X$ by $\|f\|_{X_1}$.

5. Application to the reconstruction problem in 2D-computerized tomography. We apply our abstract results of the former sections to the reconstruction problem in 2D-computerized tomography, that is, the reconstruction of a function from its line integrals. For further applications of our results in vector and local tomography we refer to [24] and [22], respectively.

The underlying operator is the *Radon transform* $\mathbf{R}$ mapping a function $f \in L^2(\Omega)$ to its line integrals. Here, Ω is the unit ball in $\mathbb{R}^2$ centered at the origin. More precisely,

$$\mathbf{R}f(s,\vartheta) := \int_{L(s,\vartheta)\,\cap\,\Omega} f(x)\; d\sigma(x). \tag{5.1}$$

The lines are parameterized by $L(s,\vartheta) = \{\tau\,\omega^\perp(\vartheta) + s\,\omega(\vartheta) \,|\, \tau \in \mathbb{R}\}$ where $s \in]-1,1[$, $\omega(\vartheta) = (\cos\vartheta, \sin\vartheta)^t$ and $\omega^\perp(\vartheta) = (-\sin\vartheta, \cos\vartheta)^t$ for $\vartheta \in]0,\pi[$. By this parameterization of lines we are dealing with the *parallel scanning geometry.*

The Radon transform maps $X = L^2(\Omega)$ continuously to $Y = L^2(Z)$ where $Z :=]-1,1[\times]0,\pi[$; see, e.g., Natterer [20, Chap. II.1]. In the appendix we will verify the following mapping property (see Theorem A.2 below):

$$\mathbf{R} : H_0^\alpha(\Omega) \to H^{\alpha+1/2}(Z) \quad \text{is continuous for any} \;\; \alpha \geq 0.$$

The involved Sobolev spaces are defined as follows. By $H_0^\alpha(\Omega)$ we denote the closure of $\mathcal{C}_0^\infty(\Omega)$, the space of infinitely differentiable functions with compact support in Ω, with respect to the norm $\|f\|_\alpha^2 = \int_{\mathbb{R}^2} \big(1 + \|\xi\|^2\big)^\alpha \, |\widehat{f}(\xi)|^2\, d\xi$. Here, $\widehat{f}$ is the Fourier transform of f.

The space $H^\beta(Z) = W_2^\beta(Z)$ is an L^2-Sobolev space defined on the rectangular domain Z; see, e.g., Wloka [25].

Since point evaluations are continuous linear functionals on $H^\beta(Z)$ for $\beta > 1$ we set $X_1 = H_0^{1/2+\kappa}(\Omega)$ and $Y_1 = H^{1+\kappa}(Z)$ for a $\kappa > 0$; cf. (2.1).

For $q, p \in \mathbb{N}$ let $h_s = 1/q$ and $h_\vartheta = \pi/p$ be the *discretization step sizes* and set $s_i = i\,h_s$, $i = -q, \dots, q$, and $\vartheta_j = j\,h_\vartheta$, $j = 0, \dots, p$. Let $\ell \in \{1, 2\}$. With this index ℓ we will be able to distinguish between two different settings using a compact notation.

To the pairs (s_i, ϑ_j) we associate the Dirac-distributions $\psi_{i,j}^{(\ell)}$ given by

$$\langle \psi_{i,j}^{(\ell)}, g\rangle_{Y_1' \times Y_1} := \varsigma_{i,j}^{(\ell)}\; g(s_i, \vartheta_j), \quad i = -q, \dots, q_\ell,\; j = 0, \dots, p_\ell,$$

for any $g \in H^{1+\kappa}(Z)$ where $q_1 = q-1$, $q_2 = q$, $p_1 = p-1$ and $p_2 = p$. The $\varsigma_{i,j}^{(\ell)}$'s are normalization factors to be defined below in (5.3). We define the mapping $\Psi_{q,p}^{(\ell)} : H^{1+\kappa}(Z) \to \mathbb{R}^{n_\ell}$ according to (2.2) using the $\psi_{i,j}^{(\ell)}$'s. The respective dimensions are $n_1 = 2qp$ and $n_2 = (2q+1)(p+1)$.

THEOREM 5.1. *The operator* $\mathbf{R}_{q,p}^{(\ell)} =: \Psi_{q,p}^{(\ell)}\mathbf{R} : H_0^{1/2+\kappa}(\Omega) \subset L^2(\Omega) \to \mathbb{R}^{n_\ell}$ *is unbounded for any* $\kappa > 0$. *Moreover,* $\mathsf{D}\big((\mathbf{R}_{q,p}^{(\ell)})^*\big) = \{0\}$.

Proof. We construct a sequence $\{f_r\}_{r\in\mathbb{N}} \subset H_0^{1/2+\kappa}(\Omega)$ with $\|f_r\|_{L^2(\Omega)} \le 1$ and $\|\mathbf{R}_{q,p}^{(\ell)} f_r\|_{\mathbb{R}^{n_\ell}} \to \infty$ as $r \to \infty$.

We will define f_r as the tensor product of two univariate functions χ_r and μ_r. Let $\{\alpha_r\}$ and $\{\beta_r\}$ be monotonically decreasing zero sequences with $0 < \alpha_r < 1$, $0 < \beta_r < 1/2$, and $\alpha_r^2 + (1-\beta_r)^2 < 1$.

Let $\chi_r \in \mathcal{C}_0^\infty(-\alpha_r, \alpha_r)$ with values in $[0, 1/\sqrt{2\alpha_r}]$ such that $\chi_r(t) = 1/\sqrt{2\alpha_r}$ for $|t| \le \alpha_r/2$. Similarly, let $\mu_r \in \mathcal{C}_0^\infty(-1+\beta_r, 1-\beta_r)$ with values in $[0, 1/\sqrt{2(1-\beta_r)}]$ such that $\mu_r(t) = 1/\sqrt{2(1-\beta_r)}$ for $|t| \le 1-2\beta_r$. Both functions can be constructed explicitly using a partition of unity; see, e.g., Wloka [25, Chap. 1.2].

For $f_r(x) := \chi_r(x_1)\,\mu_r(x_2)$ we have $0 < \|f_r\|_{L^2(\Omega)} = \|\chi_r\|_{L^2(\mathbb{R})}\,\|\mu_r\|_{L^2(\mathbb{R})} \le 1$ and $\operatorname{supp} f_r \subset [-\alpha_r, \alpha_r] \times [-1+\beta_r, 1-\beta_r] \subset \Omega$ because $\alpha_r^2 + (1-\beta_r)^2 < 1$. Thus, $f_r \in H_0^{1/2+\kappa}(\Omega)$ for any $\kappa > 0$.

Now consider $\mathbf{R} f_r$ at $s_0 = 0$ and $\vartheta_0 = 0$:

$$
\begin{aligned}
|\mathbf{R} f_r(s_0, \vartheta_0)| \;&=\; \int_{\mathbb{R}} f_r(0,t)\; dt \;=\; \chi_r(0) \int_{\mathbb{R}} \mu_r(t)\; dt \\
&\ge\; \chi_r(0) \int_{-1+2\beta_r}^{1-2\beta_r} \mu_r(t)\; dt \;=\; \frac{1-2\beta_r}{\sqrt{\alpha_r\,(1-\beta_r)}} \;\stackrel{r\to\infty}{\longrightarrow}\; \infty.
\end{aligned}
$$

Hence, $\|\mathbf{R}_{q,p}^{(\ell)} f_r\|_{\mathbb{R}^{n_\ell}} \to \infty$ as $r \to \infty$.

We are now going to verify the second statement of Theorem 5.1. To this end observe that $\lim_{r\to\infty} \mathbf{R} f_r(s_i, \vartheta_j) = 0$ if $(i,j) \ne (0,0)$. This limit holds since $\operatorname{supp} f_r$ "converges" to the line segment $L(0,0) \cap \Omega$.

The construction principle from above can be repeated for any pair (s_i, ϑ_j), $|s_i| < 1$, leading to a sequence of functions $\{f_r^{i,j}\}_{r\in\mathbb{N}} \subset H_0^{1/2+\kappa}(\Omega)$ with $\|f_r^{i,j}\|_{L^2(\Omega)} \le 1$ and

$$
\lim_{r\to\infty} \mathbf{R} f_r^{i,j}(s_k, \vartheta_l) \;=\; \begin{cases} \infty & : \quad (k,l) = (i,j), \\ 0 & : \quad \text{otherwise}. \end{cases}
$$

Assume that $0 \ne w \in \mathsf{D}\big((\mathbf{R}_{q,p}^{(\ell)})^*\big)$. Then the linear functional $f \mapsto \big\langle \mathbf{R}_{q,p}^{(\ell)} f,\, w\big\rangle_{\mathbb{R}^{n_\ell}}$ is continuous on $\mathsf{D}(\mathbf{R}_{q,p}^{(\ell)})$ with respect to the $L^2(\Omega)$-topology. With $w_{i,j} \ne 0$ we obtain

$$
\big\langle \mathbf{R}_{q,p}^{(\ell)} f_r^{i,j},\, w\big\rangle_{\mathbb{R}^{n_\ell}} \;=\; w_{i,j}\,\varsigma_{i,j}^{(\ell)}\,\mathbf{R} f_r^{i,j}(s_i, \vartheta_j) \;+\; \sum_{\substack{(k,l)\\ (k,l)\ne(i,j)}} w_{k,l}\,\varsigma_{k,l}^{(\ell)}\,\mathbf{R} f_r^{i,j}(s_k, \vartheta_l)
$$

which implies that $\lim_{r\to\infty} \big\langle \mathbf{R}_{q,p}^{(\ell)} f_r^{i,j},\, w\big\rangle_{\mathbb{R}^{n_\ell}} \;=\; \operatorname{sgn}(w_{i,j})\,\infty$. However, this unboundedness contradicts $w \in \mathsf{D}\big((\mathbf{R}_{q,p}^{(\ell)})^*\big)$. $\square$

Due to Theorem 5.1 the approximate inverse cannot be applied to the 2D-reconstruction problem

$$
\text{given } g_{q,p} \in \mathbb{R}^{n_\ell} \text{ find } f \in L^2(\Omega) \text{ such that } \; \mathbf{R}_{q,p}^{(\ell)} f = g_{q,p}.
$$

Here we are facing the situation from section 4, that is, we have to replace the "nonexisting" $S_{q,p}$ by $\Sigma_{q,p}$; compare (1.3), (1.4), and (3.7), respectively.

Canonical candidates for approximation spaces related to $\Psi^{(\ell)}_{q,p}$ are the tensor product spline spaces $V^{(\ell)}_{q,p} = S^{(\ell)}_s \otimes S^{(\ell)}_\vartheta$, $\ell = 1, 2$. Here, $S^{(\ell)}_s$ and $S^{(\ell)}_\vartheta$ are either the piecewise constant ($\ell = 1$) or linear ($\ell = 2$) spline spaces with respect to the knot sequences $\{s_i\}$ and $\{\vartheta_j\}$, respectively. As basis in $V^{(\ell)}_{q,p}$ we choose the tensor product B-spline basis

$$\left\{ B^{(\ell)}_{q,i} \otimes B^{(\ell)}_{p,j} / \varsigma^{(\ell)}_{i,j} \;\middle|\; -q \le i \le q_\ell,\ 0 \le j \le p_\ell \right\}. \tag{5.2}$$

The B-splines $B^{(\ell)}_{q,i} \in S^{(\ell)}_s$ and $B^{(\ell)}_{p,j} \in S^{(\ell)}_\vartheta$ are uniquely determined by (χ_D is the indicator function of the set D)

$$B^{(1)}_{q,i} = \chi_{[s_i, s_{i+1}[}, \qquad B^{(1)}_{p,j} = \chi_{[\vartheta_j, \vartheta_{j+1}[},$$

and

$$B^{(2)}_{q,i}(s_k) = \begin{cases} 1 : & i = k, \\ 0 : & \text{otherwise}, \end{cases} \qquad B^{(2)}_{p,j}(\vartheta_l) = \begin{cases} 1 : & j = l, \\ 0 : & \text{otherwise}, \end{cases}$$

respectively. The normalization factors $\varsigma^{(\ell)}_{i,j}$ are just the L^2-norms of the B-splines:

$$\varsigma^{(\ell)}_{i,j} := \|B^{(\ell)}_{q,i} \otimes B^{(\ell)}_{p,j}\|_{L^2(Z)}, \quad i = -q, \dots, q_\ell,\ j = 0, \dots, p_\ell. \tag{5.3}$$

Thus, the normalized tensor product B-spline basis (5.2) is an $L^2(Z)$-Riesz system where the constants in the corresponding norm equivalence do not depend on h_s or h_ϑ; compare (2.3).

We next define the interpolation operator $\Pi^{(\ell)}_{q,p} : H^{1+\kappa}(Z) \to V^{(\ell)}_{q,p}$ which links $V^{(\ell)}_{q,p}$ to $\Psi^{(\ell)}_{q,p}$:

$$\Pi^{(\ell)}_{q,p} v := \sum_{i=-q}^{q_\ell} \sum_{j=0}^{p_\ell} (\Psi^{(\ell)}_{q,p} v)_{i,j}\ B^{(\ell)}_{q,i} \otimes B^{(\ell)}_{p,j} / \varsigma^{(\ell)}_{i,j} = \sum_{i=-q}^{q_\ell} \sum_{j=0}^{p_\ell} v(s_i, \vartheta_j)\ B^{(\ell)}_{q,i} \otimes B^{(\ell)}_{p,j}.$$

Let $h = \max\{h_s, h_\vartheta\}$. Then the uniform boundedness

$$\|\Pi^{(\ell)}_{q,p} v\|_{L^2(Z)} \preceq \|v\|_{H^\alpha(Z)}, \quad \alpha > 1,$$

as well as the approximation property

$$\|v - \Pi^{(\ell)}_{q,p} v\|_{L^2(Z)} \preceq h^{\min\{\alpha, \ell\}}\ \|v\|_{H^\alpha(Z)}, \quad \alpha > 1,$$

hold true whenever the right-hand sides are finite. Both estimates are standard results from spline approximation theory; see, e.g., Schumaker [23, Chap. 12].

In the following we apply our results of section 3.2 to the 2D-reconstruction problem. In a first step we therefore construct reconstruction kernels from mollifiers using a SVD of the Radon transform. Unfortunately, a SVD of $\mathbf{R} : L^2(\Omega) \to L^2(Z)$ is not known explicitly. However, it can be shown that the Radon transform maps $L^2(\Omega)$ compactly to $L^2(\widetilde{Z}, w^{-1})$ where $\widetilde{Z} =]-1, 1[\times]0, 2\pi[$; see, e.g., Natterer [20, Chap. IV.3]. The weight function is given by $w(s) := \sqrt{1 - s^2}$ and acts on the first variable only.

Let $\left\{ \mathrm{v}_{m,l}, u_{m,l}; \sigma_m \,\middle|\, m \in \mathbb{N}_0,\ l \in \mathbb{Z},\ |l| \le m,\ m+l \in 2\mathbb{Z} \right\}$ be the singular system of $\mathbf{R} : L^2(\Omega) \to L^2(\widetilde{Z}, w^{-1})$, that is,

$$\mathbf{R}f \;=\; \sum_{m=0}^{\infty} \sum_{l=-m}^{m}{}^{\star} \sigma_m \,\langle f,\, \mathrm{v}_{m,l}\rangle_{L^2(\Omega)}\, u_{m,l}, \tag{5.4}$$

where $\star$ restricts the summation over those l's with $m + l \in 2\mathbb{Z}$.

Later on we will need an explicit representation of the σ_m's and the $u_{m,l}$'s only. We therefore give analytic expressions

$$\sigma_m \;=\; 2\sqrt{\frac{\pi}{m+1}} \qquad \text{and} \qquad u_{m,l}(s,\varphi) \;=\; \frac{1}{\pi}\, w(s)\, U_m(s)\, \mathrm{e}^{\imath\, l\, \varphi}, \tag{5.5}$$

where $U_m(s) = \sin\big((m+1)\arccos s\big)/\sin(\arccos s)$, $m \in \mathbb{N}_0$, are the *Chebyshev polynomials of the second kind*. For the $\mathrm{v}_{m,l}$'s see Louis [11] or Natterer [20].

Denoting by $\mathbf{R}^*$ and $\mathbf{R}^\#$ the adjoints of $\mathbf{R} : L^2(\Omega) \to L^2(Z)$ and $\mathbf{R} : L^2(\Omega) \to L^2(\widetilde{Z}, w^{-1})$, respectively, we have that

$$2\,\mathbf{R}^* w^{-1} u_{m,l} \;=\; \mathbf{R}^\# u_{m,l} \;=\; \sigma_m\, \mathrm{v}_{m,l}.$$

The first equality can be checked by straightforward calculations. Given a mollifier $e \in L^2(\Omega)$ normalized by $\int_\Omega e(x)\,dx = 1$ and centered about the origin we define

$$\upsilon_M \;:=\; 2 \sum_{m=0}^{M-1} \sum_{l=-m}^{m}{}^{\star} \sigma_m^{-1}\, \langle e,\, \mathrm{v}_{m,l}\rangle_{L^2(\Omega)}\, w^{-1} u_{m,l}$$

which then gives

$$\|\mathbf{R}^* \upsilon_M - e\|^2_{L^2(\Omega)} \;=\; \sum_{m=M}^{\infty} \sum_{l=-m}^{m}{}^{\star} |\langle e,\, \mathrm{v}_{m,l}\rangle_{L^2(\Omega)}|^2; \tag{5.6}$$

compare (3.11) and (3.12).

Let us assume from now on that the mollifier e is a radial function, that is, $e(x) = \mathbf{e}(\|x\|_{\mathbb{R}^2})$. Since $\langle e,\, \mathrm{v}_{m,l}\rangle_{L^2(\Omega)} = 0$ for $l \neq 0$ the representation of υ_M simplifies to

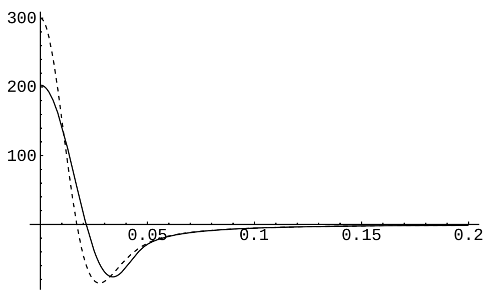
$$\upsilon_M \;=\; 2 \sum_{k=0}^{(M-1)/2} \sigma_{2k}^{-1}\, \langle e,\, \mathrm{v}_{2k,0}\rangle_{L^2(\Omega)}\, w^{-1} u_{2k,0}. \tag{5.7}$$

Hence, the reconstruction kernel does not depend on the angle ϑ. Moreover υ_M is an even function in s as so are the Chebyshev polynomials of even degree. See Figure 5.1 for an example.

Let $x_i \in \Omega$, $i = 1, \ldots, m$, be points in which we would like to reconstruct moments $\langle f, e^i\rangle_{L^2(\Omega)}$ from the data $g_{q,p}$. The mollifiers e^i are derived from e by translation and dilation:

$$e^i(\cdot) \;=\; T_1^{x_i} e(\cdot) \;:=\; \frac{1}{4}\, e\Big(\frac{\cdot - x_i}{2}\Big).$$

At the present time the choice of the dilation factor 2 seems to be artificial; however, it will become clear in the proof of Lemma 5.3 below.



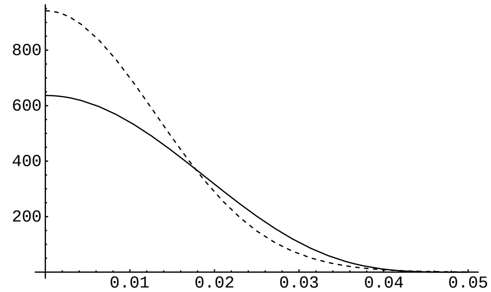


FIG. 5.1. *Reconstruction kernels (left) and radial parts of the related mollifiers (right). Solid curves: reconstruction kernel* υ_{501} (5.7) *corresponding to* $e(x) = 5\gamma^{-2}p(\|x\|_{\mathbb{R}^2}/\gamma)/\pi$ *with* $\gamma = 0.05$ *where* $p(t) = (1-t^2)^4$ *for* $t \le 1$ *and* $p(t) = 0$ *otherwise. From a numerical point of view we have* $\mathbf{R}^*\upsilon_{501} = e$*. Dashed curves: reconstruction kernel* υ (5.14) *corresponding to the Gaußian* $e(x) = (2\pi)^{-1}\gamma^{-2}\exp\big(-\gamma^{-2}\|x\|^2_{\mathbb{R}^2}/2\big)$ *with* $\gamma = 0.013$*. Please note that both kernels are negative in* $[0.2, 1]$ *and monotonically increasing.*

The invariance property

$$\mathbf{R}^* T_2^{x_i} = T_1^{x_i}\mathbf{R}^*, \quad \text{where} \quad T_2^{x_i}\upsilon(s,\vartheta) := \frac{1}{4}\,\upsilon\Big(\frac{s - x_i^t\,\omega(\vartheta)}{2}, \vartheta\Big), \tag{5.8}$$

suggests to define the reconstruction kernel υ_M^i associated with e^i by

$$\upsilon_M^i(s,\vartheta) := T_2^{x_i}\upsilon_M(s,\vartheta) = \frac{1}{4}\,\upsilon_M\Big(\frac{s - x_i^t\,\omega(\vartheta)}{2}\Big), \quad i = 1,\dots,m.$$

Thus,

$$\mathbf{R}^*\upsilon_M^i = T_1^{x_i}\mathbf{R}^*\upsilon_M \longrightarrow T_1^{x_i}e = e^i \quad \text{as } M \to \infty.$$

Remark 5.2. Thanks to the invariance property (5.8) only the kernel υ_M has to be computed and stored. The kernels for the reconstruction points x_i are simply found by the action of $T_2^{x_i}$ on υ_M.

LEMMA 5.3. *Let* $v \in H^r(Z)$, $r \ge 0$*. Then*

$$\|T_2^x v\|_{H^r(Z)} \preceq \|v\|_{H^r(Z)} \quad \textit{uniformly in } x \in \Omega. \tag{5.9}$$

Proof. The transformation $\Phi(s,\vartheta) := \big((s - x^t\,\omega(\vartheta))/2, \vartheta\big)$ maps Z to Z' bijectively where

$$Z' = \Big\{(\sigma,\varphi) \in Z \;\Big|\; \sigma \in \Big[\,\frac{-1 - x^t\,\omega(\varphi)}{2}, \frac{1 - x^t\,\omega(\varphi)}{2}\,\Big]\Big\}.$$

Moreover, Φ is a $\mathcal{C}^\infty$-diffeomorphism with $\det J\Phi(s,\vartheta) = 1/2$ where $J\Phi$ is the Jacobian of Φ. Since $T_2^x v = v\circ\Phi/4$ the assertion follows from transformation results for Sobolev norms, see; e.g., Wloka [25]. ∎

Define $E : L^2(\Omega) \to \mathbb{R}^m$ by $(Ef)_i := \langle f, e^i\rangle_{L^2(\Omega)}$, $i = 1,\dots,m$; see (3.3), and $\Sigma_{q,p}^{(\ell)} : \mathbb{R}^{n_\ell} \to \mathbb{R}^m$ by

$$\big(\Sigma_{q,p}^{(\ell)}\, b\big)_i := \big\langle b,\, G_{q,p}^{(\ell)}\,\Psi_{q,p}^{(\ell)}\,\upsilon_M^i\big\rangle_{\mathbb{R}^{n_\ell}}, \quad \ell = 1,2;$$

see (3.7). Here, $G_{q,p}^{(\ell)}$ is the Gramian matrix with respect to the B-spline basis in $V_{q,p}^{(\ell)}$. Especially $G_{q,p}^{(1)}$ is the identity matrix.

The process of evaluating $\Sigma_{q,p}^{(\ell)} g_{q,p}$ coincides with (and may be implemented exactly as) the *filtered backprojection algorithm* in computerized tomography with filter function υ_M; see, e.g., Natterer [20, Chap. V.1]. Indeed, for $\ell = 1$,

$$\big(\Sigma_{q,p}^{(1)} g_{q,p}\big)_i \;=\; \frac{\pi}{4\,q\,p} \sum_{l=-q}^{q-1} \sum_{j=0}^{p-1} g(s_l, \vartheta_j)\; \upsilon_M\Big(\frac{s_l\, - x_i^t\,\omega(\vartheta_j)}{2}\Big).$$

A reformulation of Theorem 3.12 in the present context results therefore in a novel error analysis of the filtered backprojection algorithm (Theorem 5.4 below). Compared to already known error estimates, see, e.g., Natterer [20, Th. V.1.1], we allow mild smoothness assumptions on the density distribution f. Further, the kernel υ needs only to be known approximately. The error bound reflects clearly the influence of the smoothness of f and e on the convergence rate.

THEOREM 5.4. *Let $f \in H_0^{1/2+\kappa}(\Omega)$ for $0 < \kappa \le 1$. Assume that the radial mollifier e is in $H_0^{\alpha}(\Omega)$ for $\alpha > 4 + 2\,\kappa$. Let $\lambda_\ell = \min\{1+\kappa, \ell\}$ for $\ell = 1, 2$.*

If $M = M(h) \succeq h^{-2\,\lambda_\ell/\alpha}$, then

$$\big\|\Sigma_{q,p}^{(\ell)}\, \Psi_{q,p}^{(\ell)}\, \mathbf{R} f - E f\big\|_\infty \;\preceq\; h^{\lambda_\ell}\; \|f\|_{1/2+\kappa}\; \|e\|_\alpha \quad \text{as} \quad h \to 0. \tag{5.10}$$

Proof. We follow the line of proof of Theorem 3.9 to obtain

$$\begin{aligned}
&\big|\big(\Sigma_{q,p}^{(\ell)}\, \Psi_{q,p}^{(\ell)}\, \mathbf{R} f\big)_i - \langle f, e^i\rangle_{L^2(\Omega)}\big| \\
&\quad\le\; \big|\big(\Sigma_{q,p}^{(\ell)}\, \Psi_{q,p}^{(\ell)}\, \mathbf{R} f\big)_i - \langle f, \mathbf{R}^*\upsilon_M^i\rangle_{L^2(\Omega)}\big| + \big|\langle f, \mathbf{R}^*\upsilon_M^i - e^i\rangle_{L^2(\Omega)}\big| \\
&\quad=\; \big|\big(\Sigma_{q,p}^{(\ell)}\, \Psi_{q,p}^{(\ell)}\, \mathbf{R} f\big)_i - \langle \mathbf{R} f, T_2^{x_i}\upsilon_M\rangle_{L^2(Z)}\big| + \big|\langle f, T_1^{x_i}(\mathbf{R}^*\upsilon_M - e)\rangle_{L^2(\Omega)}\big| \\
&\quad\preceq\; \|f\|_{1/2+\kappa}\, \big(h^{\lambda_\ell}\; \|\upsilon_M\|_{H^{1+\kappa}(Z)} + \|\mathbf{R}^*\upsilon_M - e\|_{L^2(\Omega)}\big),
\end{aligned}$$

where we used the invariance property (5.8) as well as (5.9). Since

$$\|(\mathbf{R}^{\#}\mathbf{R})^{-\alpha} e\|_{L^2(\Omega)} \;\preceq\; \|e\|_\alpha$$

(see Lemma A.3 below), we immediately infer from (5.6) and from the proof of Lemma 3.10 that

$$\|\mathbf{R}^*\upsilon_M - e\|_{L^2(\Omega)} \;\preceq\; \sigma_M^\alpha\; \|e\|_\alpha \;\preceq\; (M+1)^{-\alpha/2}\; \|e\|_\alpha \;\preceq\; h^{\lambda_\ell}\; \|e\|_\alpha.$$

It remains to bound $\|\upsilon_M\|_{H^{1+\kappa}(Z)}$; see (5.7). We will be guided by the proof of Lemma 3.11. Using the interpolation inequality for Sobolev norms; see, e.g., Lions and Magenes [7, Chap. 2.5], we may estimate as follows:

$$\begin{aligned}
\|w^{-1} u_{2k,0}\|_{H^{1+\kappa}(Z)} &\;\preceq\; \|w^{-1} u_{2k,0}\|_{H^1(Z)}^{1-\kappa}\; \|w^{-1} u_{2k,0}\|_{H^2(Z)}^{\kappa} \\
&\;=\; \|U_{2k}\|_{H^1(-1,1)}^{1-\kappa}\; \|U_{2k}\|_{H^2(-1,1)}^{\kappa}.
\end{aligned}$$

A bound on the Sobolev norms of the Chebyshev polynomials may be obtained by Markov's inequality (5.11); see, e.g., Lorentz [8, Chap. 3.3]: let P_r be a polynomial of degree r then

$$|P_r'(s)| \;\le\; r^2 \max_{-1\le t\le 1} |P_r(t)|, \quad |s| \le 1. \tag{5.11}$$

With $\max_{-1\le t\le 1} |U_r(t)| = r+1$ we easily find that

$$\|U_r\|_{H^1(-1,1)} \preceq (r+1)^3 \preceq \sigma_r^{-6} \quad \text{and} \quad \|U_r\|_{H^2(-1,1)} \preceq (r+1)^5 \preceq \sigma_r^{-10}$$

which result in

$$\|w^{-1} u_{2k,0}\|_{H^{1+\kappa}(Z)} \preceq \sigma_{2k}^{-2\,(3+2\kappa)}.$$

Recalling the representation (5.7) of υ_M we get

$$\begin{aligned}\|\upsilon_M\|_{H^{1+\kappa}(Z)} &\preceq \left(\sum_{k=0}^{(M-1)/2} \sigma_{2k}^{-4\alpha}\, |\langle e,\, \mathrm{v}_{2k,0}\rangle_{L^2(\Omega)}|^2\right)^{1/2} \left(\sum_{k=0}^{(M-1)/2} (2k+1)^{-2\alpha+7+4\kappa}\right)^{1/2}\\ &\preceq \|(\mathbf{R}^{\#}\mathbf{R})^{-\alpha} e\|_{L^2(\Omega)} \preceq \|e\|_\alpha.\end{aligned}$$

We used the fact that the second sum is bounded in M since $2\alpha - 7 - 4\kappa > 1$. The proof of Theorem 5.4 is now complete. ∎

Because $h = \max\{h_s, h_\vartheta\}$ it is most efficient—in view of (5.10)—to work with discretization step sizes h_s and h_ϑ which coincide: $h_s = h_\vartheta$, that is, $p = \pi\, q$. So we recovered the optimal sampling relation for the parallel scanning geometry; see, e.g., Natterer [20, Chap. III].

In the remainder of this section we comment briefly on another way to design reconstruction kernels for the Radon transform; see (5.13) below. This approach is based on the inversion formula (5.12) of the Radon transform

$$e = (2\pi)^{-1}\, \mathbf{R}^* \mathrm{I}^{-1} \mathbf{R} e \quad \text{for} \quad e \in H_0^\alpha(\Omega), \quad \alpha \ge 1/2; \tag{5.12}$$

see, e.g., Natterer [20, Chap. II.2]. The operator $\mathrm{I}^{-1} : H_0^1(-1,1) \to L^2(\mathbb{R})$ is the *Riesz potential*: $\widehat{(\mathrm{I}^{-1} f)}(\xi) = |\xi|\, \widehat{f}(\xi)$. In (5.12), the Riesz potential acts on the first variable of $\mathbf{R}e$. Motivated by (5.12) we make the ansatz $\upsilon := \mathrm{I}^{-1}\mathbf{R}e/(2\pi)$. Assuming radial symmetry of e the latter formula may be expressed as

$$\upsilon(s) = \frac{1}{\pi} \int_0^\infty \sigma\, \widehat{e}\big(\sigma\, \omega(0)\big)\, \cos(s\,\sigma)\, d\sigma; \tag{5.13}$$

compare Natterer [20, (1.5), p. 103].

For instance, let e be the Gaußian $e(x) = (2\pi)^{-1}\gamma^{-2} \exp\big(-\gamma^{-2}\, \|x\|_{\mathbb{R}^2}^2/2\big)$, $\gamma > 0$. Clearly, these mollifiers are not supported in Ω. However, for γ small, they decay fast enough to consider them elements of $H_0^{1/2}(\Omega)$. Thus,

$$\begin{aligned}\upsilon(s) &= \frac{1}{2\pi^2} \int_0^\infty \sigma\, \exp\big(-\gamma^2\,\sigma^2/2\big)\, \cos(s\,\sigma)\, d\sigma\\ &= \frac{-1}{2\pi^2\,\gamma^2} \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}\sigma}\Big(\exp\big(-\gamma^2\,\sigma^2/2\big)\Big)\, \cos(s\,\sigma)\, d\sigma.\end{aligned}$$

Now applying integration by parts and using formulæ (7.4.7) and (7.1.3) from [1] yields

$$\upsilon(s) = \frac{1}{2\pi^2\,\gamma^2}\left(1 + \sqrt{\frac{\pi}{2}}\, \frac{s}{\gamma}\, \exp\big(-s^2/(2\gamma^2)\big)\, \imath\, \mathrm{erf}\big(\imath\, s/(\sqrt{2}\,\gamma)\big)\right), \tag{5.14}$$

where $\operatorname{erf}(t) = (2/\sqrt{\pi})\int_0^t \exp(-z^2)\,dz$ is the error function. Figure 5.1 displays υ (5.14) for $\gamma = 0.013$ (dashed curves).

Remark 5.5. We recommend the filter design methods from above whenever one wants to impose certain conditions on the mollifier, e.g., nonnegativity and compact support; see Figure 5.1. The widely used Shepp–Logan filter and its noncompactly supported mollifier have frequent sign changes. To avoid artifacts in the reconstructions these oscillations require a certain fine-tuning: the dilation parameter γ (compare (5.14)) needs to be selected carefully. In contrast, the reconstructions based on the filters from Figure 5.1 are more robust with respect to the support width of the mollifier.

A. Appendix: A Sobolev space estimate of the 2D-radon transform. In this appendix we will show that the Radon transform (5.1) maps $H_0^\alpha(\Omega)$ boundedly to $H_{\mathrm{p}}^{\alpha+1/2}(\widetilde{Z})$, $\alpha \geq 0$, where $\widetilde{Z} =]-1,1[\times]0,2\pi[$. The space $H_{\mathrm{p}}^{\beta}(\widetilde{Z})$ is a Sobolev space of periodic functions. Let $g \in L^2(\widetilde{Z})$ be expressed in its Fourier series, that is,

$$g(s,\varphi) = \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \widehat{g}_{k,n}\, \mathrm{e}^{\imath(\pi k s + n\varphi)}, \quad \widehat{g}_{k,n} = \frac{1}{4\pi}\int_{\widetilde{Z}} g(s,\varphi)\, \mathrm{e}^{-\imath(\pi k s + n\varphi)}\, d\varphi\, ds.$$

Then, $g \in H_{\mathrm{p}}^{\beta}(\widetilde{Z})$, $\beta \geq 0$, iff the norm

$$\|g\|_{\mathrm{p},\beta}^2 = \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \big(1+k^2+n^2\big)^\beta\, |\widehat{g}_{k,n}|^2$$

is finite.

Remark A.1. Interpreting periodic functions in $L^2(\widetilde{Z})$ as functions defined on the torus $\mathcal{T} \subset \mathbb{R}^3$ we may identify $H_{\mathrm{p}}^{\beta}(\widetilde{Z})$ with the Sobolev space $H^\beta(\mathcal{T})$ defined on the smooth compact manifold $\mathcal{T}$ by means of local coordinates; see, e.g., Wloka [25].

In proving our main result in Theorem A.2 below we will benefit from a known Sobolev space estimate (A.1) for the Radon transform due to Louis and Natterer [14]; see also [20, Chap. II.5].

Let $H^{(\beta,0)}(\widetilde{Z})$, $\beta > 0$, be the tensor product $H_0^\beta(-1,1)\,\widehat{\otimes}\, L^2(0,2\pi)$ (for the tensor product of Sobolev spaces, see, e.g., Aubin [2]); then, for $\alpha \geq 0$,

$$\|\mathbf{R}f\|_{H^{(\alpha+1/2,0)}(\widetilde{Z})} \preceq \|f\|_\alpha \quad \text{for all } f \in H_0^\alpha(\Omega). \tag{A.1}$$

In view of Remark A.1 our estimate (A.2) below is intrinsically different from a result of Natterer which looks similar at first glance; see [20, Chap. II, Thrm. 5.3].

THEOREM A.2. *The Radon transform maps $H_0^\alpha(\Omega)$ continuously to $H_{\mathrm{p}}^{\alpha+1/2}(\widetilde{Z})$, $\alpha \geq 0$, that is,*

$$\|\mathbf{R}f\|_{\mathrm{p},\alpha+1/2} \preceq \|f\|_\alpha \quad \textit{for all } f \in H_0^\alpha(\Omega). \tag{A.2}$$

Proof. Let $g = \mathbf{R}f$. Since $(1+k^2+n^2)^\beta \leq 2^\beta\big((1+k^2)^\beta + (1+n^2)^\beta\big)$ for $\beta \geq 0$ and for all $k,n \in \mathbb{Z}$ we have that

$$\|g\|_{\mathrm{p},\alpha+1/2} \preceq A(g) + B(g)$$

with

$$A(g)^2 = \sum_{k,n\in\mathbb{Z}} (1+k^2)^{\alpha+1/2}\, |\widehat{g}_{k,n}|^2 \quad \text{and} \quad B(g)^2 = \sum_{k,n\in\mathbb{Z}} (1+n^2)^{\alpha+1/2}\, |\widehat{g}_{k,n}|^2.$$

We will bound $A(g)$ as well as $B(g)$ by a multiple of $\|f\|_\alpha$. Both relations

$$\sum_{n\in\mathbb{Z}} |\widehat{g}_{k,n}|^2 = \int_0^{2\pi} \Big|\frac{1}{2}\int_{-1}^{1} g(s,\varphi)\, \mathrm{e}^{-\imath\pi k s}\, ds\Big|^2 d\varphi$$

and

$$\sum_{k\in\mathbb{Z}} |\widehat{g}_{k,n}|^2 = \int_{-1}^{1} \Big|\frac{1}{2\pi}\int_0^{2\pi} g(s,\varphi)\, \mathrm{e}^{-\imath n \varphi}\, d\varphi\Big|^2 ds$$

follow from Parseval's identity. Hence,

$$\begin{aligned} A(g)^2 &= \frac{1}{4}\int_0^{2\pi} \sum_{k\in\mathbb{Z}} (1+k^2)^{\alpha+1/2} \Big|\int_{-1}^{1} g(s,\varphi)\, \mathrm{e}^{-\imath\pi k s}\, ds\Big|^2 d\varphi \\ &\preceq \|g\|^2_{H^{(\alpha+1/2,0)}(\widetilde{Z})} \preceq \|f\|_\alpha^2, \end{aligned}$$

where the first inequality follows from a Sobolev norm equivalence given by Natterer in [20, Chap. VII, Lem. 4.4]. The second inequality comes from (A.1).

Estimating $B(g)$ is a little bit more involved. From the singular value expansion (5.4) of $g = \mathbf{R}f$ we deduce that

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} g(s,\varphi)\, \mathrm{e}^{-\imath n\varphi}\, d\varphi &= \frac{1}{\pi}\sum_{m=0}^{\infty} \sideset{}{^\star}\sum_{l=-m}^{m} g_{m,l}\, w(s)\, U_m(s) \underbrace{\frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\imath(n-l)\varphi}\, d\varphi}_{=\,\delta_{n,l}} \\ &= \frac{1}{\pi}\sum_{\mu=0}^{\infty} g_{|n|+2\mu,n}\, w(s)\, U_{|n|+2\mu}(s) \end{aligned}$$

with $g_{m,l} = \sigma_m \langle f, \mathrm{v}_{m,l}\rangle_{L^2(\Omega)}$. Thus,

$$\begin{aligned} B(g)^2 &= \frac{1}{\pi^2}\sum_{n\in\mathbb{Z}} (1+n^2)^{\alpha+1/2} \int_{-1}^{1} \Big|\sum_{\mu=0}^{\infty} g_{|n|+2\mu,n}\, w(s)\, U_{|n|+2\mu}(s)\Big|^2 ds \\ &\le \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} (1+n^2)^{\alpha+1/2} \int_{-1}^{1} \Big|\sum_{\mu=0}^{\infty} g_{|n|+2\mu,n} \sqrt{\frac{2}{\pi}}\, w(s)\, U_{|n|+2\mu}(s)\Big|^2 w^{-1}(s)\, ds \\ &= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} (1+n^2)^{\alpha+1/2} \sum_{\mu=0}^{\infty} |g_{|n|+2\mu,n}|^2 \end{aligned}$$

because $\{\sqrt{2/\pi}\, w(\cdot)\, U_m(\cdot)\,\big|\, m \in \mathbb{N}\}$ is an orthonormal basis in $L^2(]-1,1[, w^{-1})$. Further,

$$\begin{aligned} B(g)^2 &\preceq \sum_{n\in\mathbb{Z}} (1+|n|)^{2\alpha+1} \sum_{\mu=0}^{\infty} \sigma^2_{|n|+2\mu}\, |\langle f, \mathrm{v}_{|n|+2\mu,n}\rangle_{L^2(\Omega)}|^2 \\ &\preceq \sum_{n\in\mathbb{Z}} \sum_{\mu=0}^{\infty} \sigma^{-4\alpha}_{|n|+2\mu}\, |\langle f, \mathrm{v}_{|n|+2\mu,n}\rangle_{L^2(\Omega)}|^2 \\ &\le \sum_{m=0}^{\infty} \sideset{}{^\star}\sum_{l=-m}^{m} \sigma_m^{-4\alpha}\, |\langle f, \mathrm{v}_{m,l}\rangle_{L^2(\Omega)}|^2. \end{aligned}$$

In Lemma A.3 below we bound the latter double sum by a multiple of $\|f\|_\alpha^2$ which finally proves (A.2). ∎

LEMMA A.3. *The operator* $(\mathbf{R}^\#\mathbf{R})^{-\alpha} : H_0^\alpha(\Omega) \to L^2(\Omega)$, $\alpha \geq 0$, *is continuous, that is,*

$$\text{(A.3)} \qquad \|(\mathbf{R}^\#\mathbf{R})^{-\alpha} f\|_0^2 = \sum_{m=0}^{\infty} \sideset{}{^\star}\sum_{l=-m}^{m} \sigma_m^{-4\alpha} \, |\langle f, \mathrm{v}_{m,l}\rangle_{L^2(\Omega)}|^2 \preceq \|f\|_\alpha^2$$

for all $f \in H_0^\alpha(\Omega)$.

Proof. Since $\mathcal{C}_0^\infty(\Omega)$ is dense $H_0^\alpha(\Omega)$ it suffices to consider $f \in \mathcal{C}_0^\infty(\Omega)$. Let us start with $\alpha = 2r + 1/2$, $r \in \mathbb{N}_0$. We have

$$\text{(A.4)} \qquad \langle f, \mathrm{v}_{m,l}\rangle_{L^2(\Omega)} = \sigma_m^{-1} \, \langle f, \mathbf{R}^\# u_{m,l}\rangle_{L^2(\Omega)} = \sigma_m^{-1} \, \langle \mathbf{R}f, u_{m,l}\rangle_{L^2(\widetilde{Z}, w^{-1})}.$$

Further, $\mathbf{R}f(\cdot, \varphi) \in \mathcal{C}_0^\infty(-1,1)$ for any $\varphi \in [0, 2\pi]$; see, e.g., Natterer [20]. We estimate the inner product on the right-hand side of (A.4). Let $g_\varphi(s) = \mathbf{R}f(s,\varphi)$. Integration by parts yields

$$\begin{aligned}\int_{-1}^{1} g_\varphi(s)\, U_m(s)\, ds &= \int_0^\pi g_\varphi(\cos\vartheta)\, \sin\big((m+1)\,\vartheta\big)\, d\vartheta \\ &= \frac{-1}{m+1} \int_0^\pi g_\varphi'(\cos\vartheta)\, \sin\vartheta\, \cos\big((m+1)\,\vartheta\big)\, d\vartheta,\end{aligned}$$

where we used that $g_\varphi(-1) = g_\varphi(1) = 0$. Repeating integration by parts $2r+1$-times we obtain

$$\int_{-1}^{1} g_\varphi(s)\, U_m(s)\, ds = (m+1)^{-2r-1} \int_0^\pi \rho_r(\vartheta,\varphi)\, \cos\big((m+1)\,\vartheta\big)\, d\vartheta$$

with $\rho_r(\vartheta,\varphi) = \sin\vartheta \, \sum_{i=1}^{2r+1} \big(\frac{\partial^i}{\partial s^i}\mathbf{R}f\big)(\cos\vartheta,\varphi)\, p_{i-1}(\vartheta)$ where p_{i-1} is a real trigonometric polynomial of degree $i-1$ at most. Thus,

$$\langle g, u_{m,l}\rangle_{L^2(\widetilde{Z},w^{-1})} = \pi^{-1}\,(m+1)^{-2r-1} \underbrace{\int_0^{2\pi}\!\!\int_0^\pi \rho_r(\vartheta,\varphi)\, \cos\big((m+1)\,\vartheta\big)\, d\vartheta\, \mathrm{e}^{-\imath\, l\, \varphi}\, d\varphi}_{=:\, c_{m,l}(\mathbf{R}f)}$$

implying

$$|\langle f, \mathrm{v}_{m,l}\rangle_{L^2(\Omega)}|^2 \preceq \sigma_m^{8r+2}\, |c_{m,l}(\mathbf{R}f)|^2 = \sigma_m^{4\alpha}\, |c_{m,l}(\mathbf{R}f)|^2$$

by (5.5) and (A.4). Summing up results in

$$\text{(A.5)} \qquad \sum_{m=0}^{\infty} \sideset{}{^\star}\sum_{l=-m}^{m} \sigma_m^{-4\alpha}\, |\langle f, \mathrm{v}_{m,l}\rangle_{L^2(\Omega)}|^2 \preceq \sum_{m=0}^{\infty} \sideset{}{^\star}\sum_{l=-m}^{m} |c_{m,l}(\mathbf{R}f)|^2.$$

Since $\big\{\cos(n\,\vartheta)\, \mathrm{e}^{\imath\, l\, \varphi}/\pi \,\big|\, n \in \mathbb{N},\ l \in \mathbb{Z}\big\}$ is an orthonormal system in $L^2([0,\pi]\times[0,2\pi])$ we get from the Bessel inequality that

$$\sum_{m=0}^{\infty} \sideset{}{^\star}\sum_{l=-m}^{m} |c_{m,l}(\mathbf{R}f)|^2 \leq \pi^{-2} \int_0^{2\pi}\!\!\int_0^\pi |\rho_r(\vartheta,\varphi)|^2\, d\vartheta\, d\varphi$$

$$= \pi^{-2} \int_0^{2\pi} \int_0^{\pi} \Big| \sin\vartheta \sum_{i=1}^{2r+1} \Big(\frac{\partial^i}{\partial s^i} \mathbf{R} f \Big)(\cos\vartheta, \varphi)\, p_{i-1}(\vartheta) \Big|^2 d\vartheta\, d\varphi$$

$$\preceq \int_0^{2\pi} \int_{-1}^{1} \left(\sum_{i=1}^{2r+1} \Big| \Big(\frac{\partial^i}{\partial s^i} \mathbf{R} f \Big)(s, \varphi) \Big| \right)^2 ds\, d\varphi$$

$$\preceq \|\mathbf{R} f\|^2_{H^{(\alpha+1/2,0)}(\widetilde{Z})} \preceq \|f\|^2_\alpha.$$

In the last step we used (A.1). The latter estimate together with (A.5) verifies (A.3) for $\alpha = 2r + 1/2$. For arbitrary $\alpha \geq 0$ one can use arguments from interpolation theory of Sobolev spaces; see, e.g., Lions and Magenes [7, Chap. 5.1]. ∎

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