Generalized Tableaux and Formally Well-Posed Initial Value Problems

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We generalize the notion of a tableau of a system of partial differential equations. This leads to an intrinsic definition of formally well-posed initial value problems, i.e. problems with exactly the right amount of Cauchy data. We must allow here that the data is prescribed on a flag of submanifolds. The advantage of this approach is that even for non-normal systems the data can be chosen completely arbitrarily and does not need to satisfy any constraints. The existence and uniqueness of analytic solutions is guaranteed by the Cartan-Kähler Theorem. For linear systems the uniqueness is extended to non-analytic solutions by a generalization of the Holmgren Theorem. We discuss the relation between the generalized tableaux and δ-regularity of the coordinate system and we give a rigorous definition of under- and over-determinacy.

1. Introduction

A natural question to ask about an initial value problem is: "How much Cauchy data must be prescribed?" If there is not sufficient data, we will not obtain a unique solution. If we give too much data, it must satisfy compatibility conditions. We call a problem *formally well-posed*, if exactly the right amount of data is given.

It is a bit surprising that the answer to this question for general systems of partial differential equations is hardly known. It is obvious for ordinary differential equations and *single* partial differential equations. But if we proceed to *systems* of partial differential equations, the only well-known result concerns normal systems, i.e. systems satisfying the conditions of the Cauchy-Kowalevsky Theorem. We consider the generalization to arbitrary involutive systems using the formal theory of differential equations (Pom78; Sei94a).

Although the Cartan-Kähler Theorem is well-known, especially in the context of exterior differential systems (BCG$^+$91) where it was originally derived, apparently little attention has been paid to its use in initial value problems. Especially the unusual form of prescribing Cauchy data implicitly contained in its formulation has (to our knowledge) never been studied in detail.

The standard formulation of initial value problems in a geometric framework

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considers an embedding of a submanifold of the space of independent variables of codimension 1 into an appropriate jet bundle (Gar69). However, as soon as we treat non-normal systems, this implies that the Cauchy data must satisfy compatibility conditions. Such systems are of great importance in physics, as for instance the field equations of a gauge theory can never be normal.

Such compatibility conditions on Cauchy data provide problems in numerical computations. They make the consistent initialization considerably more difficult. Empirical results show that already in the case of a system of ordinary differential equations with algebraic constraints (BCP89) most methods are very sensitive to even small inconsistencies in the initial data.

In this article we present an intrinsic description of initial value problems where the Cauchy data is completely unconstrained. This leads us to a formulation where it is prescribed on a flag of submanifolds of different dimensions. Such formulations are known from Janet-Riquier Theory (Jan20). But this approach depends decisively on the coordinate system. In a geometric setting we know of only two articles (Kor90a; Kor90b) using such an approach. But the author always assumed that the problem was formally well-posed without studying the conditions for it.

In the Cauchy-Kowalevsky Theorem one requires that the data is not given on a characteristic submanifold. We encounter similar conditions. To formulate them in a concise and intrinsic manner we must introduce a generalization of the notion of a tableau of a differential equation. The classical concept of a non-characteristic one-form must be extended to non-systatic bases of T^*X where X is the space of independent variables.

Tableaux have other interesting applications, too. We use them to propose a rigorous definition of under- and over-determinacy combining and extending previous ideas by Olver (Olv86) and Pommaret (Pom78). This definition is more consistent with the intuitive conception of an under-determined system than the standard one based on a comparison of the number of equations and unknowns.

A central problem in the formal theory is the completion of a given system to an equivalent involutive one. This requires the Cartan characters of the symbol. In most coordinate systems their determination is straightforward. But certain singular systems lead to wrong results. Using the tableaux we present an intrinsic way to compute the characters independent of the coordinate system. It is computationally more efficient than previously proposed methods.

The word "formal" means in this context that only formal power series are used without consideration of their convergence. This implies that we tackle the question of local solvability only for analytic systems. Hence we restrict our attention to integrability conditions and not to problems of the Lewy type (Lew57). Similarly all our results are purely local, although we usually omit to speak of neighborhoods etc. to make the notation simpler.

Since the formal theory of differential equations is still not very well-known, we give in the next two sections a somewhat longer introduction to it emphasizing the properties of involutive symbols and systems. The remainder of the article is organized as follows: In Section 4 we recall the definition of the tableau and of characteristics and use them to give a precise definition of under- and over-determinacy. Section 5 contains the generalization to k-tableaux and explains how they are connected with the Cartan characters. The study of initial value problems starts in Section 6 with the definition of formal well-posedness. Section 8 contains a proof of the Cartan-Kähler Theorem adapted to our purposes which is then used to extend the Holmgren Theorem to arbitrary linear systems. Section 9 considers briefly the problem of stability, before finally some conclusions are given.

2. Formal Theory

Formal theory uses a geometric approach to differential equations based on the jet bundle formalism. It is beyond the scope of this article to give a detailed introduction to the underlying theory. The interested reader is referred to the literature (Pom78; Sau89). Our presentation here is a shortened version of (Sei94a).

We will always work in a local coordinate system, although the whole theory can be expressed in a coordinate free way. Let $x_1, \ldots, x_n$ denote the independent and $u^1, \ldots, u^m$ the dependent variables. Together they form bundle coordinates for some bundle $\mathcal{E}$ over some base space X. Derivatives are written in multi-index notation $p^\alpha_\mu = \partial^{|\mu|}u^\alpha/\partial(x^1)^{\mu_1}\cdots\partial(x^n)^{\mu_n}$ where $|\mu| = \mu_1+\cdots+\mu_n$ is the length of the multi-index $\mu = [\mu_1, \ldots, \mu_n]$. Adding the derivatives p^α_μ up to order q defines a local coordinate system for the q-th order jet bundle $J_q\mathcal{E}$.

The main goal of this formalism is to identify a differential equation with an intrinsic geometric object. We define a differential equation of order q as a fibered submanifold $\mathcal{R}_q$ of the q-th order jet bundle $J_q\mathcal{E}$. This allows us to obtain results independent of any particular way of writing the equations. $\mathcal{R}_q$ can be considered as the kernel of some mapping $\Phi : J_q\mathcal{E} \to \mathcal{E}'$ for some other bundle $\mathcal{E}'$. Thus locally it is described by a system of equations

$$\mathcal{R}_q \,:\, \Phi^\tau\left(x^i, u^\alpha, p^\alpha_\mu\right) = 0\,, \quad \begin{cases} \tau=1,\ldots,p\,, \\ |\mu|\leqslant q\,. \end{cases} \tag{2.1}$$

In this geometric framework a solution is a (local) section $\sigma \in \Gamma_{loc}(\mathcal{E})$ of the bundle $\mathcal{E}$ such that its q-th prolongation satisfies $j_q(\sigma) \subset \mathcal{R}_q$.

At least some of the ideas behind the concept of involution can be understood best by considering the order by order construction of a formal power series solution. We make a power series ansatz for the general solution of the differential equation $\mathcal{R}_q$ by expanding it around some point x_0. Then we substitute this ansatz into the equations (2.1) and evaluate at x_0. This yields a system of algebraic equations for the Taylor coefficients up to order q.

For the coefficients of order $q+r$ we use the *prolonged* equations $\mathcal{R}_{q+r}$ obtained by differentiating each equation in $\mathcal{R}_q$ r times with respect to all independent variables. Since they are quasi-linear, substituting the power series ansatz into $\mathcal{R}_{q+r}$ and evaluating at x_0 yields an inhomogeneous linear system for the Taylor coefficients of order $q + r$. Those of lower order appear in its matrix and in its right hand side. Thus we can express the coefficients of order $q+r$ through them. This is the precise meaning of constructing a power series order by order.

This construction fails, if integrability conditions occur. Such equations arise usually by cross-differentiating. They are detected only after some prolongations, but pose additional conditions on the coefficients of lower order. Hence they must be known to pursue the above described procedure.

Geometrically we can see integrability conditions by considering the natural projections $\pi^{q_2}_{q_1}$ from the jet bundle $J_{q_2}\mathcal{E}$ of order q_2 onto the jet bundle $J_{q_1}\mathcal{E}$ of the lower order q_1. If an integrability condition of order q is occurring, for some order $q+r$ the projection of $\mathcal{R}_{q+r}$ on $\mathcal{R}_q$ is not surjective, i.e.

$$\mathcal{R}_q^{(r)} = \pi_q^{q+r}(\mathcal{R}_{q+r}) \subset \mathcal{R}_q\,. \tag{2.2}$$

We call a differential equation *formally integrable*, if at any order all projections are surjective, i.e. integrability conditions never arise.

For such equations it is possible to construct order by order a formal power series solution. In each step we can by the Implicit Function Theorem solve for some of the derivatives of the corresponding order. These are called *principal derivatives*. All other derivatives of the same order are *parametric* ones.

Formally integrable equations are often called involutive. This is not correct. Involution comprises formal integrability but requires additional properties which are of a more algebraic nature. To give an exact definition we have to introduce the symbol of a differential equation which is closely connected with the above presented order by order construction of power series solutions.

Definition 1. Let the differential equation $\mathcal{R}_q$ be locally described by (2.1). The solution space $\mathcal{M}_q$ of the following system of linear equations in the unknowns v_μ^α with $\alpha = 1,\dots,m,\ |\mu| = q$

$$\mathcal{M}_q\ :\ \left\{ \sum_{\alpha,|\mu|=q} \left(\frac{\partial\Phi^\tau}{\partial p_\mu^\alpha}\right) v_\mu^\alpha = 0\,, \quad \tau = 1,\dots,p \right. \tag{2.3}$$

is called the *symbol* of $\mathcal{R}_q$. (By abuse of language, we will often also call the linear system symbol instead of its solution space).

Differential geometrically seen the symbol represents a family of vector spaces over $\mathcal{R}_q$. It can be defined intrinsically as $\mathcal{M}_q = V\mathcal{R}_q \cap (S_qT^*X \otimes V\mathcal{E})$ where S_q denotes the q-fold symmetric product, T^*X the cotangent bundle of X and V the verticle bundle†. Alternatively we can use the local representation (2.1) of $\mathcal{R}_q$ and introduce the symbol map

$$\sigma\ :\ \begin{cases} S_qT^*X \otimes V\mathcal{E} & \longrightarrow \qquad V\mathcal{E}' \\ \qquad v_\mu^\alpha & \longmapsto \displaystyle\sum_{\alpha,|\mu|=q} \left(\frac{\partial\Phi^\tau}{\partial p_\mu^\alpha}\right) v_\mu^\alpha \end{cases}\ . \tag{2.4}$$

This map is intrinsically defined, i.e. independent of the choice of the local representation Φ, and its kernel is $\mathcal{M}_q$. We make the assumption that $\mathcal{M}_q$ is a vector bundle over $\mathcal{R}_q$, i.e. the rank of the symbol map σ is constant.

An important property of the symbol is that it allows for an indirect determination of the dimension of projected equations. Rank deficits in the symbol signal the occurrence of integrability conditions or identities.

Theorem 2. (Pom78) *If $\mathcal{M}_{q+1}$ is a vector bundle, then*

$$\dim \mathcal{R}_q^{(1)} = \dim \mathcal{R}_{q+1} - \dim \mathcal{M}_{q+1}\,. \tag{2.5}$$

† $J_q\mathcal{E}$ is an affine bundle over $J_{q-1}\mathcal{E}$ modeled on the vector bundle $S_qT^*X \otimes V\mathcal{E}$. This follows easily from the transformation and symmetry properties of q-th order derivatives (Pom78). One can thus consider $S_qT^*X \otimes V\mathcal{E}$ as the "vector space of all q-th order derivatives".

The multi-index $\mu = [\mu_1, \mu_2, \ldots, \mu_n]$ is said to be of *class* k, if its first non-vanishing entry is μ_k. We can order the columns of the symbol by the class of the multi-index of the corresponding v^α_μ (the ordering within a class is of no importance) and then compute a row echelon form. In this form, the class of each row is given by the class of its first non-vanishing entry, the pivot. We associate with a row of class k the *multiplicative variables* $x^1, x^2, \ldots, x^k$. The symbol is now in *solved form*. We denote by $\beta^{(k)}_q$ the number‡ of rows of class k. We will study in Section 5 under what conditions this definition is intrinsic.

The jet variables corresponding to the pivots represent one possible choice for the principal derivatives used in the order by order construction of a power series solution. The point is to find a unique way to compute all derivatives of the principal derivatives. This leads to the notion of an *involutive* symbol. An intrinsic definition of it can be obtained from the Spencer cohomology (Gol69; Pom78). We mention here only a simple criterion for involution using the multiplicative variables. It can be easily applied in concrete computations.

Definition 3. The symbol $\mathcal{M}_q$ is called *involutive*, if

$$\operatorname{rank} \mathcal{M}_{q+1} = \sum_{k=1}^{n} k\beta^{(k)}_q\,. \tag{2.6}$$

The differential equation $\mathcal{R}_q$ is called *involutive* or *in involution*, if it is formally integrable and if its symbol $\mathcal{M}_q$ is involutive.

This approach to involution via analyzing pivots is inspired by the Janet-Riquier Theory (Jan20; Rei91). According to the Cartan-Kuranishi Theorem (Kur57; Pom78), any differential equation can be completed to an involutive one by a finite number of prolongations and projections. (SSC92; Sei94a) describe an algorithm for this completion and its implementation in the computer algebra system AXIOM.

We will later need mainly the $\beta^{(k)}_q$. But to make contact with other results we recall the definition of the Cartan characters $\alpha^{(k)}_q$. They differ from the $\beta^{(k)}_q$ by a simple combinatorial factor

$$\alpha^{(k)}_q = m\binom{q+n-k-1}{q-1} - \beta^{(k)}_q \quad k = 1, \ldots, n\,. \tag{2.7}$$

$\beta^{(k)}_q$ corresponds to the number of principal derivatives of class k and order q, whereas $\alpha^{(k)}_q$ denotes the number of parametric derivatives of class k and order q.

3. Involutive Symbols and Multiplicative Variables

Definition 3 looks at first sight rather strange. But it has a straightforward explanation, which is so important that we will give it in form of a proposition for easier reference. Its proof follows from a simple analysis of the pivots in $\mathcal{M}_q$ and $\mathcal{M}_{q+1}$, respectively.

‡ This number could take different values at different points of $\mathcal{R}_q$. But like we always assume that $\mathcal{M}_q$ is a vector bundle, we always assume that its value is constant on $\mathcal{R}_q$.

Proposition 4. *If the symbol $\mathcal{M}_q$ of the differential equation $\mathcal{R}_q$ is involutive, then we obtain (in a given local representation) all independent equations of order $q+1$ of the prolongation $\mathcal{R}_{q+1}$ by differentiating each equation of $\mathcal{R}_q$ with respect to its multiplicative variables only.*

In order to study further properties of involutive symbols, we need a systematic way to choose the principal derivatives. We define a *ranking* of derivatives as a total ordering defined on the space of all derivatives p^α_μ satisfying (i) if $|\mu| < |\nu|$, then $p^\alpha_\mu < p^\alpha_\nu$, (ii) if $p^\alpha_{\mu_1} < p^\alpha_{\mu_2}$, then $p^\alpha_{\mu_1+\nu} < p^\alpha_{\mu_2+\nu}$ for any multi-index ν. If (i) holds not only for derivatives of the same dependent variable but for all, the ranking is called a *total-degree* or *orderly* ranking. In such a ranking derivatives are ordered first by their order and then by other criteria. A total-degree ranking *respects classes*, if $p^\alpha_\mu < p^\beta_\nu$ for $\operatorname{class} p^\alpha_\mu < \operatorname{class} p^\beta_\nu$.

A ranking of derivatives induces a ranking of the unknowns v^α_μ used as place holders in the symbol. From now on, we assume that the columns of the symbol are ordered using a ranking that respects classes. An important example is the inverse lexicographic ranking, a total-degree ranking. If two derivatives have the same order, we look at the first differing entry in the multi-index. The one with the higher entry is taken as higher in the ranking. Of two derivatives with the same multi-index we take the one as higher which belongs to the dependent variable with the higher index.

Proposition 5. (Pom78) *Let $\mathcal{M}_q$ be an involutive symbol in solved form, where the columns have been ordered according to a class-respecting ranking. Denote by $\mu \pm 1_i$ the multi-index obtained by adding (subtracting) 1 to μ_i.*
(i) If v^α_μ is a pivot of class i, $v^\alpha_{\mu-1_i+1_j}$ is also a pivot for all $j > i$.
(ii) If there are pivots v^α_μ of class i, then the entries μ_i of their multi-indices take all values between 1 and a maximal value.

Corollary 6. *Let $\mathcal{M}_1$ be the involutive symbol of a first-order system. Then*

$$0 \leqslant \beta_1^{(1)} \leqslant \beta_1^{(2)} \leqslant \cdots \leqslant \beta_1^{(n)} \leqslant m\,. \tag{3.1}$$

Proof. This is a direct consequence of Part (i) of Proposition 5. Let v^α_μ be a pivot of class k, i.e. $\mu_i = \delta_{ik}$. Then v^α_ν with $\nu = \mu - 1_k + 1_{k+1}$, i.e. $\nu_i = \delta_{i,k+1}$, is a pivot of class $k+1$ occurring in $\mathcal{M}_1$. Thus $\beta_1^{(k+1)} \geqslant \beta_1^{(k)}$. $\beta_q^{(n)}$ cannot be greater than m for any value of q, as only m derivatives of class n exist. ∎

This corollary holds only for first-order equations. But the corresponding result for the Cartan characters is correct at any order.

Corollary 7. *Let $\mathcal{M}_q$ be the involutive symbol of a differential equation of order q. Then*

$$\alpha_q^{(1)} \geqslant \alpha_q^{(2)} \geqslant \cdots \geqslant \alpha_q^{(n)} \geqslant 0\,. \tag{3.2}$$

Proof. For $q = 1$ this follows from (3.1) and the definition of the $\alpha_q^{(k)}$. For higher order equations one exploits that every differential equation can be transformed into a first-order one with the same Cartan characters (Pom78; Sei94a). ∎

Finally we prove a generalization of an old result by Finzi (Fin47; Olv86) for systems with the same number of equations and dependent variables.

Corollary 8. *A differential equation $\mathcal{R}_q$ has either identities or integrability conditions, if and only if $\beta_q^{(n-1)} > 0$.*

Proof. If $\beta_q^{(n-1)} > 0$, equations with non-multiplicative variables exist, as the only equations without are those of class n. According to Proposition 4 the prolongation of these equations with respect to x^n leads either to identities or to integrability conditions. For the converse, we note that if $\beta_q^{(n-1)} = 0$ all $\beta_q^{(k)}$ with $k < n-1$ vanish. This can be shown similarly to Corollary 6. Thus $\mathcal{R}_q$ contains only equations of class n and all prolonged equations are independent. ∎

4. The Tableau of a Differential Equation

Definition 9. Let $\chi = \chi_i dx^i \in T^*X$ be an arbitrary one-form over X and let $\mathcal{R}_q$ be a differential equation locally defined by the map $\Phi : J_q\mathcal{E} \to \mathcal{E}'$. The *tableau* τ_χ of $\mathcal{R}_q$ is the linear mapping

$$\tau_\chi : \begin{cases} V\mathcal{E} & \xrightarrow{\iota_\chi} & S_qT^*X \otimes V\mathcal{E} & \xrightarrow{\sigma} & V\mathcal{E}' \\ v^\alpha & \longmapsto & \chi_\mu v^\alpha & \longmapsto & \displaystyle\sum_{\alpha,|\mu|=q} \left(\frac{\partial \Phi^\tau}{\partial p_\mu^\alpha}\right) \chi_\mu v^\alpha \end{cases} \tag{4.1}$$

where $\chi_\mu = (\chi_1)^{\mu_1} \cdots (\chi_n)^{\mu_n}$.

This is a kind of "dual" definition to (BCG$^+$91) where τ_χ is called symbol and its solution space tableau. The associated matrix

$$\mathcal{T}_\alpha^\tau[\chi] = \sum_{|\mu|=q} \left(\frac{\partial \Phi^\tau}{\partial p_\mu^\alpha}\right) \chi_\mu \tag{4.2}$$

has p rows and m columns, where p is the dimension of $\mathcal{E}'$. Its entries are homogeneous polynomials of degree q in the components of χ.

Intuitively we would consider a differential equation as under-determined, if its general solution contains arbitrary functions of all independent variables, as this implies that the equation poses no restrictions at all for at least one of the dependent variables u^α.

Definition 10. The involutive equation $\mathcal{R}_q$ is said to be *under-determined* at $P \in \mathcal{R}_q$, if no one-form χ exists such that τ_χ is injective (in P). If there exists a one-form χ such that τ_χ is bijective, the equation is *well-determined* or *normal*. Otherwise the equation is called *over-determined*.

Theorem 11. *Let $\mathcal{R}_q$ be a differential equation with tableau τ_χ. In different coordinate systems we might get different values for $\beta_q^{(n)}$. Let $\tilde\beta_q^{(n)}$ be the maximum of these values. Then*

$$\max_{\chi \neq 0} \ \operatorname{rank} \tau_\chi = \tilde\beta_q^{(n)} \,. \tag{4.3}$$

Proof. By the definition of $\tilde\beta_q^{(n)}$ we can find a local description (2.1) such that

$$\frac{\partial\Phi^\tau}{\partial p^\alpha_{[0,\dots,0,q]}} = \delta^\tau_\alpha\,, \qquad \tau = 1,\dots,\tilde\beta_q^{(n)}\,. \tag{4.4}$$

The diagonal elements of the top left $\tilde\beta_q^{(n)} \times \tilde\beta_q^{(n)}$ submatrix of $\mathcal{T}[\chi]$ are now of the form $(\chi_n)^q + \dots$ and the monomial $(\chi_n)^q$ occurs nowhere else. If we choose $\chi = dx^n$ all entries vanish with the exception of these diagonal elements which become one. Hence the rank of τ_χ is at least $\tilde\beta_q^{(n)}$.

Assume now the rank be greater than $\tilde\beta_q^{(n)}$ for some one-form χ. We choose in a neighborhood of a point $x_0 \in X$ a new coordinate system $\bar x^i = \psi^i(x^j)$ where the function ψ^n satisfies

$$\frac{\partial\psi^n}{\partial x^i}(x_0) = \chi_i\,. \tag{4.5}$$

Obviously this is always possible, as χ never vanishes. For the derivatives of class n we obtain

$$\frac{\partial\bar\Phi^\tau}{\partial \bar p^\alpha_{[0,\dots,0,q]}} = \sum_{|\nu|=q}\frac{\partial\Phi^\tau}{\partial p^\alpha_\nu}\prod_{i=1}^n\left(\frac{\partial\psi^n}{\partial x^i}\right)^{\nu_i}\,. \tag{4.6}$$

At the point x_0 we can evaluate the derivatives of ψ and the right hand side becomes the matrix of the tableau τ_χ. By assumption its rank is greater than $\tilde\beta_q^{(n)}$ implying that we found a coordinate system in which more than $\tilde\beta_q^{(n)}$ equations are of class n, a clear contradiction. ■

The following corollary shows that our definition of under-determinacy is consistent with the intuitive one. If $\beta_q^{(n)} < m$, then $\alpha_q^{(n)} > 0$ implying that the general solution of $\mathcal{R}_q$ contains arbitrary functions depending on all independent variables (Sei94b).

Corollary 12. *An involutive equation $\mathcal{R}_q$ is under-determined, if and only if $\tilde\beta_q^{(n)} < m$. $\mathcal{R}_q$ is normal, if and only if $\tilde\beta_q^{(n)} = m$ and all other $\tilde\beta_q^{(k)} = 0$.*

Proof. By definition $\mathcal{R}_q$ is under-determined, if and only if no χ exists such that $\operatorname{rank}\tau_\chi = m$. By Theorem 11 this happens if and only if $\tilde\beta_q^{(n)} < m$. Similarly we get the value of $\tilde\beta_q^{(n)}$ for a normal equation. It follows from the bijectivity of the tableau that the other characters must vanish. ■

It is important to note that under-determinacy depends only on $\tilde\beta_q^{(n)}$ and not on the total number p of equations used to describe $\mathcal{R}_q$. Of course, a system which can be locally described with less equations than unknowns will always be under-determined, but the converse is not true.

Example 13. Consider the Maxwell equations in two dimensions

$$\begin{aligned} u_{tt} - v_{xt} &= 0\,,\\ u_{xt} - v_{xx} &= 0\,. \end{aligned} \tag{4.7}$$

The tableau of this system is given by the matrix

$$\mathcal{T}[\chi] = \begin{pmatrix} (\chi_1)^2 & -\chi_1\chi_2 \\ \chi_1\chi_2 & -(\chi_2)^2 \end{pmatrix} . \tag{4.8}$$

Obviously its rank is 1 for every non-vanishing one-form χ. This yields $\tilde{\beta}_2^{(2)} = 1$ and it is not possible to find a coordinate system in which two leading derivatives of class 2 occur. The tableau is never injective and the system is indeed under-determined, although there are as many equations as dependent variables. This is typical for gauge theories and related to the gauge symmetry.

Definition 14. A non-vanishing one-form $\chi \in T^*X$ for which the rank of the tableau is less than $\tilde{\beta}_q^{(n)}$ is called *systatic*. χ is a *characteristic* one-form, if τ_χ is not injective.

The term systatic was apparently coined by Pommaret (Pom78). For equations which are not under-determined systatic and characteristic are equivalent. The connection between this definition of characteristics and the classical one found in most textbooks can be seen most easily for first-order systems.

Example 15. Let us consider for simplicity a linear first-order system

$$\mathcal{R}_1 \,:\, \left\{ \sum_{i,\alpha} A_\alpha^{\tau i}(x) p_i^\alpha + \sum_\alpha B_\alpha^\tau(x) u^\alpha + C^\tau(x) = 0 \,, \quad \tau = 1, \dots, p \right. \tag{4.9}$$

and an $(n-1)$-dimensional hypersurface Σ in solved form

$$\Sigma \,:\, x^n = \phi(x^1, \dots, x^{n-1}) \,. \tag{4.10}$$

Such a surface is usually called characteristic, if prescribing the values of u^α on Σ does not suffice to compute all derivatives on the hypersurface (Joh82).

We assume that $u^\alpha\big|_\Sigma = f^\alpha(x^1, \dots, x^{n-1})$. This yields using the chain rule

$$\frac{\partial u^\alpha}{\partial x^i}\bigg|_\Sigma + \frac{\partial \phi}{\partial x^i}\frac{\partial u^\alpha}{\partial x^n}\bigg|_\Sigma = \frac{\partial f^\alpha}{\partial x^i} \,, \quad i = 1, \dots, n-1 \,. \tag{4.11}$$

Substituting these relations into the differential equation (4.9) yields a linear system for the derivatives $\partial u^\alpha / \partial x^n\big|_\Sigma$ with the matrix

$$A_\alpha^{\tau n} - \sum_{i=1}^{n-1} A_\alpha^{\tau i} \frac{\partial \phi}{\partial x^i} \,. \tag{4.12}$$

If the rank of this matrix is less than m, it is not possible to determine all derivatives on Σ. But then the one-form $\chi = dx^n - \sum_{i=1}^{n-1}(\partial\phi/\partial x^i)dx^i$ is characteristic according to Definition 14. Hence $\chi = d\Phi$ where $\Phi(x) = x^n - \phi(x^1, \dots, x^{n-1}) = 0$ describes Σ. Defining the co-normal variety $N_x^*\Sigma = \{\omega \in T_x^*X \mid \omega(T_x\Sigma) = 0\}$ as the set of all one-forms annihilating the tangent space of the surface, we see that it is generated by χ.

Finally, Theorem 11 allows us to give a simple proof of the following

Corollary 16. (Pom78) *If a one-form χ exists such that the tableau τ_χ of the differential equation $\mathcal{R}_q$ is surjective, then $\mathcal{R}_q$ is involutive.*

Proof. τ_χ can only be surjective, if the number p of equations in $\mathcal{R}_q$ is less than or equal to the number of dependent variables. The maximal rank of the tableau is then p. It follows from Theorem 11 that all equations are of class n. But this implies immediately that $\mathcal{M}_q$ is involutive and that no integrability conditions arise; thus $\mathcal{R}_q$ is involutive. ∎

5. k-Tableaux and δ-Regularity

δ-Regularity concerns our criterion (2.6) for involutive symbols. We had to introduce the class of a derivative to define the $\beta_q^{(k)}$, an obviously coordinate dependent concept. As the class depends only on the multi-index, it suffices to consider changes of the independent coordinates x^i. We can even restrict our attention to linear changes, as a general coordinate transformation affects the symbol only through its Jacobian.

Such a transformation is characterized by a matrix A with constant coefficients a_{ij}. If we perform a generic change of coordinates, i.e. without specifying the matrix A, the a_{ij} appear in the symbol in a complicated, non-linear way. Ordering the columns by class and computing a row echelon form of the symbol in this generic coordinate system yields the maximal value for $\beta_q^{(n)}$ admitted by the considered differential equation called $\tilde{\beta}_q^{(n)}$ in the last section.

We also obtain the maximal admitted values for the sums $\sum_{i=k}^n \beta_q^{(i)}$. This characterizes the intrinsically defined values of the $\beta_q^{(k)}$. We denote them by $\tilde{\beta}_q^{(k)}$. A more direct definition of them can be given using the Spencer cohomology (Pom78). Definition 3 and all the results in Section 3 apply only to the $\tilde{\beta}_q^{(k)}$. Let, in given coordinates, $\beta_q^{(k)}$ be the number of equations of class k in the symbol $\mathcal{M}_q$. The coordinate system is called *δ-regular* for $\mathcal{M}_q$, if $\beta_q^{(k)} = \tilde{\beta}_q^{(k)}$ for $k = 1, \dots, n$.

For special choices of the coefficients a_{ij} a pivot in the row echelon form may vanish and a row changes to a lower class. The corresponding matrix A leads to a δ-singular coordinate system. As the a_{ij} enter the symbol algebraically, the set of δ-singular systems can be described by an algebraic variety in the space of $n \times n$-matrices.

Example 17. δ-regularity is connected with characteristics. Consider the wave equation

$$u_{tt} - u_{xx} = 0\,. \tag{5.1}$$

The coordinate system (x, t) is obviously δ-regular, no matter how we order x and t. In a general coordinate system $\eta = \eta(x, t), \zeta = \zeta(x, t)$, (5.1) becomes

$$\begin{gathered}(\eta_t^2 - \eta_x^2)u_{\eta\eta} + 2(\eta_t\zeta_t - \eta_x\zeta_x)u_{\eta\zeta} + (\zeta_t^2 - \zeta_x^2)u_{\zeta\zeta} + \\ (\eta_{tt} - \eta_{xx})u_\eta + (\zeta_{tt} - \zeta_{xx})u_\zeta = 0\,.\end{gathered} \tag{5.2}$$

Independent of the ordering of the new coordinates, we do not find a derivative of class 2, if $\zeta_t = \pm\zeta_x$ and $\eta_t = \mp\eta_x$. But these are the equations for the two characteristics of the wave equation. It is not harmful if one of the coordinates is characteristic, as we can always choose the other one as $\bar{x}^2$. This comes from the fact that only the coordinates x^k with $\beta_q^{(k)} > 0$ are important for δ-regularity.

Since one encounters fairly often differential equations which are not given in a δ-regular coordinate system, one needs a practical tool to cope with this problem. In principle, one could simply compute in a generic coordinate system. In practice this is not feasible, as the computations simply blow up. By generalizing the notion of a tableau we obtain a method to compute all characters independent of the used coordinate system similar to Theorem 11.

The transformation law of derivatives under changes of coordinates looks rather messy in multi-index notation. We introduce some notations. Let μ be a multi-index of length q. We denote by $\mathcal{S}_\mu$ the set of all possible realizations of it with repeated indices. An element $s \in \mathcal{S}_\mu$ represents a tuple of q integers containing μ_i times the value i; e.g. with $n = 2$ one gets $\mathcal{S}_{[1,2]} = \{(1,2,2);(2,1,2);(2,2,1)\}$. We denote the k-th element of s by s_k. $r(\mu)$ is the realization of μ as a repeated index in which the entries are sorted (smallest first), i.e. $r([1,2]) = (1,2,2)$.

Definition 18. Let $\chi^{(k)}, \dots, \chi^{(n)} \in T^*X$ be $n-k+1$ linearly independent one-forms for $1 \leqslant k \leqslant n$ and let $\iota_k : N^k \hookrightarrow X$ be a submanifold such that the one-forms $\iota_k^*\chi^{(j)}$ span T^*N^k. Let the differential equation $\mathcal{R}_q$ be locally described by the map $\Phi : J_q\mathcal{E} \to \mathcal{E}'$. The *k-tableau* $\tau_{[\chi^{(k)},\dots,\chi^{(n)}]}$ is the linear mapping

$$\tau_{[\chi^{(k)},\dots,\chi^{(n)}]} : \begin{cases} S_qT^*N^k \otimes V\mathcal{E} & \xrightarrow{\iota[\chi]} & S_qT^*X \otimes V\mathcal{E} & \xrightarrow{\sigma} & V\mathcal{E}' \\ v_\nu^\alpha & \longmapsto & \displaystyle\sum_\nu C_\mu^\nu v_\nu^\alpha & \longmapsto & \displaystyle\sum_{\substack{\alpha,\nu \\ |\mu|=q}} \left(\frac{\partial \Phi^\tau}{\partial p_\mu^\alpha}\right) C_\mu^\nu v_\nu^\alpha \end{cases} \tag{5.3}$$

where ν is a multi-index with n entries of length q and class $\nu \geqslant k$ and the factor C_μ^ν is defined by

$$C_\mu^\nu = \sum_{s \in \mathcal{S}_\nu} \prod_{i=1}^q \chi_{m_i}^{(s_i)}, \qquad m = r(\mu). \tag{5.4}$$

The only difference to Definition 9 lies in the first map of (5.3) which is now parameterized by several one-forms. The multi-index ν serves as a convenient way to get local coordinates in $S_qT^*N^k \otimes V\mathcal{E}$; one could also say that we choose local coordinates in X such that N^k is given by $x^1 = const, \dots, x^{k-1} = const$. The matrix of the k-tableau is

$$\mathcal{T}_\alpha^{\tau\nu}[\chi^{(k)}, \dots, \chi^{(n)}] = \sum_{|\mu|=q} \left(\frac{\partial \Phi^\tau}{\partial p_\mu^\alpha}\right) C_\mu^\nu, \qquad \begin{cases} \text{class}\,\nu \geqslant k, \\ |\nu| = q. \end{cases} \tag{5.5}$$

It has p rows indexed by τ and mr_k columns indexed by α and ν where p is again the dimension of $\mathcal{E}'$ and $r_k = \dim S_qT^*N^k \otimes V\mathcal{E} = \binom{n-k+q}{n-k}$. The n-tableau is the usual one as defined in the last section. Since $N^1 = X$, the 1-tableau as the largest one has the same size as the symbol.

Lemma 19. *The matrix of the k-tableau contains the matrices of all ℓ-tableaux with $\ell > k$ as submatrices.*

Proof. The matrix $\mathcal{T}[\chi^{(k)},\dots,\chi^{(n)}]$ consists of r_k matrices with p rows and m columns each generated similar to $\mathcal{T}[\chi]$ in Section 4 but with different mappings ι_χ. In (5.5) all multi-indices with class $\nu \geqslant k$ are taken into account. This includes all multi-indices with class $\nu \geqslant \ell$ for $\ell > k$. (5.4) shows that for such multi-indices the entries of the k-tableau do not depend on $\chi^{(k)},\dots,\chi^{(\ell-1)}$. ■

Theorem 20. *Let $\mathcal{R}_q$ be a differential equation with characters $\tilde\beta_q^{(n)}$. Then for $1 \leqslant k \leqslant n$*

$$\max_{\chi^{(k)},\dots,\chi^{(n)}} \operatorname{rank} \tau_{[\chi^{(k)},\dots,\chi^{(n)}]} = \sum_{i=k}^{n} \tilde\beta_q^{(i)}\,. \tag{5.6}$$

Proof. The proof is a simple generalization of the one of Theorem 11. In a δ-regular coordinate system a local description of $\mathcal{R}_q$ exists such that the first $\tilde\beta_q^{(n)}$ equations are solved for derivatives of class n, the next $\tilde\beta_q^{(n-1)}$ equations for derivatives of class $n-1$ and so on. We order the columns of the k-tableau in such a way that the first m columns represent the n-tableau, the first mr_{n-1} columns the $n-1$ tableau etc. This is possible according to Lemma 19.

For $\chi^{(k)} = dx^k,\dots,\chi^{(n)} = dx^n$ the factors C_μ^ν vanish whenever $\mu \neq \nu$ and are one otherwise. Thus we eliminated all contributions by derivatives of class less than k. If the principal derivative of equation τ is p_μ^α, the column indexed by α and μ contains the first non-vanishing entry in row τ. Since our local representation was such that the symbol is in solved form, all leading derivatives are different and we have at least $\sum_{i=k}^n \tilde\beta_q^{(i)}$ linearly independent rows.

On the other hand assume that one-forms $\chi^{(k)},\dots,\chi^{(n)}$ exist such that the rank of the k-tableau is larger than the claimed value. Then we apply a coordinate transformation $\bar x^j = \psi^j(x)$ where the functions ψ^j satisfy in a point $x_0 \in X$

$$\frac{\partial \psi^j}{\partial x^i}(x_0) = \chi_i^{(j)}\,, \quad j = k,\dots,n\,. \tag{5.7}$$

Due to the linear independence of the one-forms $\chi^{(i)}$ this is always possible. The effect of this transformation on the symbol is determined by the highest order part of the transformation law for the derivatives of order q. It is given by

$$\frac{\partial p_\mu^\alpha}{\partial \bar p_\nu^\alpha} = \sum_{s\in\mathcal{S}_\nu} \prod_{i=1}^{q} \left(\frac{\partial \psi^{s_i}}{\partial x^{m_i}}\right)\,, \qquad \begin{cases} |\mu| = |\nu| = q, \\ m = r(\mu)\,. \end{cases} \tag{5.8}$$

Since we are interested in $\tilde\beta_q^{(k)},\dots,\tilde\beta_q^{(n)}$, we consider only the derivatives

$$\frac{\partial \bar\Phi^\tau}{\partial \bar p_\nu^\alpha} = \sum_{|\mu|=q} \left(\frac{\partial \Phi^\tau}{\partial p_\mu^\alpha}\right) \frac{\partial p_\mu^\alpha}{\partial \bar p_\nu^\alpha}\,, \qquad \operatorname{class}\nu \geqslant k\,. \tag{5.9}$$

But at x_0 we can evaluate the derivatives of ψ^j occurring at the right hand side when we plug in (5.8). Comparing (5.9) and (5.5) we immediately recognize the k-tableau at the right hand side. Thus we have found a coordinate system in which we have more equations of class greater than or equal to k than given by $\sum_{i=k}^n \tilde\beta_q^{(i)}$. But this contradicts the definition of the $\tilde\beta_q^{(k)}$. ■

This theorem provides the searched for intrinsic definition of the $\tilde{\beta}_q^{(k)}$. From now on we omit the tilde and always assume that $\beta_q^{(k)}$ denotes these values.

Definition 21. Let $\chi^{(i)} \in T^*X,\ i = 1, \dots, n$ be n linearly independent one-forms. They define a *non-systatic basis* of T^*X, if

$$\operatorname{rank} \tau_{[\chi^{(k)},\dots,\chi^{(n)}]} = \sum_{i=k}^{n} \beta_q^{(i)}\,, \qquad k = 1, \dots, n\,. \tag{5.10}$$

Coordinates x^i such that $dx^i = \chi^{(i)}$ are called an *associated coordinate system.*

Note that in this definition only the $\chi^{(k)}$ where $\beta_q^{(k)} > 0$ really matter. For the remaining one-forms one can choose any which complete to a basis of T^*X. Theorem 20 implies immediately the following

Corollary 22. *A coordinate system on X is δ-regular, if and only if it is associated to a non-systatic basis of T^*X.*

From (5.8) and (5.9) we see that if we write the symbol in a generic linear coordinate system, i.e. in coordinates $\bar{x}^j = \sum a_i^j x^i$, we obtain the 1-tableau by identifying a_i^j with $\chi_i^{(j)}$. Thus the k-tableaux allow us to introduce these generic coordinates step by step.

Different associated coordinate systems for the same non-systatic basis differ only by constants, namely the choice of an origin. Of course such coordinates do not exist for every non-systatic basis, as it does not necessarily satisfy the Frobenius condition. But using the idea of the proof of Theorem 20 one can transform every non-systatic basis into another one which is integrable. Namely, every coordinate system satisfying (5.7) for $j = 1, \dots, n$ is associated to the non-systatic basis $d\psi^j$. Note that the construction of the functions ψ^j does not require the solution of differential equations, as the condition applies only in one point x_0.

Example 23. The analysis of moving pseudo-spherical surfaces in $\mathbb{R}^3$ leads to the Bianchi system (JRSS95)

$$\mathcal{R}_3\ :\ \begin{cases} u_{xyt} - u_{yt}u_x \cot u + u_{xt}u_y \tan u = 0\,, \\ \left(\dfrac{u_{xt}}{\cos u}\right)_x - K(K \sin u)_t - \dfrac{u_{yt}u_y}{\sin u} = 0\,, \\ \left(\dfrac{u_{yt}}{\sin u}\right)_y + K(K \cos u)_t - \dfrac{u_{xt}u_x}{\cos u} = 0\,, \end{cases} \tag{5.11}$$

where K is a given function of t. Obviously the used coordinate system is not δ-regular no matter how we order the coordinates, for there is no derivative of class 3. We use the ordering $x^1 = t$, $x^2 = x$ and $x^3 = y$. The symbol takes after some trivial manipulations the simple form

$$\mathcal{M}_3\ :\ \Big\{\ v_{xyt} = 0\,, \quad v_{xxt} = 0\,, \quad v_{yyt} = 0\,. \tag{5.12}$$

The matrix of the 3-tableau is readily computed

$$\mathcal{T}[\chi^{(3)}] = \begin{pmatrix} \chi_1^{(3)}\chi_2^{(3)}\chi_3^{(3)} \\ \chi_1^{(3)}(\chi_2^{(3)})^2 \\ \chi_1^{(3)}(\chi_3^{(3)})^2 \end{pmatrix} . \tag{5.13}$$

We find two families of characteristic one-forms $\omega_1 = \alpha dx^1$ und $\omega_2 = \beta dx^2 + \gamma dx^3$. This implies that all three coordinate forms dx^1, dx^2, dx^3 are characteristic. Thus it is not surprising that the coordinate system is not δ-regular. The maximal rank of the 3-tableau is 1 and obtained e.g. with the choice $\chi^{(3)} = dx^1 + dx^3$.

There exist four multi-indices of length 3 and class 2, namely [0,0,3], [0,1,2], [0,2,1] and [0,3,0]. Hence the 2-tableau has four columns:

$$\mathcal{T}[\chi^{(2)},\chi^{(3)}] = \begin{pmatrix} \chi_1^{(3)}\chi_2^{(3)}\chi_3^{(3)} & \chi_1^{(2)}\chi_2^{(3)}\chi_3^{(3)} + \chi_1^{(3)}\chi_2^{(2)}\chi_3^{(3)} + \chi_1^{(3)}\chi_2^{(3)}\chi_3^{(2)} & \chi_1^{(2)}\chi_2^{(2)}\chi_3^{(3)} + \chi_1^{(2)}\chi_2^{(3)}\chi_3^{(2)} + \chi_1^{(3)}\chi_2^{(2)}\chi_3^{(2)} & \chi_1^{(2)}\chi_2^{(2)}\chi_3^{(2)} \\ \chi_1^{(3)}(\chi_2^{(3)})^2 & \chi_1^{(2)}(\chi_2^{(3)})^2 + 2\chi_1^{(3)}\chi_2^{(3)}\chi_2^{(2)} & \chi_1^{(3)}(\chi_2^{(2)})^2 + 2\chi_1^{(2)}\chi_2^{(2)}\chi_2^{(3)} & \chi_1^{(2)}(\chi_2^{(2)})^2 \\ \chi_1^{(3)}(\chi_3^{(3)})^2 & \chi_1^{(2)}(\chi_3^{(3)})^2 + 2\chi_1^{(3)}\chi_3^{(3)}\chi_3^{(2)} & \chi_1^{(3)}(\chi_3^{(2)})^2 + 2\chi_1^{(2)}\chi_3^{(2)}\chi_3^{(3)} & \chi_1^{(2)}(\chi_3^{(2)})^2 \end{pmatrix} . \tag{5.14}$$

One could compute its maximal rank by Gaussian elimination. But it is easier to substitute the non-characteristic one-form $\chi^{(3)} = dx^1 + dx^3$

$$\mathcal{T}[\chi^{(2)}, dx^1 + dx^3] = \begin{pmatrix} 0 & \chi_2^{(2)} & \chi_1^{(2)}\chi_2^{(2)} + \chi_2^{(2)}\chi_3^{(2)} & \chi_1^{(2)}\chi_2^{(2)}\chi_3^{(2)} \\ 0 & 0 & (\chi_2^{(2)})^2 & \chi_1^{(2)}(\chi_2^{(2)})^2 \\ 1 & \chi_1^{(2)} + 2\chi_2^{(2)} & (\chi_3^{(2)})^2 + 2\chi_1^{(2)}\chi_3^{(2)} & \chi_1^{(2)}(\chi_3^{(2)})^2 \end{pmatrix} . \tag{5.15}$$

The rank of this matrix is 3 whenever $\chi_2^{(2)} \neq 0$. Thus $\beta_3^{(3)} = 1$ and $\beta_3^{(2)} = 2$ and a non-systatic basis of T^*X is given by

$$\chi^{(3)} = dx^1 + dx^3\,, \quad \chi^{(2)} = dx^2\,, \quad \chi^{(1)} = dx^1\,, \tag{5.16}$$

This basis is integrable and an associated δ-regular coordinate system is $\bar{x}^3 = y + t$, $\bar{x}^2 = x$ and $\bar{x}^1 = t$. This yields the following form for the symbol

$$\mathcal{M}_3 \,:\, \Big\{\ \bar{v}_{333} + \bar{v}_{331} = 0\,, \quad \bar{v}_{332} + \bar{v}_{321} = 0\,, \quad \bar{v}_{322} + \bar{v}_{221} = 0\,, \tag{5.17}$$

where one can easily read off the $\beta_3^{(k)}$ directly.

6. Formally Well-Posed Initial Value Problems

The classical definition of a *well-posed* initial value problem is apparently due to Hadamard (Joh82). He required that the solution of the problem (i) exists for arbitrary Cauchy data, (ii) is determined uniquely by the Cauchy data and (iii) depends continuously on the Cauchy data. This is still vague, as several key words are not rigorously defined. We give a formal and precise version of this definition omitting, however, the third point.

Definition 24. An initial value problem is called *formally well-posed*, if it has a unique formal power series solution for arbitrary formal power series as Cauchy data.

In such a problem exactly the right amount of Cauchy data is prescribed. If we posed more initial conditions, we would obtain compatibility conditions for the data; with less data the solution would no longer be unique. How the solution depends on the data is not considered. Definition 24 implies the existence of a bijective mapping between the parametric coefficients of a general formal power series solution and the Taylor coefficients of the Cauchy data.

In a given (fixed) coordinate system Janet-Riquier Theory (Jan20) provides an easy method to find formally well-posed problems. Reid (Rei91) developed an algorithm which constructs such problems automatically. The idea behind this method is simple. So far we have concentrated on the principal derivatives in our analysis. According to Proposition 4 they can be obtained in a unique way by prolonging each equation with respect to its multiplicative variables only. Now we analyze in a similar way the parametric derivatives.

Reid's algorithm is designed for so-called *passive* systems. Passivity is a central concept in the Janet-Riquier Theory and defined with respect to a ranking. It lies somewhat between formal integrability and involution. We can apply the algorithm to involutive equations provided we use a δ-regular coordinate system and write the system in solved form with respect to a class-respecting ranking, as then it is also passive with respect to this ranking (Sei94a).

Instead of giving a rigorous description of Reid's algorithm we demonstrate it on a simple system. It shows that for non-normal systems one cannot prescribe all initial data on a hypersurface of codimension 1. This might be obvious for over-determined systems, but here we consider an under-determined equation.

Example 13 (cont.) Take again the Maxwell equations in two dimensions (4.7). We choose in each equation the u-derivative as principal derivative. Fig. 1 contains the Reid diagram for u. This is a simple graphical representation of the derivatives: the axes represent the independent coordinates, the dots the different derivatives. The circles indicate the pivots of the equation and the shaded area contains all principal derivatives. We need such a diagram only for u, as there are no principal v-derivatives.

In order to get all parametric derivatives, we need three initial conditions

$$u_t(x_0,t_0) = f\,, \quad u(x,t_0) = g(x)\,, \quad v(x,t) = h(x,t)\,. \tag{6.1}$$

Thus we prescribe some data on a zero-dimensional submanifold, a point (x_0, t_0), some data on a one-dimensional submanifold, the axis $t = t_0$, and some data on the whole manifold. Any other way either leads to compatibility conditions for the Cauchy data or leaves some parametric derivatives arbitrary contradicting the definition of formal well-posedness.

This is a typical feature for non-normal systems. As soon as several pivots belong to the same dependent variable, the boundary of the area with the prin-

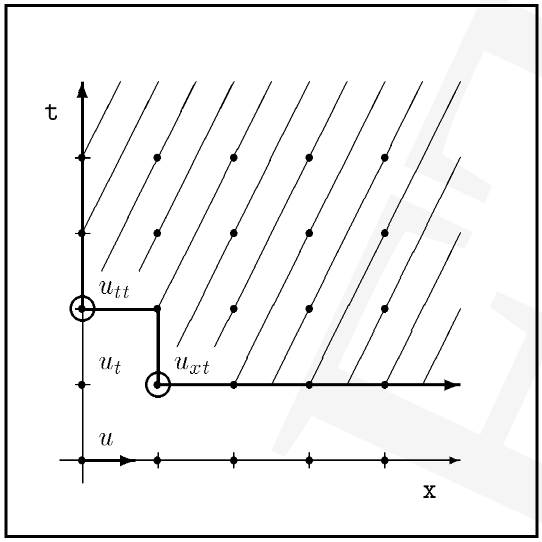

Figure 1. Reid diagram for the two-dimensional Maxwell equations.

cipal derivatives has a kind of (in general higher-dimensional) echelon form. This entails the necessity to use submanifolds of different dimensions.

The next example illustrates the importance of δ-regularity. Besides the fact that the system might not be passive, the Janet-Riquier Theory sometimes yields "initial value problems", one would not necessarily call so.

Example 25. Consider the wave equation in characteristic coordinates

$$u_{xy} = 0\,. \tag{6.2}$$

Obviously we are not dealing with a δ-regular coordinate system, but the system is nevertheless involutive and passive. Reid's algorithm leads to the "initial value problem"

$$u(x, y_0) = f(x)\,, \quad u_y(x_0, y) = g(y)\,. \tag{6.3}$$

But this represents more a kind of boundary value problem.

7. Intrinsic Description

Classically, Cauchy data are represented by an embedding of a submanifold of X with codimension 1 in an appropriate jet bundle (Gar69). This also appeals to physical intuition: the data is prescribed on a space-like manifold and the differential equation determines its evolution. For normal systems this leads to unique solutions according to the Cauchy-Kowalevsky Theorem.

But the example of the two-dimensional Maxwell equations in the last section showed that this point of view cannot be for non-normal equations. Physicists speak then of constrained systems. If Cauchy data is prescribed on a submanifold of codimension 1, it cannot be chosen arbitrarily but must satisfy constraints.

Janet-Riquier Theory shows how to avoid these constraints: initial data is prescribed on a flag of submanifolds. But it depends on a ranking to pick the principal derivatives and the submanifolds are always coordinate hypersurfaces. The k-tableaux allow for an intrinsic description of this choice.

We restrict our attention on first-order equations. We further assume that no lower order, i.e. algebraic, equations are present. They could be solved for some of the dependent variables and thus used to eliminate these. These assumptions are mainly for notational simplicity. Any system can be transformed into an equivalent first-order one, thus there is no loss of generality.

Let $\chi^{(k)}, k = 1, \dots, n$ be an integrable non-systatic basis of T^*X as defined in Section 5. We associate with it two flags of submanifolds. Let $\iota_k : X^k \hookrightarrow X$ be one-dimensional submanifolds ("coordinate axes") such that $\iota_k^*\chi^{(k)} \neq 0$. We introduce $N_k = X^1 \times \cdots \times X^{k-1}$ ($N_1 = \{x_0\}$ for an arbitrary but fixed point $x_0 \in X$) and $N^k = X^k \times \cdots \times X^n$. The N^k were already used in Definition 18. We call a flag such as N_k a *non-systatic flag of submanifolds*.

For first-order equations the k-tableau represents a mapping from $T^*N^k \otimes V\mathcal{E}$ to $\mathcal{E}'$. We are now looking for a flag of fibered submanifolds M^k of $\mathcal{E}$ such that $\emptyset \subseteq M^1 \subseteq \cdots \subseteq M^n \subseteq M^{n+1} = \mathcal{E}$ and

$$\operatorname{rank} \tau_{[\chi^{(k)},\dots,\chi^{(n)}]}\Big|_{T^*X^k \otimes VM^k} = \beta_q^{(k)} . \tag{7.1}$$

The subset condition reflects the fact that in a δ-regular coordinate system not the individual values $\beta_q^{(k)}$ become maximal but the sums $\sum_{i=k}^n \beta_q^{(i)}$.

Lemma 26. *Such a flag M^k exists for every involutive first-order equation $\mathcal{R}_1$. Furthermore one can choose M^k such that $\dim M^k = \beta_1^{(k)}$ (with $\beta_1^{(n+1)} = m$).*

Proof. We choose a coordinate system associated with the $\chi^{(k)}$. Let $(x_0, u_0) \in \mathcal{E}$ be an arbitrary but fixed point. In a neighborhood of it we can describe the N^k by the equations $x^i = x_0^i$ for $i = 1, \dots, k-1$ and we claim that the fibered submanifolds given by the equations $u^\alpha = u_0^\alpha$ for $\alpha = \beta_1^{(k)} + 1, \dots, m$ can be taken as M^k.

Proposition 5 implies that if we choose the principal derivatives according to the inverse lexicographic ranking, then with p_k^α all p_i^α with $i > k$ are principal derivatives, too. By the Implicit Function Theorem there exists a local description of $\mathcal{R}_1$ of the form

$$\Phi_k^\alpha(x, u, p) = p_k^\alpha - \varphi_k^\alpha(x^i, u^\beta, p_j^\gamma) = 0 \,, \qquad \begin{cases} \alpha = 1,\dots,\beta_1^{(k)} \,, \\ j = 1,\dots,k \,, \\ \gamma = \beta_1^{(j)}+1,\dots,m \,. \end{cases} \tag{7.2}$$

Thus principal derivatives of class k occur only for $1 \leqslant \alpha \leqslant \beta_1^{(k)}$. But restricting the k-tableau to $T^*X^k \otimes VM^k$ means nothing else but taking only the corresponding equations into account for its construction. Its rank is $\beta_1^{(k)}$. The M^k define a flag due to Corollary 6 and have the correct dimensions. ∎

From (7.2) it is easy to read off the following initial value problem:

$$
\begin{aligned}
&u^\alpha(x_0) = f^\alpha \ (= Const.)\,, \qquad && \alpha = 1, \dots, \beta_1^{(1)}\,, \\
&u^\alpha(x^1, \dots, x^k, x_0^{k+1}, \dots, x_0^n) = f^\alpha(x^1, \dots, x^k)\,, && \\
& && \alpha = \beta_1^{(k)} + 1, \dots, \beta_1^{(k+1)}\,.
\end{aligned}
\tag{7.3}
$$

In order to give an intrinsic description of these conditions, we introduce the fibered submanifolds M_k by $M^k = M_1 \times \cdots \times M_k$. Their dimensions are given by $\dim M_k = \beta_1^{(k)} - \beta_1^{(k-1)}$. Let us denote by μ_k the natural projection from $\mathcal{E}$ to M_k.† Instead of the functions f^α we use fiber preserving mappings $\rho_k : N_k \to M_k$. For a section $\sigma \in$, $_{loc}(\mathcal{E})$ (7.3) can be written as

$$
\mu_k(\sigma)\Big|_{N_k} = \rho_k\,. \tag{7.4}
$$

Proposition 27. *$\mathcal{R}_1$ together with (7.4) is a formally well-posed initial value problem, if and only if the N_k form a non-systatic flag of submanifolds and the M_k satisfy (7.1).*

Proof. The proposition is a straightforward generalization of the usual treatment of characteristic problems. Classically one defines a surface to be characteristic, if it is not possible to compute values for all derivatives given the ones tangent to the surface (cf. Example 15). If the N_k and the M_k do not satisfy the given conditions, it is similarly not possible to compute values for all derivatives. Or we must prescribe so much data that some derivatives can be obtained in different ways leading to compatibility conditions. ■

Example 13 (cont.) To make this fairly awkward construction more transparent, we detail the involved submanifolds for the two-dimensional Maxwell equations. In order to rewrite it as a first-order system, we introduce the additional dependent variables $w^2 = u_t, w^1 = u_x$ and $z^2 = v_t, z^1 = v_x$ to obtain

$$
\mathcal{R}_1 \ : \ \begin{cases}
u_t - w^2 = 0\,, & u_x - w^1 = 0\,, \\
v_t - z^2 = 0\,, & v_x - z^1 = 0\,, \\
w^2_t - z^1_t = 0\,, & w^2_x - z^1_x = 0\,, \\
w^1_t - w^2_x = 0\,, & \\
z^1_t - z^2_x = 0\,. &
\end{cases}
\tag{7.5}
$$

This yields $\beta_1^{(2)} = 5$ (left column) and $\beta_1^{(1)} = 3$ (right column). We take in each equation the first term as principal derivative. A non-systatic basis of T^*X is given by $\chi^{(1)} = dx, \chi^{(2)} = dt$.

We can now choose the submanifolds $N_3 = X$, $N_2 = \{t = t_0\}$, $N_1 = \{(x_0, t_0)\}$ and $M^3 = \mathcal{E}$, $M^2 = \{z^2 = z^2_0\}$, $M^1 = \{z^2 = z^2_0, z^1 = z^1_0, w^1 = w^1_0\}$. This can be seen as follows: The system (7.5) has been set up in such a way that it is preferable

† One should perhaps remark that in general M_k is *not* a subbundle of $\mathcal{E}$. In principle it makes no sense at all to speak about global properties of the M_k, as they exist in general only locally (like the N_k).

to choose $x^2 = t$, $x^1 = x$. Hence the values of the N_k. M^2 comes from the fact that there are no principal derivatives involving z^2. For all other dependent variables, we have a principal derivative of class 2. To get M^1 we observe that only for u, v, w^2 principal derivatives of class 1 occur.

Summarizing we get the initial value problem

$$
\begin{aligned}
&z^2(x,t) = f_1(x,t)\,, \qquad && z^1(x,t_0) = f_2(x)\,, \qquad && w^1(x,t_0) = f_3(x)\,, \\
&w^2(x_0,t_0) = f_4\,, && v(x_0,t_0) = f_5\,, && u(x_0,t_0) = f_6\,.
\end{aligned}
\tag{7.6}
$$

The initial conditions (7.4) are sufficient to select a unique formal power series solution. The natural question now is under what conditions does this formal series converge, i.e. we should look for an existence theorem. For general equations this is only possible, if everything (equations, Cauchy data, solutions) is analytic. This is the topic of the next section.

8. Cartan-Kähler and Holmgren Theorem

We restrict now to analytic equations and analytic solutions. This implies that the Cauchy data as well as the submanifolds on which they are prescribed are also analytic. The "mother of all existence theorems" in this category is the Cauchy-Kowalevsky Theorem covering normal equations.

Theorem 28. (Cauchy-Kowalevsky). *Let the functions $\phi^\alpha(x^i, u^\beta, p^\beta_j)$ and $f^\alpha(x^1, \dots, x^{n-1})$ where $\alpha = 1, \dots, m$, $i = 1, \dots, n$, $j = 1, \dots, n-1$ be (real) analytic functions of all their arguments in a neighborhood of the origin. Then the initial value problem*

$$
\begin{aligned}
p^\alpha_n &= \phi^\alpha(x^i, u^\beta, p^\beta_j)\,, \\
u^\alpha(x^1, \dots, x^{n-1}, 0) &= f^\alpha(x^1, \dots, x^{n-1})\,,
\end{aligned}
\tag{8.1}
$$

has a unique solution that is analytic in a neighborhood of the origin of X. Unique meaning that no other analytic solution exists.

It can be extended from normal equations to arbitrary involutive ones. This generalization is usually known as Cartan-Kähler Theorem. As the name already indicates, it stems originally from the Cartan-Kähler Theory of exterior differential systems (BCG+91). A first non-constructive proof within the framework of formal theory was given by Goldschmidt (Gol69) using the Spencer cohomology. We give here an adaption of the original proof following (Pom78). It allows us to generalize the Holmgren Theorem.

Theorem 29. (Cartan-Kähler). *Let the differential equation $\mathcal{R}_1$ be analytic and involutive. Then there exists one and only one analytic solution $\sigma \in\ ,_{loc}(\mathcal{E})$ of $\mathcal{R}_1$ satisfying the initial conditions (7.4) for analytic mappings ρ_k and analytic flags M_k, N_k.*

Proof. We use the local formulation of this initial value problem given by (7.2,7.3) and construct the solution step by step. In the first step, we treat only the equations of class 1:

$$p_1^\alpha = \varphi_1^\alpha(x^i, u^\beta, p_1^\gamma), \qquad \begin{cases} \alpha = 1,\dots,\beta_1^{(1)}, \\ \gamma = \beta_1^{(1)}+1,\dots,m. \end{cases} \tag{8.2}$$

If we evaluate (8.2) at $x^j = x_0^j$ for $j > 1$, we can substitute f^β (restricted to N_1) for u^β with $\beta > \beta_1^{(1)}$ and we obtain a normal system with dependent variables u^α for $\alpha = 1, \dots, \beta_1^{(1)}$, independent variable x^1 and initial conditions $u^\alpha(0) = f^\alpha$. The Cauchy-Kowalevsky Theorem yields a unique analytic solution $U^\alpha(x^1, x_0^2, \dots, x_0^n)$ satisfying $U^\alpha(x_0) = f^\alpha$.

The next step takes the equations of class 2 to extend this one-dimensional solution to a two-dimensional one $U^\alpha(x^1, x^2, x_0^3, \dots, x_0^n)$ using again the Cauchy-Kowalevsky Theorem. This inductive process continues until class n. We omit the details and go directly to step n. The other steps run similarly.

Let $U^\alpha(x^1, \dots, x^{n-1}, x_0^n)$ be a solution for all equations up to class $n-1$. The equations of class n are

$$p_n^\alpha = \varphi_n^\alpha(x^i, u^\beta, p_j^\gamma), \qquad \begin{cases} \alpha = 1,\dots,\beta_1^{(n)} \\ \gamma = \beta_1^{(j)},\dots,m, \\ j = 1,\dots,n. \end{cases} \tag{8.3}$$

They form a normal system with dependent variables u^α for $\alpha = 1, \dots, \beta_1^{(n)}$ and independent variables $x^1, \dots, x^n$. As initial conditions we take

$$u^\alpha(x^1, \dots, x^{n-1}, x_0^n) = \begin{cases} U^\alpha(x^1, \dots, x^{n-1}, x_0^n) & \text{if } 1 \leqslant \alpha \leqslant \beta_1^{(n-1)}, \\ f^\alpha(x^1, \dots, x^{n-1}) & \text{if } \beta_1^{(n-1)} < \alpha \leqslant \beta_1^{(n)}. \end{cases} \tag{8.4}$$

Since all functions are analytic, we can invoke the Cauchy-Kowalevsky Theorem. The solution yields the wanted functions $U^\alpha(x)$. u^α for $\alpha > \beta_1^{(n)}$ is unconstrained by the system (7.2) and directly given by the initial conditions (7.3).

We must still show that these functions $U^\alpha(x)$ solve the full equations of lower class, as in the previous steps we always treated some independent variables as constants. Here involution proves to be crucial. If $U^\alpha(x)$ satisfy all equations of system (7.2), the functions

$$\Delta_i^\alpha(x) = \Phi_i^\alpha(j_1(U^\beta)(x)) \tag{8.5}$$

must vanish in the considered neighborhood of x_0.

Since by assumption $\mathcal{R}_1$ is involutive, the prolongation of an equation with respect to a non-multiplicative variable must lead to a linear combination of equations obtained by differentiating with respect to multiplicative variables only (Proposition 4). Thus we find for $i < n$ (among others) the following relations between the prolonged equations

$$D_n\Phi_i^\alpha = \sum_{\substack{1\leqslant k\leqslant n, 1\leqslant l\leqslant k \\ 1\leqslant\beta\leqslant\beta_1^{(k)}}} A_{l\beta}^k(x,u,p) D_l\Phi_k^\beta + \sum_{k,\beta} B_\beta^k(x,u,p)\Phi_k^\beta, \tag{8.6}$$

as we differentiate on the left hand side with respect to a non-multiplicative variable. The ranges of the indices on the right hand side are chosen so, that only prolongations with respect to multiplicative variables are taken into account.

If we substitute the functions $U^\alpha(x)$ and their derivatives for the jet variables $u^\alpha, p_i^\alpha, \ldots$, we can easily evaluate the formal derivatives

$$(D_k\Phi_i^\alpha)(j_2(U^\alpha)(x)) = \frac{\partial}{\partial x^k}\Phi_i^\alpha(j_1(U^\alpha)(x))\,. \tag{8.7}$$

Now (8.6) becomes a normal system

$$\frac{\partial}{\partial x^n}\Delta_i^\alpha = \sum A_{l\beta}^k \frac{\partial}{\partial x^l}\Delta_k^\beta + \sum B_\beta^k \Delta_k^\beta \tag{8.8}$$

and according to our hypothesis that $U^\alpha(x^1,\ldots,x^{n-1},x_0^n)$ satisfy the equations up to class $n-1$ for $x^n = x_0^n$, we get the initial conditions

$$\Delta_j^\beta(x^1,\ldots,x^{n-1},x_0^n) = 0\,. \tag{8.9}$$

Applying for the last time the Cauchy-Kowalevsky Theorem, we get that this system has one and only one analytic solution which is trivially $\Delta_j^\beta(x) \equiv 0$. ∎

Formal integrability is not sufficient for this proof. The first part, i.e. the step by step construction of $U^\alpha(x^1,\ldots,x^n)$ could be done for any differential equation even for a not formally integrable one. It is unclear for such a system, whether the obtained U^α also solve the equations of lower class. Only involution provides the key to set up the final normal system (8.8). Nevertheless, every analytic formally integrable system has a unique analytic solution, as it will become involutive after a finite number of prolongation.

Using a technique apparently introduced by Holmgren (Hol01) it is fairly easily possible for linear systems to extend the uniqueness to non-analytic solutions.

Theorem 30. (Holmgren)**.** *Let L be a normal linear differential operator with analytic coefficients. Let $M \subset X$ be a non-characteristic submanifold of co-dimension 1. Any $\mathcal{C}^1$ solution of the initial value problem*

$$\begin{aligned} Lu &= 0\,,\\ u\big|_M &= 0 \end{aligned} \tag{8.10}$$

vanishes in a neighborhood of M.

Based on our proof of the Cartan-Kähler Theorem it is straightforward to extend this theorem to general involutive operators.

Theorem 31. *Let L be an involutive linear differential operator with analytic coefficients. Let $x_0 = N_0 \subset N_1 \subset \cdots \subset N_n = X$ be a non-systatic flag of submanifolds with $\dim N_i = i$. Then any $\mathcal{C}^1$ solution of the initial value problem*

$$\begin{aligned} &Lu = 0\,,\\ u^\alpha\big|_{N_k} = 0\,, \qquad &\alpha = \beta_1^{(k)}+1,\ldots,\beta_1^{(k+1)} \end{aligned} \tag{8.11}$$

vanishes in a neighborhood of x_0.

Proof. Let u be any solution of the considered initial value problem. $u|_{N_1}$ must satisfy (8.2) with vanishing Cauchy data. Thus according to Theorem 30 $u|_{N_1} \equiv 0$.

Similarly we can conclude following step by step the construction in the proof of Theorem 29 that $u|_{N_k} \equiv 0$ for $k = 1, \dots, n$. Each of these intermediate initial value problems has vanishing Cauchy data as a consequence of the previous step and the Holmgren Theorem is throughout applicable. ■

9. Stability

Finally, we make a few remarks on the problem of stability being omitted in our definition of a formally well-posed problem. It is especially of importance for numerical calculations, for they make sense only if the solutions of problems with "similar" Cauchy data are "similar". We have constructed in Section 7 a formally well-posed problem for any involutive differential equation. But this problem will be unstable for many equations.

Example 32. The classical example of an ill-posed problem is the Cauchy problem for the Laplace equation $u_{xx} + u_{yy} = 0$. Hadamard (CH62) was the first to show that the solution of the following sequence of initial value problems does not depend continuously on the Cauchy data:

$$u(x,0) = 0\,, \quad u_y(x,0) = f_n(x) = \frac{1}{n^2}\sin(nx)\,. \tag{9.1}$$

Obviously each problem is formally well-posed; the formal power series solutions even converge, as the solution is given by $u_n(x,y) = \sin(nx)\sinh(ny)/n^2$. Although $f_n(x) \to 0$ for $n \to \infty$, the solutions $u_n(x,y)$ do not converge to the solution of the initial value problem with $u_y(x,0) = 0$ which is $u(x,y) = 0$.

The question is how to define rigorously when the Cauchy data of two different initial value problems are close or similar. One must distinguish here between two possibilities to deform Cauchy data. In numerical analysis one considers usually the situation found in Example 32: how is the solution affected by a slight variation in the functions prescribed. But in (7.4) one must also take into account deformations of the submanifolds on which the data is given.

Shih used in his thesis (Shi94) the fact that it is possible to define topologies on the space of embeddings (GG73) to introduce a topology on the space of all Cauchy data for a given equation. He defined a problem to be well-defined, if it is an interior point of the subspace of problems having at least one solution. Thus "small" is defined to mean that there exist an open neighborhood in the used topology such that every problem in it is well-posed.

This approach tackles the second possibility of deformation neglecting the first one. It considers the initial value problem for the Laplace equation as well-posed, for it only requires the "neighboring" initial value problems have solutions. But it does not require any relation between their solutions. Shih's formulation is restricted to the standard problem with all data prescribed on a submanifold of codimension 1. Its extension to our more general formulation should be straightforward.

Restricting ourselves to this concept of stability, we can easily prove that our formulation is stable under deformation of the flag N_k. As described in Section 7 these nested submanifolds are characterized by a non-systatic basis of T^*X. Thus to study a deformation of them it suffices to analyze deformations of such bases.

This avoids the use of complicated topologies, since we can use the standard topology of $\mathbb{R}^n$.

Proposition 33. *A non-systatic basis of T^*X is stable under deformations.*

Proof. In a given coordinate system the conditions for a basis to be non-systatic can be expressed by requiring that certain subdeterminants of the k-tableaux do not vanish. But this implies that systatic bases can be characterized by the vanishing of certain rational functions of the components of the one-forms. Thus we can define a homeomorphic mapping f from the space of all bases of T^*X into an $\mathbb{R}^p$ such that $f^{-1}(0)$ gives all systatic bases. But the image of a closed set is again closed, thus the complement of $f^{-1}(0)$, the space of all non-systatic bases is open. Since transition functions are smooth, this statement holds independently of the particular coordinate system used. ■

10. Conclusion

We showed that the formal theory leads naturally to a more general type of initial value problems than the one usually considered. This is implicitly contained in the Cartan-Kähler Theorem. But to our knowledge we have given the first intrinsic description of such initial value problem. New is especially the concept of a non-systatic flag of submanifolds which generalizes the classical notion of a non-characteristic surface as it occurs e.g. in the Cauchy-Kowalevsky Theorem.

The basic tool behind this concept are the k-tableaux of a differential equation. They provide the needed independence of the coordinate system and deal with the problem of δ-regularity. Already in the Cauchy-Kowalevsky Theorem one encounters the problem that it can be applied not only to differential equations in the special solved form of Theorem 28 but also to any system that can be brought into this form by a coordinate transformation. As one can see from the proof of Theorem 11, the tableau determines whether this is possible.

The k-tableaux represent a straightforward generalization of this idea. In the case of Cauchy-Kowalevsky Theorem it suffices to consider the derivatives of class n. For our purposes we must proceed further. We have to know how many equations with a class higher than a given number can be maximally obtained. This explains the complicated form of the factor C^ν_μ in Definition 18. It captures the relevant part of the transformation law for partial derivatives under changes of coordinates.

We have demonstrated some important properties of the k-tableaux leading especially to a rigorous definition of under- and over-determinacy. It is based on previous ones by Olver (Olv86) and Pommaret (Pom78). It generalizes in a natural way the intuitive idea that in an under-determined equation some unknowns remain completely unconstrained by the differential equation.

Note the important difference between algebraic and differential equations. If in a linear system of algebraic equations some unknowns can be chosen arbitrarily, the system consists always of less equations than unknowns after the elimination of dependent equations. In the case of differential equations it is possible that some unknowns are unconstrained, although we have the same number of equations and unknowns. This is related to the possibility of gauge symmetries.

Our definition of an under-determined equation is more precise than the one given by Pommaret, as he uses essentially an intrinsic version of the classical condition that there are less equations than dependent variables. Olver considers only systems with the same number of equations and dependent variables. His definition is in so far not satisfactory, as he does not require the equation to be in involution. Thus it can happen that an over-determined equation becomes under-determined by *adding* an integrability condition!

Finally, we comment briefly on the problem of consistent initialization in numerical computations already mentioned in the introduction. As the numerical solution of general systems of partial differential equations is still hardly understood, we concentrate on systems of ordinary differential equations with algebraic constraints. According to (LPG91) *"the difficulty occurs, because, in general, the human problem solver does not know which of the variables are determined by the constraints (or the constraint derivatives) and which are arbitrary."*

The tableau formalism presented in this article can be seen as an answer to this question for arbitrary systems of partial differential equations, as it is equivalent to deciding which are the principal and which are the parametric derivatives. But this distinction can be finally done only for at least formally integrable systems. The problems of the numerical analysts with this type of systems stem mainly form the fact that they do not render their systems formally integrable. Actually, initialization methods like the graph-theoretical one presented in (Pan88) can be seen as a complicated way to analyze the symbol.

In this context it is also interesting to note that the so-called index of a differential-algebraic system (BCP89) can be intrinsically defined as the number of prolongations (and subsequent projections) needed to obtain a formally integrable system (PT93).

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