On the Geometry of generalized Severi varieties

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Chapter 1

Geometry of generalized Severi varieties

1.1 Introduction

A plane curve of degree d is a subscheme of $\mathbb{P}^2$ given by a homogeneous polynomial

$$\sum_{0 \le i,j,i+j \le d} a_{i\,j} x^i y^j z^{d-i-j}.$$

The Hilbert scheme Hilb(d) parametrizing those is isomorphic to

$$\mathbb{P}^N = \{(\ldots : a_{i\,j} : \ldots)\}, \quad N := \frac{(d+1)(d+2)}{2} - 1,$$

where the point $(\ldots : a_{i\,j} : \ldots)$ corresponds to the plane curve given by the polynomial with coefficients $a_{i\,j}$. The locus of reduced curves in Hilb(d) allows a stratification, that is a partition in finitely many Zariski locally closed subsets. Each stratum is characterized by the property to contain all curves of a certain topological singularity type. This is called the equisingular stratification of Hilb(d). In the literature, the smoothness and irreducibility of such strata are broadly investigated. But not much is known about how these loci fit together. In the present thesis we consider these questions for a certain class of strata.

At the origin of this field of research stands the so called Severi variety $V_{d,g}$. This is the locus in Hilb(d) of integral curves of given geometric genus g, which have only nodes as singularities. Therefore it is a component of an equisingular stratum. In his in 1921 published book on Algebraic Geometry, Severi gave a proof for the irreducibility of this locus ([Sev], Anhang F).

Unfortunately his arguments can not be considered satisfying . A first complete proof was given only in the eighties by J. Harris ([Har]), who in the sequel, together with S. Diaz, also investigated the geometry of the Zariski closure of $V_{d,g}$ up to codimension two ([DiaHar1] and [DiaHar2]). In the present work, we consider the locus $V_{d,g,m}$ of integral curves of geometric genus g, with an ordinary m-fold point at $(0:0:1) \in \mathbb{P}^2$ and at most nodes otherwise. This "generalized Severi variety" is a subvariety of $\overline{V}_{d,g}$ of codimension m. In Theorems 2 and 4 we generalize the results of Diaz and Harris to $V_{d,g,m}$. Furthermore, in Theorem 1 we characterize all the curves in the Zariski closure $\overline{V}_{d,g,m}$ of $V_{d,g,m}$ in Hilb(d), following work of A. Nobile ([No87]), where the case $m = 0$ is considered (this is the only published result characterizing curves in the Zariski closure of an equisingular stratum we know).

In the course of our studies we came across two problematic places in the literature. One is Fact (4.13) in [DiaHar2], which Diaz and Harris gave as statement in the category of schemes over $\mathbb{C}$. While it is correct in the category of analytic varieties, we can give a counter example for the Zariski topology. In Prop. 4.12 we prove a modified statement, which holds for schemes. The other statement in question is Remark (1.8) in [Ta1]. Tannenbaum claims, that the existence of an algebraization for the functor of semi-locally trivial deformations can be proven with the help of Artin's Algebraization Theorem. After studying thoroughly this approach, we came to the conclusion that this argument doesn't suffice. With different methods we prove the existence of an algebraization under stronger assumptions (Prop. 4.12). This algebraization exists in all cases which occur in [Ta1], Chapter 2.

At this place I want to use the occasion to thank everyone whose support has contributed to the coming into being of this dissertation. First I wish to thank my advisor Frank Herrlich for the friendly and open manner in which he at all times responds to the concerns of his students. Particularly I am grateful to him for revising my work parallel to me writing it down. Without that I wouldn't have been able to get finished in time. The many fruitful discussions which I had with Frank, Volker Braungardt and Martin Möller I remember with pleasure. For reading the proofs I thank Stefan Kühnlein and particularly Martin, whose very careful reading helped to improve the presentation considerably. Prof. C.-G. Schmidt I thank for acting as a referee. Last my gratitude is to Prof. Dr. Kronmüller, whose fellowship secured my subsistence for almost a year.

1.2 Notation

We mainly work over the field of complex numbers, the only exemption being the beginning of chapter 4, where an arbitrary algebraically closed field of characteristic zero is allowed. Let $C \subset \mathbb{P}^2$ be a plane curve. If C is reduced and $\tilde{C}$ its normalization, then let $g(C) := \dim\, H^1(\tilde{C}, \mathcal{O}_{\tilde{C}})$ denote the geometric genus of C. For $P \in \mathbb{P}^2$ let $\mu_P(C)$ be the multiplicity of C in P. If $C_i = V(f_i)$ are closed subschemes of $\mathbb{P}^2$ given by homogeneous polynomials f_i, then let $\sum_{i=0}^k n_i C_i$ ($n_i \geq 1; i = 1, \ldots, k$) denote $V(\prod_{i=0}^k f_i^{n_i})$.

By abuse of notation we don't distinguish between curves over $\mathbb{C}$ and the corresponding moduli points in the Hilbert scheme.

Let C_i be a reduced curve on the smooth complex surface S_i and p_i a singular point of C_i ($i = 1, 2$). Then the two singularities (C_i, p_i) are called topological equivalent, if there exist open neighboorhoods U_i in the analytic topology and an homeomorphism $\varphi : U_1 \to U_2$, such that $\varphi(p_1) = p_2$ and $\varphi(C_1 \cap U_1) = C_2 \cap U_2$. An equivalence class is called equisingularity type. It can be described by the system of multiplicity sequences or the resolution graph (compare [dJoPfi]). If a reduced curve C on a smooth surface has singularities $p_1, \ldots, p_n$, then the (not ordered) tuple $((C, p_1), \ldots, (C, p_n))$ of the topological equivalence classes of its singularities is called equisingularity type of C. A family $\mathcal{C} \subseteq S \times B$ of reduced curves on the surface S over the base variety B is equisingular, if for all points $b \in B$ the fibers $\mathcal{C}_b$ over b have the same equisingularity type.

1.3 Statement of results

We begin with some more notation. A triple (d, g, m) of natural numbers is called admissible iff $0 \leq g \leq \frac{(d-1)(d-2)}{2} - \frac{m(m-1)}{2}$. Hence an admissible triple satisfies $0 \leq m \leq d-1$ or $(d, g, m) = (1, 0, 1)$. $V_{d,g,m}$ is known to be nonempty iff (d, g, m) is admissible. Furthermore it is irreducible and smooth of dimension $3d+g-m-1$ (compare [HarMo], p. 30, [Har], [Ran], [GrLoShu]).

Our first result gives a characterization of the curves in the Zariski closure of $V_{d,g,m}$:

Theorem 1. *The Zariski closure* $\overline{V}_{d,g,m}$ *of* $V_{d,g,m}$ *in* $\mathrm{Hilb}(d)$ *consists of the curves* $\sum_{i=0}^{k} n_i C_i$ *where the* C_i *are integral and satisfy the following inequalities:*

$$\begin{array}{ll} (1) & \sum_{i=0}^{k} n_i \deg(C_i) = d \\ (2) & \sum_{i=0}^{k} n_i \, \mu_{(0:0:1)}(C_i) \geq m \\ (3) & \sum_{i=0}^{k} \epsilon_i (n_i \, g(C_i) - (n_i - 1)) \leq g, \quad \text{where} \quad \epsilon_i := \begin{cases} 0 & , \quad g(C_i) = 0 \\ 1 & , \quad g(C_i) > 0 \end{cases} . \end{array}$$

Next we describe the equisingular strata in the boundary of $\overline{V}_{d,g,m}$ which have codimension one. We are able to give a complete list of those strata.

First there is for any two admissible triples (d_i, g_i, m_i) $(i = 1, 2)$ with $(d_1, g_1, m_1) + (d_2, g_2, m_2) = (d, g, m)$ an irreducible stratum of codimension one which parametrizes the curves $C = C_1 + C_2$ with $C_i \in V_{d_i, g_i, m_i}$ such that C has an ordinary m-fold point at $(0:0:1) \in \mathbb{P}^2$ and at most nodes in addition.

The table on the next page gives beside the generic stratum $V_{d,g,m}$ (in the first line) a complete list of the codimension one strata corresponding to irreducible curves ($n := n(d,g,m) := \frac{(d-1)(d-2)}{2} - \frac{m(m-1)}{2} - g$ is the number of nodes of a curve in $V_{d,g,m}$).

Let $U_{d,g,m}$ be the union of the generic stratum with all the codimension one strata. Then we get

Theorem 2. *Any equisingular stratum of* $\overline{V}_{d,g,m}$ *of codimension less than or equal to 1 is contained in* $U_{d,g,m}$.

At points of $U_{d,g,m}$ we can describe the local geometry of $\overline{V}_{d,g,m}$:

Theorem 3. *Let* D *be in* $U_{d,g,m}$.

1. *If* D *has geometric genus* g *and all the branches of all the singularities are smooth, then* $\overline{V}_{d,g,m}$ *is smooth at the point corresponding to* D.

Geometric Genus	Number of Nodes	Singularity in (0:0:1)	Other Singularities
g	n	ordinary m-fold point	——
$g-1$	$n+1$	ordinary m-fold point	——
g	$n-1$	ordinary m-fold point	one cusp
g	$n-2$	ordinary m-fold point	one tacnode
g	$n-3$	ordinary m-fold point	one ordinary triple point
g	n	$m-2$ smooth branches and one cusp, the tangent cone consists of $m-1$ pairwise distinct lines	——
g	$n-1$	m smooth branches, two of them have contact of order one, the remaining ones intersect transversally	——
g	$n-m$	ordinary $(m+1)$-fold point	——

2. *$\overline{V}_{d,g,m}$ is also smooth at points corresponding to integral curves D with a singularity at $(0:0:1)$ which has $m-2$ smooth branches and one cusp such that the tangent cone consists of $m-1$ distinct lines and with only nodes otherwise.*

3. *If D has an ordinary m-fold point at $(0:0:1)$, one cusp and only nodes moreover, then $\overline{V}_{d,g,m}$ looks near D like the product of a cusp with a smooth variety.*

4. *If the geometric genus of D is $g-1$ and it has $n+1$ nodes and no other singularities, then $\overline{V}_{d,g,m}$ intersects itself in $n+1$ smooth sheets at D.*

5. *If $D = D_1 + D_2$ with $D_i \in V_{d_i,g_i,m_i}$ has an ordinary m-fold point at $(0:0:1)$ and only nodes in addition, then $\overline{V}_{d,g,m}$ intersects itself in $(D_1.D_2)$ smooth sheets at D.*

Last we want to describe the local geometry of the normalization of $\overline{V}_{d,g,m}$. This is achieved over $U_{d,g,m}$. To that end we need some more notation: the pullback of the universal family over $\mathrm{Hilb}(d)$ to $V_{d,g,0}$ is given by

$$F := \sum_{0\le i,j,i+j\le d} a_{i\,j} x^i y^j z^{d-i-j} \in \mathbb{C}[V_{d,g,0}][x,y,z].$$

We get a flat family of zerodimensional subschemes of $\mathbb{P}^2$ of length $n := n(d,g,0)$ over $V_{d,g,0}$ by setting $\mathcal{C}_{d,g,0} := V(F, F_x, F_y, F_z)$, the fiber over C corresponding to the nodes of C. This provides us with a morphism

$$\sigma : V_{d,g,0} \to \mathrm{Hilb}^n(\mathbb{P}^2) \to \mathrm{Sym}^n(\mathbb{P}^2),$$

where $\mathrm{Hilb}^n(\mathbb{P}^2)$ is the Hilbert scheme for zerodimensional subschemes of $\mathbb{P}^2$ of length n and $\mathrm{Sym}^n(\mathbb{P}^2)$ is the n-fold symmetric product of $\mathbb{P}^2$. We define $\Sigma_{d,g,0}$ to be the closure of the graph of σ in $\overline{V}_{d,g,0} \times \mathrm{Sym}^n(\mathbb{P}^2)$ and set

$$\Sigma_{d,g,m} := \Sigma_{d,g,0} \cap (\overline{V}_{d,g,m} \times \mathrm{Sym}^n(\mathbb{P}^2)).$$

Theorem 4. *We have a $\overline{V}_{d,g,m}$-morphism of $\Sigma_{d,g,m}$ onto the normalization of $\overline{V}_{d,g,m}$ which is an isomorphism over $U_{d,g,m}$. Moreover both are smooth at points over $U_{d,g,m}$.*

Chapter 2

Curves in the Zariski closure of $V_{d,g,m}$

2.1 Characterization of curves in $\overline{V}_{d,g,m}$

In this section we will give the proof of Theorem 1. If $\sum_{i=0}^{k} n_i C_i$ is in $\overline{V}_{d,g,m}$, where the C_i are integral curves, then inequalities (1) and (2) are obvious and (3) holds by Theorem (1.2) in [No87]. What is left to show is, given the inequalities, that $\sum_{i=0}^{k} n_i C_i$ belongs to $\overline{V}_{d,g,m}$. This is an easy consequence of the following lemmas (the proofs of two of them are postponed to the subsequent sections):

Lemma 2.1. *Let $C \in \mathrm{Hilb}(d)$ be an integral curve of geometric genus g and with multiplicity m in $(0:0:1)$. Then C is contained in $\overline{V}_{d,g,m}$.*

Proof: In the case $m = 0$, this is Theorem (2.1) of [No87]. Hence we can assume $m > 0$. Choose a line L through $(0:0:1)$, which is not contained in the tangent cone of C at $(0:0:1)$. C lies in a component W of the closure of the locus of reduced curves of geometric genus g in $\mathrm{Hilb}(d)$, having intersection number m with L at $(0:0:1)$. As L does not lie in the tangent cone of a generic element of W, we infer from Lemma (2.4) in [Har], that the generic element of W lies in $V_{d,g,m}$. By that the assertion follows. $\square$

Lemma 2.2. *Let (d_i, g_i, m_i) $(i = 1, \ldots, k)$ be admissible triples, not all of them equal to $(1, 0, 1)$. If (d, g, m) with $\sum_{i=0}^{k} d_i = d$, $\sum_{i=0}^{k} m_i \geq m$ and $\sum_{i=0}^{k} g_i \leq g$ is an admissible triple, then*

$$\sum_{i=0}^{k} \overline{V}_{d_i,g_i,m_i} \subseteq \overline{V}_{d,g,m}.$$

Lemma 2.3. *Let $C \in \mathrm{Hilb}(d)$ be an integral plane curve with $\mu_{(0:0:1)}(C) = m$. If $g := g(C) \geq 1$ then $n\,C$ is in $\overline{V}_{nd,ng-(n-1),nm}$ ($n \geq 1$).*

As the triple $(nd, ng-(n-1), nm)$ of Lemma 2.3 is admissible, we can apply Lemma 2.2 and the proof of Theorem 1 is finished, provided not all triples are equal to $(1,0,1)$. If all triples are equal to $(1,0,1)$ we have to consider two cases:

$k = 1$: $\overline{V}_{1,0,1}$ is the set of all lines through $(0:0:1)$; Theorem 1 holds.
$k \geq 2$: as $\overline{V}_{1,0,1} + \overline{V}_{1,0,1} \subseteq \overline{V}_{2,0,1}$ we can apply Lemma 2.2. □

2.2 Proof of Lemma 2.2

In this section we use Tannenbaum's deformation technique of smoothing nodes of a curve on a rational surface to proof Lemma 2.2 ([Ta1]; a basic example is the family $V(xy-t) \supseteq V(xy)$). Here the rational surface will be $\mathbb{P}^2_{(0:0:1)}$, the blowing-up of $\mathbb{P}^2$ in $(0:0:1)$. To apply Tannenbaum's results it is necessary that each of the curves under consideration lies in the complete linear system of a smooth curve. Whenever we need such a smooth curve, we get it the following way: the linear system will be $|d \cdot \tilde{L} - m \cdot E|$, where $\tilde{L}$ is the strict transform of a line in $\mathbb{P}^2$ and E is the exceptional divisor of the blowing-up $\pi : \mathbb{P}^2_{(0:0:1)} \to \mathbb{P}^2$. Take a curve X in $V_{d,G,m}$ with $G := \frac{(d-1)(d-2)}{2} - \frac{m(m-1)}{2}$. The strict transform $\tilde{X}$ of X will be a smooth curve with $|\tilde{X}| = |d \cdot \tilde{L} - m \cdot E|$, as required.

Lemma 2.4. *If (d,g,m) and $(d,g,m+1)$ are admissible, then*

$$\overline{V}_{d,g,m+1} \subseteq \overline{V}_{d,g,m}.$$

Proof: Let C be arbitrary in $V_{d,g,m+1}$. It suffices to show, that $C \in \overline{V}_{d,g,m}$. Denote by $\tilde{C}$ the strict transform of C. Note that $\tilde{C} + E$ has only nodes as singularities, at least one of them lying on E. Choose one node of $\tilde{C} + E$ on E and let all others be assigned in the sense of [Ta1], Def. (2.8) (the assigned nodes are those which stay nodes in the deformed curve, while all the others are smoothed). $\tilde{C} + E$ with this assignment is virtually connected (this means, that we get irreducible curves; compare [Ta1], Def. (2.12)). Let K be the canonical divisor of $\mathbb{P}^2_{(0:0:1)}$ (recall that K is linearly equivalent to $-3\cdot\tilde{L}+E$). As $K.(\tilde{C}+E)<0$ we get by Lemma (2.2), Prop. (2.11) and Thm. (2.13) of [Ta1] that $\tilde{C}+E$ is the specialization of a family of integral curves in $|d\cdot\tilde{L}-m\cdot E|$ with

$$\frac{(d-1)(d-2)}{2}-\frac{(m+1)m}{2}-g+(m+1)-1=\frac{(d-1)(d-2)}{2}-\frac{m(m-1)}{2}-g$$

nodes and no other singularities. For a curve $\tilde{F}$ of this family we have $\tilde{F}.E=(\tilde{C}+E).E=m$ and therefore $\pi(\tilde{F})\in\overline{V}_{d,g,m}$ by Lemma 2.3. □

Lemma 2.2 now follows with Lemma 2.4 by

Lemma 2.5. *Lemma 2.2 is true in the case* $\sum_{i=0}^k m_i=m$.

Proof: Let $\sum_{i=0}^k C_i$ be generic in $\sum_{i=0}^k V_{d_i,g_i,m_i}$, that is with an ordinary m-fold point in $(0:0:1)$ and only nodes as further singularities. The strict transform $\sum_{i=0}^k\tilde{C}_i$ under π has

$$\begin{aligned}&\textstyle\sum_{i=0}^k\left(\frac{(d_i-1)(d_i-2)}{2}-\frac{m_i(m_i-1)}{2}-g_i\right)+\sum_{i<j}(d_id_j-m_im_j)\\ =\;&\textstyle\frac{(d-1)(d-2)}{2}-\frac{m(m-1)}{2}-g+(g-\sum_{i=0}^k g_i+k-1)\end{aligned}$$

nodes. To $g-\sum_{i=0}^k g_i+k-1\geq 1$ of them we apply the smoothing technique of Tannenbaum: There is an i_0 such that $(d_{i_0},g_{i_0},m_{i_0})\neq(1,0,1)$. Choose for every $i\neq i_0$ a node of $C_i\cap C_{i_0}$ and further $g-\sum_{i=0}^k g_i$ nodes on $\sum_{i=0}^k\tilde{C}_i$. We assign the remaining ones (to ensure the virtual connectedness). Again $K.\sum_{i=0}^k\tilde{C}_i<0$ and as before we get by Lemma (2.2), Prop. (2.11) and Thm. (2.13) of [Ta1] that $\sum_{i=0}^k\tilde{C}_i$ is the specialization of a family of integral curves $\tilde{F}$ in $|d\cdot\tilde{L}-m\cdot E|$ with only nodes as singularities and with $\tilde{F}.E=m$, that is $\pi(\tilde{F})\in\overline{V}_{d,g,m}$ by Lemma 2.1. □

2.3 Proof of Lemma 2.3

In the sequel we show Lemma 2.3. The central idea stems from A. Nobile, who proved the case $m=0$ ([No87], Thm. (2.6)): choose an unramified covering $\tilde{D}\to\tilde{C}$ of degree n of the normalization $\tilde{C}$ of C (note that the genus of $\tilde{D}$ is $\tilde{g}:=ng-(n-1)$) and consider it as an element of the space $\mathcal{M}_{\tilde{g}}(nd)$ of maps from smooth curves of genus $\tilde{g}$ to $\mathbb{P}^2$ such that the push-forward of the fundamental class has degree nd. Then by dimension estimates the generic push-forward is a reduced curve.

As we prescribe multiplicity nm in $(0:0:1)$, we have to give some additional arguments. It will be helpful to consider curves of $\overline{V}_{nd,\tilde{g},0}$ which intersect the line $L := V(x) \subseteq \mathbb{P}^2$ with order $\geq nm$ at $(0:0:1)$. If $\sum_{0\leq i,j,i+j\leq nd} a_{i\,j} x^i y^j z^{d-i-j}$ is the equation of a curve, then to intersect L at $(0:0:1)$ with order $\geq nm$ is equivalent to $a_{0\,0} = \cdots = a_{0\,nm-1} = 0$. Without loss of generality we may assume, that L is not contained in the tangent cone of C at $(0:0:1)$.

For a precise definition of $\mathcal{M}_g(d)$ we refer the reader to [No87]. We only recall those of its properties, that we want to use:

1. The existence is known for maps from regular curves of genus $g \geq 1$ to $\mathbb{P}^2$.

2. It is an irreducible quasi-projective variety.

3. $$\dim(\mathcal{M}_g(d)) \geq \begin{cases} 3d+g-1 & \text{if } \; g \geq 2 \\ 3d+1 & \text{if } \; g = 1 \end{cases}$$

4. We have a naturally defined morphism $\Psi_{d,g} : \mathcal{M}_g(d) \to \mathrm{Hilb}(d)$, which pointwise is the push-forward of the fundamental cycle.

5. The Zariski closure of $\Psi_{d,g}(\mathcal{M}_g(d))$ is $\overline{V}_{d,g,0}$.

6. $$\dim(\Psi^{-1}_{nd,\tilde{g}}(nC)) = \begin{cases} 0 & \text{if } \; g \geq 2 \\ 1 & \text{if } \; g = 1 \end{cases}$$

In the following we set

$$\mathcal{M} := \mathcal{M}_{\tilde{g}}(nd), \quad \Psi := \Psi_{nd,\tilde{g}} \quad \text{and} \quad \mathcal{M}_m := \Psi^{-1}(V(a_{0\,0}, \ldots, a_{0\,nm-1})),$$

where

$$V(a_{0\,0}, \ldots, a_{0\,nm-1}) \subseteq \mathrm{Hilb}(nd) = \{(\ldots : a_{i\,j} : \ldots)\}.$$

Note that

$$\dim(\mathcal{M}_m) \geq \begin{cases} n(3d+g-m-1) & \text{if } \; g \geq 2 \\ n(3d-m)+1 & \text{if } \; g = 1 \end{cases}.$$

Lemma 2.6. *Let* $(\tilde{E}, \varphi_{\tilde{E}}) \in \mathcal{M}_m$ *be generic* ($\tilde{E}$ *a smooth curve,* $\varphi_{\tilde{E}} : \tilde{E} \to \mathbb{P}^2$ *a morphism). Then* $E := \Psi((\tilde{E}, \varphi_{\tilde{E}}))$ *is an integral curve, which intersects* L *at* $(0:0:1)$ *with order* $\geq nm$.

Proof: As the image of a smooth curve is irreducible, it suffices to show that the generic element E of $\Psi(\mathcal{M}_m))$ is reduced. Assume that E is not reduced. Therefore it is of the form $k \cdot E'$ with $k \geq 2$ and E' an integral plane curve of degree $\frac{nd}{k}$. The geometric genus g' of E' is $\leq \frac{n}{k}(g-1)+1$ by Riemann-Hurwitz. It intersects L with order $m' \geq \frac{nm}{k}$ at $(0:0:1)$.

Note that the locus in $\mathrm{Hilb}(nd)$ of such curves lies in a set $H(k,m)$ of dimension $3\frac{nd}{k} + g' - 1 - m'$: the universal family over $\mathrm{Hilb}(\frac{nd}{k})$ is given by

$$F(a_{i\,j}, x, y, z) := \sum_{0 \leq i, j, i+j \leq \frac{nd}{k}} a_{i\,j} x^i y^j z^{\frac{nd}{k} - i - j}.$$

The family of curves of degree nd given by F^k induces a finite morphism $\kappa : \mathrm{Hilb}(\frac{nd}{k}) \to \mathrm{Hilb}(nd)$. We define $H(k,m)$ to be the Zariski closure of the image of $V(k,m) := \overline{V}_{\frac{nd}{k}, g', 0} \cap V(a_{0\,0}, \ldots, a_{0\,m'-1})$ under κ. The dimension of $V(k,m)$ is $3\frac{nd}{k} + g' - 1 - m'$ by Lemma (2.4) in [Har].

But this leads to a contradiction:

$$\begin{aligned} 0 &\leq \dim(H(k,m)) - \dim(\Psi(\mathcal{M}_m))) \\ &\leq n(\tfrac{1}{k} - 1)(3d + g - m - 1) - 1 \; < \; 0 \end{aligned}$$

$\square$

Recall that we assumed that L is not contained in the tangent cone to C at $(0:0:1)$. Therefore L can not be contained in the tangent cone at $(0:0:1)$ of a generic element E of $\Psi(\mathcal{M}_m))$. By Lemma (2.4) in [Har] E has an ordinary nm-fold point in $(0:0:1)$ and only nodes otherwise. This proves Lemma 2.3. $\square$

Chapter 3

The equisingular strata in codimension one

In this section we determine the equisingular strata in codimension one of the generalized Severi variety $\overline{V}_{d,g,m}$. Our proof is based on the observation that

$$\overline{V}_{d,g,m} \cap V(a_{0\,0}, \ldots, a_{0\,d}) = \overline{V}_{d-1,\tilde{g},m-1} + L,$$

where

$$\tilde{g} := \begin{cases} g & \text{if} \quad g \leq G \\ G & \text{if} \quad g \geq G \end{cases}$$

($L := V(x)$ and $G := \frac{(d-2)(d-3)}{2} - \frac{(m-1)(m-2)}{2}$ is the maximal genus, that a curve of degree $d-1$ with ordinary $(m-1)$-fold point at $(0:0:1)$ can have). This follows directly from Theorem 1, as $a_{0\,0} = \cdots = a_{0\,d} = 0$ is equivalent to

$$L = V(x) \subseteq V(\sum_{0 \leq i,j,i+j \leq d} a_{i\,j} x^i y^j z^{d-i-j}) \quad (a_{i\,j} \in \mathbb{C}).$$

Therefore we can prove Theorem 2 by induction on the multiplicity m in $(0:0:1)$: if we intersect the Zariski closure E of an equisingular stratum of codimension one in $\overline{V}_{d,g,m}$ with $\overline{V}_{d-1,\tilde{g},m-1} + L$, then we get either all of $\overline{V}_{d-1,\tilde{g},m-1} + L$ or we get a Zariski closed subset $V \subseteq \overline{V}_{d-1,\tilde{g},m-1}$ such that $V + L$ is of codimension one. As the case $m = 0$ is treated in [DiaHar1], we only have to care about the step $m-1 \to m$.

Before we begin with the discussion, we explain an argument, that will be implicitely used several times([KasSchl]): Let D be a reduced curve on a nonsingular surface X (we apply it only to $\mathbb{P}^2$ and $\mathbb{P}^2_{(0:0:1)}$) and let

$\mathcal{D} \subseteq X \times B$ be a flat family of curves on X over an analytic variety B with special fiber $\mathcal{D}_b \cong D$ over some $b \in B$. Then for a singularity p of D there exist open analytic neighbourhoods $U \subseteq B$ of b and $A \subseteq X$ of p and a morphism $\varphi : U \to \mathrm{Def}$ into the versal deformation space Def for singularities analytically equivalent to (D, p), such that there is an isomorphism onto its image $U \times A \to U \times \mathbb{A}^2$ which is compatible with the embeddings of the families $\mathcal{D}$ and $\varphi^*\mathcal{C}$ (pull-back of the versal family $\mathcal{C}$):

$$\begin{array}{ccccc}
\mathcal{D} \cap (U \times A) & \hookrightarrow & U \times A & \to & U \\
\downarrow & \square & \downarrow & & \downarrow \mathrm{id}_U \\
\varphi^*\mathcal{C} & \hookrightarrow & U \times \mathbb{A}^2 & \to & U \\
\downarrow & \square & \downarrow & \square & \downarrow \varphi \\
\mathcal{C} & \hookrightarrow & \mathrm{Def} \times \mathbb{A}^2 & \to & \mathrm{Def}
\end{array}$$

This has the consequence, that singularities of $\mathcal{D}|_U$ near p can only be singularities which occur in the versal family. In the versal families of node, cusp, ordinary triple point or tacnode for example, all other fibers contain at most nodes.

There is a second argument, that we need a few times:

Remark 3.1. *Given an integral curve $C \subseteq \mathbb{P}^2$, there are only finitely many lines through $(0 : 0 : 1)$, which are contained in the tangent cone to some point of C.*

Proof: The projection from the point $(0 : 0 : 1)$ to $\mathbb{P}^1 \cong V(z) \subseteq \mathbb{P}^2$ provides us with a morphism from the normalization of C to $\mathbb{P}^1$, which has only finitely many ramification points. Our statement follows, as every line through $(0 : 0 : 1)$, which is contained in the tangent cone to some point of C induces at least one ramification point. $\square$

Now we begin with the proof of Theorem 2. Let $V \subseteq \overline{V}_{d-1,\tilde{g},m-1}$ be as above. Note that for a family of plane curves over an irreducible base variety there exists a Zariski open subset over which the fibers are equisingular to the generic fiber ([Z], p. 213). Therefore we are able to choose a curve $B \subseteq E$ which runs through a generic subset of E and meets a generic subset of $V + L$ in a point b. Pull back the universal family of Hilb(d) to B and let $D := C + L$ be the fiber over b. We investigate which deformations of D are possible. That the strata we tell are the only ones lying in $\overline{V}_{d,g,m}$ can be checked in each case with Theorem 1.

Case 0: $V = \overline{V}_{d-1,\tilde{g},m-1}$ (hence $C \in V_{d-1,\tilde{g},m-1}$ and D has an ordinary m-fold point at $(0:0:1)$ and only nodes elsewhere). Here we have to distinguish two possibilities: if $\tilde{g} = g$, then $V + L$ has codimension two in $\overline{V}_{d,g,m}$. As we demand multiplicity $\geq m$ at $(0:0:1)$, the fibers over B near b still have an ordinary m-fold point in $(0:0:1)$ by [dJoPfi], Thm. 6.4.3. Hence the only thing that can happen, is the smoothing of nodes. But if we smooth a node, we are either in the generic stratum or outside $\overline{V}_{d,g,m}$. Therefore $E = \overline{V}_{d-1,g,m-1} + \overline{V}_{1,0,1}$. If $\tilde{g} < g$, then we can smooth $g - \tilde{g}$ nodes and get $E = \overline{V}_{d,g-1,m}$.

Next we consider the case, that V contains the Zariski closure of one of the equisingular strata of the list.

Case 1: $C \in V_{d_1,g_1,m_1} + V_{d_2,g_2,m_2}$ ($(d_1, g_1, m_1) + (d_2, g_2, m_2) = (d-1, \tilde{g}, m-1)$). As D is generic, we have that $C := C_1 + C_2$ has an ordinary $(m-1)$-fold point at $(0:0:1)$ and only nodes otherwise. Moreover, D has an ordinary m-fold point at $(0:0:1)$ and only nodes otherwise by Remark 3.1. As we demand multiplicity $\geq m$ at $(0:0:1)$, the fibers over B near b still have an ordinary m-fold point in $(0:0:1)$ by [dJoPfi], Thm. 6.4.3. Hence the only thing that can happen, is the smoothing of nodes. By Lemma (2.2) in [Ta1] we have to smooth $g - \tilde{g} + 1$ nodes to get a curve which belongs to an equisingular stratum of codimension 1. Hence we get $E = \overline{V}_{d_1',g_1',m_1'} + \overline{V}_{d_2',g_2',m_2'}$, where $(d_1', g_1', m_1') + (d_2', g_2', m_2') = (d, g, m)$ are admissible.

Case 2: C is integral of geometric genus $\tilde{g} - 1$, has an ordinary $(m-1)$-fold point at $(0:0:1)$ and only nodes elsewhere. By Remark 3.1 D has an ordinary m-fold point and elsewhere only nodes. Smoothing $g - \tilde{g} + 1$ nodes leads to $E = \overline{V}_{d,g-1,m}$.

Case 3: C is integral of geometric genus $\tilde{g}$, with ordinary $(m-1)$-fold point at $(0:0:1)$, with one additional singularity which is either a cusp or an ordinary triple point or a tacnode and at most nodes elsewhere. In analogy to the second case, we get the stratum of integral curves of geometric genus g with ordinary m-fold point, with either a cusp or an ordinary triple point or a tacnode and at most nodes moreover.

Case 4: C is integral of geometric genus $\tilde{g}$. D has only nodes outside $(0:0:1)$ and one of the following additional singularities at $(0:0:1)$:

- $m-2$ smooth branches and one cusp, the tangent cone consists of

$m-1$ pairwise distinct lines

- m smooth branches, two of them having contact of order one, the remaining intersect transversally
- ordinary $(m+1)$-fold point

We take the strict transform of the family over B in $B \times \mathbb{P}^2_{(0:0:1)}$. Again we have to smooth $g - \tilde{g} + 1$ nodes. We get integral curves of geometric genus g, of degree d, with only nodes outside $(0:0:1)$ and one of the above singularities at $(0:0:1)$.

What remains to be discussed, is that the generic element of V lies in $V_{d-1,\tilde{g},m-1}$. It may be special in the way it intersects L.

Case 5: D has an ordinary m-fold point at $(0:0:1)$ and only nodes otherwise. This case does not occur (compare case 0).

Case 6: $(C.L)_{(0:0:1)} = m$ (hence L lies in the tangent cone of C at $(0:0:1)$). Consider the strict transform of the family over B in $B \times \mathbb{P}^2_{(0:0:1)}$. We get a curve with at most nodes outside $(0:0:1)$ and which has at $(0:0:1)$ a singularity with m smooth branches, two of them having contact of order one, the remaining ones are intersecting transversally.

Case 7: $(C.L)_{(0:0:1)} = m-1$ and there are other points p on L with $(C.L)_p \geq 2$. By Lemma (2.5) in [Har] there is only one point $p \neq (0:0:1)$ on L with $(C.L)_p = 2$. Hence D has an ordinary triple point or a tacnode at p. If we smooth nodes on the strict transform, we get a curve with ordinary m-fold point at $(0:0:1)$, either an ordinary triple point or a tacnode in addition and elsewhere only nodes.

Finally by Lemma (2.4) in [Har] the locus of curves in $\overline{V}_{d-1,\tilde{g},m-1} + L$ with $(C.L)_{(0:0:1)} > m$ has codimension ≥ 2.

We have now proven, that there can be no other strata of codimension one, than those of our list. It remains to be shown, that they are not of greater codimension. This is clear for $V_{d_1,g_1,m_1} + V_{d_2,g_2,m_2}$, $V_{d,g,m}$, $V_{d,g-1,m}$ and for $V_{d,g,m+1}$. Later we will see that it is also true for the strata, which have either a cusp, a tacnode or an ordinary triple point beside the m-fold point and the nodes (Remark 5.4). That the remaining two strata have codimension one follows from the proof to Lemma (2.4) in [Har]. The proof of Theorem 2 is now complete. $\square$

Chapter 4

Deformations of curves with simple singularities

4.1 Some deformation theory

In the following we discuss the claims of the first chapter of [Ta1] and give the lacking proofs. At the beginning of this section we work over an arbitrary algebraically closed field k of characteristic 0.

Let X be a smooth irreducible surface, $D \subseteq X$ a reduced curve and A a finite local artinian k-scheme. Then we define contravariant functors from the category **Art** of local artinian k-algebras to the category **Set** of sets by

$\mathrm{H}_{X,D}(A)$:=$\{$ subschemes of $X \times \mathrm{Spec}(A)$ flat over A, inducing D on $X\}$

(the functor of infinitesimal embedded deformations) and by

$\mathrm{H}'_{X,D}(A)$:= $\{$ $\overline{D} \in \mathrm{H}_{X,D}(A)$ which are locally trivial deformations of $D\}$.

Definition 4.1. *Let* $\mathrm{Sing}(D) := \{p_1, \dots, p_n\}$ *be the singular points of* D *and* $S \subseteq \mathrm{Sing}(D)$ *a fixed subset. We say that* $\overline{D} \in \mathrm{H}_{X,D}(A)$ *is semi-locally trivial with respect to* S *if for every open affine* $U \subseteq X \setminus S$ *the deformation* $\overline{D} \cap U \times_{\mathrm{Spec}\, k} \mathrm{Spec}(A)$ *of* $D \cap U$ *is equivalent to the trivial deformation.*

Remark 4.2. *For* $S = \emptyset$ *we get the definition of "locally trivial deformation".*

By setting

$\mathrm{H}^S_{X,D}(A)$:=$\{$ $\overline{D} \in \mathrm{H}_{X,D}(A)$, $\overline{D}$ semi-locally trivial with respect to $S\}$

we get for every S a functor.

Let $N_D := T^1(D|X, \mathcal{O}_D)$ be the normal sheaf of D in X (for the notation see [LiSchl]). Using the Lichtenbaum-Schlessinger cotangent complex define $N'_D := \mathrm{Ker}(N_D \to T^1(D|k, \mathcal{O}_D))$. If we choose an open affine cover $\{U_i\}$ of D such that each singularity of D is contained in only one U_i, then

$$N_D|U_i \cap U_j = N'_D|U_i \cap U_j \quad \text{for} \quad i \neq j$$

by Theorem 3.1.5 (5) in [LiSchl]. It follows that the sheaves

$$N_D^S(i) := \begin{cases} N_D|U_i & , U_i \cap S \neq \emptyset \\ N'_D|U_i & , U_i \cap S = \emptyset \end{cases}$$

glue to a sheaf N_D^S on D, which does not depend on the chosen covering.

Proposition 4.3. *The functor $\mathrm{H}^S_{X,D}$ is pro-representable with tangent space $\mathrm{H}^S_{X,D}(\mathbb{C}[\epsilon]) = H^0(D, N_D^S)$ ($\mathbb{C}[\epsilon]$ is the ring of dual numbers). It is smooth if $H^1(D, N_D^S) = 0$.*

After typing our proof, we discovered, that a proof for the smoothness similar to ours can be found in the literature ([GrLo], Thm. (3.6)).

Proof: Let $\{U_i\}$ be an open affine cover of X such that each U_i contains at most one of the singularities of D. Set $\mathrm{H} := \mathrm{H}^S_{X,D}$, $\mathrm{H}_i := \mathrm{H}^{S\cap U_i}_{U_i, D\cap U_i}$ and $\mathrm{H}_{i,j} := \mathrm{H}^{S\cap U_i \cap U_j}_{U_i\cap U_j, D\cap U_i \cap U_j}$. For $A, A', A'' \in \mathbf{Art}$ we get a commutative diagram with exact rows:

$$\begin{array}{ccccc} \mathrm{H}(A' \times_A A'') & \hookrightarrow & \prod_i \mathrm{H}_i(A' \times_A A'') & \rightrightarrows & \prod_{i,j} \mathrm{H}_{i,j}(A' \times_A A'') \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{H}(A') \times_{\mathrm{H}(A)} \mathrm{H}(A'') & \hookrightarrow & \prod_i \mathrm{H}_i(A') \times_{\mathrm{H}(A)} \mathrm{H}(A'') & \rightrightarrows & \prod_{i,j} \mathrm{H}_{i,j}(A') \times_{\mathrm{H}(A)} \mathrm{H}(A'') \end{array}$$

As the H_i and $\mathrm{H}_{i,j}$ are pro-representable by [Wa1] 1.4.4, 1.4.7 and 1.4.8, the pro-representability of H follows with the diagram by Schlessinger's criterion ([Wa1], p. 532 or [Schl] Theorem 2.11). The same references give the identification of the tangent space with $H^0(D, N_D^S)$.

The proof of smoothness is almost the same as in the case $S = \mathrm{Sing}(D)$ (compare [Mum], p. 157-159), so we only give the lacking arguments. It suffices to show that given $q : A \to \overline{A} \in \mathbf{Art}$, $\overline{A}$ a quotient of A by a principal ideal (η) with $\eta\, m_A = 0$ (m_A the maximal ideal of A), for every $\overline{D} \in \mathrm{H}(\overline{A})$ there is $\mathcal{D} \in \mathrm{H}(A)$ extending $\overline{D}$, i.e. $\mathrm{H}(q)(\mathcal{D}) = \overline{D}$. Let $\{U_i\}$ be an open affine cover of X as above, $F_i^{(0)}$ and $\overline{F}_i$ equations for D on U_i and for $\overline{D}$ on $\overline{U}_i := U_i \times \overline{A}$ and let $\overline{G}_{ij} \in \Gamma(U_i \cap U_j, (\mathcal{O}_X \otimes \overline{A})^*)$ with $\overline{F}_i = \overline{G}_{ij}\overline{F}_j$. As the H_i are smooth (once again [Wa1] 1.4.4, 1.4.7), we get $\mathcal{D}_i \in \mathrm{H}_i(A)$ with equations F_i such that $F_i \equiv \overline{F}_i \bmod (\eta)$. Furthermore we can choose $G_{ij} \in \Gamma(U_i \cap U_j, (\mathcal{O}_X \otimes A)^*)$ with $G_{ij} \equiv \overline{G}_{ij} \bmod (\eta)$. Then $F_i - G_{ij}\, F_j = h_{ij}\, \eta$ with $h_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_X)$ and we have to show that for suitable choice of F_i and G_{ij} the h_{ij} are equal to 0.

Now by the proof in [Mum] we know that the G_{ij} may be chosen such that $G_{ij}\, G_{jk} = G_{ik}$ and that $\{\frac{h_{ij}}{F_i^{(0)}}\}$ is a 1-Čech cocycle. As $H^1(D, N_D^S) = 0$ we get $\frac{f_i}{F_i^{(0)}} \in N_D^S(U_i) \subseteq T^1(\mathcal{O}_D|\mathcal{O}_X, \mathcal{O}_D)(U_i) \cong (\mathcal{O}_X(D)/\mathcal{O}_X)(U_i)$ such that $\frac{h_{ij}}{F_i^{(0)}} = \frac{f_j}{F_j^{(0)}} - \frac{f_i}{F_i^{(0)}} \bmod \mathcal{O}_X(U_i \cap U_j)$. Therefore we find $g_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_X)$ with $g_{ij}\, F_j^{(0)} = h_{ij} + f_i - f_j\, G_{ij}^{(0)}$, where $G_{ij}^{(0)} \equiv G_{ij} \bmod m_A$. Then one achieves that $h'_{ij} = 0$ for $F'_i := F_i + f_i\, \eta$ and $G'_{ij} := G_{ij} + g_{ij}\, \eta$. A straightforward computation shows $G'_{ij}\, G'_{jk} = G'_{ik}$.

Finally, we have to verify that the F'_i are trivial deformations if $U_i \cap S = \emptyset$. In that case the image of $\frac{f_i}{F_i^{(0)}}$ in $T^1(D \cap U_i|k, \mathcal{O}_D(U_i))$ vanishes by definition. From the exactness of the cotangent complex we get a derivation $d_i \in \mathrm{Der}_k(\mathcal{O}_X(U_i), \mathcal{O}_D(U_i))$ which maps to $\frac{f_i}{F_i^{(0)}}$ under

$$a_2 : \mathrm{Der}_k(\mathcal{O}_X(U_i), \mathcal{O}_D(U_i)) \cong T^0(\mathcal{O}_X|k, \mathcal{O}_D)(U_i) \to T^1(\mathcal{O}_D|\mathcal{O}_X, \mathcal{O}_D)(U_i)$$

(compare [Wa1] 1.4.4, 1.4.8). Checking the construction of a_2 we find $a_2(d_i) = \frac{d_i(F_i^{(0)})}{F_i^{(0)}}$, that is it exists $\partial_i \in \mathrm{Der}_k(\mathcal{O}_X(U_i), \mathcal{O}_X(U_i))$ with $\partial_i(F_i^{(0)}) \equiv f_i \bmod (F_i^{(0)})$ and we get $F'_i = e(\eta, \partial_i)F_i$, where the automorphism $e(\eta, \partial_i)$ deforms F_i trivially ([Wa1] p. 536 bottom). $\square$

Let X be projective. In the following we assume that $\mathrm{H}_{X,D}^S$ is smooth. This has the consequence that it is pro-represented by a pro-couple $(\hat{P}, \xi)$, where $\hat{P}$ is the completion of a finitely generated polynomial ring $P = k[x_i]$ with respect to the maximal ideal $m_P := (x_i)$ ([Schl], Rem. 2.10 and Thm. 2.11). Pro-represented by $(\hat{P}, \xi)$ means that we get every $\mathcal{C} \in \mathrm{H}_{X,D}^S(A)$ from

$\xi = (\xi^{(n)}) \in \lim_n \mathrm{H}^S_{X,D}(\hat{P}/m^n_{\hat{P}})$ by a unique local morphism $\alpha : \hat{P} \to A$. More precisely α factorizes over $\alpha_n : \hat{P}/m^n_{\hat{P}} \to A$ for n with $m^n_A = 0$ and $\mathcal{C} = \mathrm{H}^S_{X,D}(\alpha_n)(\xi^{(n)})$. Furthermore the universal property of the Hilbert scheme for curves on X yields the effectiveness of $(\hat{P}, \xi)$, that is ξ is actually a family on $\mathrm{Spec}(\hat{P})$. Our aim will be to find an algebraization of $(\hat{P}, \xi)$. By this we mean a flat family $\mathcal{C} \hookrightarrow X_B$ of curves on X (where $B := \mathrm{Spec}(R)$ is of finite type over k), together with a k-rational point b on B such that we have an isomorphism $\hat{\mathcal{O}}_{B,b} \cong \hat{P}$ under which $\mathcal{C}$ pulls back to ξ. Unfortunately we haven't been able to prove the existence of an algebraization based on the technique of Artin's Algebraization Theorem ([A73], p. 68 or more generally [A69]), as Tannenbaum suggests. We failed to show that there exists a finitely generated sub-k-algebra $\varphi : R \hookrightarrow \hat{P}$ and a family ξ' semi-locally trivial with respect to S on $\mathrm{Spec}(R)$ such that it induces ξ under the assumptions of [Ta1]. The reason is that we might be in a situation as in the following example: $\psi : k[x, y] \to k[[t]]$, with $x \mapsto t$ and $y \mapsto \sin(t)$. In our setting: over the complex numbers there exists an analytic subspace of the Hilbert scheme which represents the functor of semi-locally trivial deformations, but we don't know it to be locally closed in the Zariski topology. Therefore it suggests itself to allow only singularities for which equisingular and equianalytic deformations coincide. These are the simple singularities (equal to the ADE-singularities). Furthermore we require that X is a rational surface. Under these prerequisites we can shut in the scheme theoretic image of $\mathrm{Spec}(\hat{P})$ in the Hilbert scheme in a Zariski closed subset of dimension $h^0(N^S_D)$. From that we will get an algebraization in the next section.

In order to achieve this we consider the flattening stratification (see [Mum], p. 55) of the component Hilb of the Hilbert scheme which contains D with respect to the coherent sheaf $T^1(\mathcal{U}|\mathrm{Hilb}, \mathcal{O}_{\mathcal{U}})$ ($\mathcal{U}$ denotes the universal family over Hilb). The strata which contain curves with only simple singularities will be equisingular strata, as for those the flatness of $T^1(\mathcal{U}|\mathrm{Hilb}, \mathcal{O}_{\mathcal{U}})$ is equivalent to equisingularity. There are only finitely many equisingular strata and they are locally closed in the Zariski topology ([GrLo], Prop. (2.1)).

Definition 4.4. *A stratum Σ of the flattening stratification of* Hilb *with respect to $T^1(\mathcal{U}|\mathrm{Hilb}, \mathcal{O}_{\mathcal{U}})$ is called* relevant *with respect to D and S, if there* *exists an integral algebraic curve $C \subseteq \mathrm{Hilb}$ with the following properties:*

1. *it contains D*
2. *all but finitely many points of C lie in the regular locus of the reduced scheme Σ^{red}*
3. *let $p_1, \dots, p_k$ be the singular points of D not contained in S; we require that the curve D^* obtained by pulling back $\mathcal{U}$ to the generic point of C has singular points $p_1^*, \dots, p_k^*$ which specialize to $p_1, \dots, p_k$ and are equisingular to them*

Lemma 4.5. *Let X be a smooth projective rational surface, D a reduced curve on it with only simple singularities, $S \subseteq \mathrm{Sing}(D)$ a fixed subset and $H^1(D, N_D^S) = 0$. Then $\dim(\Sigma) \leq h^0(N_D^S)$ for all strata Σ which are relevant with respect to D and S.*

Before we begin with the proof we introduce some notation and make some preliminary remarks. Let $\mathcal{C} \subseteq X_B$ be a family of curves on X over a smooth base B. From the sequence of $\mathcal{O}_B$-modules

$$\mathcal{O}_B \to \mathcal{O}_{X_B} \to \mathcal{O}_{\mathcal{C}}$$

we get an exact sequence

$$T^0(\mathcal{O}_{X_B}|\mathcal{O}_B, \mathcal{O}_{\mathcal{C}}) \to T^1(\mathcal{O}_{\mathcal{C}}|\mathcal{O}_{X_B}, \mathcal{O}_{\mathcal{C}}) \to T^1(\mathcal{O}_{\mathcal{C}}|\mathcal{O}_B, \mathcal{O}_{\mathcal{C}}) \to 0$$

(compare [LiSchl]). The subschemes of the form $\mathrm{Spec}(k[x,y]_g)$ give a base of the topology of the rational surface X ($k[x,y]_g := \{\frac{f}{g^k} | f \in k[x,y], k \in \mathbb{N}\}$). If $\mathrm{Spec}(A)$ is an affine open subset of B, then $\mathrm{Spec}(A[x,y]_g)$ is an affine open subset of X_B and as $R := A[x,y]_g$ is regular in codimension one, we find $F \in R$ such that $\mathcal{C} \cap \mathrm{Spec}(R)$ is isomorphic to $\mathrm{Spec}(R/(F))$. By [Wa1] Prop. 1.4.4 and (1.4.8) our exact sequence restricted to $\mathrm{Spec}(R)$ looks like

$$\mathrm{Hom}_R(\Omega_{R|A}, R/(F)) \to \mathrm{Hom}_R((F)/(F^2), R/(F)) \twoheadrightarrow \mathrm{Hom}_R((F)/(F^2), R/(F, F_x, F_y)).$$

From this we draw some conclusions that we will use in the proof of the lemma:

1. For every closed point $b \in \mathrm{Spec}(A)$ corresponding to a maximal ideal $m \trianglelefteq A$ we get (with $k = k(b) := A/m$) a commutative diagram

$$\begin{array}{ccccc} T^0(\mathcal{O}_{X_B}|\mathcal{O}_B, \mathcal{O}_\mathcal{C}) \otimes k & \rightarrow & T^1(\mathcal{O}_\mathcal{C}|\mathcal{O}_{X_B}, \mathcal{O}_\mathcal{C}) \otimes k & \twoheadrightarrow & T^1(\mathcal{O}_\mathcal{C}|\mathcal{O}_B, \mathcal{O}_\mathcal{C}) \otimes k \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ T^0(\mathcal{O}_X|k, \mathcal{O}_{\mathcal{C}_b}) & \rightarrow & T^1(\mathcal{O}_{\mathcal{C}_b}|\mathcal{O}_X, \mathcal{O}_{\mathcal{C}_b}) & \twoheadrightarrow & T^1(\mathcal{O}_{\mathcal{C}_b}|k, \mathcal{O}_{\mathcal{C}_b}) \end{array}.$$

2. $T^1(\mathcal{O}_\mathcal{C}|\mathcal{O}_{X_B}, \mathcal{O}_\mathcal{C})$ is a flat $\mathcal{O}_B$-module because $\mathcal{O}_\mathcal{C}$ is.

3. The support V of $T^1(\mathcal{O}_\mathcal{C}|\mathcal{O}_B, \mathcal{O}_\mathcal{C})$ is finite over the base B (the fibers are quasi-finite).

Remark 4.6. *We restrict ourselves to rational surfaces because for those we arrive easily at the preceding conclusions.*

Let B be the normalization of the curve C in Definition 4.4 and let $\mathcal{C}$ be the pullback of $\mathcal{U}$ to B. By removing points of B we may assume that for all $b \in B$ and for every specialization $p \in \mathcal{C}_b$ of a p_i^* the singularity at p is equisingular to p_i^* and that the family over $B \setminus \{d\}$ is formally locally trivial, d a point mapping to D. Let $V_S := \cup \overline{p_i^*}$ be the union of the Zariski closures of the p_i^* in $\mathcal{C}$. Then $V_S \subseteq V$ and $V \setminus V_S$ is closed in $\mathcal{C}$. Hence we get two new sheaves on X_B by setting

$$T^S := T^1_S(\mathcal{O}_\mathcal{C}|\mathcal{O}_B, \mathcal{O}_\mathcal{C}) := \begin{cases} T^1(\mathcal{O}_\mathcal{C}|\mathcal{O}_B, \mathcal{O}_\mathcal{C}) & \text{on } \mathcal{C} \setminus (V \setminus V_S) \\ 0 & \text{on } \mathcal{C} \setminus V_S \end{cases}$$

and

$$N^S_\mathcal{C} := \mathrm{Ker}(T^1(\mathcal{O}_\mathcal{C}|\mathcal{O}_{X_B}, \mathcal{O}_\mathcal{C}) \rightarrow T^S).$$

Note that for every $b \in B$ the $\mathcal{O}_X$-module

$$T^S(b) := T^1_S(\mathcal{O}_{\mathcal{C}_b}|k(b), \mathcal{O}_{\mathcal{C}_b}) := \begin{cases} T^1(\mathcal{O}_{\mathcal{C}_b}|k(b), \mathcal{O}_{\mathcal{C}_b}) & \text{on } \mathcal{C}_b \setminus (V \setminus V_S) \\ 0 & \text{on } \mathcal{C}_b \setminus V_S \end{cases}$$

is isomorphic to the restriction T^S_b of T^S to $\mathcal{C}_b$ and that

$$N^S_{\mathcal{C}_b} := \mathrm{Ker}(T^1(\mathcal{O}_{\mathcal{C}_b}|\mathcal{O}_X, \mathcal{O}_{\mathcal{C}_b}) \rightarrow T^S(b))$$

is isomorphic to $N^S_{\mathcal{C},b}$.

Proof of 4.5: T^1_S is a flat $\mathcal{O}_B$-module by definition of the stratification. As the same holds for $T^1(\mathcal{O}_\mathcal{C}|\mathcal{O}_{X_B}, \mathcal{O}_\mathcal{C})$ we infer that $N^S_\mathcal{C}$ is also a flat $\mathcal{O}_B$-module. Semicontinuity ([Ha] III, Thm. 12.8) tells us that

$$\dim_k H^0(N^S_{\mathcal{C},b}) \leq \dim_k H^0(N^S_{\mathcal{C},d})$$

(d the point on B which maps to D). From the exact sequence

$$0 \to N'_{\mathcal{C}_b} \to N^S_{\mathcal{C}_b} \to T^1(\mathcal{O}_{\mathcal{C}_b}|k, \mathcal{O}_{\mathcal{C}_b})/T^S(b) \to 0$$

we get

$$\dim_k H^0(N'_{\mathcal{C}_b}) \leq \dim_k H^0(N^S_{\mathcal{C}_b}).$$

But $N'_{\mathcal{C}_b}$ is the sheaf describing locally trivial deformations of $\mathcal{C}_b$. As being simple is an open condition, the singularities of $\mathcal{C}_b$ are all simple. By the following Remark 4.8 we conclude that there is an open subscheme of Σ which is a component of the scheme representing the functor of formally locally trivial deformations of curves with the same singularity type and the same Hilbert polynomial as those of $\mathcal{C}_b$ for b generic. Hence the dimension of the tangent space to Σ at $\mathcal{C}_b$ is equal to $\dim_k H^0(N'_{\mathcal{C}_b})$ and we conclude

$$\dim(\Sigma) \leq \dim_k H^0(N'_{\mathcal{C}_b}) \leq \dim_k H^0(N^S_D).$$

□

To formulate the next statement we introduce a functor $\mathcal{J}_{P,\tau}$ from the category of algebraic k-schemes to **Set**. Here $\tau = (f_1, \ldots, f_s)$ $(f_i \in k[x,y])$ is a singularity type, where the f_i describe simple singularities and P is a polynomial. A reduced curve C on X is said to be of singularity type τ, if for $\mathrm{Sing}(C) = \{p_1, \ldots, p_s\}$ we have $\hat{\mathcal{O}}_{C,p_i} \cong k[[x,y]]/(f_i)$ $(i = 1, \ldots, s)$. We define

$$\mathcal{J}_{P,\tau}(T) := \left\{ \begin{array}{l} \text{flat relative Cartier divisors } \mathcal{D} \hookrightarrow X \times T \text{ which are} \\ \text{formally locally trivial at all } t \in T \text{ and such that all} \\ \text{geometric fibers are of type } \tau \text{ with Hilbert polynomial } P \end{array} \right\}.$$

Lemma 4.7. *Let X be a smooth projective rational surface, τ a simple singularity type. Then $\mathcal{J}_{P,\tau}$ is representable by a subscheme of* Hilb.

Proof: Combine the proof of Thm. 3.3.5 in [Wa1] with the arguments in the proof of the proposition on p. 91 in [GrKar]. □

Remark 4.8. *The subscheme of* Hilb *representing $\mathcal{J}_{P,\tau}$ consists of open subschemes of those strata of the flattening stratification of* Hilb *with respect to $T^1(\mathcal{U}|\mathrm{Hilb}, \mathcal{O}_{\mathcal{U}})$ which contain closed points C of singularity type τ.*

Proposition 4.9. *Let X be a smooth projective rational surface over the complex numbers, D a reduced curve on it with only simple singularities and $S \subseteq \mathrm{Sing}(D)$ a fixed subset. Then the following holds:*

1. *$H^1(N_D^S) = 0$ if and only if $\mathrm{H}_{X,D} \to \prod_{p \in \mathrm{Sing}(D) \setminus S} \mathcal{D}ef_{(D,p)}$ is surjective ($\mathcal{D}ef_{(D,p)}$ denotes the functor of isomorphism classes of deformations of the analytic germ (D,p)).*
2. *If $H^1(N_D^S) = 0$ then there exists a submanifold $M := M_{X,D}^S$ of Hilb containing D such that: if $\mathcal{H}_{M,D}$ denotes the local ring of holomorphic functions on M at D, then the morphism $\mathcal{O}_{\mathrm{Hilb},D} \to \hat{P}$ induced by the universal property factorizes over the natural morphism $\mathcal{O}_{\mathrm{Hilb},D} \hookrightarrow \mathcal{H}_{\mathrm{Hilb},D} \twoheadrightarrow \mathcal{H}_{M,D}$ and the completion with respect to the maximal ideal $\hat{\mathcal{H}}_{M,D}$ is isomorphic to $\hat{P}$.*

Proof: alteration of [GrLo] Prop. 3.11; compare also [GrKar]. □

Proposition 4.10. *Let X be a smooth projective rational surface over the complex numbers, D a reduced curve on it with only simple singularities, $S \subseteq \mathrm{Sing}(D)$ a fixed subset and $H^1(D, N_D^S) = 0$. Then there exists an integral subscheme Z_D^S of Hilb of dimension $\dim_{\mathbb{C}} H^0(N_D^S)$ over which $\mathrm{Spec}(\hat{P_S}) \to \mathrm{Hilb}$ factorizes.*

Proof: Let M be the manifold of Prop. 4.9 and $\{\Sigma\}$ the flattening stratification of Hilb with respect to $T^1(\mathcal{U}|\mathrm{Hilb}, \mathcal{O}_{\mathcal{U}})$. Let $\{\sigma\}$ be the stratification of M consisting of the connected components of the $\Sigma \cap M$. If for some σ D lies in the closure $\overline{\sigma}$ with respect of the analytic topology on M, then the unique Σ, which contains σ, is relevant. Thus the component M' of $M \setminus \cup_{D \notin \overline{\sigma}} \overline{\sigma}$ containing D still satisfies the property of Prop. 4.9. Let Z_D^S be the Zariski closure of M'. As M' is covered by relevant strata the dimension of Z_D^S is equal to $\dim_{\mathbb{C}} H^0(N_D^S) = \dim(M')$ by Lemma 4.5. Then the natural morphism $\eta : \mathcal{O}_{Z_D^S,D} \to \mathcal{H}_{M',D}$ is injective: any element of $\mathcal{O}_{Z_D^S,D}$ can be written as quotient $\frac{f}{g}$ of two polynomials in the coordinate ring of $\mathbb{A}^N \subseteq \mathbb{P}^N \cong \mathrm{Hilb}$. As $\eta(\frac{f}{g}) = 0$ iff $f = 0$ on an open subset U of D in M', $f = 0$ on the Zariski closure Z_D^S of U and hence in $\mathcal{O}_{Z_D^S,D}$. Set $I := \mathrm{Kernel}(\mathcal{O}_{\mathrm{Hilb},D} \to \hat{P_S})$. Then from the following commutative diagram we get $\mathcal{O}_{Z^S_D,D} \cong \mathcal{O}_{\mathrm{Hilb},D}/I$ and hence the assertion:

$$\begin{array}{ccccc} I & \to & \mathcal{O}_{\mathrm{Hilb},D} & \to & \hat{P}_S \\ & \swarrow & \downarrow & & \updownarrow\cong \\ \mathcal{O}_{Z^S_D,D} & \to & H_{M',D} & \to & \hat{\mathcal{H}}_{M',D} \end{array}$$

□

4.2 Correction of Fact (4.13) in [DiaHar2]

In this section we work again over the field of complex numbers. Let D be a reduced projective curve of degree d in $\mathbb{P}^2$ and let $p_1, \ldots, p_n$ be the singular points of D. Let S be a fixed subset of $p_1, \ldots, p_n$. We define a sheaf of ideals $J(D,S)$ on $\mathbb{P}^2$ as follows: for any open set $U \subseteq \mathbb{P}^2$,

$$J(D,S)(U) := \left\{ \begin{array}{l} f \in \mathcal{O}_{\mathbb{P}^2}(U) \textit{ such that } \forall p_i \in U \cap (\mathrm{Sing}(D) \setminus S) \\ \textit{the image of } f \textit{ in the complete local ring of } \mathbb{P}^2 \\ \textit{in } p_i \textit{ lies in the Jacobian ideal of } D \textit{ at } p_i \end{array} \right\}.$$

Remark 4.11. *Recall that if $U_i \cap S = \emptyset$ then $N^S_D(U_i)$ is the image of*

$$a_2 : \mathrm{Der}_k(\mathcal{O}_{\mathbb{P}^2}(U_i), \mathcal{O}_D(U_i)) \to T^1(\mathcal{O}_D|\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_D)(U_i) \cong (\mathcal{O}_{\mathbb{P}^2}(D)/\mathcal{O}_{\mathbb{P}^2})(U_i),$$

where $a_2(d_i) = \frac{d_i(F_i)}{F_i}$ for an equation F_i for D on U_i. Hence N^S_D is isomorphic to $\mathcal{O}_D(d) \otimes J(D,S)$.

If $H^1(D, \mathcal{O}_D(d) \otimes J(D,S)) = 0$, then fact (4.13) in [DiaHar2] claims the existence of a subscheme Z of $\mathrm{Hilb}(d)$ with the following properties:

1. it contains the point q corresponding to D
2. Z is reduced and smooth with tangent space naturally identified with $H^0(D, \mathcal{O}_D(d) \otimes J(D,S))$
3. the restriction $\mathcal{C}$ of the universal family to Z is formally locally trivial with respect to S at q in the Zariski topology
4. if Z' is any other subscheme of $\mathrm{Hilb}(d)$ satisfying 1. and 2. then, in some Zariski neighborhood of q, $Z' \subseteq Z$.

It is at least misleading to speak here of a subscheme as the following example illustrates: take $D \in V_{d,g-1,0}$ such that it has $n \geq 2$ nodes $p_1, \ldots, p_n$ and let $S := \{p_2, \ldots, p_n\}$. By applying Tannenbaum's technique to D ($H^1 = 0$ by Prop. 5.2), we see that $\overline{V}_{d,g,0}$ near q looks like n smooth sheets intersecting in $V_{d,g-1,0}$, each of them corresponding to the smoothing of one of the nodes of D. Obviously, the restriction of the universal family to $\overline{V}_{d,g,0}$ is not formally locally trivial with respect to S at q in the Zariski topology, but (viewed analytically) the restriction to the sheet corresponding to the smoothing of p_1 is. So it suggests itself to replace "subscheme" by "analytic subspace" (see Prop. 4.9). We give a proof for a modified statement in the category of schemes (this is the announced algebraization for the pro-couple $(\hat{P}, \xi)$):

Proposition 4.12. *Let X be a smooth projective rational surface, D a reduced curve on it with only simple singularities, $S \subseteq \mathrm{Sing}(D)$ a fixed subset and $H^1(D, N_D^S) = 0$. Then there exists an integral subscheme Z of the Hilbert scheme together with a closed point q on the normalization $\tilde{Z}$, such that the pullback of the universal family to* $\mathrm{Spec}(\hat{\mathcal{O}}_{\tilde{Z},q})$ *defines a pro-couple for $\mathrm{H}^S_{X,D}$. In particular $\tilde{Z}$ is smooth at q with tangent space naturally identified with $H^0(D, N_D^S)$.*

Proof: Let $(\hat{P}, \xi)$ be a pro-couple for $\mathrm{H}^S_{X,D}$, $\mathrm{Spec}(R_0) \subseteq \mathrm{Hilb}$ an open affine subset of the Hilbert scheme such that ξ is induced by the morphism $\varphi_0 : R_0 \to \hat{P}$. Set $R := R_0/\mathrm{Ker}(\varphi_0)$ (R is integral). Prop. 4.10 implies $Z_D^S = \overline{\mathrm{Spec}(R)}$. Let $\varphi : R \to \hat{P}$ be the morphism induced by φ_0 and $m_R := \varphi^{-1}(m_{\hat{P}})$. First we prove

$$(*) \quad m_R \hat{P} = m_{\hat{P}}.$$

Let $\eta_j : \hat{P} \cong \mathbb{C}[[x_i]] \to \mathbb{C}[\varepsilon]$ be the morphism onto the dual numbers declared by

$$\eta_j(x_i) := \begin{cases} \varepsilon & , \text{ if } i = j \\ 0 & , \text{ if } i \neq j \end{cases} .$$

Due to the versality of $(\hat{P}, \xi)$, the family on $\mathrm{Spec}(\mathbb{C}[\varepsilon])$ induced by η_j from ξ is not the constant family. On the other hand we also get it via $\eta_j \circ \varphi$

from the universal family on Hilb. Hence $\eta_j \circ \varphi$ has to be surjective for every j, that is the generators x_i of $m_{\hat{P}}$ are contained in the image of φ.

Next let $\tilde{R}$ be the integral closure of R in its quotient field. Then we can extend φ to $\tilde{\varphi} : \tilde{R} \to \hat{P}$, as $\hat{P}$ is normal. The prime ideal $m_{\tilde{R}} := (\tilde{\varphi})^{-1}(m_{\hat{P}})$ lies over m_R, so it is maximal. We extend $\tilde{\varphi}$ to $\Phi : \tilde{R}_{m_{\tilde{R}}} \to \hat{P}$, where $\tilde{R}_{m_{\tilde{R}}}$ denotes the localization of $\tilde{R}$ at $m_{\tilde{R}}$. Let $\hat{R}$ be the completion of $\tilde{R}_{m_{\tilde{R}}}$ with respect to its maximal ideal and $\hat{\Phi} : \hat{R} \to \hat{P}$ the extension of Φ. Then $\hat{\Phi}$ is surjective by $(*)$. We can take $Z = Z_D^S$ and q the point in $\mathrm{Spec}(\tilde{R})$ corresponding to $m_{\tilde{R}}$, if we show that $\mathrm{Ker}(\hat{\Phi}) = (0)$ holds. It follows from Prop. 4.10 that $\dim(R) = \dim(\hat{P})$. Hence we get

$$\dim(\hat{R}) = \dim(R) = \dim(\hat{P}) = \dim(\hat{R}/\mathrm{Ker}(\hat{\Phi})),$$

from which we infer $\mathrm{Ker}(\hat{\Phi}) = (0)$. $\square$

Chapter 5

On the geometry of $\overline{V}_{d,g,m}$

5.1 The local geometry of $\overline{V}_{d,g,m}$

Our aim for this section is to describe the local geometry of $\overline{V}_{d,g,m}$ at least at the points of $U_{d,g,m}$.

We begin with a general observation which holds for any projective rational surface X over the field of complex numbers. Let $N'_D := N^{\emptyset}_D$ be the sheaf describing formally locally trivial deformations of a reduced curve D on X and assume $h^1(N'_D) = 0$. Then by Prop. 4.9 the morphism of functors $\mathrm{H}_{X,D} \to \prod_{p\in \mathrm{Sing}(D)} \mathcal{D}ef_{(D,p)}$ is surjective. That means that we have a morphism $\psi = (\dots, \psi_p, \dots)$ of the analytic germ of Hilb at D to the product of the versal deformation spaces $\prod_{p\in\mathrm{Sing}(D)} \mathrm{Def}_{(D,p)}$ of the singularities, which is smooth at D. For a partition $\mathrm{Sing}(D) = S^{arb} \cup S^{es} \cup S^{ea} \cup S^{eg}$ we can therefore describe the subgerm corresponding to arbitrary deformations of the $p \in S^{arb}$, equisingular deformations of the $p \in S^{es}$, equianalytic deformations of the $p \in S^{ea}$ and equigeneric deformations of the $p \in S^{eg}$: it is the preimage under ψ of the product of the respective loci in the versal deformation spaces. More precisely, because ψ is smooth, it has smooth fibers and the germ is isomorphic to the product of the target germ with the smooth fiber.

In the sequel we use the versal deformation spaces of the node, the cusp, the tacnode and the ordinary triple point to describe the local geometry of $\overline{V}_{d,g,m}$. Therefore we give them here as examples:

- As versal deformation space $\mathrm{Def}_{(V(xy),(0,0))}$ of the node $V(xy) \subseteq \mathbb{A}^2$ we may take the affine line $\mathbb{A}^1$ with family $V(xy - t) \subseteq \mathbb{A}^2 \times \mathbb{A}^1$. The equisingular and equigeneric locus coincide: for $t = 0$ the fiber is a node. For $t \neq 0$ the fibers are smooth.

- In the case of the cusp $V(y^2 - x^3) \subseteq \mathbb{A}^2$ we take $\mathrm{Def}_{(V(y^2-x^3),(0,0))} := \mathbb{A}^2$ with family $V(y^2 - x^3 - ax - b) \subseteq \mathbb{A}^2 \times \mathbb{A}^2$. The equisingular locus is the point $(a, b) = (0, 0)$, with the cusp as fiber. The equigeneric locus is $V(4a^3 + 27b^2)$, where for $(a, b) \neq (0, 0)$ the fiber is a node. The fibers over $D(4a^3 + 27b^2)$ are smooth.
- The tacnode $V(y^2 - x^4) \subseteq \mathbb{A}^2$ has versal deformation space $\mathbb{A}^3$ with versal family $V(y^2 - x^4 - ax^2 - bx - c) \subseteq \mathbb{A}^2 \times \mathbb{A}^3$. The equigeneric locus is $V(b, a^2 - 4c)$. The fibers over $V(b, a^2 - 4c)$ have two nodes, if $a \neq 0$ and the tacnode sits over $(0, 0, 0)$.
- The ordinary triple point $V(x^2y + xy^2)$ has versal deformation space $\mathbb{A}^2$ with versal family $V(x^2y + xy^2 + txy + sx) \subseteq \mathbb{A}^2 \times \mathbb{A}^2$. The equigeneric locus is given by $s = 0$. The fibers over the equigeneric locus have three nodes for $t \neq 0$ and the ordinary triple point at $(0, 0)$.

We will apply this to the study of the scheme $\overline{V}'_{d,g,m}$, which is the closure of the locus of integral curves of geometric genus g on $\mathbb{P}^2_{(0:0:1)}$ in $\mathrm{Hilb}(d, m)$. Here $\mathrm{Hilb}(d, m)$ is the Hilbert scheme of curves linearly equivalent to the divisor $d \cdot \tilde{L} - m \cdot E$, where $\tilde{L}$ is the strict transform of a line in $\mathbb{P}^2$ and E is the exceptional divisor. Note that $\mathrm{Hilb}(d, m)$ is canonically isomorphic to the linear system $|d \cdot \tilde{L} - m \cdot E|$ ([Mum] p. 94 and 96). The reason for investigating $\overline{V}'_{d,g,m}$ is, that nailing down the ordinary m-fold point at $(0 : 0 : 1)$ yields

Lemma 5.1. *$\overline{V}'_{d,g,m}$ is naturally isomorphic to $\overline{V}_{d,g,m}$.*

Proof: We can naturally identify $\mathrm{Hilb}(d, m)$ with the linear subspace $V(a_{i\,j}|0 \leq i, j, i + j < m)$ of $\mathrm{Hilb}(d) = \{(\cdots : a_{i\,j} : \ldots)\}$: The pull back of the universal family $\mathcal{U} \subseteq \mathbb{P}^2 \times \mathrm{Hilb}(d)$ to $V(a_{i\,j}|0 \leq i, j, i + j < m)$ is given by the polynomial

$$\sum_{0 \leq i,j \,\wedge\, m \leq i+j \leq d} a_{i\,j} x^i y^j z^{d-i-j}.$$

Restricting it to $D(z) \times V(a_{i\,j}|0 \leq i, j, i + j < m)$ gives

$$\sum_{0 \leq i,j \,\wedge\, m \leq i+j \leq d} a_{i\,j} x^i y^j.$$

The strict transform under $(u,v) \mapsto (v, uv)$ is

$$\sum_{0\leq i,j \,\wedge\, m\leq i+j\leq d} a_{i\,j} u^j v^{i+j-m}.$$

But this is just the family on $\mathrm{Hilb}(d,m) \cong |d \cdot \tilde{L} - m \cdot E|$ restricted to the open affine subset we've considered. Now the claim follows, as $\overline{V}_{d,g,m} \subseteq V(a_{i\,j}|0 \leq i,j, i+j < m)$ maps to $\overline{V}'_{d,g,m} \subseteq \mathrm{Hilb}(d,m)$ under this identification (see Thm. 1). $\square$

Let $D \in |d\cdot\tilde{L}-m\cdot E|$ be a reduced curve of geometric genus g with n nodes and c cusps as only singularities and irreducible components $D_1, \ldots, D_k$. Further let $\tilde{D} = \tilde{D}_1 \cup \cdots \cup \tilde{D}_k$ be the normalization of D, let $\pi : \tilde{D} \to D$ be the normalization map and let R be its ramification divisor. From [Ta2], p. 172-173 it follows that $h^1(N'_D) = 0$ if

$$(*) \quad (K.D_i) + \deg(R|_{\tilde{D}_i}) < 0 \quad \text{for} \quad i = 1, \ldots, k$$

(K the canonical divisor on $\mathbb{P}^2_{(0:0:1)}$). In that case, the germ of $\overline{V}'_{d,g,m}$ near D looks like the germ at zero of

$$V(4a_1^3 + 27b_1^2, \ldots, 4a_c^3 + 27b_c^2, t_1, \ldots, t_n) \subseteq \mathbb{A}^{3d+g-m-1+c+n}$$

where

$$\mathbb{A}^{3d+g-m-1+c+n} \cong \prod_{i=1}^{c} \mathrm{Def}_{(V(y^2-x^3),(0,0))} \times \prod_{j=1}^{n} \mathrm{Def}_{(V(xy),(0,0))} \times \mathbb{A}^{3d+g-m-1-c}$$

(for all singularities we allow equigeneric deformations, that is $S^{eg} := \mathrm{Sing}(D)$).

Proposition 5.2. *For irreducible D and $c \in \{0,1\}$ condition $(*)$ is always satisfied.*

Proof: One has $K \sim -3\cdot\tilde{L}+E$, $D \sim d\cdot\tilde{L}-m\cdot E$ and $\deg(R) = c$. Hence $(K.D) + \deg(R) = -3d + m + c < 0$ $\square$

Proposition 5.3. *Let $D \in |d \cdot \tilde{L} - m \cdot E|$ be integral, corresponding to a plane curve with ordinary m-fold point at $(0:0:1)$, either one tacnode or one ordinary triple point and only nodes otherwise. Then $h^1(N'_D) = 0$ holds.*

Proof: We consider only the tacnode, as the computation for the ordinary triple point is similar. Let $\nu : \tilde{D} \to D$ be the normalization map. Then we have a short exact sequence

$$0 \to N'_D \to \nu_*\nu^* N'_D \to \mathcal{Q} \to 0$$

of sheaves on D, where $\mathcal{Q}$ is a skyscraper sheaf whose support is the tacnode (the cohomology of $\nu^* N'_D$ is easier to compute). Locally we may assume $N'_D \cong \mathbb{C}[x,y]/(f) \otimes (f_x, f_y)$, where f is a local equation for D (compare Remark 4.11). To investigate the local behaviour near the tacnode, we are allowed to consider $\mathbb{C}[x,y]/(y^2+x^4)$ and its normalization $\mathbb{C}[u]\oplus\mathbb{C}[v]$, where $x \mapsto (u,v)$ and $y \mapsto (iu^2, -iv^2)$ determine the inclusion corresponding to ν. Then $\nu^* N'_D$ corresponds to the ideal $((u^2,0),(0,v^2))$ and the image of N'_D in $\nu_*\nu^* N'_D$ corresponds to the subvectorspace $\mathbb{C}\cdot(iu^2,-iv^2) + ((u^3,0),(0,v^3))$ therein. Hence $\nu^* N'_D$ is isomorphic to

$$\nu^*\mathcal{O}_D(D)(-2t_1 - 2t_2 - p_1 - q_1 - \ldots - p_n - q_n),$$

where t_1 and t_2 are the preimages of the tacnode and the p_i and q_i are the preimages of the nodes. From

$$\begin{aligned}
& 2g - 2 - \deg(\nu^* N'_D(-t_1 - t_2)) \\
= \;& [(d-1)(d-2) - m(m-1) - 2(n+2)] - 2 - [d^2 - m^2 - 6 - 2n] \\
= \;& -[3d - 2 - m] < 0
\end{aligned}$$

we infer one after another

$$\begin{aligned}
h^1(\nu^* N'_D) &= h^1(\nu^* N'_D(-t_1 - t_2)) \;=\; 0, \\
h^0(\nu^* N'_D) &= h^0(\nu^* N'_D(-t_1 - t_2)) + 2
\end{aligned}$$

and that $H^0(\nu^* N'_D)$ contains an element which in $\mathbb{C}[u] \oplus \mathbb{C}[v]$ looks like $(u^2 + u^3\, g(u), v^2 + v^3\, h(v))$. Therefore $H^0(\nu_*\nu^* N'_D)$ surjects onto $H^0(\mathcal{Q})$ and the long exact cohomology sequence yields $h^1(N'_D) = 0$. □

Remark 5.4. *The loci of nodal curves in $|d\cdot\tilde{L} - m\cdot E|$ with either one cusp, one tacnode or one ordinary triple point have codimension one in $\overline{V}'_{d,g,m}$, as the equianalytic locus has codimension one in the equigeneric locus in the versal deformation space of each of those singularities.*

Remark 5.5. *In the same way we want to describe the geometry of* $\overline{V}'_{d,g,m}$ *near* D *where* $D \in |d \cdot \tilde{L} - m \cdot E|$ *is a reduced curve of geometric genus* $g-1$ *with* $n+1$ *nodes and no other singularities. First assume that* D *is irreducible. Then condition* $(*)$ *holds and the germ of* $\overline{V}'_{d,g,m}$ *at* D *is isomorphic to the germ at* $(0,\dots,0)$ *of*

$$V(t_it_j|i \neq j) \subseteq \mathbb{A}^{3d+g-m-1+n},$$

where

$$\mathbb{A}^{3d+g-m-1+n} \cong \prod_{j=1}^{n+1} \mathrm{Def}_{(V(xy),(0,0))} \times \mathbb{A}^{3d+g-m-1-1}$$

(we allow that any specified node is deformed arbitrarily, while the remaining ones are deformed equisingularly). $V(t_it_j|i \neq j) \subseteq \mathbb{A}^{n+1} \cong \prod_{j=1}^{n+1} \mathrm{Def}_{(V(xy),(0,0))}$ *is the union of the* t_i*-axes, each axis corresponding to the deformation of the respective specified node.*

Remark 5.6. *Secondly we consider* D *with two irreducible components* D_1 *and* D_2*. Again* $(*)$ *is satisfied and hence the germ of* $\overline{V}'_{d,g,m}$ *at* D *is isomorphic to the germ at* $(0,\dots,0)$ *of*

$$V(\prod_{p_i,p_j \in D_1 \cap D_2,\, p_i \neq p_j} t_it_j) \subseteq \mathbb{A}^{3d+g-m-2+(D_1.D_2)},$$

where this time

$$\mathbb{A}^{3d+g-m-2+(D_1.D_2)} \cong \prod_{p \in D_1 \cap D_2} \mathrm{Def}_{(V(xy),(0,0))} \times \mathbb{A}^{3d+g-m-2}$$

(we allow that any specified node in $D_1 \cap D_2$ *is deformed arbitrarily, while the remaining ones are deformed equisingularly). We see that in both cases* $\overline{V}'_{d,g,m}$ *near such a* D *looks like the transversal intersection of* $n+1$*, respectively* $(D_1.D_2)$ *smooth sheets.*

Finally we investigate the geometry of $\overline{V}_{d,g,m}$ at points D, where D is an integral curve of geometric genus g with the property that all the branches of all of its singularities are smooth. It's a classical result, that $\overline{V}_{d,g,0}$ is smooth at those points ([AC83], p. 486-487). We will show that the same is true for $\overline{V}_{d,g,m}$. We use the theory from [Ho1] and therefore work in the category of complex manifolds. Our notation is as follows:

$$\begin{array}{rl}
\tilde{D} & \text{normalization of } D \\
\varphi : \tilde{D} \to \mathbb{P}^2_{(0:0:1)} & \text{the map to the strict transform of } D \\
\beta : \mathbb{P}^2_{(0:0:1)} \to \mathbb{P}^2 & \text{blowing up of } \mathbb{P}^2 \\
\theta_X & \text{tangent sheaf of } X \\
N_\varphi & := \varphi^* \theta_{\mathbb{P}^2_{(0:0:1)}} / \theta_{\tilde{D}} \\
N_{\beta\circ\varphi} & := \beta^* \varphi^* \theta_{\mathbb{P}^2} / \theta_{\tilde{D}}
\end{array}$$

The idea of the proof is, that a complete deformation of φ is smooth and gives locally an embedding into a complete deformation of $\beta \circ \varphi$, which is locally isomorphic to $\overline{V}_{d,g,0}$.

As φ is unramified it follows from [Ta2], p. 172-173 that $h^0(\tilde{D}, N'_\varphi) = 3d + g - m - 1$ and that $h^1(\tilde{D}, N'_\varphi) = 0$. Hence Thm. 2.1 and Thm. 3.1 of [Ho1] give us a family $(\mathcal{C}_m, \mathcal{D}_m, \pi_m, \Phi_m)$ of morphisms to $\mathbb{P}^2_{(0:0:1)}$, $m \geq 1$, of dimension $3d + g - m - 1$ with the following properties (terminology see [Ho1] §1):

- it is complete at a point 0 of $\mathcal{D}_m$
- $\pi_m^{-1}(0) = \tilde{D}$ and $\Phi_m|_{\pi_m^{-1}(0)} = \varphi$
- the diagram

$$\begin{array}{ccccc}
\tilde{D} & \to & \mathcal{C}_m & \stackrel{\Phi_m}{\to} & \mathbb{P}^2_{(0:0:1)} \times \mathcal{D}_m \\
\downarrow & & \pi_m \downarrow & & \downarrow \\
\{0\} & \to & \mathcal{D}_m & = & \mathcal{D}_m
\end{array}$$

  is commutative

$\beta \circ \varphi$ is also unramified. As in the case $m \geq 1$ we get a complete family $(\mathcal{C}_0, \mathcal{D}_0, \pi_0, \Phi_0)$ of morphisms to $\mathbb{P}^2$ with analogous properties (in particular $\pi_0^{-1}(0) = \tilde{D}$ and $\Phi_0|_{\pi_0^{-1}(0)} = \beta \circ \varphi$).

As $\overline{V}_{d,g,0}$ is smooth at D we may assume $\mathcal{D}_0 \subseteq \overline{V}_{d,g,0}$: let $\tilde{V}_{d,g,0}$ denote the open subset of $\overline{V}_{d,g,0}$ whose closed points correspond to integral curves of geometric genus g, let $\tilde{C}_{d,g,0}$ be the normalization of the pull back of the family over $\overline{V}_{d,g,0}$ (note that this is a flat family whose fibers are smooth irreducible curves of genus g ([DiaHar2], Thm. (2.5)). Let π denote the morphism $\tilde{C}_{d,g,0} \to \tilde{V}_{d,g,0}$ and let Φ be the morphism of $\tilde{C}_{d,g,0}$ to $\mathbb{P}^2 \times \tilde{V}_{d,g,0}$. By the completeness of $\mathcal{D}_0$ we get a morphism from a complex analytic neighbourhood U of D in $\tilde{V}_{d,g,0}$ to $\mathcal{D}_0$ under which $(\mathcal{C}_0, \mathcal{D}_0, \pi_0, \Phi_0)$ pulls back to the restriction of $(\tilde{C}_{d,g,0}, \tilde{V}_{d,g,0}, \pi, \Phi)$. This morphism is injective between complex manifolds of the same dimension and therefore an isomorphism on its image.

We will show that $\mathcal{D}_m$ embeds into $\mathcal{D}_0$ which entails

Proposition 5.7. *$\overline{V}_{d,g,m}$ is smooth at points D, where D is an integral curve of geometric genus g with the property that all the branches of each of its singularities are smooth.*

Proof: First we argue why it would help us that $\mathcal{D}_m$ embeds into $\mathcal{D}_0$: let $\tilde{V}_{d,g,m}$ denote the open subset of $\overline{V}'_{d,g,m}$ whose closed points correspond to integral curves of geometric genus g, let $\tilde{C}_{d,g,m}$ be the normalization of the pull back of the family over $\overline{V}'_{d,g,m}$, let $\tilde{\pi}_m$ be the morphism $\tilde{C}_{d,g,m} \to \tilde{V}_{d,g,m}$ and let $\tilde{\Phi}_m$ be the morphism of $\tilde{C}_{d,g,m}$ to $\mathbb{P}^2_{(0:0:1)} \times \tilde{V}_{d,g,m}$. Pull $(\tilde{C}_{d,g,m}, \tilde{V}_{d,g,m}, \tilde{\pi}_m, \tilde{\Phi}_m)$ back to a desingularization X of $\tilde{V}_{d,g,m}$. For a preimage $x \in X$ of D we find a morphism of an open neighbourhood U_m of x to $\mathcal{D}_m$, such that the tuple restricted to U_m is equivalent to the pull back of $(\mathcal{C}_m, \mathcal{D}_m, \pi_m, \Phi_m)$. Therefore we have a commutative diagram

$$\begin{array}{ccc} U_m & \to & \tilde{V}_{d,g,m} \\ \downarrow & & \downarrow \\ \mathcal{D}_m & \to & \overline{V}_{d,g,0} \end{array}$$

and $\tilde{V}_{d,g,m} \subseteq \mathcal{D}_m$ holds in a neighboorhood of D. If $\iota : \mathcal{D}_m \to \tilde{V}_{d,g,0}$ would be an embedding at D, we were finished as they have the same dimension. But this holds, as the differential $\delta : T_{\mathcal{D}_m,0} \to T_{\mathcal{D}_0,0}$ is an inclusion (follows from the next lemma). $\square$

By [Ho1], Thm. 3.1 the characteristic maps $\tau_m : T_{\mathcal{D}_m,0} \to H^0(\tilde{D}, N_\varphi)$ and $\tau_0 : T_{\mathcal{D}_0,0} \to H^0(\tilde{D}, N_{\beta\circ\varphi})$ are isomorphisms (for the definition of τ see [Ho1] (1.3)). Let $\eta : H^0(\tilde{D}, N_\varphi) \to H^0(\tilde{D}, N_{\beta\circ\varphi})$ be the natural map.

Lemma 5.8. *The diagram*

$$\begin{array}{ccc} T_{\mathcal{D}_m,0} & \xrightarrow{\tau_m} & H^0(\tilde{D}, N_\varphi) \\ \delta \downarrow & & \downarrow \eta \\ T_{\mathcal{D}_0,0} & \xrightarrow{\tau_0} & H^0(\tilde{D}, N_{\beta\circ\varphi}) \end{array}$$

is commutative and η is an inclusion.

Remark 5.9. *The proof shows the functoriality of the characteristic map.*

Proof: We infer the injectivity of η from the injectivity of the sheaf morphism $N_\varphi \to N_{\beta\circ\varphi}$. It suffices to check this on an open covering of $\tilde{D}$. We do that for one affine open subset U to give the idea. Locally on $\mathbb{P}^2_{(0:0:1)}$ the blowing up is given by $\beta(u,v) := (v, uv)$. That is $\theta_{\mathbb{P}^2_{(0:0:1)}} \to \beta^*\theta_{\mathbb{P}^2}$ is given by

$$\begin{array}{ccc} \frac{\partial}{\partial u} & \mapsto & v \cdot \frac{\partial}{\partial y} \\ \frac{\partial}{\partial v} & \mapsto & \frac{\partial}{\partial x} + u \cdot \frac{\partial}{\partial y} \end{array}.$$

As u and v pull back to nonzerodivisors in $\mathcal{O}_{\tilde{D}}(\varphi^{-1}(U))$ the assertion follows.

Before we start to proof the commutativity, we introduce some more notation. We have a commutative diagram

$$\begin{array}{ccccc} \mathcal{D}_m & \longleftarrow & \mathcal{C}_m & \xrightarrow{\Phi_m} & \mathbb{P}^2_{(0:0:1)} \times \mathcal{D}_m \\ h\downarrow & \square & H\downarrow & & \downarrow \beta \times h \\ \mathcal{D}_0 & \longleftarrow & \mathcal{C}_0 & \xrightarrow{\Phi_0} & \mathbb{P}^2 \times \mathcal{D}_0 \end{array},$$

where h and H are the morphisms induced by the completeness of $\mathcal{D}_0$. As the left hand square is cartesian, we may choose local coordinates x_i on $\mathcal{D}_m$, y_j on $\mathcal{D}_0$, y_j, z_k on $\mathcal{C}_0$ and x_i, z_k on $\mathcal{C}_m$ such that

$$H((x_i, z_k)_{i,k}) = (H_j((x_i, z_k)_{i,k}), H_k((x_i, z_k))_{i,k})_{j,k} = (h_j((x_i, z_k)_{i,k}), z_k)_{j,k}$$

holds. Further we choose local coordinates v_μ on $\mathbb{P}^2_{(0:0:1)}$ and w_ν on $\mathbb{P}^2$. Then we get

$$\tau_0 \circ \delta(\frac{\partial}{\partial x_i}) = \sum_j \frac{\partial h_j}{\partial x_i}(0)\, \tau_0(\frac{\partial}{\partial y_j}),$$

where

$$\tau_0(\frac{\partial}{\partial y_j}) = \sum_\nu \frac{\partial \Phi_{0,\nu}}{\partial y_j} \circ i_0 \cdot \frac{\partial}{\partial w_\nu}$$

by definition (i_0 denotes the inclusion of $\tilde{D}$ into $\mathcal{C}_0$). On the other hand we have

$$\eta \circ \tau_m(\frac{\partial}{\partial x_i}) = \sum_\mu \frac{\partial \Phi_{m,\mu}}{\partial x_i} \circ i_m \cdot \eta(\frac{\partial}{\partial v_\mu})$$

with

$$\eta(\frac{\partial}{\partial v_\mu}) = \sum_\nu \frac{\partial \beta_\nu}{\partial v_\mu} \circ \varphi \cdot \frac{\partial}{\partial w_\nu}$$

by definition (i_m denotes the inclusion of $\tilde{D}$ into $\mathcal{C}_m$). Hence $\tau_0 \circ \delta = \eta \circ \tau_m$ is equivalent to

$$\sum_j \frac{\partial h_j}{\partial x_i}(0) \cdot \frac{\partial \Phi_{0,\nu}}{\partial y_j} \circ i_0 = \sum_\mu \frac{\partial \Phi_{m,\mu}}{\partial x_i} \circ i_m \cdot \frac{\partial \beta_\nu}{\partial v_\mu} \circ \varphi$$

for all i and all ν. This results from the following computation:

$$\begin{aligned}
\sum_j \tfrac{\partial h_j}{\partial x_i}(0) \cdot \tfrac{\partial \Phi_{0,\nu}}{\partial y_j} \circ i_0 &= \sum_j \tfrac{\partial h_j}{\partial x_i}(0) \cdot \tfrac{\partial \Phi_{0,\nu}}{\partial y_j} \circ H \circ i_m \\
&= \sum_j \tfrac{\partial H_j}{\partial x_i}(0) \cdot \tfrac{\partial \Phi_{0,\nu}}{\partial y_j} \circ H \circ i_m \\
&= \tfrac{\partial(\Phi_{0,\nu} \circ H)}{\partial x_i} \circ i_m \\
&= \tfrac{\partial((\beta \times h)_\nu \circ \Phi_m)}{\partial x_i} \circ i_m \\
&= \sum_\mu \tfrac{\partial \Phi_{m,\mu}}{\partial x_i} \circ i_m \cdot \tfrac{\partial \beta_\nu}{\partial v_\mu} \circ \Phi_m \circ i_m \\
&= \sum_\mu \tfrac{\partial \Phi_{m,\mu}}{\partial x_i} \circ i_m \cdot \tfrac{\partial \beta_\nu}{\partial v_\mu} \circ \varphi
\end{aligned}$$

□

Remark 5.10. *Note that we have proved Theorem 3 in this section: 1. is Proposition 5.7, 2. and 3. follow from Proposition 5.2, 4. is Remark 5.5 and 5. is Remark 5.6.*

5.2 The normalization of $\overline{V}_{d,g,m}$

In this section we give the proof of Theorem 4. First we introduce a scheme $\Sigma_{d,g,m}$ which is, as we will see, almost the normalization of $\overline{V}_{d,g,m}$. Let $\sigma : V_{d,g,o} \to \mathrm{Hilb}^n(\mathbb{P}^2) \to \mathrm{Sym}^n(\mathbb{P}^2)$ be the morphism that assigns to $C \in V_{d,g,0}$ the cycle $p_1 + \cdots + p_n$ of its $n := \frac{(d-1)(d-2)}{2} - g$ nodes. More precisely: let

$$F(x, y, z, a_{ij}) := \sum_{0 \leq i,j,i+j \leq d} a_{ij} x^i y^j z^{d-i-j}$$

be the polynomial describing the universal curve over $\mathrm{Hilb}(d)$. Then the pull back of the family $V(F, F_x, F_y, F_z)$ to $V_{d,g,o}$ is a flat family of zero dimensional length n schemes on $\mathbb{P}^2$. Hence we get a unique morphism to the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{P}^2)$ for such families. The morphism to the n-fold symmetric product of $\mathbb{P}^2$ is canonical. We define $\Sigma_{d,g} := \Sigma_{d,g,0}$ to be the closure of the graph of σ in $\overline{V}_{d,g,0} \times \mathrm{Sym}^n(\mathbb{P}^2)$. Now $\Sigma_{d,g,m}$ shall be given by the requirement that

$$\begin{array}{ccc} \Sigma_{d,g,m} & \hookrightarrow & \overline{V}_{d,g,m} \times \mathrm{Sym}^n(\mathbb{P}^2) \\ \downarrow & \Box & \downarrow \\ \Sigma_{d,g} & \hookrightarrow & \overline{V}_{d,g} \times \mathrm{Sym}^n(\mathbb{P}^2) \end{array}$$

is a cartesian diagram. Let $V'_{d,g,m} \subseteq \overline{V}'_{d,g,m}$ be the locus of integral curves of geometric genus g with only nodes. In analogy to the construction of $\Sigma_{d,g}$ we define $\Sigma'_{d,g,m}$ to be the closure of the graph of the map $V'_{d,g,m} \to \mathrm{Hilb}^n(\mathbb{P}^2_{(0:0:1)}) \to \mathrm{Sym}^n(\mathbb{P}^2_{(0:0:1)})$ in $\overline{V}'_{d,g,m} \times \mathrm{Sym}^n(\mathbb{P}^2_{(0:0:1)})$.

Lemma 5.11. *We have a commutative diagram*

$$\begin{array}{ccc} \Sigma'_{d,g,m} & \hookrightarrow & \overline{V}'_{d,g,m} \times \mathrm{Sym}^n(\mathbb{P}^2_{(0:0:1)}) \\ \varsigma \downarrow & & \downarrow \\ \Sigma_{d,g,m} & \hookrightarrow & \overline{V}_{d,g,m} \times \mathrm{Sym}^n(\mathbb{P}^2) \end{array}$$

and there exists an open subset U of $\Sigma_{d,g,m}$ containing all strata of $U_{d,g,m}$ such that $\varsigma^{-1}(U) \to U$ is an isomorphism.

Proof: We define U to be the union of two open sets: The first one is the intersection of $\Sigma_{d,g,m}$ with $\overline{V}_{d,g,m} \times \mathrm{Sym}^n(\mathbb{P}^2 \setminus \{(0:0:1)\})$. Here ς restricts to an isomorphism as

$$\overline{V}'_{d,g,m} \times \mathrm{Sym}^n(\mathbb{P}^2_{(0:0:1)} \setminus E) \to \overline{V}_{d,g,m} \times \mathrm{Sym}^n(\mathbb{P}^2 \setminus \{(0:0:1)\})$$

is an isomorphism. Secondly we take the preimage U_2 in $\Sigma_{d,g,m}$ of the locus $W_2 \subseteq \overline{V}_{d,g,m}$ of integral curves of geometric genus g all of whose singularities have only smooth branches. Let W'_2 be the corresponding locus in $\overline{V}'_{d,g,m}$. Then we have a commutative diagram

$$\begin{array}{ccc} \varsigma^{-1}(U_2) & \to & W'_2 \\ \varsigma \downarrow & & \downarrow \\ U_2 & \to & W_2 \end{array}$$

all of whose arrows are isomorphisms: the horizontal ones, because they are bijective onto smooth varieties (Prop. 5.7), the one on the right hand side by Lemma 5.1.

By the principle of conservation of number ([dJoPfi], Thm. 6.4.3) the only strata of $U_{d,g,m}$ not in the first set are the one with ordinary $(m+1)$-fold point in $(0:0:1)$ and the one where the singularity in $(0:0:1)$ has m smooth branches, two of them having first order contact. But they are both contained in U_2. $\square$

Next, we want to show, that $\Sigma_{d,g,m}$ is the normalization of $\overline{V}_{d,g,m}$ at points D, where D is integral of geometric genus g with ordinary m-fold point at $(0:0:1)$, one cusp and only nodes apart from that. This follows with Lemma 5.1 and Lemma 5.11 from

Proposition 5.12. *Let $D \in \overline{V}'_{d,g,m}$ be an integral curve of geometric genus g with one cusp and only nodes otherwise. Then $\Sigma'_{d,g,m}$ is the normalization of $\overline{V}'_{d,g,m}$ at D and is smooth there.*

Proof: We work in the complex analytic category. Let $\mathcal{C}'_{d,g,m} \subseteq \overline{V}'_{d,g,m} \times \mathbb{P}^2_{(0:0:1)}$ be the family over $\overline{V}'_{d,g,m}$. Without loss of generality, we may assume that the cusp sits at $(0,0) \in \mathbb{A}^2 \subseteq \mathbb{P}^2_{(0:0:1)}$. The theorem of [KasSchl] gives us open neighbourhoods U of D and $A \subseteq \mathbb{A}^2 \subseteq \mathbb{P}^2_{(0:0:1)}$ of the cusp, a morphism $\varphi : U \to V(4a^3 + 27b^2)$ (which is smooth, recall section 5.1), and an isomorphism $U \times A \to U \times \mathbb{A}^2$ onto its image, which respects the embedded families:

$$\begin{array}{ccccc}
V & \hookrightarrow & V(4a^3+27b^2) \times \mathbb{A}^2 & \to & V(4a^3+27b^2) \\
\uparrow & \square & \uparrow & \square & \uparrow \\
\varphi^* V & \hookrightarrow & U \times \mathbb{A}^2 & \to & U \\
\uparrow & \square & \uparrow & & \\
\mathcal{C}'_{d,g,m} \cap (U \times A) & \hookrightarrow & U \times A & &
\end{array}$$

(where $V := V(4a^3 + 27b^2, y^2 - x^3 + ax + b)$). The crucial observation is that the variety

$$(*) \quad \begin{aligned} &\{((a,b),(x_0,y_0)) : V(y^2 - x^3 + ax + b) \quad \text{is singular at} \quad (x_0,y_0)\} \\ &= \{((a,b),(x_0,y_0)) : y_0 = 0, a = -3x_0^2, b = 2x_0^3\} \cong \mathbb{A}^1 \end{aligned}$$

is just the normalization of $V(4a^3 + 27b^2)$. Therefore the variety U^* in the commutative diagram

$$\begin{array}{ccccc}
U^* & \hookrightarrow & U \times A & \to & U \\
\downarrow & \square & \downarrow & & \downarrow \varphi \\
\mathbb{A}^1 & \hookrightarrow & V(4a^3+27b^2) \times \mathbb{A}^2 & \to & V(4a^3+27b^2)
\end{array}$$

is smooth because φ is. Furthermore it is isomorphic to the normalization $\tilde{U}$ of U: it is normal, so there is a unique morphism to $U^* \to \tilde{U}$. Conversely, as $\tilde{U}$ is normal, we get a unique morphism from $\tilde{U}$ to $\mathbb{A}^1$ and hence to U^*. By the universal property of normalization, these have to be isomorphisms.

Let D' be the preimage of D in $\Sigma'_{d,g,m}$. Define $\Sigma^*_{d,g,m}$ by the cartesian diagram

$$\begin{array}{ccc} \Sigma^*_{d,g,m} & \hookrightarrow & U \times ((\mathbb{A}^2)^n \setminus \Delta) \\ \downarrow & \square & \downarrow \\ \Sigma'_{d,g,m} & \hookrightarrow & \overline{V}'_{d,g,m} \times \mathrm{Sym}^n(\mathbb{P}^2_{(0:0:1)}) \end{array},$$

where $\Delta := \{(x_1, \ldots, x_n) \in (\mathbb{A}^2)^n | \exists\, i \neq j : x_i = x_j\}$ is the coincidence locus. We choose a neighbourhood W of a preimage of D' in such a way, that $W \to \Sigma'_{d,g,m}$ is an isomorphism onto its image (note that the arrow on the right hand is étale). Let $\pi : W \to U \times \mathbb{A}^2$ be the projection, where the points of $\mathbb{A}^2$ correspond to the position of the cusp. π factorizes over U^*, as we may choose W small enough. But this map is injective between varieties of the same dimension, so it is an isomorphism onto its image, as U^* is smooth. $\square$

From the subsequent proposition, it follows in a completely analogous way, that $\Sigma_{d,g,m}$ is the normalization of $\overline{V}_{d,g,m}$ at points corresponding to curves of geometric genus $g-1$, which have an ordinary m-fold point at $(0:0:1)$ and at most nodes elsewhere and which are either integral or are reduced with two irreducible components.

Proposition 5.13. *Let $D \in \overline{V}'_{d,g,m}$ be of geometric genus $g-1$ with only nodes and such that it is either integral or reduced with two irreducible components. Then $\Sigma'_{d,g,m}$ is the normalization of $\overline{V}'_{d,g,m}$ at D and is smooth there.*

Proof: This is essentially the same proof as in Prop. 5.12. Without loss of generality, we may assume that all nodes lie in $\mathbb{A}^2 \cong \mathbb{P}^2_{(0:0:1)} \setminus E$. The theorem of [KasSchl] gives us open neighbourhoods U_i of D and open neighbourhoods $A_i \subseteq \mathbb{A}^2 \subseteq \mathbb{P}^2_{(0:0:1)}$ of the $n+1$ nodes, morphisms $\varphi_i : U_i \to \mathrm{Def}_i := \mathrm{Def}_{(V(xy),(0,0))}$, and isomorphisms $\alpha_i : U_i \times A_i \to U_i \times \mathbb{A}^2$ onto their images, which respect the embedded families. Let $U \subseteq \cap_i U_i$ be an open neighbourhood of D.

$$\begin{array}{ccccc} U \times A_i & \stackrel{\alpha_i}{\to} & U \times \mathbb{A}^2 & \stackrel{\varphi_i \times \mathrm{id}_{\mathbb{A}^2}}{\longrightarrow} & \mathrm{Def}_i \times \mathbb{A}^2 \\ & & \downarrow & & \downarrow \\ & & U & \stackrel{\varphi_i}{\to} & \mathrm{Def}_i \end{array}$$

Recall from section 5.1, that

$$\Phi := (\varphi_1, \ldots, \varphi_k) : U \to V(t_i t_j | i \neq j \wedge i, j \in I) \subseteq \prod_{i \in I} \mathrm{Def}_i$$

is smooth, where $I := \{1, \ldots, k\}$ is an index set enumerating all nodes in the case that D is integral and enumerating those in $D_1 \cap D_2$ if $D = D_1 \cup D_2$. Let D' be a preimage of D in $\Sigma'_{d,g,m}$. We find an open neighbourhood $W \subseteq U \times ((\mathbb{A}^2)^n \setminus \Delta)$ of a preimage of D', which maps isomorphically onto its image in $\Sigma'_{d,g,m}$. We may assume that W is contained in $U \times \prod_{i \in N \setminus \{j\}} A_i$, where N is an index set for all nodes of D and where $j \in I$ is the index of the node that does not occur in the cycle of D'. Observe that $p_i \circ \alpha_i(W) = \{(0,0)\}$, where $p_i : U_i \times \mathbb{A}^2 \to \mathbb{A}^2$ is the projection onto the second factor. Let $\pi : W \to U$ be the projection onto the first factor. We infer that $\varphi_i \circ \pi = 0$ for $i \neq j$. Consequently π factorizes over $V_j := \Phi^{-1}(V(t_i | i \neq j)$. $\Phi|_{V_j}$ is smooth, as Φ is. Therefore V_j is smooth and hence W is, as π is an injective morphism between varieties of the same dimension. $\square$

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