Response of Josephson junction arrays near the quantum phase transition

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We study the superconductor-insulator transition of a 2-dimensional Bose-Hubbard model, considering as a specific example, an array of Josephson junctions. Within a coarse-graining approach we derive an effective free-energy functional from which we determine the phase diagram. At zero temperature it consists of a superconducting phase and Mott-insulating lobes. The phase boundaries of some of these lobes display reentrant behaviour as a function of temperature. Next, we evaluate the electromagnetic response functions of the system. The real part of the longitudinal conductivity is characterized by an excitation gap, whereas the imaginary part describes a capacitor. In an ideal system, under certain conditions a universal conductance is found at the transition. If we add low frequency dissipation to the model a different value of the universal conductance is found, but still it is independent of the strength of the dissipation. Qualitatively differing results are obtained for frustrated and unfrustrated systems. We also discuss the Hall conductance of the system.

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I. INTRODUCTION

Interacting Bose systems have attracted considerable interest in the past few years. Several predictions[1–3] on the superconductor-insulator (S-I) transition as a function of the system parameters, disorder, chemical potential, and magnetic field have been verified in experiments on granular films of superconducting material[4,5]. More recently the same transition has been studied in fabricated, regular arrays of Josephson junctions[6–8], where the parameters can be controlled, and disorder is less important. In high quality junction arrays the charges on islands change only in discrete quanta due to tunneling of Cooper pairs. The tunneling of single electrons is frozen out at low temperatures. This makes junction arrays an ideal experimental system to test the concepts of the Bose-Hubbard model and of the S-I transition. The Josephson coupling E_J of the junction array is equivalent to the hopping term t in the Bose-Hubbard model; the inverse capacitance matrix defines the charging energy scale E_C and describes the interaction of particles; a gate voltage V_x applied between the ground plane and the array replaces the chemical potential μ.

The S-I transition of a Josephson junction array (or Bose-Hubbard model) can be described by a Ginzburg-Landau (G-L) free-energy functional, derived from the original Hamiltonian within the so-called coarse-graining approach[9,10]. The coefficients depend on the ratio of Josephson coupling E_J and charging energy E_C, and also on the value of a gate voltage V_x applied between the ground plane and the array. This G-L theory, in contrast to the standard time-dependent Ginzburg-Landau description of (bulk) superconductors, does not describe a relaxation process. Rather the second order time derivative is essential, leading to nontrivial response functions[11]. For instance, a universal conductance may appear at the transition[1], instead of the fluctuation conductivity found in the framework of the standard G-L theory of bulk superconductors.

In the present paper we extend our earlier results[10,11] (see also[12]) to finite temperatures and consider the effect of low frequency dissipation on the response functions. At zero temperature the phase diagram as a function of E_J/E_C and V_x consists of a superconducting phase and Mott-insulating lobes. A finite value of V_x, in general, breaks particle-hole symmetry. The dependence of the system properties on V_x is periodic and in many respects reminiscent of the dependence of classical arrays on the magnetic frustration f. We, therefore, denote the former by 'charge frustration'. At finite temperature the phase boundary of some of the Mott-insulating lobes is reentrant. At higher temperatures the dependence on the applied gate voltage is washed out and the phase boundary approaches the Kosterlitz-Thouless result.

We also present further properties of the response functions as a function of temperature, dissipation, magnetic frustration and applied gate voltage (charge frustration). At zero temperature, in the insulating phase the real part of the conductivity is governed by a gap in the excitation spectrum, equal to the energy for the creation of particle-hole pairs. The imaginary part is that of a capacitor. In unfrustrated (magnetic and charge) arrays (with particle-hole symmetry) the energy gap and the inverse capacitance vanish continuously at the phase transition, implying a vanishing Coulomb gap in the I-V curves. At the transition the conductance takes a universal value. For general frustration (magnetic or charge) the excitation energy and Coulomb gap remain finite up to the transition and jump discontinuously to zero as the superconducting phase is entered. At finite temperature, free charge carriers (particle- and hole-like) are activated. Their excitation energies and densities depend on the applied gate voltage. We find a nonzero Hall conductance if the densities of particle-and hole-like excitations are different. The lower of their excitation energies vanishes continuously at the transition; characterized by a dynamic critical exponent $z = 1$ or $z = 2$ for unfrustrated or frustrated systems, respectively. The presence of the free carriers turns the system into a perfect conductor, i.e. the conductivity has a Drude form with infinite scattering time. Low frequency dissipation 'regularizes' the perfect conductivity at finite temperature and leads to a superconducting-resistive phase transition. It also reduces the value of the universal conductance of the $T = 0$ transition. Remarkably, this new universal conductance is independent of the strength of dissipation.

In the following section we briefly outline the coarse-graining approach to derive the G-L free-energy functional. We present the phase diagram of the Josephson junction array with finite range interaction of the charges, extending our earlier work to finite temperatures. In section 3 we study the conductivity for several distinct cases, including the effect of low frequency dissipation. We conclude with a discussion.

II. THE PHASE DIAGRAM

We consider a square array of Josephson junctions. In terms of the excess Cooper pair charges (2e) on the islands Q_i and the phases φ_i of the superconducting order parameters it is described by the Hamiltonian

$$H = \frac{1}{2}\sum_{i,j}(Q_i - Q_x)C^{-1}_{ij}(Q_j - Q_x) - \sum_{<i,j>} E_J \cos(\varphi_i - \varphi_j - A_{ij}) \ . \qquad (1)$$

The scale for the Josephson coupling is E_J. The Coulomb interaction of the charges on the islands is described by an *inverse* capacitance matrix C^{-1}_{ij}. The capacitance matrix contains in the diagonal the self capacitance of the islands C_0 and nearest neighbor terms describing the junction capacitances C_1. Hence $C_{ii} = C_0 + 4C_1, C_{ij} = -C_1$ for i and j nearest neighbors, and $C_{ij} = 0$ otherwise. A characteristic scale for the interaction is set by the charging energy $E_C = \frac{1}{2}e^2 C^{-1}_{ii}$.

Magnetic frustration f is introduced by a vector potential

$$A_{ij} = \frac{2\pi}{\Phi_0}\int_i^j \vec{A}\cdot d\vec{l} \ , \ \ \sum\!\!\!\!\!\!\bigcirc A_{ij} = 2\pi f \ . \qquad (2)$$

We also allow for a homogeneous 'charge frustration' or 'external' charges $Q_{x,i} = Q_x$ on the islands. Their value can be controlled by an overall gate voltage V_x applied between the array and the substrate. In general this introduces a term $V_x \sum_i Q_i$ into the Hamiltonian, where $\sum_i Q_i$ is the net charge which has traversed the voltage source. Clearly this corresponds to $Q_x = C_0 V_x$ in eq. (1).

The Hamiltonian (1) is equivalent to that of a Bose-Hubbard model, provided that the mean number of bosons per site is large and certain amplitude fluctuations can be neglected[10]. The Josephson coupling term corresponds to the hopping term. The inverse capacitance matrix C^{-1}_{ij} describes the interaction, which in a general has a finite range. The applied gate voltage V_x corresponds to the chemical potential for the bosons.

If $C_0 = 0$ the charges interact logarithmically, as do the vortex excitations contained in (1). Then duality arguments imply[1,13] that at the superconductor-insulator transition the resistance of the array is given by the quantum value $R_Q = h/4e^2 = 6.45k\Omega$. If $C_0 \neq 0$, the case we consider here, the duality is broken and the resistance at the transition in general will be different.

In order to study the model further we make use of the so-called 'coarse-graining' approximation developed by Doniach[9]. The essence of this approach is to introduce a complex order parameter field ψ, whose expectation value is proportional to that of $\exp(i\varphi)$. As long as ψ is small, i.e. close to the onset of phase coherence, the Hamiltonian (1) reduces to an effective Ginzburg-Landau theory. The derivation has been presented before[10] and we only state the result. The partition function

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\{-F[\bar{\psi},\psi]\} \qquad (3)$$

is governed by a G-L functional

$$F[\psi] = \int_0^\beta d\tau \int d^2r \Big\{ \epsilon \mid \psi \mid^2 + \kappa \mid \psi \mid^4 + \gamma \mid (\vec{\nabla} + \frac{2\pi i}{\Phi_0}\vec{A})\psi \mid^2 + \lambda\bar{\psi}\partial_\tau\psi + \zeta \mid \partial_\tau\psi \mid^2 \Big\}. \qquad (4)$$

The coefficients depend on the frequency dependent phase-phase correlation function[10]

$$g(\omega_\mu, q_x) = \frac{8E_C}{Z_0}\sum_{\{q_i\}} \frac{\exp[-2e^2\beta\sum_{i,j}(q_i - q_x)C^{-1}_{ij}(q_j - q_x)]}{[4E_C]^2 - [4e^2\sum_j C^{-1}_{0j}(q_j - q_x) - i\omega_\mu]^2}$$

$$\text{where} \quad Z_0 = \sum_{\{q_i\}} \exp[-2e^2\beta\sum_{i,j}(q_i - q_x)C^{-1}_{ij}(q_j - q_x)] \qquad (5)$$

as follows

$$\begin{aligned} \gamma &= g^{-1}(0,q_x)/8E_C = (\epsilon + E_J/E_C)/4 \\ \epsilon &= g^{-1}(0,q_x)/2E_C - E_J/E_C \\ \lambda &= i\partial_{\omega_\mu} g^{-1}(\omega_\mu, q_x)\big|_{\omega_\mu=0}/2E_C \\ \zeta &= \partial^2_{\omega_\mu} g^{-1}(\omega_\mu, q_x)\big|_{\omega_\mu=0}/4E_C \end{aligned} \qquad (6)$$

The sums in (5) run over all integer charge configurations $q_i = Q_i/2e = 0, \pm1, ...$ on each site. Since the properties of the system are periodic in $q_x = Q_x/2e$ with period 1 we can restrict in the explicit formulas below q_x to the range $-1/2 \le q_x \le 1/2$. Within the coarse graining approach the coefficient of the fourth order term in the G-L equation turns negative if the interaction is sufficiently long range[14]. This may indicate a tendency of the system to show a first oder transition in this limit, or it reflects simply a weakness of the present approach. In the following we will consider the case where the interaction decays sufficiently fast to avoid this problem. For on-site interactions and $q_x = 0$ we have $\kappa = 7E_J^2/32E_C^3$.

Notice that the correlation function $g(\omega_\mu, q_x)$ depends only on the combination $(2eV_x + i\omega_\mu)$ as required by gauge invariance[15]. The time derivatives in Eq. (4) arise if we expand in ω_μ. The coefficient of the first order time derivatives λ is nonzero only for $q_x \neq 0$. It multiplies a time derivative with respect to the imaginary time τ and must *not* be interpreted as a dissipative term. The coefficient λ vanishes on the lines in the phase diagram where the system exhibits particle-hole symmetry. If the particle-hole symmetry is broken, and $\lambda \neq 0$ a nonvanishing Magnus force on vortices[8,16] and a finite Hall conductance may arise.

For on-site interaction only (self-capacitance limit, $C_0 \gg C_1$) we find at $T = 0$

$$\begin{aligned} \epsilon &= 1 - 4q_x^2 - E_J/E_C \\ \lambda &= q_x/E_C \\ \zeta &= 1/16E_C^2 \end{aligned} \tag{7}$$

In this case $g^{-1}(\omega_\mu, q_x)$ is a 2^{nd} order polynomial in ω_μ, and the frequency expansion of the G-L functional is exact.

The mean-field phase boundary is given by the condition $\epsilon_f \equiv \epsilon + \gamma 2\pi f = 0$. For on-site interaction it reduces to

$$\left.\frac{E_J}{E_C}\right|_{cr} = (1 - 4q_x^2)(1 + \frac{\pi}{2}f) \tag{8}$$

for $-1/2 \le q_x \le 1/2$ and periodic beyond. In Fig. 1a we show for this limit the $T = 0$ phase boundary between the insulating and the superconducting phase as a function of q_x and E_J/E_C. Mott insulating lobes for small E_J/E_C are separated by regions of superconducting phase at larger E_J/E_C. A similar lobe structure had been predicted before for the Bose-Hubbard model[2]; in the Josephson junction array it is perfectly periodic in q_x with period 1.

The Mott insulating phase inside the lobe is characterized by an excitation gap for adding or removing bosons (Cooper pairs). From the G-L functional (4) we find at $k = 0$ two excitation frequencies $\omega^\pm = (\pm\lambda + \sqrt{\lambda^2 + 4\zeta\epsilon})/2\zeta$. In the limit $E_J = 0$ for on-site interaction this reduces simply to $\omega^\pm = 4E_C(1 \pm 2q_x)$. It is reassuring to see that in the limit $E_J = 0$, where the excitation gap can be determined directly from the electrostatic part in the Hamiltonian (1), the result obtained within the coarse graining approximation coincides with the exact result. We, furthermore, notice that a hopping process in the ground state, which creates a pair of excitations, costs the sum of the single particle excitation energies $\Sigma = \omega^+ + \omega^-$. The energy Σ is independent of q_x. Throughout the lobes in Fig. 1a it is given by $\Sigma = 8E_C\sqrt{1 - E_J/E_C}$.

From the excitation spectrum we can obtain the dynamic critical exponent z. On the particle-hole symmetry lines, $\lambda = 0$, the excitation energies vanish proportional to $\sqrt{\epsilon}$. Hence the product of critical exponents is equal to $z\nu = 1/2$. Within the mean field approximation used here $\nu = 1/2$, which implies $z = 1$. The transition is known to be in the universality class of the 3D XY model and better estimates for the critical exponents exist. It turns out that $z = 1$ and $\nu \sim 0.67$. In the general case (where $\lambda \neq 0$) the lowest excitation energy vanishes linearly in ϵ. Hence, in the mean field approximation we have $z = 2$. In ref.[2] it was argued that this is the exact result, since in frustrated systems the transition is described by mean field exponents.

The nature of the phase transition in the symmetry points between the lobes differs from the rest. At $q_x = \frac{1}{2}$ and $E_J = 0$ the phase transition is first order, and the free-energy functional (4) ceases to be a proper description. At these points a mapping onto a spin model provides more insight[17].

For a general interaction, i.e. a general capacitance matrix, the phase diagram acquires more structure. For definiteness let us consider on-site and a weaker nearest-neighbor (n.n.) interaction, i.e. the *inverse* capacitance matrix is restricted to diagonal and n.n. terms. We, furthermore, restrict $C_{ij}^{-1} \le C_{ii}^{-1}/4$, thus avoiding unphysical instabilities. In this case the relevant ground state charge configurations are i) the state in which each island has an equal number of bosons and ii) the two degenerate 'checkerboard' configurations where neighboring sites are occupied with n or n+1 bosons, respectively. Accordingly the phase diagram consists of two types of insulating lobes, those with homogeneous charge distributions centered around integer values of q_x, and checkerboard configurations centered around half-integer values of q_x. If we define $W = 1 + 4C_{01}^{-1}/C_{00}^{-1}$ the coefficients that describe an 'integer lobe' are

$$\begin{aligned} \epsilon &= 1 - 4W^2q_x^2 - E_J/E_C \\ \lambda &= Wq_x/E_C \\ \zeta &= 1/16E_C^2 \end{aligned} \tag{9}$$

The phase boundary of integer lobe is limited by $E_J = E_C(1 - 4W^2q_x^2)(1 + \frac{\pi}{2}f)$. The analogue for the 'half-integer lobe' is

$$\begin{aligned} \epsilon &= 2b_+b_-/(b_+ + b_-) - E_J/E_C \\ \lambda &= \frac{2}{E_C}\frac{h_+b_-^2 + h_-b_+^2}{(b_+ + b_-)^2} \\ \zeta &= \frac{1}{8E_C}\left[1 - \frac{2b_+b_-}{(b_+ + b_-)^2} + \frac{(h_+b_- - h_-b_+)^2}{(b_+ + b_-)^3}\right] \end{aligned}$$

where $b_+ = [1 - 4h_+^2]$, $b_- = [1 - 4h_-^2]$, $h_+ = [1 - Wq_x]$

and $h_- = [W(1-q_x)-1]$. The half-integer lobe extends at $q_x = \frac{1}{2}$ to $E_J = E_C[1-4(1-W/2)^2]$. An example for the phase diagram with on-site and nearest-neighbor interaction is shown in Fig. 1b. For a more general capacitance matrix (and hence longer range interaction) the lobe structure is more complicated[10]. If we include next nearest neighbour interactions the possibility for a supersolid arises[18,17].

At finite, but low temperatures charge fluctuations are suppressed exponentially. As a result the critical value of $(E_J/E_C)_{cr}$ at $q_x = 0$ depends only weakly on T[3]. For on-site interaction the leading temperature dependence of the phase boundary follows from

$$\left(\frac{E_J}{E_C}\right)_{cr} = 1 + \frac{8}{3} e^{-4E_C/T} \ , \tag{10}$$

For a complete picture we evaluated the correlation function Eq.(5) numerically, and for finite range interaction employing Monte-Carlo techniques. For short range interactions between the charges it is sufficient to study small system sizes. Numerical problems arise due to the nonanalytical behaviour of the correlation function at certain values of q_x. The phase boundary for different temperatures is shown in Fig. 1a for on-site interactions and in Fig. 1b for on-site and nearest neighbour interactions. As is clear from Fig.1b and 1c the phase boundary near $q_x = \frac{1}{2}$, i.e. near the tips of the half-integer lobes is reentrant. In contrast the phase boundary of the lobes at integer q_x is not reentrant. This difference may be due to domain walls between the two equivalent checkerboard charge configurations of the half-integer lobes, which may influence the transition.

Several papers have dealt with the question whether the phase boundary is reentrant or not. In some cases reentrance appeared to be related to approximations used in the calculation[19], in others it appeared to depend on the range of the interaction[3]. However, the dependence on q_x and related, the difference between integer and noninteger lobes had not been realized before.

Similarly as the phase boundary the coefficients in the G-L functional (4) depend on the temperature on a scale set by E_C (except in the regime of q_x where the $T = 0$ phase boundary extends to $E_J = 0$). Below we will find that the reponse functions are temperature dependent on smaller energy scales. Restricting our attention to low enough temperatures we, therefore, can take the coefficients of the G-L equation to be constants.

III. CONDUCTIVITY NEAR THE TRANSITION

From the G-L model (4) we can evaluate explicitly the frequency dependent response to an electromagnetic field in the different phases. This sheds light on the origin of the universal conductance predicted at the transition[1], and we obtain further qualitative and quantitative results. In order to do so we study the imaginary time correlation function

$$\sigma_{\mu\nu} = \frac{\hbar}{\omega_\nu} \int d^2r \int d\tau \frac{\delta^2 \ln Z}{\delta A_\mu(\tau,\vec{r})\delta A_\nu(0)} e^{i\omega_\nu \tau + i\vec{q}\cdot\vec{r}} \ , \tag{11}$$

where Z is the partition function (3). The variational derivative yields

$$\sigma_{\mu\nu}(i\omega_\nu,\vec{q}) = \frac{\pi}{R_Q\omega_\nu}\Big[4\gamma\langle\bar{\psi}(0)\psi(0)\rangle\delta_{\mu\nu}$$
$$-2\int d^2r \int_0^\beta d\tau \langle J_\mu(\tau,\vec{r})J_\nu(0)\rangle e^{i\omega_\nu\tau + i\vec{q}\cdot\vec{r}}\Big], \tag{12}$$

where the current is

$$J_\mu(\tau,\vec{r}) = 4\gamma\left[\frac{1}{2i}(\bar{\psi}\nabla_\mu\psi - \psi\nabla_\mu\bar{\psi}) - \frac{2\pi}{\Phi_0}A_\mu \mid \psi \mid^2\right] . \tag{13}$$

We consider first the insulating phase ($\epsilon > 0$) in zero magnetic field. In this case the transverse component $\sigma_{xy}(i\omega_\nu, \vec{q} = 0) \equiv \sigma_H(i\omega_\nu)$ (the Hall conductivity) vanishes by symmetry. The longitudinal component $\sigma_{xx}(i\omega_\nu, \vec{q} = 0) \equiv \sigma(i\omega_\nu)$ becomes, in Gaussian approximation, after a partial integration of the diamagnetic part

$$\sigma(i\omega_\nu) = \frac{1}{8R_Q\omega_\nu}\int_0^\infty dk k^3 \frac{1}{\beta}\sum_\mu G_{\omega_\mu,k}[G_{\omega_\mu,k} - G_{\omega_\mu+\omega_\nu,k}], \tag{14}$$

where

$$G_{\omega_\mu,k} = \frac{1}{r_k + i\lambda\omega_\mu + \zeta\omega_\mu^2} \quad , \quad r_k = \epsilon + k^2/4. \tag{15}$$

The sum over Matsubara frequencies in (14) is readily performed by contour integration, with the result

$$\sigma(i\omega_\nu) = \frac{1}{8R_Q\zeta^2}\int dk k^3 \Big\{\frac{\beta v(\beta,\omega^+,\omega^-)}{\Sigma_k^2}\frac{1}{\omega_\nu}$$
$$+\frac{u(\beta,\omega^+,\omega^-)}{\Sigma_k^3}\left(\frac{1}{\omega_\nu + i\Sigma_k} + \frac{1}{\omega_\nu - i\Sigma_k}\right)\Big\}. \tag{16}$$

Here the k-dependent excitation energies are $\omega^\pm = (\pm\lambda + \sqrt{\lambda^2 + 4\zeta r_k})/2\zeta$, $\Sigma_k = \omega^+ + \omega^-$. The functions u and v are defined in terms of the Bose distribution function $N(\omega) = (\exp(\beta\omega) - 1)^{-1}$ as

$$u = N(\omega^+) + N(\omega^-) + 1$$
$$v = [N(\omega^-) + 1]N(\omega^-) + [N(\omega^+) + 1]N(\omega^+) \tag{17}$$

In order to extract the conductivity as a function of real frequencies we perform the analytic continuation to real frequencies. This amounts to setting $i\omega_\nu \rightarrow \omega + i\delta$. The identity $\lim_{\delta\rightarrow 0}\frac{1}{a\pm i\delta} = P\frac{1}{a} \mp i\pi\delta(a)$ finally yields the real and imaginary parts of the conductivity.

There is some discussion on this point, related to the order of analytic continuation and summation over Matsubara frequencies. In Ref.[12] the analytical continuation is performed before the summation over Matsubara frequencies. In this case the first term of Eq. (16) is absent. However, as we will show below, this term has a physical interpretation.

A. Zero temperature

At zero temperature the function v vanishes and $u = 1$. Hence we are left with

$$\sigma(\omega) = \frac{1}{8R_Q\zeta^2}\int dk \frac{k^3}{\Sigma_k^3}\Big\{\frac{2i\omega}{\omega^2-\Sigma_k^2} + \pi\delta(\omega-\Sigma_k) + \pi\delta(\omega+\Sigma_k)\Big\}. \quad (18)$$

which reduces to[20]

$$\mathrm{Re}\,\sigma(\omega) = \frac{\pi}{8R_Q}\left(1-\frac{\omega_c^2}{\omega^2}\right)\theta(\omega^2-\omega_c^2)$$
$$\mathrm{Im}\,\sigma(\omega) = \frac{1}{8R_Q}\left\{\frac{-2\omega_c}{\omega} + \left(1-\frac{\omega_c^2}{\omega^2}\right)\ln\left|\frac{\omega-\omega_c}{\omega+\omega_c}\right|\right\} \quad (19)$$

The real and imaginary part of the response function are shown in Fig. 2. The real part vanishes below a threshold frequency $\omega < \omega_c$, as we expect for the Mott insulating phase. The threshold frequency coincides with the sum of the excitation gaps for a particle and hole like excitation $\omega_c = \Sigma(k=0) = \sqrt{\lambda^2 + 4\zeta\epsilon}/\zeta$. Above the threshold frequency ω_c particle-hole excitations can be created, and the real part of the conductivity is finite. In the language of the G-L functional it means that propagating Josephson plasmon modes can be excited[9]. Notice that the conductivity at zero temperature depends on ω/ω_c and a universal constant only.

The threshold frequency is shown for different parameters in Fig. 3. For the case of on-site interaction it is given throughout the lobe by $\omega_c = 8E_C\sqrt{1-E_J/E_C}$ independent of q_x. In general the sum of the excitation energies remains finite up to the transition at $\epsilon = 0$, where it reaches the value $\omega_c = \lambda/\zeta$. Only on the particle-hole symmetry lines $\lambda = 0$, i.e. at the tips of the integer lobes, as well as, for finite range interactions, the tip of the half-integer lobes the gap vanishes as $\sqrt{\epsilon}$ when the superconducting phase boundary is approached.

We want to stress this result: At zero temperature the applied electromagnetic field only produces excitations in pairs (particle *and* hole). The sum of the excitation energies enters into the conductivity, whereas the lower of the single particle excitation energies shows the critical behavior characterized by the dynamic critical exponent z.

At the transition, on the particle-hole symmetry lines, where the gap in the response function vanishes, a finite dc conductance equal to

$$\sigma^* = \frac{\pi}{8R_Q} \quad (20)$$

emerges. This response is the universal conductivity found by Cha et al.[1]. In the general case the gap remains finite at the transition and $\mathrm{Re}\sigma(\omega=0) = 0$.

The imaginary part of the conductivity can be expanded at low frequencies $\omega \ll \omega_c$, with the result $\mathrm{Im}\sigma(\omega) = \omega C_{eff}$. This implies that the system behaves as a capacitor with effective capacitance

$$C_{eff} = \frac{1}{6\omega_c R_Q}. \quad (21)$$

Thus we expect on the insulating side of the transition in the I-V curves of junction arrays the phenomenon of 'Coulomb blockade' up to a voltage scale given by the 'Coulomb gap' $(2e)^2/2C_{eff}$. This means no current is flowing for voltage smaller than a threshold voltage which scales with e/C_{eff}. If $\lambda = 0$ the effective capacitance diverges near the transition as $\epsilon^{-1/2}$, and the transition to the superconducting state is marked by a vanishing Coulomb gap. If $\lambda \neq 0$ the particle-hole symmetry is broken and the capacitance remains finite up to the transition. This means the Coulomb gap vanishes discontinuously as we enter the superconducting phase (Fig. 3a).

B. Finite temperature

At low temperatures $T \ll \min\{\omega^+, \omega^-\}$ the conductivity is

$$\mathrm{Re}\,\sigma(\omega) = \pi\rho_D\,\delta(\omega) + \frac{\pi\theta(\omega^2-\omega_c^2)}{8R_Q}\left(1-\frac{\omega_c^2}{\omega^2}\right)\times$$
$$\times(1+2\cosh(\frac{\beta\lambda}{2\zeta})e^{-\beta|\omega|/2}), \quad (22)$$
$$\mathrm{Im}\,\sigma(\omega) = \rho_D/\omega + \omega C_{eff}, \quad (23)$$
$$\text{where}\quad C_{eff} = \frac{(1+12\frac{T^2}{\omega_c^2}(e^{-\beta\omega^+}+e^{-\beta\omega^-}))}{6\omega_c R_Q}, \quad (24)$$

where the Drude weight $\rho_D = T(e^{-\beta\omega^+}+e^{-\beta\omega^-})/R_Q$ was introduced. In the real part two contributions can be distinguished. The second term in Eq.(22) generalizes (19). It still describes the simultaneous excitation of a particle *and* a hole. However, at finite temperature the absorption and emission processes are characterized by Bose functions. Indeed the temperature dependent coefficient can be written as $\{[1+N(\omega^+)][1+N(\omega^-)] - N(\omega^+)N(\omega^-)\}\delta_{\omega^++\omega^-,|\omega|}$.

The first term in Eq.(22) describes the coupling of the external field to thermally excited particles *or* holes. It is proportional to their total density $\exp(-\beta\omega^+) + \exp(-\beta\omega^-)$. The frequency dependence of this term arises from $1/(\omega+i\delta)$ with vanishing δ, i.e. it is an ordinary Drude conductivity in the limit where the scattering time is infinite. It may appear peculiar that the phase which is insulating at $T = 0$ turns into a perfect conductor at finite temperature. It arises due to the absence of a low frequency dissipation or disorder in our model. Hence the thermal charge excitations are freely accelerated. Below we will show that the inclusion of dissipation regularizes this zero frequency contribution. Although the system is a perfect conductor it is not a superconductor, since it shows no Meissner effect. In order to check this we write eq.(12) for small but finite $\vec{q}$ in the zero frequency limit as

$$\langle j_\mu(\vec{q})\rangle = K_{\mu\nu}(\vec{q})A_\nu(\vec{q}) \ . \tag{25}$$

After expansion in $\vec{q}$ the dia- and paramagnetic parts in $K_{\mu\nu}$ cancel to leading order, i.e. $K = O(q^2)$, which implies a vanishing superfluid density and, therefore the absence of a Meissner effect[21,22].

The scale of the cross-over temperature to classical behaviour is set by $T \approx \min\{\omega^+,\omega^-\}$. Note that at the transition $\min\{\omega^+,\omega^-\} = 0$. This means that at any finite temperature the transition is in a sense classical. For $T \gg \omega_c$ we find

$$\begin{aligned}\mathrm{Re}\ \sigma(\omega) &= \pi\rho_D\,\delta(\omega) + \frac{\pi T}{2R_Q|\omega|}\frac{\omega^2-\omega_c^2}{(\omega^2-\lambda^2/\zeta^2)}\theta(\omega^2-\omega_c^2)\\ \mathrm{Im}\ \sigma(\omega) &= \rho_D/\omega + \frac{T\omega}{4R_Q\omega_c^2}\ , \quad \omega \ll \omega_c\end{aligned} \tag{26}$$

The excitation gap persists. However, the frequency dependence is changed. On the particle-hole symmetry lines $\lambda = 0$ the low frequency conductivity at the transition diverges rather than being universal. At high temperatures $T \gg \min(\omega^+,\omega^-)$ the Drude weight, i.e. the prefactor of the $1/(\omega + i\delta)$, in the absence of dissipation and disorder diverges near the transition as $-T\ln[\min\{\omega^+,\omega^-\}/T]/R_Q$.

C. Magnetic field effects

The effect of a magnetic field, if we ignore commensurability effects, is also described by the G-L functional (4) (In Refs.[23,24] commensurability effects have been included in the coarse-graining approach). In this case we can take the magnetic field into account by expanding the order parameter in Hermite polynomials that diagonalize the free energy. Thus the momentum integrals in (14) are replaced by a sum over 'Landau levels' n and eq. (14) is replaced by

$$\begin{aligned}\sigma(i\omega_\nu) = \frac{(4\gamma\pi f)^2}{2R_Q\beta\omega_\nu}\sum_{n=0}^{\infty}\sum_{\mu}(n+1)\big[2G_{\omega_\mu,n}G_{\omega_\mu,n+1}\\ -G_{\omega_\mu+\omega_\nu,n}G_{\omega_\mu,n+1} - G_{\omega_\mu,n}G_{\omega_\mu+\omega_\nu,n+1}\big]\end{aligned} \tag{27}$$

where

$$G_{\omega_\mu,n} = \frac{1}{\zeta\omega_\mu^2 + i\lambda\omega_\mu + r_n}\ ,\ \ r_n = 4\gamma\pi f n + \epsilon_f \tag{28}$$

For finite magnetic fields the mean field phase transition is determined by $\epsilon_f \equiv \epsilon + \gamma 2\pi f = 0$. We expect the Hall conductance to be nonzero[25]. Hence we also consider $\sigma_{xy}(i\omega_\nu) \equiv \sigma_H(i\omega_\nu)$

$$\begin{aligned}\sigma_H(i\omega_\nu) = \frac{i(4\gamma\pi f)^2}{2R_Q\omega_\nu}\sum_{n=0}^{\infty}(n+1)\times\\ \frac{1}{\beta}\sum_\mu\left[G_{n,\omega_\mu}G_{n+1,\omega_\mu+\omega_\nu} - G_{n,\omega_\mu+\omega_\nu}G_{n+1,\omega_\mu}\right],\end{aligned} \tag{29}$$

Proceeding along the same lines as in the zero field case we find the real and imaginary parts of the conductivity

$$\begin{aligned}\mathrm{Re}\,\sigma(\omega) = \frac{\pi(4\gamma\pi f)^2}{2R_Q\zeta^2}\sum_n\frac{n+1}{\omega_n\omega_{n+1}}\times\\ \left\{\frac{u_n+u_{n+1}}{\Sigma_n}\delta(\mid\omega\mid - \Sigma_n) + \frac{u_n-u_{n+1}}{\Delta_n}\delta(\mid\omega\mid-\Delta_n)\right\}\end{aligned} \tag{30}$$

$$\begin{aligned}\mathrm{Im}\,\sigma(\omega) = \frac{(4\gamma\pi f)^2}{R_Q\zeta^2}\sum_n\frac{n+1}{\omega_n\omega_{n+1}}\times\\ \left\{\frac{u_n+u_{n+1}}{\omega^2-\Sigma_n^2}\frac{\omega}{\Sigma_n} + \frac{u_n-u_{n+1}}{\omega^2-\Delta_n^2}\frac{\omega}{\Delta_n}\right\}\ .\end{aligned} \tag{31}$$

Here we introduced $\omega_n = \omega_n^+ + \omega_n^-$, $\Delta_n = \omega_{n+1}^+ - \omega_n^+$, $\Sigma_n = \omega_{n+1}^+ + \omega_n^-$ and $\omega_n^\pm = (\mp\lambda + \sqrt{\lambda^2 + 4\zeta r_n})/2\zeta$. The temperature enters through the function $u_n = N(\omega_n^+) + N(\omega_n^-) + 1$. As is clear from (31), the excitation gap frequency ω_c is now given by $\omega_c = \Sigma_{n=0}$. Even on the particle-hole symmetry line $\lambda = 0$, the gap remains finite $\sqrt{\pi f/\zeta}$ up to the transition (see Fig. 4). This implies that magnetic frustration, similarly as charge frustration, prevents the appearance of the universal (zero frequency) conductance at the transition. On the other hand, the effective capacitance may still diverge. For strong magnetic fields f (close to the transition) we can replace the sum over Landau levels by the first (divergent) term. Hence at zero temperature and $\lambda = 0$ the effective capacitance reduces to

$$C_{eff} = \frac{\sqrt{\zeta}}{4R_Q\sqrt{\epsilon_f}}\ . \tag{32}$$

For temperatures $T \gg \omega_c$ the effective capacitance depends on the critical field f_{cr}. For a large range of parameters it is inversely proportional to the field, i.e. $C_{eff} \sim f^{-1}$.

For small field or far from the transition, $f \ll \epsilon_f, \lambda^2/4\zeta$, the sum over Landau levels can be substituted by an integral. In this way we find corrections to the $f = 0$ results. For instance (19) and (21) are replaced at zero temperature by

$$\begin{aligned}\mathrm{Re}\ \sigma(\omega) &= \frac{\pi}{8R_Q}\Theta(\omega^2-\omega_c^2)\left(1-\frac{\omega_0^2}{\omega^2}+\frac{(4\gamma\pi f)^2}{\zeta^2\omega^4}\right)\\ \mathrm{Im}\ \sigma(\omega) &= \frac{\omega}{6R_Q\omega_c}\left(1+6(1-\frac{\omega_0^2}{\omega_c^2})+\frac{3}{10}\frac{(4\gamma\pi f)^2}{\zeta^2\omega_c^4}\right).\end{aligned} \tag{33}$$

D. The Hall conductance

The real and imaginary part of the transverse conductivity are

$$\mathrm{Re}\,\sigma_H(\omega) = \tag{34}$$

$$\frac{(4\gamma\pi f)^2}{R_Q\zeta^2}\sum_n \frac{(n+1)(w_{n+1}-w_n)}{\omega_n\omega_{n+1}} \tag{35}$$

$$\left\{\frac{1}{\Sigma_n^2-\omega^2}-\frac{1}{\Delta_n^2-\omega^2}\right\}$$

$$\mathrm{Im}\,\sigma_H(\omega) = \tag{36}$$

$$\frac{\pi(4\gamma\pi f)^2}{2R_Q\zeta^2}\sum_{n,\pm}\frac{(n+1)(w_{n+1}-w_n)}{\omega_n\omega_{n+1}}$$

$$\left\{\pm\frac{1}{\Sigma_n}\delta(\omega\mp\Sigma_n)\mp\frac{1}{\Delta_n}\delta(\omega\mp\Delta_n)\right\}. \tag{37}$$

The function $w_n = N(\omega_n^-) - N(\omega_n^+)$ governs the temperature dependence. At zero temperature the Hall conductance vanishes. This follows from the fact that at $T = 0$ no excitations are present. As expected, it is also zero at the particle-hole symmetry points where $(w_n - w_{n+1})\mid_{\lambda=0}= 0$. A nonvanishing Hall conductance σ_T arises in a magnetic field at finite temperature if the density of hole-like and particle-like excitations differ, i.e. if the particle-hole symmetry is broken.

In comparison to the longitudinal conductivity (31) the behavior of the real and imaginary parts has been interchanged: The imaginary part of the Hall conductance exhibits an excitation gap and the real part is finite at zero frequency

$$\begin{aligned}\mathrm{Re}\,\sigma_H(\omega=0) &= \frac{(4\gamma\pi f)^2}{R_Q\zeta^2}\sum_n\frac{(n+1)(w_{n+1}-w_n)}{\omega_n\omega_{n+1}}\times\\ &\quad\left(\frac{1}{\Sigma_n^2}-\frac{1}{\Delta_n^2}\right)\\ &= R_Q^{-1}\sum_n(n+1)(w_n-w_{n+1})\end{aligned} \tag{38}$$

Close to the transition where $\epsilon_f \ll \min\{\lambda T, \lambda^2/2\zeta\}$ the Hall conductivity diverges as $\mathrm{Re}\ \sigma_H(\omega = 0) = R_Q^{-1}T\lambda\epsilon_f^{-1}$. In the opposite limit, i.e. $T \ll \min\{\omega_0^+,\omega_0^-\}$, we can approximate (38) for small fields by an integral, with the result

$$\mathrm{Re}\,\sigma_H(\omega=0)= \frac{\zeta\omega_0}{4\gamma\pi f\beta R_Q}\Big[\exp(-\beta\omega_0^+)-\exp(-\beta\omega_0^-)\Big],$$

$$\omega_0 = \zeta^{-1}\sqrt{\lambda^2+4\zeta\epsilon_f} \tag{39}$$

which is odd in λ as well as in f. Thus we see that far from the transition the Hall conductance is inversely proportional to the magnetic field and proportional to the difference in density of activated particle and hole like carriers.

E. The influence of dissipation

So far in our model no low frequency dissipation was included. The only source of dissipation is the creation of particle and hole like excitations, related to the excitation gap in the nonsuperconducting phase. We found a perfect dc conductivity at finite temperatures (proportional to $\frac{i}{\omega}+\pi\delta(\omega)$). Now we will show that the inclusion of a phenomenological low frequency dissipation regularizes the divergent Drude conductivity.

Generalizing the approach of Caldeira and Leggett[26] we can account for damping in an imaginary time formalism by an extra term $\eta\mid\omega_\mu\mid$ in the free energy. The inclusion of this term changes the analytic properties of the Matsubara sums and some care is needed when evaluating them. The analytic continuation now yields

$$\begin{aligned}\sigma(\omega) &= \frac{1}{16\pi R_Q\omega}\int_{-\infty}^{\infty}dz\int_0^{\infty}\frac{dkk^3}{1-e^{-\beta z}}\left[G^R(z)-G^A(z)\right]\\ &\times\left[G^R(z)+G^A(z)-G^R(z+\omega)-G^A(z-\omega)\right],\end{aligned} \tag{40}$$

where the advanced and retarded Green's functions $G^{A/R}(z) = (r_k + \lambda z - \zeta z^2 \pm i\eta z)^{-1}$ were introduced. From these real time Green's functions we immediately obtain the 'density of states' as shown in Fig. 4. For weak dissipation, i.e. $\eta E_C \ll \frac{1}{4}$, it is peaked around the two excitation frequencies $\omega^\pm$, whereas for $\eta E_C \gg \frac{1}{4}$ the largest contribution appears at zero frequency.

The k-integration in Eq.(40) can be performed analytically. The Bose distribution function, and therefore the conductivity, is seperated conveniently in a $T = 0$ part and a finite temperature part. The z-integration was done numerically, the results for different cases are shown in Fig. 5. Note that, although smeared, the gap structure is still visible. At finite temperatures a contribution to the real part centered around zero frequency appears. This is reminiscent of the perfect conductivity $\delta(\omega)$ peak in eq.(22) for the case without dissipation.

For finite temperatures, but lower than the gap frequency, the height of the zero frequency peak in the real part of the conductivity shows activated behavior. In the limit $T \ll \omega_c$ on the symmetry line $\lambda = 0$ and for weak damping $\eta \ll \epsilon, \zeta$ the result is

$$\sigma(\omega=0) = \frac{8}{\pi R_Q}\frac{\pi\zeta}{\eta\beta}\exp(-\frac{1}{2}\beta\omega_c) \tag{41}$$

This demonstrates that the inclusion of dissipation regularizes the singular behavior in Eq. (22). Note, the surprising result that at zero temperature, although dissipation is present, the dc conductivity vanishes $\sigma(\omega = T = 0, \eta \neq 0) = 0$.

At the zero temperature transition the dc conductivity $\tilde\sigma$ may be evaluated directly from Eq.(40). Again a universal value emerges, but with a different value

$$\tilde\sigma = \frac{1}{8R_Q}\left(\frac{\pi}{2}-\frac{2}{\pi}\right) = 0.117\frac{1}{R_Q} = 0.3\sigma^*, \tag{42}$$

independent of the strength of the dissipation η. This important and perhaps surprising result can be understood as a consequence of hyperuniversality[27,1]: At a continuous phase transition certain amplitudes, as the conductivity at a $T = 0$ transition in two dimensions, are universal constants. The inclusion of dissipation in the Caldeira-Legett sense changes the universality class. Again a universal conductivity is found, independent of the strength of the dissipation, but with a different value. Fig. 6 demonstrates how this new universal conductance arises as we approach the transition. Shown is a one parameter family of curves that depends on the value of $\eta^2/\epsilon\zeta$. At zero temperature the conductivity is a function of ω/ω_c and $\eta^2/\epsilon\zeta$ only. The horizontal curve (V) corresponds to either infinite damping or the response at the transition. All curves cross at the value $\sigma = \tilde{\sigma}$.

It is interesting to compare $\tilde{\sigma}$ with the results of Monte Carlo simulations on *disordered* Bosons[28], where a very similar value of the universal conductance equal to $\sigma_{MC} = (0.14 \pm 0.03)/R_Q)$ was reported. It is not clear at this stage whether the agreement is a coincidence or whether it indicates a link between disorder and the model for low frequency dissipation used by us.

At finite temperature the dc conductance at the transition is infinite, similar as in the absence of dissipation. In the high temperature limit we can study directly the real-time dependent Ginzburg-Landau equation corresponding to the free energy (4)[15]

$$\left(\epsilon - \gamma \left[\vec{\nabla} + \frac{2\pi i}{\Phi_0}\vec{A}(\vec{r},t)\right]^2 + (\eta - i\lambda)\partial_t + \zeta\partial_t^2\right)\psi = \xi \, . \tag{43}$$

Here we wrote both the time derivative arising due to gauge invariance with coefficient λ and the dissipative term proportional to η. On the right appears a Langevin force ξ with power spectrum $< \xi\xi >_\omega = 2\eta T$. The fluctuation conductivity can be derived using the ordinary Kubo formula. In this way we can derive the high temperature results quoted above. They differ from the standard fluctuation conductivity results for superconductors[29] due to the second order time derivative.

F. Response in the SC phase

In the superconducting phase the conductivity can be evaluated along the same lines as in the insulating phase. Now ϵ is negative and the order parameter is finite $\langle|\psi|^2\rangle = \langle\rho_s\rangle = \mid \epsilon \mid /2\kappa$. Writing $\psi = \sqrt{\rho_s}e^{i\phi}$ and taking into account phase fluctuations only we find the free energy

$$F[\rho,\phi] = \int_0^\beta d\tau \int d^2r\rho_s \times \left\{\frac{1}{4}(\nabla_\mu\phi + \frac{2\pi}{\Phi_0}A_\mu)^2 + \zeta(\partial_\tau\phi)^2 + i\lambda\partial_\tau\phi\right\} . \tag{44}$$

The term $i\lambda\partial_\tau\phi$ is a total derivative and does not influence the response functions. Using (11) we find

$$\sigma_{\mu\nu}(\vec{q}, i\omega_\nu) = \frac{\pi\rho_s}{R_Q\omega_\nu}\left(-\delta_{\mu\nu} + \frac{q_\mu q_\nu}{4\zeta\omega_\nu^2 + q^2}\right) \tag{45}$$

The analytic continuation to real frequencies yields for $q \rightarrow 0$

$$\sigma_{\mu\nu}(\vec{q} = 0, \omega) = -\frac{\pi\rho_s}{R_Q}\delta_{\mu\nu}\left(\frac{i}{\omega} + \pi\delta(\omega)\right) , \tag{46}$$

which implies perfect conductivity, and for $\omega \rightarrow 0$

$$\langle j_\mu\rangle = -\frac{\pi\rho_s}{R_Q}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)A_\nu , \tag{47}$$

which demonstrates the presence of a Meissner effect[21].

IV. DISCUSSION

We analyzed the Bose-Hubbard model describing the S-I transition of Josephson junction arrays. It reduces to a G-L free-energy functional (4), differing from the standard one for bulk superconductors by the absence of low frequency dissipation and relaxation. Rather than the first order time derivative, characteristic for the ordinary G-L equation, the second order time derivative is crucial.

First we investigated the phase diagram for finite temperatures and finite range of the interaction. Reentrant behaviour is found for the 'half-integer' lobes, whereas no reentrant behaviour arises for the integer lobes. This is presumably related to the presence or absence of domain-wall in the two cases.

Secondly we derived the frequency dependent conductivity of the system using the G-L free energy. The real and imaginary part are governed by a threshold frequency ω_c and an effective capacitance C_{eff}, respectively. The threshold frequency is the sum of the excitation energies of a particle and a hole-like excitation, which is independent of q_x in the integer lobes. The effective capacitance is related to the Coulomb gap $(2e)^2/2C_{eff}$, which has been found in many small capacitance junction systems and is responsible for the insulating behavior at voltages below $2e/C_{eff}$. If particle-hole symmetry exists, $\lambda = 0$, and if $f = 0$ the threshold frequency and the Coulomb gap vanish at the transition. In general ($\lambda \neq 0$ or $f \neq 0$) both remain finite up to the transition. (This in contrast to the disordered case where the particle-hole asymmetry scales to zero at the transition[16].) On the other hand, the excitation gap for a *single* excitation (particle or hole-like, whichever lies lower depending on the chemical potential) vanishes at the transition and is governed by the dynamic critical exponent z. At finite temperatures this gap energy determines the density of the majority type excitations. However, this critical energy scale does not influence the $T = 0$ conductivity.

We analyzed the problem in a mean field approximation. In this way we can obtain explicit results, for instance the complete frequency dependence of the conductivity. Moreover we reproduce the correct value of the dynamic critical exponent z. This is not so surprising, since it follows essentially from symmetry arguments. It is $z = 1$ for a system with particle-hole symmetry $\lambda = 0$ (which follows from the equivalence of space and time derivatives in this limit) and $z = 2$ in the general case $\lambda \neq 0$. Both agree with the analysis of ref.[2]. On the other hand, the exponent ν is $\nu = 1/2$ in the man field approximation, which is correct only for $\lambda \neq 0$. In general non-Gaussian corrections can also modify the universal conductance. Right at the transition the fourth order term in the Ginzburg-Landau free energy should have the most pronounced effect. But even there Monte Carlo simulations of Ref.[1] yield results which differ from the mean field value by only 30 %. This gives us confidence into the quality of our results.

The two kinds of frustration, external charge q_x and the magnetic frustration f, turned out to have a very similar effect on the response functions, although the free energy from which they were derived is not self-dual for finite-range interactions between charges. The nature of the phase transition and the response function differ in the presence or absence of charge or magnetic frustration[2]. Only for integer or half-integer values of q_x (i.e. at the tips of the lobes) *and* for integer f, i.e. at the point of maximal symmetry, does the threshold frequency for the real part of the conductivity vanish at the transition, leading to a universal value. The Coulomb gap vanishes at the transition, either continuously in the unfrustrated or with a jump in the charge frustrated case.

The Hall conductivity also reflects the different nature of the phase transition in the presence or absence of particle-hole symmetry. A nonzero value is obtained only at finite temperatures when the particle-hole symmetry is broken.

Without low frequency dissipation thermally activated carriers can be freely accelerated by an electric field, resulting in perfect conductivity without Meissner effect. The inclusion of low frequency dissipation regularizes this singular behaviour and yields a Drude-like contribution. The dc conductivity at the transition $\tilde{\sigma}$ is smaller than in the case without dissipation but still universal and independent of the amount of dissipation. It is $\tilde{\sigma} \sim 0.12/R_Q$, which is very close to the Monte Carlo result $(0.14 \pm 0.03)/R_Q$)[28].

The calculated response should be visible in experiments on junction arrays where the effect of disorder and dissipation is negligible at low temperatures. The measured Coulomb gap as a function of magnetic frustration shows qualitative agreement[30] with the results presented here. The threshold frequency in the excitation spectrum has not yet been verified. It would require high frequency measurements.

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[1] M.P.A. Fisher, G. Grinstein, and S.M. Girvin, Phys. Rev. Lett. **64**, 587 (1990); M.-C. Cha, M.P.A. Fisher, S.M. Girvin, M. Wallin, and A.P. Young, Phys. Rev. B **44**, 6883 (1991)

[2] M.P.A. Fisher, B.P. Weichman, G. Grinstein, and D.S. Fisher, Phys. Rev. B **40**, 546 (1989); M.P.A. Fisher, Phys. Rev. Lett. **65**, 923 (1990)

[3] K.B. Efetov, Sov. Phys. JETP **51**, 1015 (1980)

[4] B.G. Orr, H.M. Jaeger, A.M. Goldman, and C.G. Kuper, Phys. Rev. Lett. **56**, 378 (1986); H.M. Jaeger, D.B. Haviland, B.G. Orr, and A.M. Goldman, Phys. Rev. B **40**, 182 (1989); D.B. Haviland, Y. Liu, and A.M. Goldman, Phys. Rev. Lett. **62**, 2180 (1989); Y. Liu, K.A. Greer, B. Nease, D.B. Haviland, G. Martinez, J.W. Haley, and A.M. Goldman, Phys. Rev. Lett. **67**, 2068 (1991)

[5] A.F. Hebard and M.A. Paalanen, Phys. Rev. Lett. **65**, 927 (1990)

[6] H.S.J. van der Zant, F.C. Fritschy, W.E. Elion, L.J. Geerligs, and J.E. Mooij, Phys. Rev. Lett. **69**, 2971 (1992); H.S.J. van der Zant, L.J. Geerligs, and J.E. Mooij, Europhys. Lett. **19**, 541 (1992)

[7] L.J. Geerligs, M. Peters, L.E.M. de Groot, A. Verbruggen, and J.E. Mooij, Phys. Rev. Lett. **63**, 326 (1989); J.E. Mooij, B.J. van Wees, L.J. Geerligs, M. Peters, R. Fazio, and G. Schön, Phys. Rev. Lett. **65**, 645 (1990)

[8] R. Fazio, A. van Otterlo, G. Schön, H. van der Zant, and J.E. Mooij, Helvetica Physica Acta **65**, 228 (1992)

[9] S. Doniach, Phys. Rev. B **24**, 5063 (1981)

[10] C. Bruder, R. Fazio, A. Kampf, A. van Otterlo, and G. Schön, Physica Scripta **42**, 159 (1992)

[11] A.van Otterlo, R. Fazio and G. Schön, to appear in Jap. J. Appl. Phys.

[12] A.P. Kampf and G.T. Zimanyi, Phys. Rev. B **47** 279 (1993)

[13] R. Fazio and G. Schön, Phys. Rev. B **43**, 5307-5320 (1991)

[14] J.G. Kissner and U. Eckern, preprint (submitted to Z. Phys. B)

[15] A.G. Aronov and A.I. Larkin, submitted to Phys. Rev. Lett.; A.G. Aronov and A.B. Rapoport, submitted to Europhys. Lett.

[16] M.P.A. Fisher, Physica **A177**, 553 (1991)

[17] C. Bruder, R. Fazio, G. Schön, Phys. Rev. B **47** 342 (1993)

[18] see for instance: K-S. Liu and M.E. Fisher, J. Low Temp. Phys **10** 655 (1973)

[19] A.P. Kampf and G. Schön, Physica **152**, 239 (1988), (note that in this article charges were assumed to take continuous values)

[20] As a result of the integration over the first Brillouin zone a cut-off in Re σ appears at $\omega_{cut} = 8\gamma\sqrt{\epsilon/\zeta + \pi^2/4\zeta}$. Therefore no sum rules are violated. Since this cut-off does not influence the low frequency behavior, we will neglect it in the following. Note that the corresponding cut-off with magnetic field would lead to an n_{max} in Eq.(27) equal to $\mathrm{int}(\pi/4f + 1)$.

[21] see for instance B. Schrieffer, Superconductivity (Benjamin, New York, 1964)
[22] for a recent discussion of the difference between perfect conductivity and superconductivity see D.J. Scalapino, S.R. White, and S.C. Zhang, Phys. Rev. Lett. **68**, 2830 (1992)
[23] A.P. Kampf and G. Schön, Phys. Rev. B **37**, 5954 (1988)
[24] E. Granato and J.M. Kosterlitz, Phys. Rev. Lett. **65**, 1267 (1990)
[25] recently the Hall effect in ordinary superconductors has been reinvestigated in Ref.[15]
[26] A.O. Caldeira and A.J. Leggett, Ann. Phys. (N.Y.) **149**, 347 (1983)
[27] K. Kim and P.B. Weichman, Phys. Rev. B **43**, 13583 (1991)
[28] E.S. Sorensen, M. Wallin, S.M. Girvin and A.P Young, Phys. Rev. Lett **69**, 828 (1992)
[29] see for instance D.R. Tilley and J.B. Parkinson, J. Phys. C **2**, 2175 (1969)
[30] Herre van der Zant, private communication.

FIG. 1. (a) Phase diagram at $T/E_C = 0$, 0.2, 0.4, 0.6, 0.8, 1.0 for bosons with on-site interaction (or a junction array with self capacitance only) as a function of E_J/E_C and charge frustration q_x. (b) Same, now with both on-site and nearest neighbor interaction. The relative strength of the interactions is set by the parameter $W \equiv 1 + 4C_{01}^{-1}/C_{00}^{-1} = 4/3$. (c) The phase boundary with both on-site and nearest neighbour interactions for $W = 4/3$ and $q_x = 1/2$ clearly demonstrates reentrance as a function of temperature.

FIG. 2. The real and imaginary parts of the frequency dependent conductivity at the symmetry point, $\lambda = f = 0$.

FIG. 3. (a) Excitation gap (or inverse capacitance) vs. ϵ for both the nonfrustrated and the frustrated case at $T = 0$. I: $\lambda = f = 0$, II: $\lambda = 0$, $\pi f = \frac{1}{2}$, III: $\lambda = \frac{1}{2}$, $f = 0$. (b) Excitation gap on the phase boundary where $\epsilon = 0$ as a function of charge frustration (on-site interaction: curve I, also n.n. interaction, $W \equiv 1 + 4C_{01}^{-1}/C_{00}^{-1} = 4/3$: curve II) and magnetic frustration (curve III).

FIG. 4. The density of states N as a function of frequency. In this example we choose $\epsilon = \frac{1}{4}$, $\lambda E_C = -0.1$ and $\eta E_C = 0$, 0.2, 0.8 for curves I, II and III respectively.

FIG. 5. (a) The real (even) and imaginary (odd) parts of the conductivity at zero temperature in the presence of dissipation. I: $\eta^2/\epsilon\zeta = 0.01$, II: $\eta^2/\epsilon\zeta =$. (b) The real and imaginary part of the conductivity at finite temperature $T = 0.2\omega_c$ and finite dissipation $\eta^2/\epsilon\zeta = 0.1$.

FIG. 6. The real part of the conductivity for $\eta^2/\epsilon\zeta = \infty$,1000,10,0.1,0, denoted by I, II, II, IV and V respectivly. Note that all the curves cross each other at $\sigma = \tilde{\sigma}$.