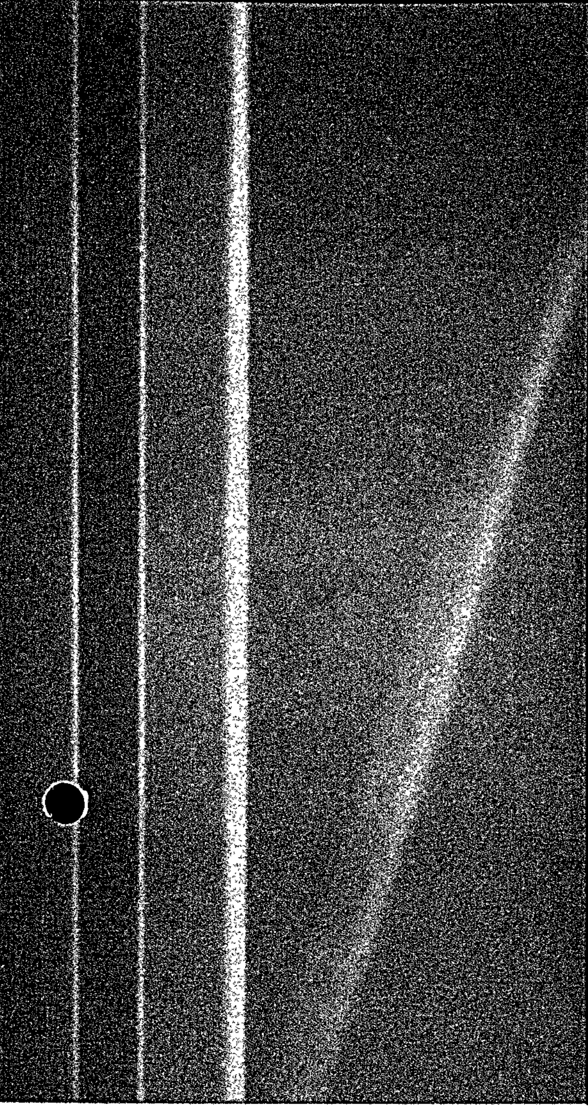

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Some Remarks on the Effect of a Nonuniform Temperature Distribution on the Temperature Dependence of Resonance Absorption

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The influence of a non-uniform temperature distribution on the absorption of neutrons in a purely absorbing resonance of an isolated lump is studied. It is shown that in practical situations the absorption in the lump depends only on its average temperature and not on whether its temperature distribution is uniform or not.

The criticality, on one hand, and the initial time behavior of heterogeneous reactors following a large instantaneous reactivity addition, on the other, depend to a great extent on the increase in resonance absorption with increasing temperature. Therefore, the temperature dependence of resonance absorption is one of the important quantities determining both the criticality and safety of heterogeneous reactors. This temperature dependence has been the subject of much study in the past, but in all previous work the temperature distribution was uniform in the absorber lump, quite in contrast to the state of affairs in an actual reactor. The effects of non-uniformity in the temperature distribution may be important, and it is the purpose of this paper to discuss them.

We shall only consider the case of absorption in an isolated lump (i.e., no Dancoff effect) by a purely absorbing resonance (i.e., $\Gamma\gamma \approx \Gamma$). Furthermore, we shall ignore potential scattering altogether. Thus we shall not consider either monoenergetic scattering or scattering with moderation in the absorber lump. Under these conditions the effective resonance integral of the absorber is given by the equation

$$NVI\phi = \int_{\text{res}} du \int_S dS \int_{\mathbf{n}\cdot\boldsymbol{\omega}>0} d\omega \, \frac{\cos\vartheta}{\pi} \frac{\phi}{4} \cdot \left(1 - \exp\left\{-N\sigma_0 \int_0^{l(\mathbf{r}_S,\boldsymbol{\omega})} \psi[\theta(s\boldsymbol{\omega}), x]\, ds\right\}\right) \quad (1)$$

where

N	=	the atomic density of absorber atoms in the lump
V	=	the lump volume
I	=	the effective resonance integral
ϕ	=	the flux in the moderator, taken as uniform in space and lethargy
u	=	the lethargy
dS	=	the differential element of the lump surface, S
$\mathbf{r}_S$	=	the position of a point on the lump surface
$\mathbf{n}$	=	the inward normal to S at $\mathbf{r}_S$
$\boldsymbol{\omega}$	=	a unit vector at $\mathbf{r}_S$
ϑ	=	the angle between $\boldsymbol{\omega}$ and $\mathbf{n}$
σ_0	=	the peak resonance cross section in the absence of Doppler broadening
$l(\mathbf{r}_S, \boldsymbol{\omega})$	=	the length of the chord at $\mathbf{r}_S$ in direction $\boldsymbol{\omega}$
$\psi(\theta, x)$	=	the Doppler broadened line shape given below in (2)
θ	=	the ratio of the natural width of the resonance, Γ, to the Doppler width, $(4E_0kT/A)^{1/2}$
E_0	=	the resonance energy
kT	=	the temperature of the absorber nuclei in energy units
A	=	the mass number of the absorber nuclei
x	=	the energy measured from exact

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resonance in units of the half-width, i.e., $x = 2(E - E_0)/\Gamma$

s = the distance along the chord $l(\mathbf{r}_S, \boldsymbol{\omega})$

and where $\psi(\theta, x)$ is given by

$$\psi(\theta, x) = \frac{\theta}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{\exp\left[-(\theta^2/4)(x-y)^2\right]}{1+y^2}\, dy \qquad (2)$$

By using the inequality (*1*)

$$\int f(v) e^{-v}\, dv \geqq \exp\left(-\int f(v) v\, dv\right) \qquad (3)$$

where $f(v)$ is any normalized probability frequency function, we can rewrite (1) as

$$NVI\phi \leqq \frac{S\phi}{4} \int_{\mathrm{res}} du \left(1 - \exp\left\{-N\sigma_0 \int_S \frac{dS}{S} \cdot \int_{\mathbf{n}\cdot\boldsymbol{\omega}>0} d\omega \int_0^{l(\mathbf{r}_S,\boldsymbol{\omega})} ds \frac{\cos\vartheta}{\pi} \psi[\theta(s\boldsymbol{\omega}), x]\right\}\right) \qquad (4)$$

As we shall see later, the difference between the RHS of (1) and the RHS of (4) is actually rather small. Hence, in what follows we shall drop the $<$ –sign. The exponent in (4) can be rewritten by interchanging the order of the S and $\boldsymbol{\omega}$ integrations:

$$\int_S \frac{dS}{S} \int_{\mathbf{n}\cdot\boldsymbol{\omega}>0} d\omega \int_0^{l(\mathbf{r}_S\boldsymbol{\omega})} ds \frac{\cos\vartheta}{\pi} \psi[\theta(s\boldsymbol{\omega}), x] = \frac{1}{\pi S} \int d\omega \int_S dS \int_0^{l(\mathbf{r}_S,\boldsymbol{\omega})} ds \cdot \cos\vartheta\; U(\mathbf{n}\cdot\boldsymbol{\omega})\; \psi[\theta(s\boldsymbol{\omega}), x] \qquad (5)$$

where $U(y) = 0$ if $y < 0$ and $U(y) = 1$ if $y > 0$. The last two integrals over S and s give exactly the volume integral of ψ over V. Thus finally we have

$$\frac{1}{\pi S} \int d\omega \int_S dS \int_0^{l(\mathbf{r}_S,\boldsymbol{\omega})} ds \cos\vartheta\, U(\mathbf{n}\cdot\boldsymbol{\omega}) \cdot \psi[\theta(s\boldsymbol{\omega}), x] = \bar{l}\,\overline{\psi(\theta, x)} \qquad (6)$$

where the bar over ψ denotes a spatial average over V and $\bar{l} = 4V/S$. Substituting (6) into (4) now gives

$$I = (N\bar{l})^{-1} \int_{\mathrm{res}} du \{1 - \exp[-N\sigma_0 \bar{l}\,\overline{\psi(\theta, x)}]\} \qquad (7a)$$

$$= (N\bar{l})^{-1} \frac{\Gamma}{E_0} \int_0^{\infty} dx \{1 - \exp[-N\sigma_0 \bar{l}\,\overline{\psi(\theta, x)}]\} \qquad (7b)$$

In the last step we have used the relation $-du = dx(\Gamma/2E)$.

The difference between the RHS of (4) and the RHS of (1) due to the use of inequality (3) can be calculated in the case of no Doppler effect at all, i.e., $\psi = (1 + x^2)^{-1}$, by the method of Gurevich and Pomeranchouk (*2*). These authors find that the ratio of the two RHS's is $\overline{l^{1/2}}/\bar{l}^{1/2}$ where the bars denote averages with respect to the chord-length distribution

$$f(l) = \int_S \frac{dS}{S} \int_{\mathbf{n}\cdot\boldsymbol{\omega}>0} d\omega \frac{\cos\vartheta}{\pi} \delta\,[l - l(\mathbf{r}_S, \boldsymbol{\omega})] \qquad (8)$$

and where $\bar{l}$ still also equals $4V/S$. The factor $\overline{l^{1/2}}/\bar{l}^{1/2}$ is respectively equal to 0.980, 0.975, and 0.943 for spheres, cylinders, and slabs. Thus only a small error is introduced into (4) when it is taken with the equality sign.

That this last conclusion does not depend very strongly on the precise shape of the line can be seen by repeating the above calculation for a line with the shape $(1 + x^n)^{-1}$, $n > 1$. In this case, the ratio of the RHS of (*1*) to that of (*4*) is given by $\overline{l^{1/n}}/\bar{l}^{1/n}$. For spheres and slabs this ratio has a shallow minimum near $n = 2$, rising to unity for $n = 1$ and $n = \infty$. The values at the minimum are essentially those given in the previous paragraph. Thus we expect that for spheres and cylinders especially, the error arising from the use of (3) in (1) is neither very great nor varies rapidly with the shape of the line.

According to (2)

$$\overline{\psi(\theta, x)} = \int f(\theta)\, d\theta \cdot \frac{\theta}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{\exp[-(\theta^2/4)(x-y)^2]}{1+y^2}\, dy$$
$$= \frac{\bar{\theta}}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{dy}{1+y^2} \int \frac{f(\theta)\theta}{\bar{\theta}} \cdot \exp[-(\theta^2/4)(x-y)^2]\, d\theta \qquad (9)$$

where $f(\theta')\,d\theta'$ is that fraction of the lump volume in which θ lies between θ' and $\theta' + d\theta'$. Now $f(\theta)\,\bar{\theta} = 2g(\theta^2)$, where the latter function, g, is then a possible *normalized* distribution of θ^2. In this case, it follows from (9) and (3) that

$$\overline{\psi(\theta, x)} \geqq \frac{\bar{\theta}}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{\exp\left[-\frac{\langle\theta^2\rangle}{4}(x-y)^2\right]}{1+y^2}\, dy \qquad (10)$$

Here the braces denote an average with respect to g. However,

$$\langle\theta^2\rangle = \int \theta^2 g(\theta^2)\, d\theta^2 = \int \frac{\theta^3 f(\theta)\, d\theta}{\bar{\theta}} = \frac{\overline{\theta^3}}{\bar{\theta}} \qquad (11)$$

where the bars, as usual, denote space averages.

From (10) and (11) it follows that

$$\overline{\psi(\theta, x)} \geqq c^{-1}\psi(c\bar{\theta}, x) \qquad (12a)$$

$$c^2 = \overline{\theta^3}/\bar{\theta}^3 \qquad (12b)$$

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From (12) and (7) it then follows that

$$I \gtrsim c^{-1} I_{\text{unif}} (\bar{l} \rightarrow c^{-1}\bar{l}, \quad \theta \rightarrow c\bar{\theta}) \tag{13}$$

where I_{unif} is the resonance integral of a lump of the same material and shape, with a size given by $c^{-1}\bar{l}$, and at a *uniform* temperature $c\bar{\theta}$. The $\sim$ sign occurs rather than an equality sign to indicate that (13) holds as an inequality only within the (small) error arising from dropping the $<$ sign in (4).

f-Distributions are conceivable in which c does not differ very much from unity, although θ varies over a reasonably wide range of values. (We shall see an example of such a distribution shortly.) If $c - 1$ is small, however, the right- and left-hand sides of (12a) cannot be very different. For, indeed, their integrated difference is given by

$$\int [\overline{\psi(\theta, x)} - c^{-1} \psi(c\bar{\theta}, x)] \, dx = \frac{c-1}{c} \pi \geqq 0 \tag{14}$$

and since the integrand is positive everywhere, it is clear that if $c - 1 \ll 1$, $\overline{\psi(\theta, x)}$ and $c^{-1}\psi(c\bar{\theta}, x)$ cannot be very different at all. In such a case we would expect that a valid approximation to (13) would be the *equality*

$$I = c^{-1} I_{\text{unif}} (l \rightarrow c^{-1}\bar{l}, \theta \rightarrow c\bar{\theta}) \approx I_{\text{unif}} (\bar{\theta}) \tag{15}$$

where $I_{\text{unif}} (\bar{\theta})$ is the resonance integral of the actual lump being considered at a uniform temperature $\bar{\theta}$.

In order to pursue this study further and, in particular, test the accuracy of (15), some reasonable f-distribution must be assumed. Let us therefore imagine that in a cylindrical fuel rod in a power reactor the fission heat source is effectively spatially uniform. In this case the temperature distribution in the rod will be parabolic, i.e.,

$$T(r) = T_0 + (T_1 - T_0)(r/R)^2 \tag{16}$$

where r is the radial coordinate in the rod, R is the radius, and T, T_0, and T_1 are, respectively, the temperatures at any point, at the center, and at the boundary. If T_1 is not very different from T_0, the corresponding distribution of θ is also parabolic, i.e.,

$$\theta(r) = \theta_0 + (\theta_1 - \theta_0)(r/R)^2 \tag{17}$$

From (17) it follows that

$$f(\theta) = (\theta_1 - \theta_0)^{-1} \qquad \theta_0 \leqq \theta \leqq \theta_1 \tag{18a}$$

$$g(\theta^2) = (\theta_1^2 - \theta_0^2)^{-1} \qquad \theta_0^2 \leqq \theta^2 \leqq \theta_1^2 \tag{18b}$$

i.e., f and g are both uniform distributions. We will assume henceforth for the sake of argument that the distributions (18) hold exactly; fortunately, our final conclusions do not depend critically on the *details* of these f- and g-distributions. With respect to general form (18) is, of course, representative.

From (18a) it follows that

$$c = \left(\frac{\theta_3}{\bar{\theta}_3}\right)^{1/2} = \left[\frac{2(\alpha^2 + 1)}{(\alpha + 1)^2}\right]^{1/2}; \alpha = \frac{\theta_1}{\theta_0} \tag{19}$$

For $\alpha = \sqrt{2}$, which *corresponds to a ratio* $T_0/T_1 = 2$, c differs from unity by less than 1.5%; for $\alpha = \sqrt{3/2}$, $c = 1.005$. Hence, under the anticipated circumstance that in practical cases α is not too large, we expect (15) to be valid.

We can test this conclusion for the interesting case of extreme Doppler broadening (i.e., $\theta \ll 1$) for which Wigner *et al.* (3) have suggested the use of the approximate line shape

$$\psi(\theta, x) = \frac{\theta\sqrt{\pi}}{2} \exp [-(\theta^2/4) x^2] \tag{20}$$

Equation (20) is particularly appropriate for those resonances which contribute strongly to the temperature coefficient of the resonance integral. From (20) and (18) it follows that

$$\overline{\psi(\theta, x)} = \sqrt{\pi} \frac{\exp[-(\theta_0^2/4)x^2] - \exp[-(\theta_1^2/4)x^2]}{(\theta_1 - \theta_0) x^2} \tag{21}$$

Inserting (21) into (7b) and rearranging one obtains

$$I = \frac{\pi}{2} \sigma_0 \frac{\Gamma}{E_0} \frac{1 + \alpha}{\sqrt{\pi}\,\beta} \int_0^\infty \left(1 - \exp\left\{-\beta \exp(-z^2) \cdot \frac{1 - \exp[-(\alpha^2 - 1) z^2]}{(\alpha^2 - 1) z^2}\right\}\right) dz \tag{22}$$

where $\beta = N\sigma_0\bar{l}(\sqrt{\pi}/2)\bar{\theta}$ and $\alpha = (\theta_1/\theta_0)$. When $\alpha = 1$, $I = I_{\text{unif}} (\bar{\theta})$.

Typical values for the parameters are $N = 0.05$ barns^{-1} cm^{-1} for metallic U, $\bar{l} = 2$ cm, $\bar{\theta} = 0.05$, $\sigma_0 = 5000$ barns. For these values $\beta \approx 22$. Numerical integration of (22) for the four cases, $\alpha = 1, \sqrt{2}$; $\beta = 10, 30$ gives the following results: For $\beta = 10$, I and $I_{\text{unif}} (\bar{\theta})$, i.e., $I(\alpha = \sqrt{2})$ and $I(\alpha = 1)$, are within about 1.5% of one another, the former being larger. For $\beta = 30$, the corresponding difference is about 2.5%. Thus for these conditions our expectation that (15) would be accurate is verified.

Based on the foregoing analysis we are inclined to consider the approximation $I = I_{\text{unif}} (\bar{\theta})$ as a fairly accurate one[1]. For practical purposes it is virtually

[1] This approximation is originally due to G. M. Roe, "Resonance Absorption of Neutrons in Doppler Broadened Resonances," KAPL-1241 (October, 1954) Roe assumed that for slabs, I was given by $I_{\text{unif}} (\theta_e)$, where θ_e was determined by the requirement that the integrals $\int_0^\infty \overline{\psi(\theta, x)}^2 \, dx$ and $\int_0^\infty \psi^2(\theta_e, x) \, dx$ be the same. This condition gave θ_e approximately equal to $\bar{\theta} - (3/2)[(\overline{\theta^2} - \bar{\theta}^2)/(1 + 4\bar{\theta})]$. The form

identical with the oft-repeated prescription that the actual resonance integral is equal to that of the same lump uniformly held at the average temperature, since the effective temperature determined from $\bar{\theta}$ is usually not very different from the average temperature. The virtue of the present note is that it indicates that this prescription is, in fact, quite accurate, at least in the special case treated.

of the second term depends on how θ_e is determined and is to some extent arbitrary. Thus the real meaning of Roe's result is that if the second term is not too large, $\theta_e = \bar{\theta}$ to a good order of approximation.

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REFERENCES

1. G. H. Hardy, J. E. Littlewood, and G. Polya, "Inequalities," p. 137, theorem 187. Cambridge Univ. Press, London and New York, 1934.
2. I. I. Gurevich and I. Y. Pomeranchouk, *Proc. 1st Intern. Conf. Peaceful Uses Atomic Energy, Geneva* **5**, 466 (1955).
3. E. P. Wigner, E. Creutz, H. Jupnik, and T. Snyder, *J. Appl. Phys.* **26**, 260 (1955).

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