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A Variational Approach to the Time-Dependent Slab

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With 2 Figures in the Text

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Abstract. In this paper we discuss the asymptotic behaviour of a neutron pulse injected into an infinite slab of finite thickness. For illustration, a monokinetic transport of neutrons with isotropic scattering has been chosen. Two integrals of the transport equation and the variational method are used in order to obtain the dependence of the extrapolated endpoint upon the slab thickness. The extrapolated end-point x_0 is obtained as the largest positive root of a given transcendental equation. It is suggested that the asymptotic mode may exist within the slab no matter how thin the slab is.

The expression for the angular distribution of the leaking neutrons has been improved by one iteration of the integral transport equation. The results show a characteristic angular peaking of the neutrons leaking from slabs of extremely small thicknesses.

I. Introduction

In the theory of pulsed neutron experiments one is interested in finding the asymptotic behaviour of a burst of neutrons injected into a finite assembly. Usually, the asymptotic decay constant (α) is expressed in terms of the space-eigenvalues, B, of the corresponding infinite-medium problem. A difficulty arises when one tries to relate the infinite-medium buckling (B^2) to the physical dimensions of a finite medium, for the appropriate boundary conditions have to be taken into account. So far this problem has been attacked by applying the P_1 or P_3 approximation and using MARSHAK's or MARK's boundary conditions [1] bis [6]. However, some doubt has been expressed as to whether the P_N method with $N=1$ or 3 is sufficiently convergent to give reliable results in this case [7]. As the P_N method in the higher order approximation requires rather laborious procedures, it seems impractical to investigate the convergence by simply increasing the order of the approximation. One must therefore apply some other method.

In the present paper we shall describe the application of a variational method. As a starting point, we want to investigate the basic transport properties of the neutron distribution decaying in a finite system. For this reason, we confine ourselves to the simplest case: monokinetic transport of neutrons with isotropic scattering in a one-dimensional infinite slab of finite thickness. However, the same method can be generalized to take into account the velocity dependency and the general scattering law.

Consider a one-dimensional slab of finite thickness, $2a$, with the origin of the coordinate system placed in

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the middle of the slab. The neutron flux distribution $\Phi(x, \mu, t)$, satisfies the following transport equation:

$$\left.\begin{aligned}&\frac{1}{v}\frac{\partial}{\partial t}\Phi(x,\mu;t)+\mu\frac{\partial}{\partial x}\Phi(x,\mu;t)+\Sigma_t\cdot\Phi(x,\mu;t)\\&=\int_{-1}^{1}\Sigma(\mu'\to\mu)\,\Phi(x,\mu';t)\,d\mu'=\frac{\Sigma_s}{2}\int_{-1}^{1}\Phi(x,\mu';t)\,d\mu',\end{aligned}\right\}\quad(1)$$

where Σ_s and Σ_t are the scattering and the total macroscopic cross-section, respectively; μ is the cosine of an angle between the neutron velocity vector and the positive x-axis and v is the neutron velocity.

As the slab is surrounded by vacuum, the solution of the above integro-differential Eq. (1) must satisfy the following boundary conditions:

$$\left.\begin{aligned}\Phi(-a,\mu;t)&=0 \quad\text{for}\quad \mu>0,\\ \Phi(a,\mu;t)&=0 \quad\text{for}\quad \mu<0.\end{aligned}\right\}\quad(2)$$

Furthermore, because of the mirror-symmetry with respect to the origin of the coordinate system, we require:

$$\Phi(x,-\mu;t)=\Phi(-x,\mu;t). \quad(3)$$

We are interested in the asymptotic solution, existing long after the injection, which can be written as

$$\Phi(x,\mu;t)=\Psi(x,\mu)\,e^{-\alpha t} \quad(4)$$

where the asymptotic decay constant α and the corresponding time-independent angular flux, $\Psi(x,\mu)$, are solutions of the following integro-differential equation:

$$\left.\begin{aligned}&\mu\frac{\partial}{\partial x}\Psi(x,\mu)+\left(\Sigma_t-\frac{\alpha}{v}\right)\Psi(x,\mu)\\&=\int_{-1}^{1}\Sigma(\mu'\to\mu)\,\Psi(x,\mu')\,d\mu'=\frac{\Sigma_s}{2}\int_{-1}^{1}\Psi(x,\mu')\,d\mu'.\end{aligned}\right\}\quad(5)$$

The boundary conditions (2) and the symmetry condition (3) are valid also for $\Psi(x,\mu)$.

We do not lose generality if we neglect the absorption of neutrons, since introducing the absorption (Σ_a) is equivalent to changing the asymptotic decay constant α by a constant amount $v\Sigma_a$. For this reason, we shall deal with nonabsorbing medium only, i.e. $\Sigma_t=\Sigma_s$.

It has been known for some time [8], [1] that Eq. (5) with $\Sigma_t=\Sigma_s$ is identical to the stationary transport equation for a multiplying medium. To see this analogy, we define an effective scattering cross-section, Σ^*, and an effective number of secondaries per collison, c^*, defined as

$$\Sigma^*=\Sigma_s-\frac{\alpha}{v} \quad\text{and}\quad c^*=\frac{\Sigma_s}{\Sigma^*}. \quad(6)$$

Eq. (5) then transforms into:

$$\mu\frac{\partial}{\partial x}\Psi(x,\mu)+\Sigma^*\Psi(x,\mu)=c^*\frac{\Sigma^*}{2}\int_{-1}^{1}\Psi(x,\mu')\,d\mu'$$

which is the form of a stationary transport equation for a multiplying medium. From this analogy, we can extract some initial information about the asymptotic decay constant. It is evident that, for a stationary distribution in a finite slab, a smaller thickness is required when the number of secondaries per collison is increased. The greater production of neutrons must be compensated for by larger escape from the system. This means that for smaller slabs the time-decay constant α will be larger [see Eq. (6)]. Furthermore, we see from Eq. (6) that the asymptotic decay constant α must be less than $v\Sigma_s$, since the number of secondaries per collision must be a positive and finite number. Hence

$$\alpha<v\Sigma_s. \quad(7)$$

In the next step we transform the integro-differential Eq. (5) into the integral form by integrating over x and taking into account the boundary conditions (2). We obtain:

$$\Psi(x,\mu)=\mathfrak{K}\langle\Psi(x,\mu)\rangle, \quad(8)$$

where the linear integral operator $\mathfrak{K}$ is defined as:

$$\left.\begin{aligned}&\mathfrak{K}\langle\Psi(x,\mu)\rangle\\&=\int_{\eta(\mu)}^{x}\frac{dx'}{\mu}\left\{\int_{-1}^{1}\Sigma(\mu\to\mu')\,\Psi(x';\mu')\,d\mu'\right\}e^{-\frac{x-x'}{\mu}\Sigma^*};\end{aligned}\right\}\quad(9)$$

$\eta(\mu)=-a$ for $\mu>0$ and $=+a$ for $\mu<0$.

The condition (7) for the asymptotic decay constant α can also be obtained from Eq. (9). To assure that, for $\mu\to 0$, the integrand on the right-hand side of the Eq. (9) remains finite, the effective cross-section Σ^* must be positive. Eq. (6) then gives $\alpha<v\Sigma_s$.

The solution of Eqs. (8), (9) satisfies the symmetry condition (3). We find $\Psi(x,-\mu)$ and $\Psi(-x,\mu)$ to be solutions of the same homogeneous integral equation, i.e. they are proportional to each other. Because of the continuity reasons at $x=0$ and $\mu=0$, this proportionality constant must be taken to be equal to unity. The condition (3) is then satisfied.

Generally, one should try to solve the integral Eq. (8) for the eigenvalue α and the associated angular flux distribution $\Psi(x,\mu)$ for each given slab thickness, $2a$. One is also interested in finding higher discrete and continuous eigensolutions which describe the approach to the asymptotic state of the neutron pulse. A rather complete review of the exact mathematical methods in connection with these problems has recently been given by WING [9]. However, one is inclined to believe that the exact mathematical methods can not as yet satisfactorily describe the behaviour of the neutron distribution for the majority of practical cases, especially when dealing with velocity-dependent problems. In this paper, we shall try to solve the integral Eq. (8) approximately using the variational method. The whole procedure can easily be extended to the velocity-dependent problems.

As a first step, we transform the homogeneous integral Eq. (8) into a nonhomogeneous integral equation by writing the angular distribution $\Psi(x,\mu)$ as

$$\Psi(x,\mu)=\Psi_{as}(x,\mu)-\Psi'(x,\mu) \quad(10)$$

where $\Psi_{as}(x,\mu)$ is some appropriate known function. It is convenient to use the infinite-medium solution for this purpose; this solution will be referred to as the asymptotic solution.

II. The asymptotic solution

We search for the solution of the integro-differential Eq. (5) corresponding to an infinite medium with an Ansatz form [8] like:

$$\Psi(x,\mu)\approx g(\mu)\,e^{-iBx}, \quad(11)$$

where $g(\mu)$ and B are the eigen function and the eigen value, respectively, of the following homogeneous integral equation:

$$\left.\begin{aligned}&(\Sigma^* - iB\mu)\, g(\mu)\\ &= \int_{-1}^{1} \Sigma(\mu\to\mu')\, g(\mu')\, d\mu' = \frac{\Sigma_s}{2}\int_{-1}^{1} g(\mu')\, d\mu' .\end{aligned}\right\} \quad (12)$$

This equation has to be solved for all possible eigen solutions. However, we shall take into account only the lowest real eigenvalue B.

The desired solution of Eq. (12) with isotropic scattering is well known [8], [1]:

$$g(\mu) = \frac{\text{const}}{\Sigma^* - iB\mu}. \quad (13)$$

The eigenvalue B is obtained, for each given asymptotic time constant α (i.e. for given Σ^*), from the transcendental equation:

$$\Sigma^* = \frac{B}{\tan(B/\Sigma_s)} \quad \text{or} \quad \frac{\alpha}{v} = \Sigma_s - \frac{B}{\tan(B/\Sigma_s)}. \quad (14)$$

In order to use expression (11) with Eq. (13) for our $\Psi_{as}(x,\mu)$ we first have to find a symmetric combination of the forms (11) associated with a given B. We see that for each given pair $[B, g(\mu)]$ a pair $[-B, g(-\mu)]$ is also a solution of the Eq. (12). Therefore, we shall write $\Psi_{as}(x,\mu)$ as

$$\Psi_{as}(x,\mu) = C_1 g(\mu) e^{-iBx} + C_2 g(-\mu) e^{iBx} \quad (15)$$

where C_1 and C_2 are arbitrary real constants. To assure that $\Psi_{as}(x,\mu)$ has the required symmetry properties with respect to the origin of the coordinate system (3), we take $C_1 = C_2 = C/2$ and write Eq. (15) as

$$\Psi_{as}(x,\mu) = C\,[g_1(\mu)\cos(Bx) + Bg_2(\mu)\sin(Bx)], \quad (16)$$

where $g_1(\mu)$ and $g_2(\mu)$ are real functions, defined by:

$$\left.\begin{aligned}2g_1(\mu) &= g(\mu) + g(-\mu), \quad g_1(-\mu) = g_1(\mu),\\ 2(iB)g_2(\mu) &= g(\mu) - g(-\mu), \quad g_2(-\mu) = -g_2(\mu)\end{aligned}\right\} \quad (17)$$

and where C is a positive real number whose value depends on the "source" distribution, i.e., on the contribution of neutrons decaying into the asymptotic state.

From Eqs. (13) and (17) we obtain:

$$g_1(\mu) = \frac{\Sigma_s \Sigma^*}{\Sigma^{*2} + B^2\mu^2}, \quad g_2(\mu) = \frac{\mu}{\Sigma^*} g_1(\mu). \quad (18)$$

The value of the constant in Eq. (13) has been chosen such that, in the limit $B \to 0$, $g_1(\mu) \to 1$.

Knowing $\Psi_{as}(x,\mu)$ we can write Eq. (10) as

$$\left.\begin{aligned}&\Psi(x,\mu)\\ &= C\,[g_1(\mu)\cos(Bx) + Bg_2(\mu)\sin(Bx) - h(x,\mu)],\end{aligned}\right\} \quad (19)$$

where $\Psi'(x,\mu) = C \cdot h(x,\mu)$.

The asymptotic solution (16) may describe approximately the actual flux distribution in regions which are several mean free paths away from the boundary of the slab. It is known [10], [11] that the problem of a finite medium is equivalent to the infinite medium problem providing suitable fictitious sources are introduced on the boundaries of a finite medium. The strength and the angular distribution of these sources are determined by requiring the collision density of the infinite-medium solution to vanish identically outside the slab region. The flux distribution within the slab is then equal to the sum of the general, infinite-medium, solution and the particular solution corresponding to the fictitious source distribution. The latter decays exponentially with the distance from the source location; hence, several mean free paths away from the boundary of the slab the actual flux distribution is essentially given by the asymptotic solution, which, in our case, will assume the form (16). The contribution to the actual flux of the fictitious sources is represented by the function $h(x,\mu)$ (19) which, therefore, is significantly different from zero only in the vicinity of the boundaries.

From Eq. (8) we obtain an integral equation for the function $h(x,\mu)$ if Eq. (19) is used for $\Psi(x,\mu)$:

$$h(x,\mu) = f(\mu)\, e^{-\frac{x}{\mu}\Sigma^*} + \mathfrak{K}\langle h(x,\mu)\rangle, \quad (20)$$

where

$$\left.\begin{aligned}f(\mu) &= [g_1(\mu)\cos(Ba) - Bg_2(\mu)\sin(Ba)]\times\\ &\times e^{-\frac{a}{\mu}\Sigma^*} = f(-\mu), \quad \mu > 0.\end{aligned}\right\} \quad (21)$$

We have arrived at an inhomogeneous integral Eq. (20) which will be useful for the variational method, but we have simultaneously introduced an auxiliary quantity B whose relation to the slab thickness has yet to be determined. To find this relation we use two integrals of the transport equation.

III. K-integrals of the transport equation

Though we do not know the angular flux distribution we can find two angular integrals of the flux whose space variation can be found. The two integrals are defined as:

$$K_i(x) = \int_{-1}^{1} \Psi(x,\mu)\, g_i(\mu)\, \mu\, d\mu, \quad i = 1, 2. \quad (22)$$

Multiplying either the integro-differential Eq. (5) or the integral Eq. (8) with $g_i(\mu)$ ($i = 1, 2$), then integrating over μ and using integral equations for $g_i(\mu)$ [similar to Eq. (12)], we obtain the identities:

$$\frac{d}{dx} K_1(x) = B^2 K_2(x) \quad \text{and} \quad \frac{d}{dx} K_2(x) = -K_1(x)$$

or

$$\frac{d^2}{dx^2} K_i(x) + B^2 K_i(x) = 0, \quad i = 1, 2. \quad (23)$$

These equations are valid for every x including the boundary region.

From the definition (22) and the symmetry condition (3) we obtain $K_1(x)$ and $K_2(x)$ as an odd and even function, respectively. The solution of Eq. (23) then takes the form:

$$\left.\begin{aligned}K_1(x) &= \text{const}\cdot B\cdot\sin(Bx),\\ K_2(x) &= \text{const}\cdot\cos(Bx).\end{aligned}\right\} \quad (24)$$

The value of the constant in the above equations depends on the initial conditions. These have already been taken into account when we wrote the asymptotic solution (16). To assure the same initial conditions, we use the asymptotic solution (16) for $\Psi(x,\mu)$ in Eqs. (22) and we obtain the value of the constant equal to $C \cdot \alpha_{12}$, where α_{12} is one of the integrals of the following form:

$$\alpha_{ik} = 2\int_0^1 g_i(\mu)\, g_k(\mu)\, \mu\, d\mu, \quad i,k=1,2. \tag{25}$$

Eqs. (24) then become:

$$K_1(x) = C \cdot \alpha_{12} B \cdot \sin(Bx), \tag{26a}$$

$$K_2(x) = C \cdot \alpha_{12} \cos(Bx). \tag{26b}$$

Using Eqs. (26), (22) and (19) we can obtain two conditions to be fullfilled for all x by the exact solution for $h(x,\mu)$. These two conditions have the following form:

$$\int_{-1}^1 h(x,\mu)\, g_1(\mu)\, \mu\, d\mu = 0, \tag{27a}$$

$$\int_{-1}^1 h(x,\mu)\, g_2(\mu)\, \mu\, d\mu = 0. \tag{27b}$$

In regions where the asymptotic solution predominates, the above conditions are trivial since the function $h(x,\mu)$ itself vanishes in such regions.

The conditions (27) are equivalent to the well known orthogonality relations for the eigen solutions $g(\mu)$ of the integral Eq. (12)[1].

We shall need the conditions (27) for $x=a$. The form of the function $h(x,\mu)$ for $x=a$ and $\mu<0$ can be obtained either from the integral Eq. (20) or from Eq. (19) with boundary conditions for $\Psi(x,\mu)$:

$$h(a,\mu) = g_1(\mu)\cos(Ba) + g_2(\mu) B \cdot \sin(Ba), \quad \mu<0.$$

Using this form of $h(a,\mu)$ for the negative part of μ-range, we can rewrite the conditions (27) as:

$$\left.\begin{aligned} &2\int_0^1 h(a,\mu)\, g_1(\mu)\, \mu\, d\mu \\ &\qquad = -\alpha_{12} B \sin(Ba) + \alpha_{11}\cos(Ba), \end{aligned}\right\} \tag{30a}$$

$$\left.\begin{aligned} &2\int_0^1 h(a,\mu)\, g_2(\mu)\, \mu\, d\mu \\ &\qquad = \alpha_{22} B \sin(Ba) - \alpha_{12}\cos(Ba), \end{aligned}\right\} \tag{30b}$$

where the constants α_{11}, α_{12}, and α_{22} are defined by Eq. (25). If the function $h(a,\mu)$ for $\mu>0$ were known (i.e. if the distribution of leaking neutrons were known) either of the above two equations would determine the parameter B, associated with given boundary condititons. We shall use a variational method to obtain an approximate expression for the left sides of Eqs. (30) which will give us an approximate relation between the buckling B^2 and the slab thickness $2a$.

[1] We multiply Eq. (27b) by iB, and add the resulting equating to the Eq. (27a). Using the definitions (17) we obtain:

$$\int_{-1}^1 h(x,\mu) g(\mu) \mu\, d\mu = 0. \tag{28}$$

Furthermore, say there would exist a complete set of the eigen functions $g_B(\mu)$ of the homogeneous integral Eq. (12) with the corresponding complete set of the eigen values B. Then, one could write the function $h(x,\mu)$ as:

$$h(x,\mu) = \sum_B c' g_B(\mu)\, e^{-iBx} + c'' g_B(-\mu)\, e^{iBx}], \tag{29}$$

where the summation (or the integration) goes over all higher discrete (or continuous) eigen values of Eq. (12). The lowest eigen value of Eq. (12), giving the buckling B^2, is excluded from the above summation since our function $h(x,\mu)$ must vanish in the asymptotic region. Put the development (29) into the Eq. (28). The condition (28) is satisfied for all x only when

$$\int_{-1}^1 g(\mu)\, g_B(\mu)\, \mu\, d\mu = 0.$$

That is, all higher eigen functions $g_B(\mu)$ must be orthogonal to the lowest eigen function $g(\mu)$. Such orthogonality conditions for the eigen functions of the transport operator can also be obtained directly from Eq. (12) [15], [13].

IV. The variational method

The advantage of the variational method is that we can connect the integrals on the left sides of the Eqs. (30) directly to the stationary value of the chosen functional; and that the error in the stationary functional is of the second order in magnitude. By this method we can obtain a fairly accurate dependence of B upon slab half-thickness a.

The form of the functional is similar to that used in the case of the stationary Milne's problem [12]. We define the functional operator $I\langle h\rangle$ by:

$$\left.\begin{aligned} I\langle h(x,\mu)\rangle = \int_{-a}^{a}\int_{-1}^{1} h^+(x,\mu)\Big[2f(\mu)\, e^{-\frac{x}{\mu}\Sigma^*} + \\ + \mathfrak{K}\langle h(x,\mu)\rangle - h(x,\mu)\Big]\, dx\, d\mu, \end{aligned}\right\} \tag{31}$$

where the function $h^+(x,\mu)$ represents an abbreviation of the expression:

$$h^+(x,\mu) = \int_{-1}^1 \Sigma(-\mu \to \mu')\, h(x,\mu')\, d\mu'. \tag{32}$$

Using well known methods, we can prove that the above functional (31) attains its stationary value I_{st} when the function $h(x,\mu)$ is a solution of the integral Eq. (20). The form of the stationary functional is of particular interest:

$$I_{st} = \int_{-a}^{a}\int_{-1}^{1} h^+(x,\mu)\, e^{-\frac{x}{\mu}\Sigma^*} f(\mu)\, dx\, d\mu. \tag{33}$$

To find the connection between I_{st} and the left sides of Eqs. (30), we multiply Eq. (30a) by $\cos(Ba)$, Eq. (30b) by $-B \cdot \sin(Ba)$ and add up the resulting equations. Using Eq. (21) we can write the final equation as

$$\left.\begin{aligned} &2\int_0^1 h(a,\mu)\, f(\mu)\, e^{\frac{a}{\mu}\Sigma^*} \mu\, d\mu \\ &\qquad = (\alpha_{11} + B^2\alpha_{22})\cos^2(Ba) - B^2\alpha_{22}. \end{aligned}\right\} \tag{34}$$

Furthermore, from Eqs. (20) and (32) we obtain,

$$h(a,\mu) = f(\mu)\, e^{-\frac{a}{\mu}\Sigma^*} + e^{-\frac{a}{\mu}\Sigma^*}\int_{-a}^{a} h^+(x',-\mu)\, e^{\frac{x'}{\mu}\Sigma^*} dx',$$

so that Eq. (34) may be rewritten as:

$$(\alpha_{11} + B^2\alpha_{22})\cos^2(Ba) = B^2\alpha_{22} + 2\int_0^1 f^2(\mu)\, \mu\, d\mu + $$
$$+ 2\int_0^1 f(\mu)\int_{-a}^{a} h^+(x,-\mu)\, e^{\frac{x}{\mu}\Sigma^*} d\mu\, dx$$

or

$$(\alpha_{11} + B^2\alpha_{22})\cos^2(Ba) - B^2\alpha_{22} - \beta - I_{st} = 0 \tag{35}$$

where β is defined by:

$$\beta = 2\int_0^1 f^2(\mu)\, \mu\, d\mu$$
$$= 2\int_0^1 [g_1(\mu)\cos(Ba) - g_2(\mu) B\sin(Ba)]^2 e^{-\frac{2a}{\mu}\Sigma^*} \mu\, d\mu.$$

The stationary functional I_{st} must still be determined. For this purpose, we use a variational trial function which is linear in the variational parameters A_i. The best form of the trial function $\tilde{h}(x,\mu)$ would be given by Eq. (29), i.e.

$$\tilde{h}(x,\mu)=\sum_i A_i[g_{1,B_i}(\mu)\cos(B_i x)+g_{2,B_i}(\mu)B_i\sin(B_i x)],$$

where the summation runs over all higher eigen values B_i and where $g_{1,B_i}(\mu)$ and $g_{2,B_i}(\mu)$ are defined similarly as $g_1(\mu)$ and $g_2(\mu)$, respectively, Eq. (17). However, as we do not yet know the higher eigen values B_i and the corresponding eigen functions $g_{B_i}(\mu)$ we have to use more simple trial functions, as for instance:

$$\left.\begin{aligned}&\tilde{h}(x,\mu)\\&=2A_1 e^{-\frac{a}{l}}\left[g_1(\mu)\cosh\left(\frac{x}{l}\right)+g_2(\mu)\,B\sinh\left(\frac{x}{l}\right)\right].\end{aligned}\right\}\quad(36)$$

Here, $g_1(\mu)$ and $g_2(\mu)$ correspond to the lowest eigen function and l is some arbitrary length $\leq 1/\Sigma_s$. The parameter l^{-1} should be close to the real part of the first higher eigen value B_2. We shall take $l^{-1}=\Sigma_s$.

In the above form of the trial function, we have taken into account the symmetry property of the function $h(x,\mu)$ and the fact that $h(x,\mu)$ is different from zero only in the vicinity of the boundaries.

The variational parameter A_1 should be determined from the equation:

$$\frac{\partial}{\partial A_1}I\langle\tilde{h}(x,\mu)\rangle=0. \quad (37)$$

Using Eq. (36) for $\tilde{h}(x,\mu)$ we obtain the following result:

$$A=y_1/y_2, \quad (38)$$

where y_1 and y_2 are definite integrals whose form is given in the Appendix. The stationary functional I_{st} (33) is now given by:

$$I_{st}=2y_1A_1=2y_1^2/y_2. \quad (39)$$

The angular distribution of the leaking neutrons may be determined by one iteration of the integral (20) using the variational trial function (36) with (38) as a first trial. One obtains the following result:

$$\left.\begin{aligned}\Psi(a,\mu)=C\{&g_1(\mu)\cos(Ba)+\\&+g_2(\mu)B\sin(Ba)-h(a,\mu)\},\quad \mu>0\end{aligned}\right\}\quad(40)$$

where

$$h(a,\mu)=f(\mu)e^{-\frac{a}{\mu}\Sigma^*}+A_1\left[\frac{F(-\mu)}{\Sigma^*+\mu/l}\left(1-e^{-\frac{2a}{l}-\frac{2a}{\mu}\Sigma^*}\right)+\right.$$
$$\left.+\frac{F(\mu)}{\Sigma^*-\mu/l}\left(e^{-\frac{2a}{l}}-e^{-\frac{2a}{\mu}\Sigma^*}\right)\right]$$

with

$$F(\mu)=\Sigma^* g_1(\mu)+B^2\mu\, g_2(\mu)+B\mu\, g_1(\mu)-B\Sigma^* g_2(\mu).$$

The final expression for the leakage angular flux is:

$$\left.\begin{aligned}\Psi(a,\mu)=C\Bigg\{&g_1(\mu)\cos(Ba)\left(1+e^{-\frac{2a}{\mu}\Sigma^*}\right)+\\&+g_2(\mu)B\sin(Ba)\left(1-e^{-\frac{2a}{\mu}\Sigma^*}\right)-\\&-A_1\left[\frac{F(\mu)}{\Sigma^*-\frac{\mu}{l}}\left(e^{-\frac{2a}{l}}-e^{-\frac{2a}{\mu}\Sigma^*}\right)+\right.\\&\left.+\frac{F(-\mu)}{\Sigma^*+\frac{\mu}{l}}\left(1-e^{-\frac{2a}{l}-\frac{2a}{\mu}\Sigma^*}\right)\right]\Bigg\}\end{aligned}\right\}\quad(41)$$

V. The extrapolated end-point

The relation between the parameter B and the slab thickness is usually expressed in terms of the extrapolated end-point x_0. One assumes the existence of the asymptotic region within the slab and one defines x_0 as the distance beyond the physical boundaries of the slab at which the asymptotic total flux, if extrapolated across the boundaries, would go to zero. Integrating Eq. (16) over μ we obtain:

$$\Psi_{as}(x)=\int_{-1}^{1}\Psi_{as}(x,\mu)\,d\mu\propto\cos(Bx), \quad (42)$$

where B is the lowest positive real solution of the Eq. (35). By definition, $\Psi_{as}(a+x_0)=0$ or

$$B=\frac{\pi}{2(a+x_0)}. \quad (43)$$

The extrapolated end-point x_0 may depend on the slab thickness. However, it approaches a finite limit for infinitely large slab thicknesses. Therefore, the parameter B becomes infinitely small when a is infinitely large.

In case the asymptotic region is not achieved within the slab one still can assign a meaning to the extrapolated endpoint x_0 defined by Eq. (43). Since the function $K_2(x)$ (26b) and the total asymptotic flux (42) have the same space-dependence, one could define x_0 by requiring that the extrapolated $K_2(x)$ be zero at the extrapolated end-point. However, in such cases (i.e. for small slabs), the definition of x_0 no longer has any physical meaning.

VI. Discussion of the numerical results

To obtain numerical results we have to solve the transcendental algebraic Eq. (35) for each given slab halfthickness (a). Only the smallest positive root (B) is required. We solve this algebraic equation numerically using the iterational procedure. The first guess for B may be obtained from Eq. (43) where the well-known value $0.71/\Sigma_s$ may be used as the first value for x_0. Using this B, we calculate Σ^* from Eq. (14), $g_1(u)$ and $g_2(u)$ from Eqs. (18) and finally calculate $F(a,x_0)$ which is the left side of the Eq. (35). The function $F(a,x_0)$ is positive for large values of x_0 (say for $x_0\Sigma_s\approx 1$) and becomes negative for smaller x_0. The position of the largest zero of this function gives us the extrapolated end-point x_0 for each given a.

Results for the extrapolated end-point x_0, the corresponding lowest space-eigen value B and for the asymptotic time decay constant α are given in Table 1. Fig. 1 represents the dependence of x_0 upon slab thickness d $(=2a)$ or upon the buckling B^2. The extrapolated end-point (x_0) is practically independent of the slab-dimensions when the slab-thickness is greater than 3 to 4 mean free paths. For smaller slabs the extrapolated end-point starts to increase rather rapidly.

A problem arises in cases of very thin slabs, say for $a\Sigma_s<0.3$, since the negative portion of the function $F(a,x_0)$ becomes very small and new zero positions (with smaller x_0) appear having the singularity of a tangent function. For this reason, it is practically impossible to find a solution for the extrapolated end-point in the limit of very small slab thicknesses. This also means that by following this procedure, we can not

Table 1. *The lowest space eigen value B, the extrapolated end-point x_0, and the asymptotic time decay constant α as functions of the slab thickness d. Monokinetic transport theory, isotropic scattering*

$d \cdot \Sigma_s$	$\frac{B}{\Sigma_s}$	$x_0\,\Sigma_s$	$\frac{\alpha}{v\Sigma_s}$
6.00	0.4234	0.7101	0.06048
5.00	0.4893	0.7102	0.08111
4.00	0.5795	0.7105	0.11454
3.00	0.7104	0.7113	0.17414
2.50	0.8005	0.7123	0.22331
2.00	0.9162	0.7145	0.2968
1.80	0.9720	0.7160	0.3368
1.60	1.0347	0.7181	0.3852
1.40	1.1052	0.7213	0.4447
1.20	1.1846	0.7260	0.5183
1.00	1.2732	0.7338	0.6095
0.90	1.3203	0.7397	0.6622
0.80	1.3677	0.7485	0.7184
0.70	1.4128	0,7619	0.7749
0.65	1.4332	0.7710	0,8016

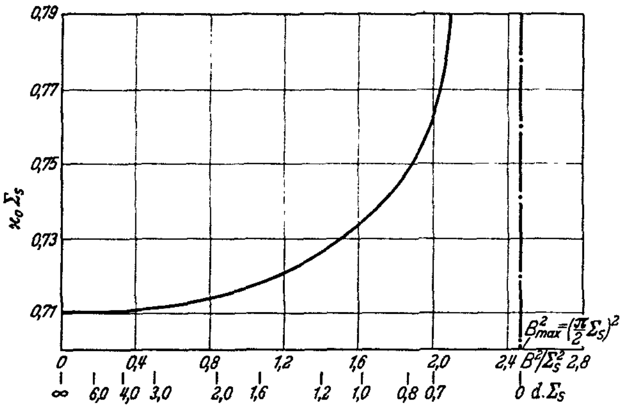

Fig. 1. The extrapolated end-point x_0 vs. buckling B^2 for slab geometry. Monokinetic transport theory with isotropic scattering

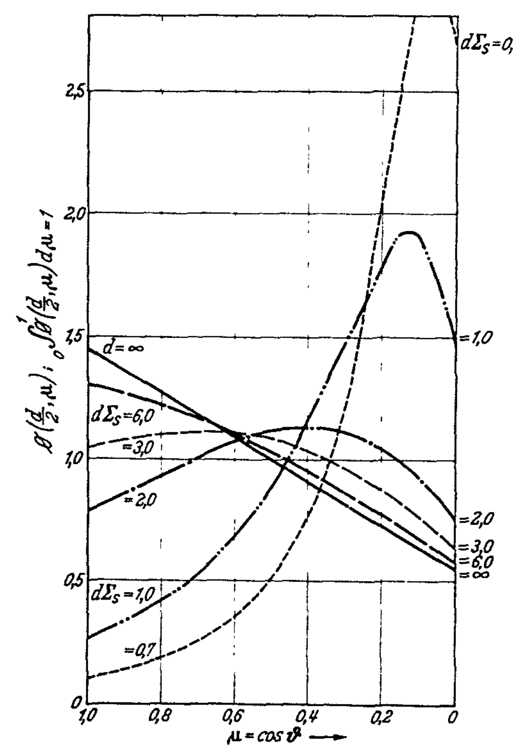

Fig. 2. The time-asymptotic angular flux of neutrons leaking into the vacuum for different slab thicknesses

say for certain how the approach to the limiting value $v\Sigma_s$ of the asymptotic decay constant (α) proceeds.

In the limit $\alpha \to v\Sigma_s$, Eq. (14) gives $\Sigma^* \to 0$ and $B \to \pi\Sigma_s/2$, so $g_1(\mu) \to 0$ and $g_2(\mu) \to \Sigma_s/\mu B^2$. Furthermore, Eq. (43) shows for this limiting case that $a + x_0 \to 1/\Sigma_s$. We see that the largest possible real buckling is equal to $B^2_{\max} = (\pi\Sigma_s/2)^2$.

We still have to determine the critical slab half-thickness (a_c) for which the asymptotic decay constant α achieves its limiting value. The left side of the Eq. (35) is identically equal to zero when $\Sigma^* \to 0$. This prevents us from calculating the critical slab thickness. However, we can still obtain some information from the expression for the stationary functional (33), (39). The limiting form of the stationary functional (33), i.e. the form of the functional (33) for $\Sigma^* \to 0$, is:

$$I_{st} \to -2B^2\alpha_{22} \cdot \sin^2(Ba),$$

where $B^2 = (\pi\Sigma_s/2)^2$ and $\alpha_{22} = \frac{-2}{B^4}\Sigma_s^2 \lim_{\mu\to 0}(\ln\mu)$, which is a divergent value.

Since the stationary functional must always be finite, the critical half-thickness a_c must be zero. The limiting value of the extrapolated end-point would then be $1.0/\Sigma_s$. The above information suggests that the lowest mode may always exist within the slab, no matter how thin the slab is. This apparently agrees with NELKIN's suggestion [14] that the asymptotic mode may exist within the slab even for thicknesses smaller than some critical value. However, since our result can not be considered as a proof, the fundamental question concerning the existence of the fundamental mode may still be considered as unresolved.

The results for the angular leakage flux show the characteristic behaviour of the angular distribution of neutrons leaking into the vacuum (Fig. 2). In cases of thick slabs, the angular distribution is an ordinary one, more or less similar to a cosine distribution. However, the maximum of the distribution starts to shift toward more glancing directions when the slab thickness is smaller than about 4 mean free paths. For extremely small slab thicknesses ($< 1.0/\Sigma_s$) the maximum of the angular distribution becomes strongly pronounced and its direction is almost parallel to the surface of the slab. The explanation of such behaviour may be the following: Up to the time the asymptotic state has been established within the slab only those neutrons remain within the slab which have been moving and which have been scattered approximately parallel to the surface of the slab. Other neutrons have escaped from the slab during the approach to the time-asymptotic state.

There should not be much difficulty in detecting this angular peaking experimentally.

VII. Appendix

To find the dependence of the functional $I\langle\tilde{h}\rangle$ (31) upon the variational parameter A_1, we use the variational trial function (36) for $\tilde{h}(x,\mu)$. The integration over x can be carried out immediately, whereas it is more convenient to perform the integration over μ numerically.

We obtain the following expression for $I\langle\tilde{h}\rangle$:

$$I = 4A_1 y_1 - 2A_1^2(y_{23} - y_{22} - y_{21})$$

with

$$y_1 = \int_0^1 [g_1(\mu)\cos(Ba) - g_2(\mu)\, B\sin(Ba)]\, L(\mu)\, \mu\, d\mu,$$

where

$$L(\mu) = \frac{F(\mu)}{\Sigma^* - \frac{\mu}{l}}\left(e^{-\frac{2a}{l}} - e^{-\frac{2a}{\mu}\Sigma^*}\right) + \\ + \frac{F(-\mu)}{\Sigma^* + \frac{\mu}{l}}\left(1 - e^{-\frac{2a}{l} - \frac{2a}{\mu}\Sigma^*}\right);$$

$$y_{21} = \int_0^1 \frac{F(-\mu)}{\Sigma^* + \frac{\mu}{l}}\left\{\frac{l}{2}F(\mu)\left(1 - e^{-\frac{4a}{l}}\right) + \\ + 2aF(-\mu)\, e^{-\frac{2a}{l}} - e^{-\frac{2a}{l}}\mu L(\mu)\right\} d\mu,$$

$$y_{22} = \int_0^1 \frac{F(\mu)}{\Sigma^* - \frac{\mu}{l}}\left\{\frac{l}{2}F(-\mu)\left(1 - e^{-\frac{4a}{l}}\right) + \\ + 2aF(\mu)\, e^{-\frac{2a}{l}} - \mu L(\mu)\right\} d\mu,$$

$$y_{23} = \int_0^1 \left\{2\,[F(\mu)\, g^-(\mu) + F(-\mu)\, g^+(\mu)]\, a e^{-\frac{2a}{l}} + \\ + \frac{l}{2}[F(\mu)\, g^+(\mu) + F(-\mu)\, g^-(\mu)\right\} d\mu,$$

where

$$g^+(\mu) = g_1(\mu) + Bg_2(\mu) \quad \text{and} \quad g^-(\mu) = g_1(\mu) - Bg_2(\mu).$$

The optimal value of the variational parameter A_1 is obtained from Eq. (37), i.e.

$$\frac{\partial I}{\partial A_1} = 0 = 4 \cdot y_1 - 4 \cdot A_1(y_{23} - y_{22} - y_{21})$$

or

$$A_1 = y_1/(y_{23} - y_{22} - y_{21}) = y_1/y_2.$$

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