

KFK-393

**KERNFORSCHUNGSZENTRUM
KARLSRUHE**

Dezember 1965

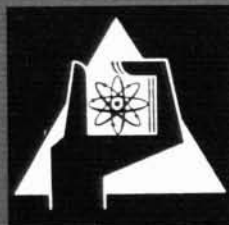
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Institut für Reaktorentwicklung

FORTRAN-Programme for Evaluation of Besselfunctions
of Complex Arguments

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Complex Arguments

by

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1. Abstract

A digital computer programme has been developed which will evaluate the zero and first order Bessel functions of the first and the second kind for complex arguments ($z = x + i y$). The zero order Bessel functions are the fundamental solutions of the differential equation

$$(1) \quad z \frac{d^2 w}{d z^2} + \frac{d w}{d z} + z w = 0$$

The first order Bessel functions are the fundamental solutions of the equation

$$(2) \quad z^2 \frac{d^2 w}{d z^2} + z \frac{d w}{d z} + (z^2 - 1) w = 0$$

The programme will be described. Results of sample problems and the FORTRAN listing of the code will be given.

2. Description of the Programme

2.1 The programme has been written as a FORTRAN subroutine called BESSEL. It will compute

J_0 the zero order Bessel function of the first kind
 J_1 the first order Bessel function of the first kind
 Y_0 the zero order Bessel function of the second kind
 Y_1 the first order Bessel function of the second kind

of the complex argument $z = x + i y$. These functions are defined by the following equations for small and large absolute values of z , respectively:

$$(3) \quad J_0 = \sum_{k=0}^{\infty} \frac{1}{k! k!} \left(\frac{i z}{2} \right)^{2k}$$

$$(4) \quad J_0 = \sqrt{\frac{2}{\pi z}} \left[\cos\left(z - \frac{\pi}{4}\right) \cdot P_0(z) - \sin\left(z - \frac{\pi}{4}\right) \cdot Q_0(z) \right]$$

$$(5) \quad J_1 = \frac{z}{2} \sum_{k=0}^{\infty} \frac{1}{k! (k+1)!} \left(\frac{iz}{2} \right)^{2k}$$

$$(6) \quad = \sqrt{\frac{2}{\pi z}} \left[\cos\left(z - \frac{3\pi}{4}\right) \cdot P_1(z) - \sin\left(z - \frac{3\pi}{4}\right) \cdot Q_1(z) \right]$$

$$(7) \quad Y_0 = \frac{2}{\pi} J_0(z) \ln \frac{\gamma z}{2} - \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{2}{k! k!} \left(\frac{iz}{2} \right)^{2k} \left(\sum_{n=1}^k \frac{1}{n} \right)$$

$$(8) \quad = \sqrt{\frac{2}{\pi z}} \left[\sin\left(z - \frac{\pi}{4}\right) \cdot P_0(z) + \cos\left(z - \frac{\pi}{4}\right) \cdot Q_0(z) \right]$$

$$(9) \quad Y_1 = \frac{2}{\pi} J_1(z) \ln \frac{\gamma z}{2} - \frac{2}{\pi z} - \frac{z}{2\pi} \sum_{k=0}^{\infty} \frac{1}{k! (k+1)!} \left(\frac{iz}{2} \right)^{2k} \left(\sum_{n=1}^k \frac{1}{n} + \sum_{n=1}^{k+1} \frac{1}{n} \right)$$

$$(10) \quad = \sqrt{\frac{2}{\pi z}} \left[\sin\left(z - \frac{3\pi}{4}\right) \cdot P_1(z) + \cos\left(z - \frac{3\pi}{4}\right) \cdot Q_1(z) \right]$$

Here we have used the abbreviations

$$P_n(z) = 1 - \frac{(n,2)}{(2z)^2} + \frac{(n,4)}{(2z)^4} - \frac{(n,6)}{(2z)^6} + \dots$$

$$Q_n(z) = \frac{(n,1)}{(2z)} - \frac{(n,3)}{(2z)^3} + \frac{(n,5)}{(2z)^5} - \dots$$

with $n = 0$ and 1 respectively and

$$(n,m) = \frac{(4n^2 - 1^2) (4n^2 - 3^2) (4n^2 - 5^2) \dots (4n^2 - (2m - 1)^2)}{2^{2m} \cdot m!}$$

$$\gamma = 1.781072418 \text{ (Euler's constant)}$$

See: Jahnke-Emde-Lösch: Tafeln höherer Funktionen, 6. Auflage, 1960, Teubner

2.2 The series expansions of equations (3) through (10) will be evaluated until the error made by neglecting the following term will be less than 10^{-6} or 10^{-7} . The value 10^{-6} will be used if the control number N is positive, the value 10^{-7} will be used if N is negative. In the programme the check will be made on the absolute error if the absolute value of the series expansion is less than 1, and on the relative error if this value is greater than 1.

2.3 Since the Bessel functions of a complex argument $z = x + i y$ with a large absolute value of y would be a very large number of the order of $e^{|y|}$, the programme will compute the functions J_0 , J_1 , Y_0 and Y_1 only if $|y| < 100$. For $|y| > 100$ the programme instead will give the values of

$$J_0 \cdot e^{-|y|}, \quad J_1 \cdot e^{-|y|}, \quad Y_0 \cdot e^{-|y|} \text{ and } Y_1 \cdot e^{-|y|}$$

in order to avoid exponent overflow.

2.4 The programme BESSEL will call several of the standard FORTRAN functions such as the exponential and trigonometric functions.

2.5 An error stop ERROR KENN IN BESSEL will occur if the control number N is not equal to ± 1 , ± 2 or ± 3 . An error stop ERROR BETR IN BESSEL will occur if the absolute value of $|z|$ is less than 10^{-49} and if Bessel functions of the second kind ($N = \pm 2$ or ± 3) are to be computed.

An error stop will occur if the real part of the complex number $z = x + i y$ is larger than can be handled by the trigonometric functions SINP (x) and CØSF (x). No other errors will be detected by the programme.

3. Usage of the Programme

There are three different ways to use the programme.

3.1 Bessel Functions of the First Kind

CALL BESSEL (X, Y, N, A, B, C, D)

X and Y are floating point variables or constants

N is a fixed point variable or constant of the value +1 or -1

A, B, C, D are floating point variables.

The programme will calculate

$$A + i B = J_0 (X + i Y)$$

$$C + i D = J_1 (X + i Y)$$

3.2 Bessel Functions of the Second Kind

CALL BESSEL (X, Y, N, E, F, G, H)

X and Y are floating point variables or constants

N is a fixed point variable or constant of the value +2 or -2

E, F, G, H are floating point variables

The programme will calculate

$$\left. \begin{aligned} E + i F &= Y_0 (X + i Y) \\ G + i H &= Y_1 (X + i Y) \end{aligned} \right\} \text{ see 2.3}$$

3.3 Bessel Functions of the First and Second Kind

CALL BESSEL (X, Y, N, A, B, C, D, E, F, G, H)

X and Y are floating point variables or constants

N is a fixed point variable or constant of the value +3 or -3

A through H are floating point variables

The programme will calculate

$$\left. \begin{aligned} A + i B &= J_0 (X + i Y) \\ C + i D &= J_1 (X + i Y) \\ E + i F &= Y_0 (X + i Y) \\ G + i H &= Y_1 (X + i Y) \end{aligned} \right\} \text{ see 2.3}$$

4. Sample Problems

For definitions of X, Y and A through H see page 5. (E-01 equals 10^{-1})

X	Y	A	B	C	D
1.	0.	7.65198E-01	0.	4.40051E-01	0.
0.	1.	1.26607	0.	0.	5.65159E-01
10.	0.	-2.45936E-01	0.	4.34727E-02	0.
0.	10.	2.81571E+03	0.	0.	2.67099E+03

X	Y	E	F	G	H
1.	0.	8.82570E-02	0.	-7.81213E-01	0.
0.	1.	-2.68032E-01	1.26607	-5.65159E-01	3.83186E-01
10.	0.	5.56712E-02	0.	2.49015E-01	0.
0.	10.	0.	2.81571E+03	-2.67099E+03	0.

5. Listing of the Programme

```
C      BESELFUNKTION                                02550
C      E.G.SCHLECHTENDAHL   IRE   KERNFORSCHUNG KARLSRUHE 02550
SUBROUTINE BESSEL (XB,YB,KEN ,BJOR,BJOI,BJ1R,BJ1I,BNOR,BNOI,BN1R,
1BN1I)                                02550
  X=ABSF(XB)                                02550
  Y=YB* SIGNF(1.,XB)                        02550
  IF(KEN)1501,1502,1503                    02550
1501 KENN=-KEN                               02550
  EXA=1.E-14                                02550
  GO TO 1504                                02550
1503 KENN=KEN                               02550
  EXA=1.E-12                                02550
1504 IF(KENN-3)1505,1505,1502              02550
1502 PRINT 1506,KEN,X,Y                    02550
1506 FORMAT(21H ERROR KENN IN BESSELI12,2E16.8) 02550
  CALL EXIT                                 02550
1505 Z=X**2+Y**2                            02550
  IF(Z-49.)20,21,21                        02550
  20 PRO=1.                                  02550
  PIO=0.                                    02550
  PR1=0.                                    02550
  PI1=0.                                    02550
  A=0.5*X                                  02550
  B=0.5*Y                                  02550
  24 XN=1.                                  02550
  U=1.                                      02550
  V=0.                                      02550
  25 FO=1./XN                               02550
  Z=(U*A-B*V)*FO                          02550
  V=(V*A+B*U)*FO                          02550
  PR1=PR1+Z                                02550
  PI1=PI1+V                                02550
  U=(B*V-Z*A)*FO                          02550
  V=(-B*Z-A*V)*FO                         02550
  PRO=PRO+U                                02550
  PIO=PIO+V                                02550
  XN=XN+1.                                  02550
  QR1=PR1**2+PI1**2                        02550
  QRO=PRO**2+PIO**2                        02550
  QRO=MAX1F(1.,QRO)                        02550
  QR1=MAX1F(1.,QR1)                       02550
  IF((U**2+V**2)/MIN1F(QRO,QR1)-EXA )26,25,25 02550
  26 BESJOR=PRO                             02550
  BESJOI=PIO                               02550
  BESJ1R=PR1                              02550
  BESJ1I=PI1                              02550
  GO TO (1000,1004,1004),KENN              02550
  21 ZR=0.5*X/Z                             02550
  ZI=-0.5*Y/Z                             02550
  A=1.                                      02550
  B=0.                                      02550
  XN=0.                                    02550
  N=1                                       02550
  FO=1.                                    02550
  F1=1.                                    02550
  PRO=1.                                    02550
  PIO=0.                                    02550
  PR1=1.                                    02550
  PI1=0.                                    02550
  QRO=0.                                    02550
```

```

      QI0=0.                                02550
      QR1=0.                                02550
      QI1=0.                                02550
3     XN=XN+1.                              02550
      Z=(2.*XN-1.)**2                      02550
      FO=-FO*0.25*Z/XN                     02550
      F1=(-0.25*Z+1.)*F1/XN               02550
      Z=A*ZR-B*ZI                           02550
      B=A*ZI+B*ZR                           02550
      A=Z                                    02550
      IF(N)1,2,1                             02550
1     QRO=QRO+FO*A                          02550
      QIO=QIO+FO*B                          02550
      QR1=QR1+F1*A                          02550
      QI1=QI1+F1*B                          02550
      N=0                                     02550
      GO TO 3                                02550
2     N=1                                    02550
      A=-A                                    02550
      B=-B                                    02550
      PRO=PRO+FO*A                          02550
      PIO=PIO+FO*B                          02550
      PR1=PR1+F1*A                          02550
      PI1=PI1+F1*B                          02550
      Z=FO**2*(A**2+B**2)                   02550
      IF(Z-1.E-12)4,3,3                     02550
4     Z=SQRTF(X**2+Y**2)                    02550
      U=SQRTF(MAX1F(0.,0.5*(Z+X)))           02550
      V=SIGNF(SQRTF(MAX1F(0.,0.5*(Z-X))),-Y) 02550
2006  U=0.79788456*U/Z                      02550
      V=0.79788456*V/Z                      02550
      Z=X-0.78539816                         02550
      A=SINF(Z)                              02550
      B=COSF(Z)                              02550
      IF(ABSF(Y)-100.)6,7,7                 02550
7     CH=0.5                                02550
      SH=0.5*SIGNF(1.,Y)                    02550
      GO TO 8                                02550
6     Z=EXPF(Y)                              02550
      CH=(Z+1./Z)*C.5                       02550
      SH=Z-CH                                02550
8     ZR=B*CH*PRO+A*SH*PIO-A*CH*QRO+B*SH*QIO 02550
      ZI=B*CH*PIO-A*SH*PRO-A*CH*QIO-B*SH*QRO 02550
      BESJOR=U*ZR-V*ZI                      02550
      BESJOI=U*ZI+V*ZR                      02550
      ZR=A*CH*PR1-B*SH*PI1+B*CH*QR1+A*SH*QI1 02550
      ZI=B*SH*PR1+A*CH*PI1+B*CH*QI1-A*SH*QR1 02550
      BESJ1R=U*ZR-V*ZI                      02550
      BESJ1I=U*ZI+V*ZR                      02550
      GO TO (1000,1005,1005),KENN           02550
1000  BJOR=BESJOR                           02550
      BJCI=BESJOI                           02550
      BJ1R=BESJ1R*SIGNF(1.,XB)              02550
      BJ1I=BESJ1I*SIGNF(1.,XB)              02550
      GO TO (1002,1002,1003),KENN           02550
1001  IF(XB)1700,1701,1701                 02550
1701  BJOR=UO                               02550
      BJOI=VO                               02550
      BJ1R=U1                               02550
      BJ1I=V1                               02550
1002  RETURN                               02550
```

```
1003 IF(XB)1700,1801,1801                                02550
1801 BNOR=UO                                             02550
      BNOI=VO                                             02550
      BN1R=U1                                             02550
      BN1I=V1                                             02550
      GO TO 1002                                          02550
1005 ZR=A*CH*PRO-B*SH*PIO+B*CH*QRO+A*SH*QIO           02550
      ZI=A*CH*PIO+B*SH*PRO+B*CH*QIO-A*SH*QRO           02550
      UO=U*ZR-V*ZI                                       02550
      VO=U*ZI+V*ZR                                       02550
      ZR=-B*CH*PR1-A*SH*PI1+A*CH*QR1-B*SH*QI1         02550
      ZI=-B*CH*PI1+A*SH*PR1+A*CH*QI1+B*SH*QR1         02550
      U1=U*ZR-V*ZI                                       02550
      V1=U*ZI+V*ZR                                       02550
      GO TO(1000,1001,1000),KENN                         02550
1004 Z=X**2+Y**2                                         02550
      IF(Z-1.E-40)3001,3002,3002                        02550
3001 U1=1.E30*X                                         02550
      V1=1.E30*Y                                         02550
      Z=U1**2+V1**2                                       02550
      IF(Z-1.E-38)5000,5001,5001                        02550
5001 U1=-0.63661977E+30*(U1/Z)                          02550
      V1=+0.63661977E+30*(V1/Z)                          02550
      A=ABSF(X)                                           02550
      B=ABSF(Y)                                           02550
      IF(-A+B)5003,5003,5004                              02550
5003 ARG=ATANF(B/A)                                      02550
      GO TO 5006                                          02550
5004 ARG=1.5707964-ATANF(A/B)                            02550
5006 IF(-Y)5008,5008,5009                               02550
5009 ARG=-ARG                                            02550
5008 VO=ARG*0.63661977                                   02550
      UO=0.63661977*(LOGF(0.89053624*SQRTF(Z))-69.077553) 02550
3003 GO TO (1000,1001,1000),KENN                        02550
3002 U=(X**2-Y**2)*0.25                                  02550
      V=X*Y*0.5                                           02550
      FOR=1.                                              02550
      FOI=0.                                              02550
      FIR=0.5*X                                           02550
      FII=C.5*Y                                           02550
      SOR=0.                                              02550
      SOI=0.                                              02550
      SIR=C.5*FIR                                         02550
      SII=0.5*FII                                         02550
      BO=1.                                              02550
      BI=1.                                              02550
      A=0.                                              02550
      XV=0.                                              02550
3004 XV=XV+1.                                           02550
      A=A+1./XV                                           02550
      FOO=XV**2                                           02550
      FO1=(-U*FOI-V*FOR)/FOO                              02550
      FOR=(-FOR*U+FOI*V)/FOO                              02550
      FOI=FO1                                             02550
      FOO=XV*(XV+1.)                                       02550
      FO1=(-FII*U-FIR*V)/FOO                              02550
      FIR=(-FIR*U+FII*V)/FOO                              02550
      FII=FO1                                             02550
      GOR=FOR*A                                           02550
      GOI=FOI*A                                           02550
      FOO=A+0.5/(XV+1.)                                   02550
```

```
G1R=F1R*F00 02550
G1I=F1I*F00 02550
SOR=SOR+GOR 02550
SOI=SOI+GOI 02550
S1R=S1R+G1R 02550
S1I=S1I+G1I 02550
BO=MAX1F(B0,SOR**2+SOI**2) 02550
B1=MAX1F(B1,S1R**2+S1I**2) 02550
IF((GOR**2+GOI**2)/BO-EXA )3005,3004,3004 02550
3005 IF((G1R**2+G1I**2)/B1-EXA )3006,3004,3004 02550
3006 A=ABSF(X) 02550
      B=ABSF(Y) 02550
      R=SQRTF(A**2+B**2) 02550
4001 IF(-A+B)4003,4003,4004 02550
4003 ARG=ATANF(B/A) 02550
      GO TO 4006 02550
4004 ARG=1.5707964-ATANF(A/B) 02550
4006 IF(-Y)4008,4008,4009 02550
4009 ARG=-ARG 02550
4008 BO=LOGF(0.89053624*R) 02550
      U0=(BESJOR*BO-BESJOI*ARG-SOR)*0.63661977 02550
      V0=(BESJOI*BO+BESJOR*ARG-SOI)*0.63661977 02550
      U1=(BESJ1R*BO-BESJ1I*ARG-S1R-X/Z)*0.63661977 02550
      V1=(BESJ1I*BO+BESJ1R*ARG-S1I+Y/Z)*0.63661977 02550
      GO TO 3003 02550
5000 TYPE 5010,KEN,X,Y 02550
5010 FORMAT(21H ERROR BETR IN BESSELI12,2E16.8) 02550
      CALL EXIT 02550
1700 A=SIGNF(2.,+YB) 02550
      U0=U0-A*BESJOI 02550
      V0=V0+A*BESJOR 02550
      U1=-U1+A*BESJ1I 02550
      V1=-V1-A*BESJ1R 02550
      GO TO (1801,1701,1801),KENN 02550
```