KERNFORSCHUNGSZENTRUM

KARLSRUHE

September 1966

KFK 483

Institut für Experimentelle Kernphysik

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I. Introduction

The slotted iris structure has been proposed by Giordano[1)]for the acceleration of protons and this structure has been investigated subsequently by a number of authors. However, expressions for the quality factor and the shunt impedance have not been published so far and a systematic optimization of this structure has not been performed. This may be due to the fact that a rigorous treatment is not feasible and also computer programs which have been developed for structures with rotation symmetry around the axis cannot be applied.

The slotted iris structure is of particular interest for a superconducting high energy linear accelerator for protons. Because of its relative mechanical simplicity it seems possible to coat this structure with a superconducting layer and to cool away the dissipated power. In order to learn more about the properties of the slotted iris structure extensive measurements have been performed by Eschelbacher[2)] in this laboratory.

For an interpretation of these results approximate expressions for the relevant quantities will be derived in this paper. For this purpose a cavity consisting of a number of cells will be treated as a chain of individual cells with the coupling introduced additionally. Since it is hard to achieve resonance coupling in a conventional slotted iris structure we shall consider here only weak coupling. In this case the π-mode seems to be most advantageous (Smith[3)]) and hence only this mode will be discussed. However, most of the results can be applied to a $\pi/2$ mode structure with resonant coupling.

II. Computation of the Fields and the Resonance Frequency

First the field in an individual cell with drift tubes but without coupling slots will be calculated. The geometrical dimensions of the cell are defined in fig. 1. The thickness of the cell walls is neglected.

A rigorous solution of Maxwell's equation cannot be found analytically for such a cavity. An approximate treatment for reentrant cavities based on a variation principle has been developed by Bernier[4]. Independent solutions for the regions $r \leq a$ and $a \leq r \leq R$ are deduced and these are matched approximately at the boundary $r = a$.
For the acceleration of particles we are interested only in a mode with an axial electric field. For a cavity without drift tubes the simplest is the TM_{010} mode for which

$$E_z = -iZ_0 J_0(kr); \quad E_r = E_\varphi = 0 \qquad (1)$$

and

$$H_\varphi = J_1(kr); \quad H_r = H_z = 0$$

where $Z_0 = \sqrt{\mu_0/\varepsilon_0} = 377\Omega$ and $J_0(kr)$ and $J_1(kr)$ are the Bessel functions. For the wave number we have $k = 2\pi/\lambda = \omega/c$.

If drift tubes are inserted the field lines are distorted. However, if the cavity is short ($L/R < 1$) this distortion will not change the main features of this mode. Then the solutions for the fields in the two regions are+)

$r \leq a$

$$E_z = -iAZ_0 J_0(kr)$$

$$H_\varphi = AJ_1(kr) \qquad (2)$$

$a \leq r \leq R$

$$E_z = -iZ_0 L_0(kr)$$

$$H_\varphi = L_1(kr)$$

- 3 -

+) The normalization and the units are irrelevant, since they do not enter into the frequency, coupling coefficient and Q-value.

The functions L_n are linear combinations of Bessel and Neuman functions. They are defined by

$$L_n(kr) = J_n(kr)N_0(kR) - N_n(kr)J_0(kR) \tag{3}$$

Obviously one has $L_0(kR) = 0$ since the electric field vanishes at the cavity wall.

If the gap between the drift tubes disappears ($g = 0$) the boundary condition on the inner cavity wall requires

$$L_0(ka) = 0 \tag{4}$$

The solution of (4) yields the resonance frequency $\omega_0 = kc$ of the cavity. If we write $k = \tau_{01}/R$ then τ_{01} is the first zero of the function $L_n(x)$ which depends on the ratio a/R. τ_{01} has been tabulated[5] but for the range of a/R which is of interest here a good approximation is given by

$$kR = \tau_{01} = \frac{1}{1-\frac{a}{R}} \left\{ 2{,}40 + \frac{0{,}737\left(\frac{a}{R}\right)}{0{,}977\left(\frac{a}{R}\right) + 0{,}023} \right\} \tag{4a}$$

For drift tubes with a finite gap the condition (4) cannot be satisfied rigorously. A good approximation is obtained[6], if the solutions (2) are matched at r = a by requiring that the electric potential and the currents are continuous which gives

$$\begin{aligned} gAJ_0(ka) &= L \cdot L_0(ka) \\ AJ_1(ka) &= L_1(ka) \end{aligned} \tag{5}$$

Solving for A one obtains

$$A = \frac{L_1(ka)}{J_1(ka)} = \frac{L_0(ka)}{J_0(ka)} \cdot \frac{L}{g} \tag{6}$$

and finally

$$\frac{L_1(ka)}{L_0(ka)} = \frac{J_1(ka)}{J_0(ka)} \cdot \frac{L}{g} \tag{7}$$

Since the condition (4) is now not satisfied at the surface of the drift tubes (r = a) the matching according to (5) corresponds to the introduction of an effective drift tube radius. If this approximation is not accurate enough the full variation procedure has to be performed[4)6)].

Equation (7) can be written

$$\frac{\dfrac{N_0(kR)}{J_0(kR)} - \dfrac{N_1(ka)}{J_1(ka)}}{\dfrac{N_0(kR)}{J_0(kR)} - \dfrac{N_0(ka)}{J_0(ka)}} = \frac{L}{g} \tag{8}$$

This relation determines now the resonance frequency $kc = \omega_0$. An explicite solution is not possible, however. Therefore the left side of (8) has been calculated for given values of a/R as a function of $kR = 2\pi R/\lambda$. The results are displayed in fig. 2. For given geometrical dimensions of an isolated cell the appropriate resonance frequency can be taken from this figure.

The resonance frequency ω_q of the q-mode of a chain of coupled cells is given by[7)]

$$\left(\frac{\omega_{(q)}}{\omega_0}\right)^2 = \frac{1}{1 + \varkappa \cos\varphi} \tag{9}$$

where $\omega_0 = \omega_{(\pi/2)}$ is the frequency of an isolated cell and $\varkappa$ is the coupling coefficient[+)]. For the π-mode the phase shift between ad-

- 5 -

+) Sometimes a different definition of the coupling coefficient is used resulting in the dispersion relation

$$(\omega_{(q)}/\omega_0)^2 = 1/1 + k(1-\cos\varphi)$$

and one has $k/(1+k) = -\varkappa$ which for weak coupling becomes $k \approx -\varkappa$.

jacent cells is $\varphi = \pi$ and

$$\omega^2_{(\pi)} = \frac{\omega^2_0}{1-\varkappa} \tag{10}$$

The coupling coefficient $\varkappa$ can be determined experimentally from the width of the dispersion diagram according to (9). A theoretical expression will be derived in section V.

III. Volume and Surface Integrals

In order to compute the quality factor, the shunt impedance and other quantities various integrals over the fields are required. These will be calculated in this section neglecting the coupling of the cells.

The electric energy W_e contained in the cavity is given by+)

$$2\,W_e = \int_V |E^2|\,dV = 2\pi g \int_0^a A^2 J_0^2 r dr + 2\pi L \int_a^R L_0^2 r dr = \tag{11}$$

$$= \pi R^2 L \left\{ L_1^2(kR) - (\tfrac{a}{R})^2 \left[L_0^2(ka) + L_1^2(ka) - \tfrac{g}{L} A^2 (J_0^2(ka) + J_1^2(ka)) \right] \right\}$$

$$= \pi R^2 L\, L_0^2(ka) \left\{ \frac{L_1^2(kR)}{L_0^2(ka)} - (\tfrac{a}{R})^2 (1 - \tfrac{L}{g}) \; (1 - \frac{L}{g} \frac{J_1^2(ka)}{J_0^2(ka)}) \right\}$$

In an analogous way one finds for the magnetic energy

$$2\,W_m = \int_V |H^2|\,dV =$$

$$\pi R^2 L \left\{ L_1^2(kR) - (\tfrac{a}{R})^2 [L_1^2(ka) - L_0(ka) L_2(ka) - \tfrac{g}{L} A^2 (J_1^2(ka) - J_0(ka) J_2(ka))] \right\} =$$

$$\pi R^2 L L_0^2(ka) \left\{ \frac{L_1^2(kR)}{L_0^2(ka)} - (\tfrac{a}{R})^2 [\frac{L_1^2(ka)}{L_0^2(ka)} (1 - \tfrac{g}{L}) - \frac{L_2(ka)}{L_0(ka)} + \frac{L}{g} \frac{J_2(ka)}{J_0(ka)}] \right\} \tag{12}$$

One notices that W_e and W_m are not exactly equal as it should be. This is a consequence of the approximations (5). If the gap disappears

- 6 -

+) Here the relation $\int_a^b Z_n^2(x) x\,dx = \frac{x^2}{2} [Z_n^2 - Z_{n-1} Z_{n+1}]_a^b$ is used where Z stands for J_n, N_n and L_n and $Z_{-1}(x) = -Z_1(x)$.

(g = 0) one obtains $W_e = W_m = (\pi L/2)\left\{R^2 L_1^2(kR) - a^2 L_1^2(ka)\right\}$ as one expects.

The power dissipated in the walls of the cavity is defined by $P_\sigma = \frac{R_S}{2}\int_S |H|^2 dS$ where the integral has to be extended over the inner surface S of the cavity. Neglecting the central beam hole one finds

$$\frac{P_\sigma}{R_S} = R(L+R)L_1^2(kR) + a^2[-L_1^2(ka)+L_0(ka)L_2(ka)+J_1^2(ka)+J_0^2(ka)]+aL_1^2(ka) =$$

$$\frac{2W_m}{L} + \pi R^2 L_0^2(ka)\left\{\frac{L}{R}\frac{L_1^2(kR)}{L_0^2(ka)} + \frac{aL}{R^2}\frac{L_1^2(ka)}{L_0^2(ka)}(1-\frac{g}{L}) \right. \quad (13)$$

$$\left. +(\frac{a}{R})^2\left[\frac{J_0^2(ka)+J_1^2(ka)}{L_0^2(ka)} - \frac{g}{L}\frac{L_1^2(ka)}{L_0^2(ka)}\left(1-\frac{J_0(ka)J_2(ka)}{J_1^2(ka)}\right)\right]\right\}$$

These expressions are rather involved and hence we shall consider the special case that the drift tube diameter 2a is small compared to the cavity diameter 2R or more precisely $(ka/2) \ll 1$. For most practical cases this approximation will be sufficiently accurate. Then the Bessel functions can be expanded and only the leading terms will be maintained. We shall use especially the approximation

$$\frac{L_1(ka)}{L_0(ka)} = \frac{L}{g}\,\frac{J_1(ka)}{J_0(ka)} \approx \frac{L}{g}\cdot\frac{ka}{2} \quad (14)$$

and for $kR > 1$ one may use the asymptotic form

$$\frac{L_1(kR)}{L_0(ka)} \approx \frac{2}{\pi kR} \quad (15)$$

It is further expedient to introduce the quantity

$$q = \frac{ak}{2} = \frac{a}{R}\cdot\frac{kR}{2} \quad (16)$$

where kR can be taken from fig. 2.

For the case $(ka/2) \ll 1$ the integrals can be written in the following form

$$W_e = \frac{2R^2L}{\pi(kR)^2} \left\{ 1+\pi^2 q^2 (\frac{L}{g} - 1)(1-\frac{L}{g}q^2) \right\}$$

$$W_m = \frac{2R^2L}{\pi(kR)^2} \left\{ 1+\pi^2 q^2 (\frac{L}{g} - 1)[1-q^2(\frac{L}{g} + 1)] \right\} \qquad (17)$$

$$\frac{P\sigma}{R_S} = \frac{2W_m}{L} + \frac{4RL}{\pi(kR)^2} \left\{ 1+\pi^2 q^2 \frac{2L}{g}(\frac{L}{g} - 1)(\frac{R}{a}q^2 + \frac{R}{L}) \right\}$$

Of course, these expressions are valid only if $q^2L/g < 1$ since W_e and W_m must be positive.

IV. The Equivalent Circuit

For some purposes it is advantageous to consider the equivalent circuit of one cell, which consists essentially of a capacitance C and an inductivity $\mathcal{L}$ connected in parallel. In order to define these quantities two points of the cavity have to be chosen as poles. Since particles will be accelerated by the electric field on the axis of the cavity we shall choose the centers of the front walls of the drift tubes as poles assuming that there is no beam hole.

The capacitance is defined by the relation $W_e = \frac{1}{2} CU^2$ from which one infers $C = 2\, W_e / \left| \int E_z dz \right|^2$. With $\int E_z dz = Ag = L\, L_0(ka)/J_0(ka)$ one obtains

$$C = \frac{\varepsilon_0 \pi R^2}{L} J_0^2(ka) \left\{ \frac{L_1^2(kR)}{L_0^2(ka)} + (\frac{a}{R})^2 (\frac{L}{g} - 1)(1 - \frac{J_1^2(ka)}{J_0^2(ka)} \frac{L}{g}) \right\} \qquad (18)$$

This expression can be interpreted as the sum of three capacities

$$C = C_c - C_g' + C_g$$

with
$$C_c = \frac{\varepsilon_0 \pi}{L} R^2 J_0^2(ka) \frac{L_1^2(ka)}{L_0^2(ka)} \qquad (19)$$

$$C_g' = \frac{\varepsilon_0 \pi a^2}{L} J_0^2(ka)[1-\frac{g}{L} \frac{L_1^2(ka)}{L_0^2(ka)}]$$

$$C_g = \frac{\varepsilon_0 \pi a^2}{g} [1-\frac{L}{g} \frac{J_1^2(ka)}{J_0^2(ka)}]$$

C_g is obviously the capacity of the drift tubes, whereas C_c is the capacity of the empty cavity. C_g' has to be subtracted from C_c because of the volume filled by the drift tubes. The second terms in the brackets take into account the distortion of the fields.

The inductance is defined by $\mathcal{L} = \left|\int_\Sigma \vec{H}\, d\vec{\Sigma}\right|^2/2W_m$ where the area Σ is half the cross section of the cavity. From Maxwell's equation one can deduce that $\int E_z dz = - i\omega \int_\Sigma \vec{H} d\vec{\Sigma}$ and hence

$$\mathcal{L} = \frac{\mu_0 L}{\pi (kR)^2 J_0^2(ka)} \left\{ \frac{L_1^2(kR)}{L_0^2(ka)} - (\frac{a}{R})^2 \frac{L_1^2(ka)}{L_0^2(ka)} (1-\frac{g}{L})[1+\frac{g}{L}(1-(\frac{2}{ka})^2)] \right\}^{-1} \qquad (20)$$

It can easily be seen that this expression can be decomposed in the following way

$$\frac{1}{\mathcal{L}} = \frac{1}{\mathcal{L}_c} - \frac{1}{\mathcal{L}_d'} + \frac{1}{\mathcal{L}_d}$$

with
$$\mathcal{L}_c = \frac{\mu_0}{\pi k^2 J_0^2(ka)} \frac{L}{R^2} \frac{L_0^2(ka)}{L_1^2(kR)}$$

$$\mathcal{L}_d' = \frac{\mu_0}{\pi k^2 J_0^2(ka)} \frac{L}{a^2} \frac{L_0^2(ka)}{L_1^2(ka)} \frac{1}{1+\frac{g}{L}[1-(\frac{2}{ka})^2]} \qquad (21)$$

$$\mathcal{L}_d = \frac{\mu_0}{\pi k^2 J_1^2(ka)} \frac{g}{a^2} \frac{1}{1+\frac{g}{L}[1-(\frac{2}{ka})^2]}$$

Here $\mathcal{L}_c$ is the inductance of the empty cavity and $\mathcal{L}_d$ the inductance of the drift tubes.

For the case $(a/R) \ll 1$ and $kR > 1$ one finds

$$C = \frac{\varepsilon_0 4}{\pi L \;\; k^2} \left\{ 1+\pi^2 q^2 (\frac{L}{g} - 1)(1-\frac{L}{g} q^2) \right\} \qquad (22)$$

$$\mathcal{L} = \frac{\mu_0 L \pi}{4} \left\{ 1+\pi^2 q^2 (\frac{L}{g} - 1)[1-q^2(\frac{L}{g} + 1)] \right\}^{-1} \qquad (23)$$

Of course C and $\mathcal{L}$ satisfy the relation $\mathcal{L}C = 1/\omega_0^2$. Equation (22) and (23) are not very useful as they stand because they contain k which has to be determined as solution af equ. (8). What one would like to have, however, is a simple expression that gives the dependence of ω_0 on k(g=L) of the cavity without drift tubes and additional terms for the drift tubes, which do not change k(g=L).

Such a relation can be obtained on the basis of the following consideration. If drift tubes are installed into a cavity the main change is caused by the capacity of the drift tubes. The inductance on the other hand is not varied appreciably. This is because part of the displacement current is converted into a current in the drift tubes but this does not change drastically the distribution of the magnetic field which anyway is small near the cavity axis. Then equ. (22) and (23) can be simplified by putting

$$C = \frac{\varepsilon_0 4}{\pi L k^2} \left\{ 1+ K\, q^2 (\frac{L}{g} - 1) \right\}$$

$$\mathcal{L} = \frac{\mu_0 \pi L}{4}$$

and one obtains with k(g = L) = 2,40 and q = (a/R)(2,40/2)

$$kR = \omega_0 R/c = \frac{2,4}{\sqrt{1+Kq^2(\frac{L}{g}-1)}} \qquad (24)$$

The constant K has to be determined empirically. It replaces π^2 in equ.(22) and therefore one expects that it has a value between 1 and

10. Indeed it is found that a very good approximation is achieved with K = 2,74. The values of R/λ calculated in this way are presented in fig. 2 as dashed lines. It can be seen that for the range $0,1 \leq (a/R) \leq 0,3$ the agreement with the values obtained from equ. (8) is astonishingly good.

V. The Coupling Coefficient

In this section the influence of the coupling slots will be considered and an expression for the coupling coefficient will be derived.

A small hole in the cavity wall distorts the electric and magnetic fields. This effect can be calculated, if the hole is replaced by a layer of electric and magnetic dipoles. The distribution of these dipoles can be calculated by satisfying the boundary conditions[8].

If two cavities are coupled by such a small hole the coupling coefficient as defined in equ. (9) can then be computed and one obtains[9][10][6]

$$\varkappa = p \frac{E_c^2}{2W_e} - m \frac{H_c^2}{2W_m} = \varkappa_e - \varkappa_m \qquad (25)$$

Here E_c and H_c are the electric and magnetic fields that would exist at the center of the hole, if the hole were absent. The polarisabilities p and m depend on the shape of the hole and very crudely they are proportional to $s^{3/2}$ where s is the area of the hole. $\varkappa$ is the ratio of the energy $W_\varkappa$ stored in the coupling element and the total energy $W = W_e = W_m$ in the cavity.

In a cavity without drift tubes $E_c \sim J_0(kr)$ and $H_c \sim J_1(kr)$. Therefore a hole on the axis (beam hole) will produce only electric coupling ($\varkappa > 0$) whereas a hole close to the cylinder wall will result mainly in magnetic coupling ($\varkappa < 0$). Somewhere inbetween will pass through zero. Hence in order to get a most efficient coupling the coupling slots should be as far away from the axis as possible.

For a circular beam hole with radius ϱ one has

$$p = \frac{2\varrho^3}{3} \qquad m = \frac{4\varrho^3}{3} \tag{26}$$

and hence the magnetic coupling is twice the electric. For a rectangular hole with the dimensions h and d and with the magnetic field parallel to h one obtains

$$p = m = \tilde{\tau}\, h^2 d \tag{27}$$

Here the electric and magnetic coupling are equally strong but still opposite in sign. The constant $\tilde{\tau}$ is $\pi/16$ if $h,d \ll \lambda$. Since this is not true in most cases $\tilde{\tau}$ will be considered as an adjustable parameter of order 1.

The case of the rectangular slot in a homogeneous magnetic field can easily be adapted to a wedgelike opening (see fig. 1) in a cylindrical cavity. Because of the cylindrical symmetry the magnetic field is again parallel (or perpendicular) to the edges of the slot and hence equ. (27) can be applied. If the slot width d is not small compared to the radius of the cavity the electric and magnetic field averaged over the slot area should be inserted into equ. (25). Putting $h = r\alpha$ and using (27) equ. (25) is converted into

$$\varkappa = \tilde{\tau}\, \frac{\alpha^2}{2W} \left\{ \int_{r_1}^{r_2} r^2 E^2(kr)\,dr - \int_{r_1}^{r_2} r^2 H^2(kr)\,dr \right\} \tag{28}$$

The integrals can only be computed numerically. However, in order to study the general behaviour of $\varkappa$ one can replace $(r/R)^2$ under the integral by $[2r/(r_1 + r_2)] \cdot [(r_1 + r_2)/2R]^2$. This approximation is not too bad as long as $(r_2 - r_1) \ll R$. Making this substitution one arrives at the following expression for the coupling coefficient

$$\varkappa = \frac{\tilde{\tau}\,\alpha^2}{2\pi (kR)^2}\, \frac{r_1 + r_2}{2L} \cdot \frac{L_0(kr_1)[L_0(kr_2)+L_2(kr_2)](\frac{r_2}{R})^2 - L_0(kr_1)[L_0(kr_1)+L_2(kr_1)](\frac{r_1}{R}}{L_1^2(kR)\left\{1 + 3.95(\frac{a}{R})^2 \; (\frac{L}{g}-1)\right\}} \tag{29}$$

One notices that $\varkappa$ is simply proportional to the square of the slot angle. Further $\varkappa L$ is essentially constant since the correction terms containing L/g are small in most cases. The dependence on the slot width (r_2 - r_1) and the slot position is given by the numerator. The general features of this dependence has been discussed already. If for a fixed r_2 the slot width is increased by reducing r_1 the coupling coefficient increases first, passes through a maximum and may even become negative. Some examples will be discussed in section VII. The reason for the appearance of a maximum is of course that the magnetic coupling is partly compensated by the electric.

Equ. (29) describes the coupling of one slot. If n slots are used the total coupling is given by

$$\varkappa_{tot} = n\,\varkappa + \varkappa_b$$

where

$$\varkappa_b = 1.58\left(\frac{b}{R}\right)^3 \cdot \frac{R}{L}$$

is the coupling of the beam hole (radius b) without drift tubes. If drift tubes are inserted the electric field at the hole is reduced and this might be taken into account by applying a factor $(1 + l/b)^{-2}$. However, in most cases $\varkappa_b$ can be neglected.

VI Quality Factor and Shunt Impedance

The damping D of a cavity is defined as the ratio of the power loss per period and the total energy stored in the cavity.

$$D = \frac{P}{\omega W} = \frac{P_\sigma + P_\varkappa}{\omega W} = \frac{1}{Q} \qquad (32)$$

The inverse of D is the quality factor Q. The total power loss is the sum of the ohmic losses P_σ in the cavity walls and the losses through the coupling holes $P_\varkappa$.

An expression for P_σ was given already in equ. (13) and (17). From this one obtains for the attenuation D due to omic losses

$$D_\delta = \frac{P_\delta}{\omega W} = \frac{2R_s}{\mu_o \omega L} \left\{ 1 + \frac{L}{R} \frac{\frac{L_1^2(kR)}{L_o^2(ka)} + \frac{a}{R}\frac{L_1^2(ka)}{L_o^2(ka)}(1-\frac{g}{L}) + \frac{a^2}{RL}\left[\frac{J_o^2(ka)+J_1^2(ka)}{L_o^2(ka)} - \frac{g}{L}\frac{L_1^2(ka)}{L_o^2(ka)}\left[1-\frac{J_o(ka)J_2(ka)}{J_1^2(ka)}\right]\right]}{\frac{L_1^2(kR)}{L_o^2(ka)} - (\frac{a}{R})^2 \frac{L_1^2(ka)}{L_o^2(ka)}(1-\frac{g}{L}) - \left[\frac{L_2(ka)}{L_o(ka)} - \frac{L}{g}\frac{J_2(ka)}{J_o(ka)}\right]} \right\} \quad (33)$$

For the normal skin effect one has $R_s = \mu_0 \omega s/2$ where the skin depth s is determined by

$$s = \sqrt{\frac{2}{\omega\mu\sigma}} = \frac{42}{2\pi}\sqrt{\frac{1}{f/Hz}}\ \text{cm} \qquad (34)$$

The last term is valid for copper at room temperature. For a superconducting structure the expression for R_s is more complicated[11] and $R_s \sim \omega^2$ instead of $R_s \sim \sqrt{\omega}$ for normal conduction.

If again the approximation $(ak/2) \ll 1$ is introduced one finds

$$D_\sigma = \frac{2R_s}{\mu_o \omega L}\left\{1 + \frac{L}{R}\cdot\frac{1+\pi^2 q^2 \frac{2L}{g}(\frac{L}{g}-1)[\frac{R}{a}q^2+\frac{R}{L}]}{1+\pi^2 q^2 (\frac{L}{g}-1)[1-q^2(\frac{L}{g}+1)]}\right. \qquad (35)$$

for a cavity without drift tubes $(q \to 0)$ one gets the well-known expression

$$D_\sigma = \frac{2R_s}{\mu_o \omega L}\ (1+\frac{L}{R}) = s(\frac{1}{L}+\frac{1}{R}) \qquad (36)$$

The next step is to calculate the attenuation caused by the coupling slots. This could be done by describing the influence of the slots by an appropriate distribution of electric and magnetic dipoles (see section V). Then the dissipated power $P_{\mathcal{H}} = (R_s/2)\int_S |H|^2 dS$ could be determined if the surface integral is computed for $H = H_0 + H_D$ where H_0 is the field in the cavity without coupling slots and H_D is the dipole field. Here we shall follow a simpler procedure by considering the equivalent circuit of the coupling element. The ratio of the energy dissipated in the coupling element to the total energy is given by $R_{\mathcal{H}}/\omega \mathcal{L}_{\mathcal{H}}$ where $R_{\mathcal{H}}$ and $\mathcal{L}_{\mathcal{H}}$ are the equivalent resistance and inductivity of the coupling element, respectively. For the inductivity of the slot one has approximately

$$\mathcal{L}_{\varkappa} = \frac{\mu_0}{4\pi}\, d \qquad (37)$$

where d is the dimension of the slot perpendicular to the magnetic field produced by the equivalent magnetic dipole. If one further identifies $R_{\varkappa}$ with R_s one obtains $R_{\varkappa}/\omega\mathcal{L}_{\varkappa} \approx (s/2d)$ and one finally finds for n slots

$$P_{\varkappa} = \frac{4\pi}{\mu_0}\frac{R_s}{d}\, n\,|W_{\varkappa}| = 2\pi n\omega\,\frac{s}{d}\,|W_{\varkappa}| \qquad (38)$$

and

$$D_{\varkappa} = \frac{P_{\varkappa}}{\omega W} = \frac{4\pi}{\omega\mu_0}\cdot\frac{R_s}{d}\,|\varkappa| = \frac{2\pi s}{d}\,|\varkappa| \qquad (39)$$

Expression (39) demonstrates that for a constant slot width d the damping D is directly proportional to the coupling coefficient $\varkappa$.

According to equ. (27) $\varkappa \sim h^2 d$ and hence $D_{\varkappa}$ is independent of d. This is plausible since the damping $D_{\varkappa}$ originates from the deflection of the currents by the slots in the front walls of the cavity. This deflection results in longer current paths and therefore a higher dissipation of power. However, the radial component of the current path is not changed by the coupling hole but rather the additional path in azimuthal direction increases the damping. Therefore one expects that only the slot angle (determined essentially by h) but not the slot width d influences the damping $D_{\varkappa}$. This becomes obvious for the limiting case of a very narrow slot in radial direction. It does not distort the current distribution and hence its length does not effect the damping. On the other hand according to (28) $D_{\varkappa}$ should be proportional to α^2 and this is in very good agreement with the experimental results (see section VII).

The total attenuation and the quality factor are obtained from $1/Q = D_{\sigma} + D_{\varkappa}$ where D_{σ} and $D_{\varkappa}$ are given by (33) and (39), respectively. For typical cases both contributions to the attenuation are of the same order of magnitude ($\sim 10^{-4}$ for copper cavities) and therefore for an optimization both have to be taken into account.

The shunt impedance per unit length Z of an accelerating structure is defined by the square of the maximum energy gain divided by the power loss each per unit length of the structure

$$Z = \frac{[\int_0^L E_z(r=o)dz]^2}{P \cdot L} \cdot T^2 = \frac{2T^2}{L}\frac{Q}{\omega C} = \frac{2T^2}{L}\omega \mathcal{L} Q \qquad (40)$$

$$\text{where} \quad T = \int_0^L E_z \sin\frac{\pi z}{L}\,dz / \int_g E_z dz = (2L/\pi g)\ \sin\ (\pi g/2L) \qquad (41)$$

is the transit time factor for the π-mode. If $E_z(r = 0)$ is inserted into (40) one may write

$$\frac{Z_0}{Z} = \frac{J_0^2(ka)}{L_0^2(ka)} \cdot \frac{P_\delta + P_{\varkappa}}{L} \qquad (42)$$

were P_δ and $P_{\varkappa}$ are given by equ. (13) and (38) respectively.

Using the approximations for $(ak/2) \ll 1$ one obtains

$$\frac{1}{Z} = \frac{\pi R^2 R_s}{Z_0^2 T^2}\left(\frac{L_1(kR)}{L_0(ka)}\right)^2 \left\{\frac{1}{L}(1+A) + \frac{1}{R}(1+B) + (1+A)\frac{2\pi|\varkappa|}{d}\right\} \qquad (43)$$

with $\quad A = Kq^2(\frac{L}{g} - 1)$

$\quad B = Kq\frac{2L}{g}(\frac{L}{g} - 1)(\frac{R}{a}q^2 + \frac{R}{L})$

For a cavity without drift tubes and no coupling slots one finds the well-known relation

$$\frac{Z_0}{Z} \approx \frac{s}{T^2}(1 + \frac{R}{L}) \qquad (44)$$

For constant g/L one infers from equ. (43) that 1/Z should be proportional to $|\varkappa|$ and also proportional to R/L apart from constant terms.

VII. Comparison with measurements

In the following we shall compare the results of the calculations with the experimental data obtained by Eschelbacher at 760 MHz. Because of the approximations introduced in the theory a 10 percent agreement will be considered as satisfactory. The main purpose of this work was to get a better qualitative understanding of the phenomena whereas more accurate numerical calculations are necessary for an actual design study.

a) Resonance frequency

The dependence of the resonance frequency ω_o of an isolated cell on the length and diameter of the drift tubes is described by the simple formula (24) for $a/R \ll 1$. The constant K = 2.74 has been chosen such that the best agreement with equ. (8) is obtained for a/R = 0.225 (compare fig. 2). Some measurements for a structure with this value of a/R and different gap widths are shown in fig. 2. The frequency ω_o was determined by fitting equ. (10) to the experiments which yields ω_o and $\varkappa$. As can be seen the data for ω_o fall on the theoretical curve for $g/L \gtrsim 0.6$ in fig. 2 whereas for larger gap widths the experimental frequencies are lower than the calculated ones. This means that the drift tube diameter should be replaced by an effective diameter which is about 10% larger at $g/L \approx 0.5$ and about 25% larger at $g/L \approx 0.3$. Since, however, such small gap widths are not practical because of sparking the interesting range of g/L is around and above 0.5 where our approximation is sufficiently good.

b) Coupling coefficient

According to equ. (28) the coupling coefficient is expected to be proportional to the square of the slot length; i.e. $\varkappa \sim \alpha^2$. Some measurements of $\varkappa$ as a function of α^2 are presented in fig. 3. Indeed there is a linear dependence and only for very large angles the experimental numbers deviate from the straight line. This is presumably caused by an interaction between the four slots.

Equ. (28) predicts that $\varkappa . L$ should not change if the cell length L is varied but g/L and a/R are kept constant. That this is indeed so within the accuracy of the present calculations is shown by fig. 4. In a similar way it can be shown that the dependence of $\varkappa$ on g/L is rather weak, since according to fig. 2 k does not change drastically as long as $g/L > 0.4$ and hence the functions $L_i(kr)$ in equ. (28) or (29) vary only little. In addition the changes in the numerator and denominator are in the same direction and hence the overall change of $\varkappa$ is small. This is verified by the measurements (see fig. 25 of Eschelbacher[2]).

Finally one has to discuss the dependence of $\varkappa$ on the slot width and slot position. This is essentially given by the integral $\int r^2[E^2 - H^2]dr$ which has to be taken over the slot width. Since measurements for very large values of d have been performed equ. (29) is too bad an approximation and the integral was computed by graphical methods inserting equ. (2) for the fields. The results are shown in fig. 5. As is seen the theoretical curves reproduce quite well the general behaviour of the experimental results. In the case of no drift tubes (g/L = 1) $\varkappa$ rises sharply with increasing slot width, passes a maximum and drops off. The reason is that near the axis the electric coupling becomes more important and since it is opposite in sign it cancels partly the magnetic coupling. For long drift tubes the electric field is reduced considerably and as a consequence $\varkappa$ decreases only slightly. The absolute magnitude of the experimental results is in some disagreement with the predicted values, however, this could be improved by adjusting μ or a slight change of k.

c) Quality factor

Using equ. (36) and (39) one may write

$$D = s\left[\frac{1}{L} + \frac{1}{R}\,\frac{1 + B}{1 + A} + \frac{2\pi\,|\varkappa|}{d}\right] = 1/Q \qquad (44a)$$

where A and B have been defined in (43). From this relation one expects that $1/Q$ is proportional to $\mathcal{X}$ and consequently proportional to α^2. Measurements for a special geometry are shown in fig. 6. For values of $\alpha^2 < 1.5$ the prediction is realized very nicely. The slope of the broken curve was calculated from a measured value $\mathcal{X} = 16\%$ at $\alpha = 1$. The intersection of both straight lines with the coordinate axis cannot be calculated reliably since a demountable model was used and hence it is to be expected that the measured Q values are low by a given factor. Indeed for $\mathcal{X} = 0$ one finds $1/Q \approx 1.10^{-4}$ whereas for an ideal copper cavity one expects $1/Q = (s/L) + (s/R) = 0.4 \cdot 10^{-4}$.

Since, as we have seen, $\mathcal{X} \sim 1/L$ holds with good accuracy one expects that $D \sim 1/L$ or $D \sim \mathcal{X}$ if $\mathcal{X}$ is changed by varying L but keeping g/L fixed. As fig. 7 shows this expectation is also born out by the measurements. The straight lines connecting the points for a given slot width all intersect the coordinate axis at the same point which is given by $1/Q = s/R$ since in this case $\mathcal{X} \to 0$ corresponds to $L \to \infty$. The point obtained by extrapolating the measurements is $0.2 \cdot 10^{-4}$ whereas for an ideal copper cavity one expects $0.16 \cdot 10^{-4}$. This agreement is somewhat better because losses caused by bad joints are not important for the two front plates of the cavity. If the coupling coefficient is varied by changing the slot width the relation between $1/Q$ and $\mathcal{X}$ is more complicated. This follows from $|\mathcal{X}|/d$ being a complex function of r_1 and r_2 according to (28). However, in a first approximation one expects a linear relationship also in this case. This cannot be tested since measurements only for two slot widths have been performed. However, if the dashed lines in fig. 7 which correspond to L = const. are extrapolated to the coordinate axis the resulting values of $1/Q - 0.2 \cdot 10^{-4} = (0.97\ ;\ 0.70;\ 0.48)10^{-4}$ should according to equ. (44a) be proportional to $1/L = (1.01;\ 0.74; 0.56)10^{-2}$ which is approximately the case.

Summarizing one might say that the approximate treatment of a reentrant cavity with coupling slots as presented here describes such a structure with reasonable accuracy. It should also be pointed out that the relations derived for the resonance frequency, the Q value and the shunt impedance are independent of the type of coupling and hence these results can be applied immediately to a $\pi/2$ mode structure with resonance coupling.

VIII. Optimalization of the Shunt Impedance

In order to keep the rf power low for a normal linear accelerator or in order to reduce the cooling power in the case of a superconducting accelerator it is important to make the shunt impedance as high as possible. This optimalization should be carried out in two steps.

1) Optimalization for fixed v/c

First we consider a single cell or a tank consisting of identical cells designed for a given particle velocity. As a consequence of the accelerating condition $2L = ß\lambda_0$ ($ß = v/c$; λ_0 = vacuum ware length) the cell length L is fixed. Because of practical reasons the tank radius R will be kept constant over the whole length of the accelerator or at least over a large section.

Furthermore the coupling coefficient $\varkappa$ will be chosen by other considerations than the optimalization of the shunt impedance. It determines the bandwidth of the structure and hence the mode stability and also the group velocity. Consequently we shall assume here that $\varkappa$ is determined by these requirements and is kept constant over the whole accelerating structure. This has the additional advantage that the resonance frequency is not altered by a change of $\varkappa$ which simplifies the discussion considerably.

If we use equ. (43) as a basis for our optimalization one has to make 1/Z as small as possible by a proper choice of g/L and a/R. For practical reasons one wants to have the same resonance frequency for all tanks. According to equ. (24) this implies that

$$A = 2{,}74 \quad q^2\left(\frac{L}{g} - 1\right)$$

has to be constant. As we shall see later a maximum for the shunt impedance is obtained for $g/L \sim 0.5$. An inspection of fig. 2 shows that for the region $0.3 < g/L < 1$ a variation of g/L can easily be compensated by a small change of the drift tube diameter i.e. of q^2. Since the necessary variation of g/L to obtain the maximum Z for each ß is comparatively small it is always possible to keep A = const (and hence ω_0 = const) by a proper choice of the drift tube diameter a with $0.1 < a/R < 0.3$.

It is convenient to bring equ. (43) into the form

$$\frac{1}{Z} = C_0\left(\frac{x}{\sin\frac{\pi}{2}x}\right)^2\left\{C_1 + \frac{C_2}{x} + \frac{C_3}{\sqrt{x(1-x)}}\right\} \qquad (45)$$

$$x = g/L$$

where the constants C_0, C_1, C_2 and C_3 are defined by

$$C_0 = \frac{\pi^3}{4}\cdot\frac{R^2R_s}{Z_0^2}\left(\frac{L_1(kR)}{L_0(ka)}\right)^2$$

$$C_1 = \frac{1}{L}(1 + A) + \frac{1}{R} + (1 + A)\frac{2\pi|\partial\ell|}{d} \qquad (46)$$

$$C_2 = \frac{A}{L}$$

$$C_3 = 1{,}2\sqrt{\frac{A}{K}}\cdot\frac{A}{R}$$

The three constants C_1, C_2 and C_3 are of approximately the same magnitude and hence all three terms must be taken into account. The value of x for which 1/Z has a minimum can of course easily be found by differentiating equ. (45) and putting it equal to zero. However, we shall restrict ourselves to a qualitative discussion.

The factor $(x/\sin\frac{\pi}{2}x)$ has a minimum for x = 0 whereas C_2/x accepts its lowest value at x = 1. Furthermore the term $C_3/\sqrt{x(1-x)}$ has a minimum at x = 1/2. Therefore neglecting the transit time factor and the other terms the highest shunt impedance would be obtained for g/L = 0.5. The term C_2/x which becomes more important for small β will shift the minimum to somewhat higher values of g/L. The transit time factor on the other hand favours smaller values of g/L. In order to determine g/L precisely the actual numerical values of the constants have to be used. Moreover for an actual design study the more accurate equ. (42) should be used. However, since the maximum of Z is rather flat $g/L \approx 0.5$ is a reasonable value in most cases. The precise value of g/L will then be determined also by other considerations, e.g. sparking properties.

2) Optimalization of the whole structure

Once the relative maximum of Z has been obtained by determining g/L (which at the same time yields a definite value of a/R because of the frequency condition) the question arises how an absolute maximum can be achieved by an adequate choice of R, ω and $\varkappa$.

The selection of an advantageous radius R, and intimately connected to it the choice of the frequency, is influenced by two considerations. It follows from (46) that the constants C_0C_1, C_0C_2 and C_0C_3 increase with increasing R. Therefore as far as the geometry is concerned a small radius is preferable which implies of course a higher frequency ω_0. On the other hand the surface resistivity R_s decreases for superconducting cavities with decreasing ω and hence a lower frequency is more advantageous. The final choice cannot be made without cost estimates since the cost of the structure with its cryogenic shields is a strong function of R and L. Such considerations would be beyond the scope of this paper, however.

Because of phase stability and phase acceptance and in favour of a simple rf system it will certainly be most favourable to use the same frequency for the whole accelerator. However, it might turn out that a change of R at one or two points could be profitable. The right frequency can be obtained by changing a/R in the proper way.

A last remark concerns the coupling coefficient. In order to keep the problems of mode stability and tolerance in hand it seems that a constant $\varkappa$ over the whole structure will be expedient. This can easily be realized. If for example protons are injected at an energy of 150 MeV (β = o.5) and if they are accelerated to energies above 1 GeV ($\beta > 0.9$) than L will vary by a factor of about 2 and as a consequence $\varkappa$ changes by the same factor. However, this change can easily be compensated since $\varkappa \sim \alpha^2$ (see equ. (28)) and therefore a comparatively small variation of the slot width will suffice.

The author would like to thank Mr. H. Eschelbacher and Mr. L. Szecsi for helpful discussions, hints for useful references and computational assistance. He also appreciates an illuminating discussion with Dr. Lapostolle clarifying the significance of different definitions of coupling coefficients.

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Figure Captions

Fig. 1 Dimensions of cavity and coupling slots

Fig. 2 The resonance frequency of a single cell ω_o as a function of g/L (g gap width, L cell length) for different ratios a/R (a drift tube diameter, R cell diameter). Dashed lines calculated from equ. (24).

Fig. 3 The coupling coefficient $\varkappa$ as function of the slot angle α for a/R = 0.226 and g/L = 0.5. Full line is drawn through points, broken line was calculated according to (28).

Fig. 4 The coupling coefficient $\varkappa$ as function of the cell length L (in units of R) for two values of the slot width d and two extreme drift tube lengths l.

Fig. 5 The coupling coefficient $\varkappa$ as function of the slot width $d = r_2 - r_1$ (r_2 = 115 mm, R = 144 mm). The curves are calculated with equ. (28) putting γ = 1.

Fig. 6 1/Q as a function of the slot angle α. The full curve is drawn through the experimental points. The broken curve was calculated using $\varkappa$ = 16% at α = 1.

Fig. 7 1/Q as a function of $\varkappa$. Here $\varkappa$ was varied by changing L whereas all other parameters are kept constant.

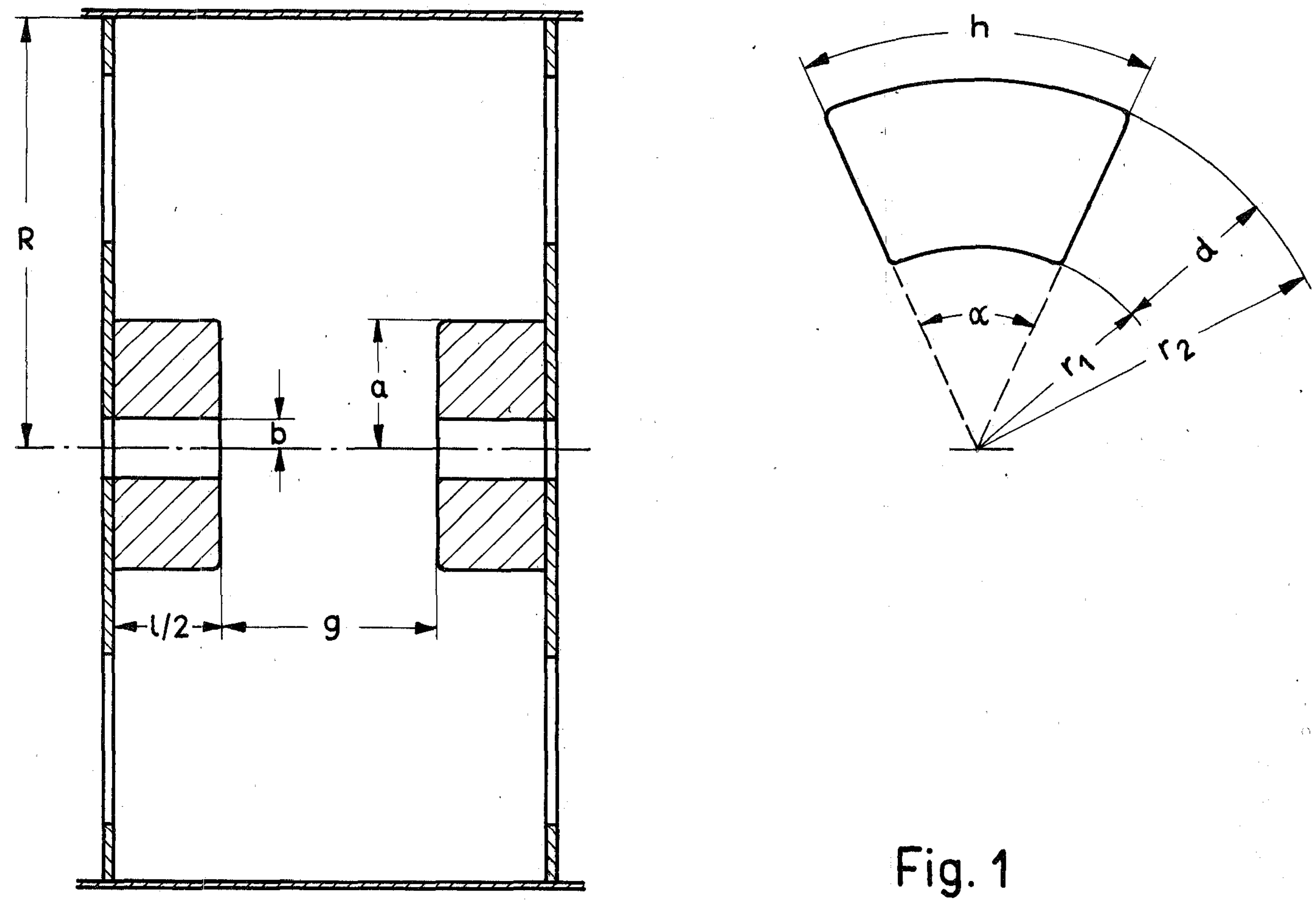
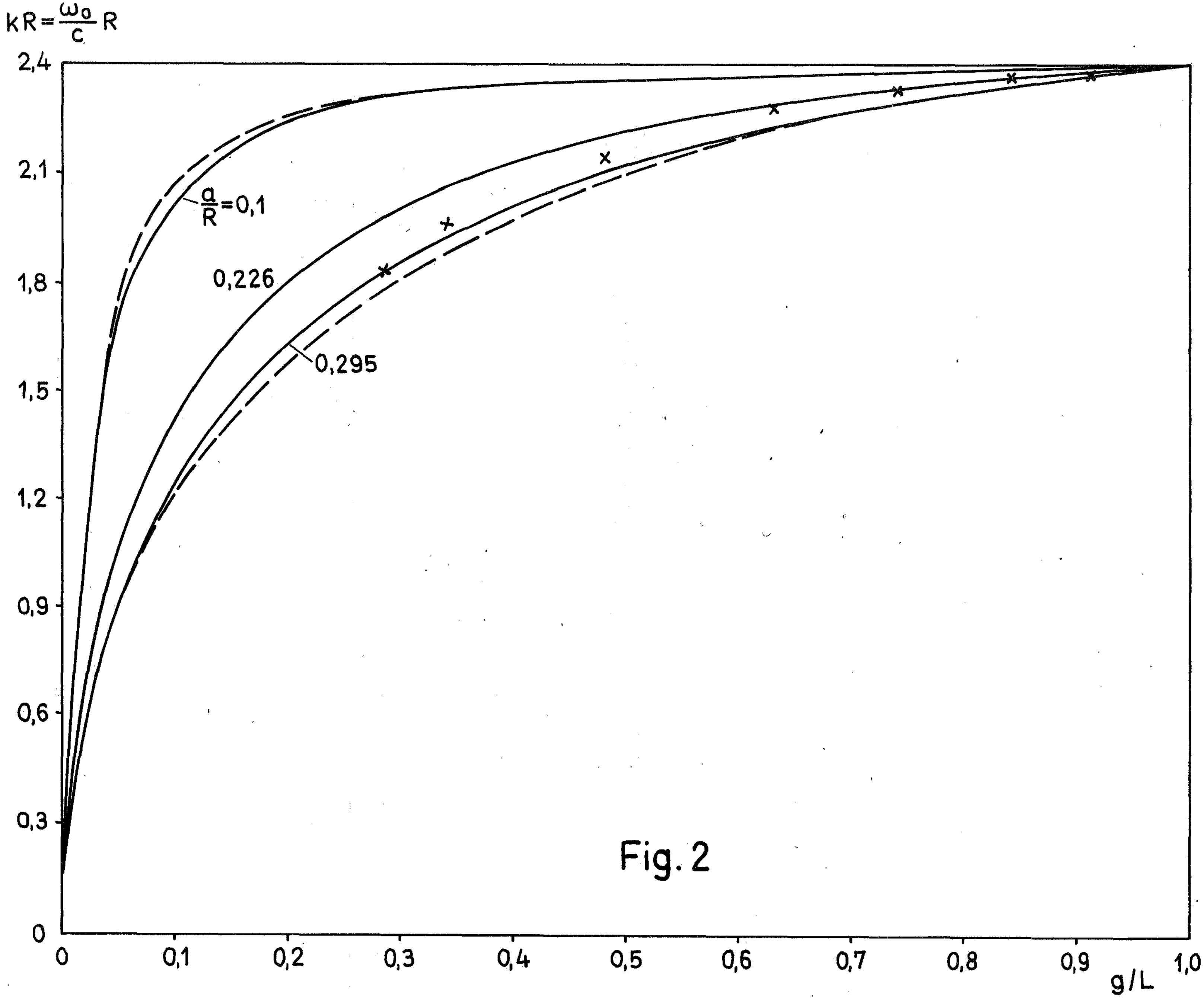


Fig. 1



kR = ωo/c R
2,4
2,1
1,8
1,5
1,2
0,9
0,6
0,3
0
a/R = 0,1
0,226
0,295
0
0,1
0,2
0,3
0,4
0,5
0,6
0,7
0,8
0,9
1,0
g/L

Fig. 2

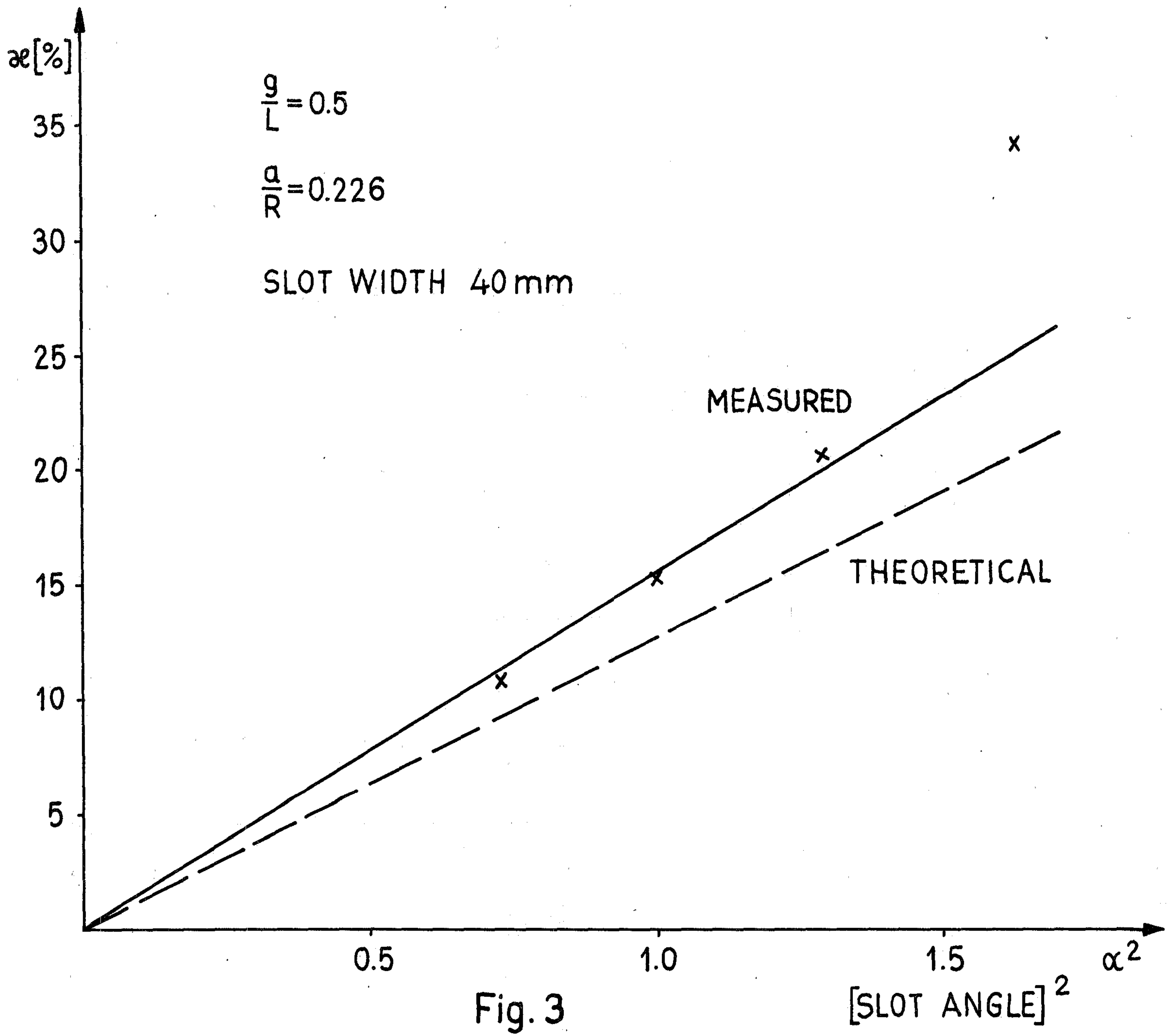
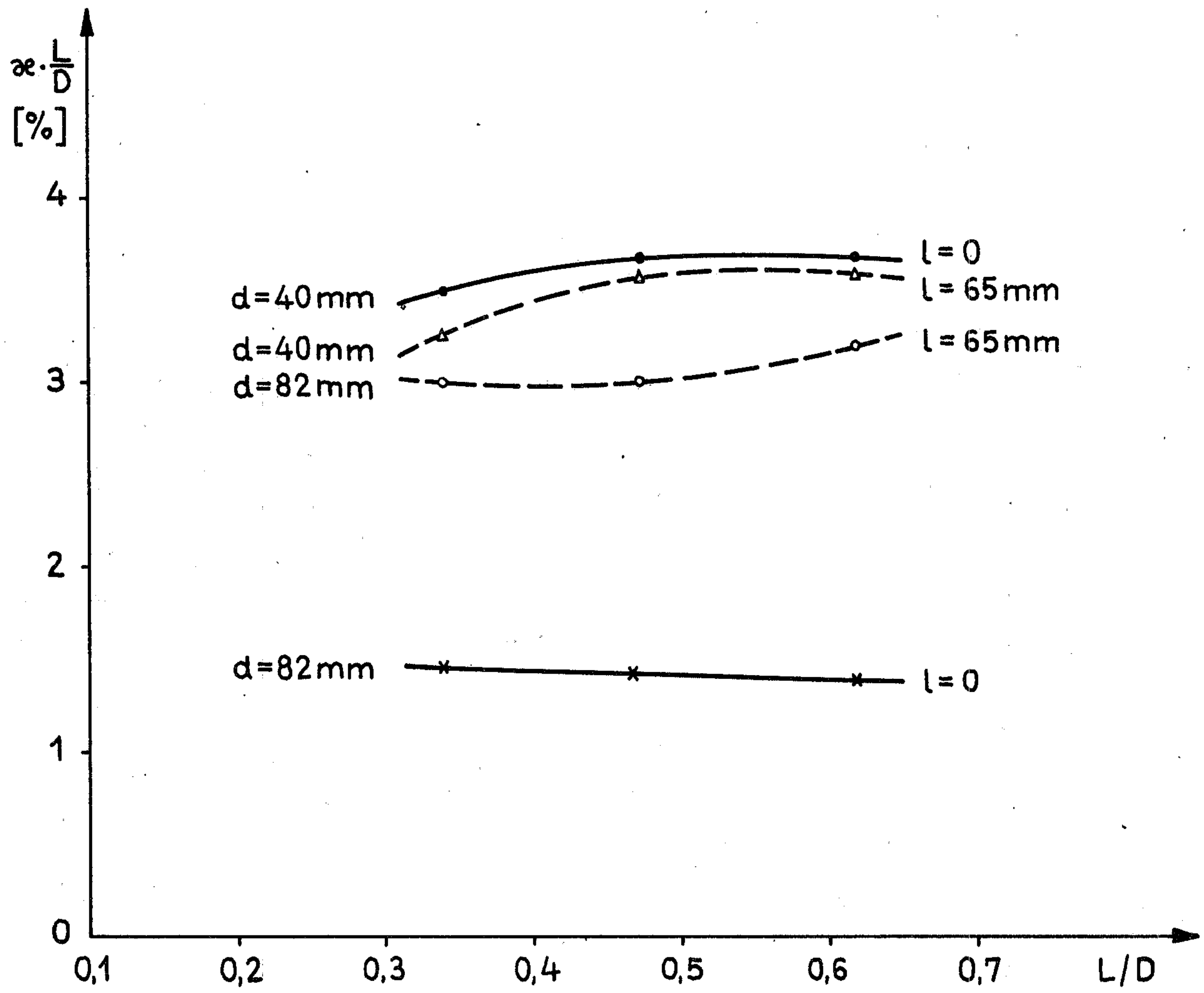

æ[%]
35
30
25
20
15
10
5
g/L = 0.5
a/R = 0.226
SLOT WIDTH 40 mm
MEASURED
THEORETICAL
0.5
1.0
1.5
α²
[SLOT ANGLE]²

Fig. 3



æ·L/D
[%]
4
3
2
1
0
0,1
0,2
0,3
0,4
0,5
0,6
0,7
L/D
d=40mm
d=40mm
d=82mm
d=82mm
l=0
l=65mm
l=65mm
l=0

Fig. 4

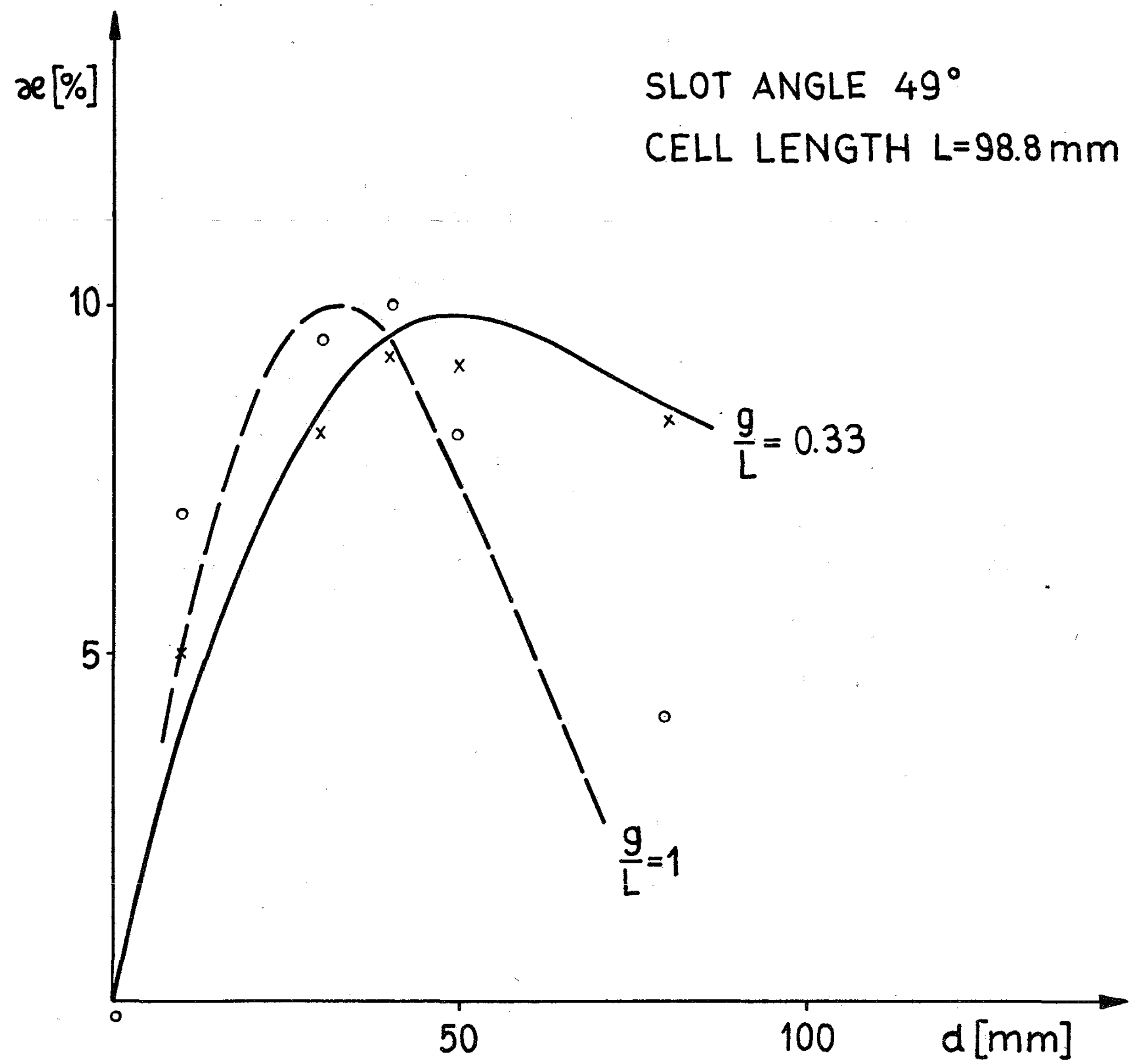


Fig. 5

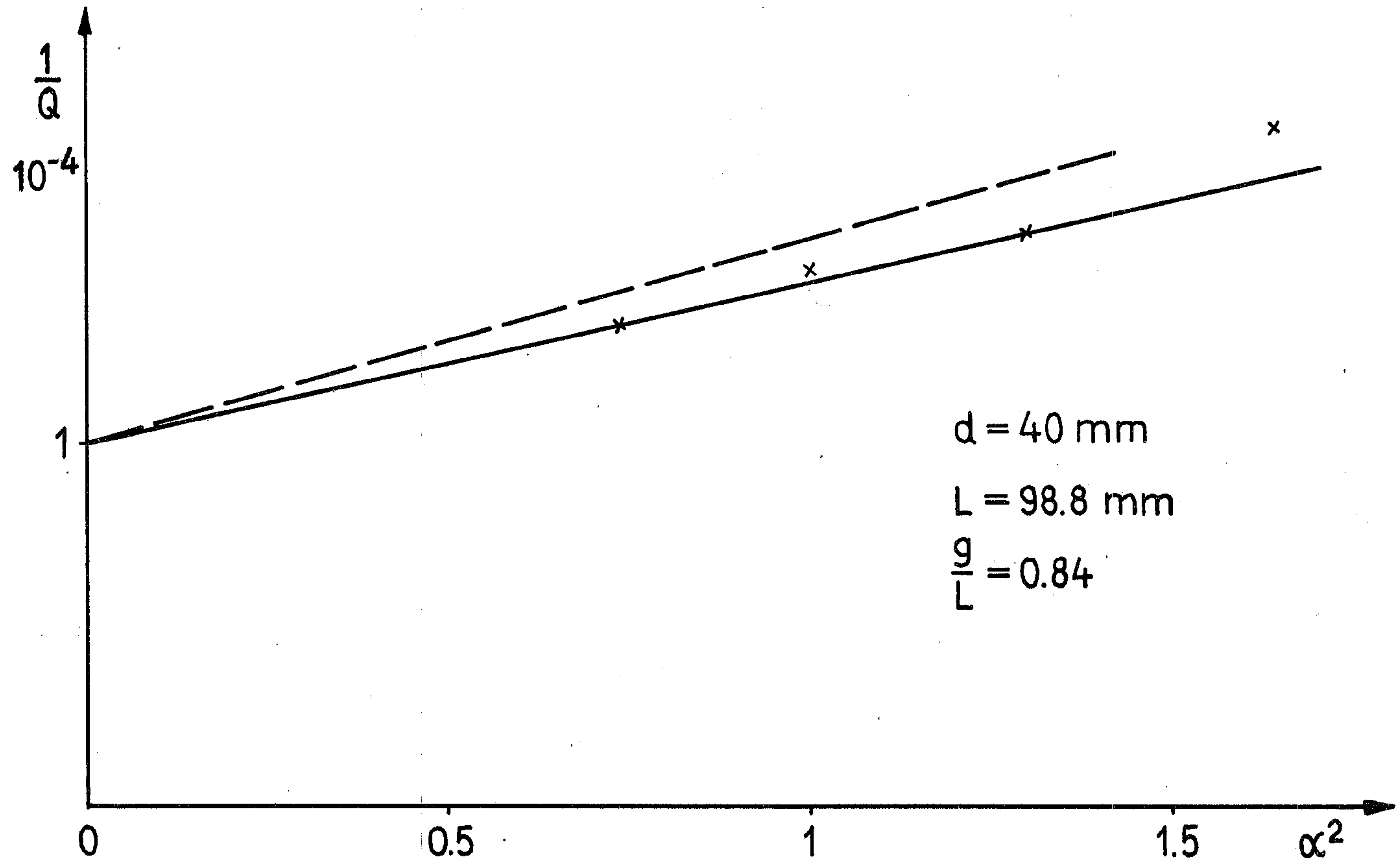
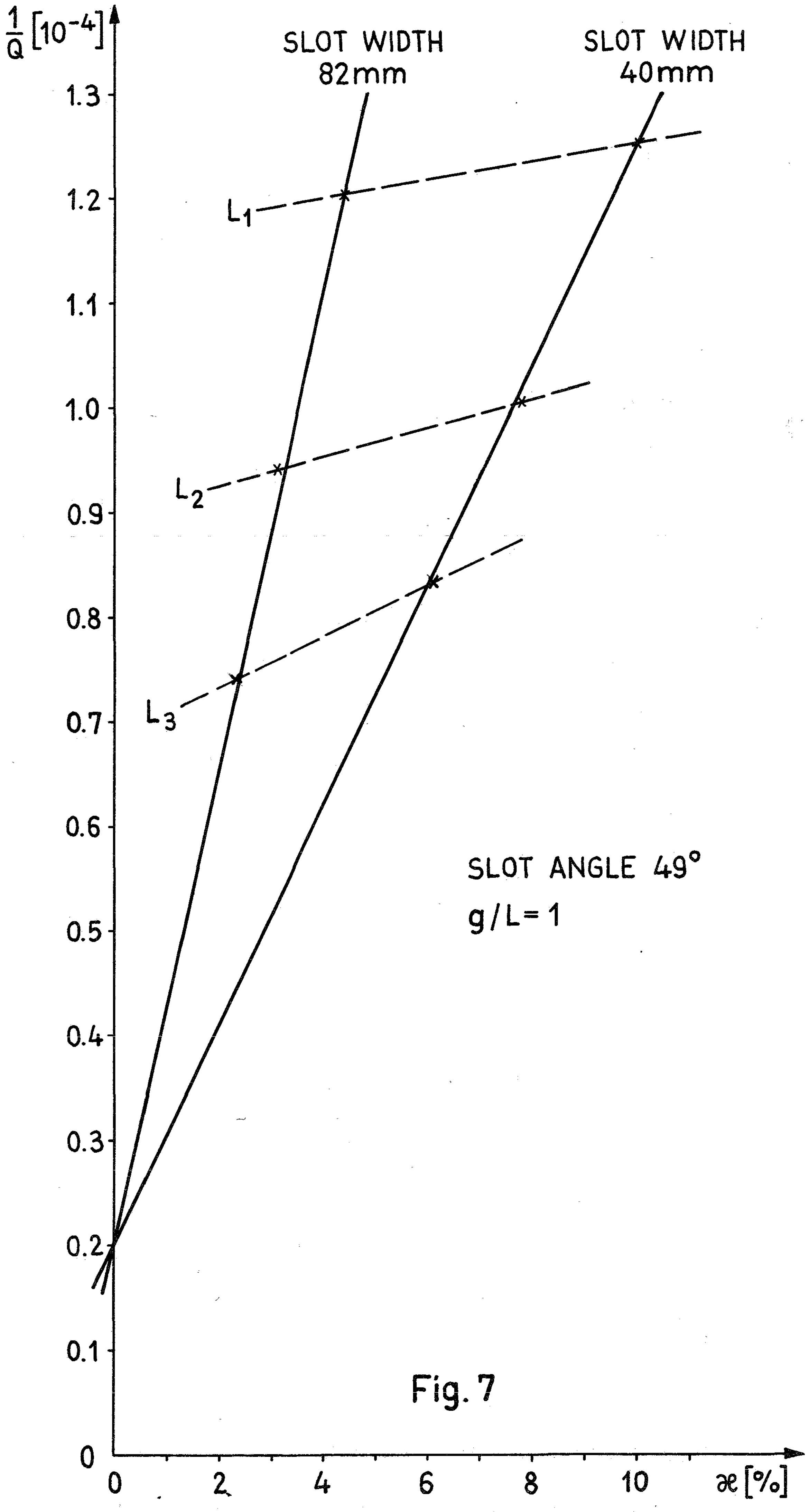


Fig. 6



1/Q [10^-4]
SLOT WIDTH
82mm
SLOT WIDTH
40mm
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
L1
L2
L3
SLOT ANGLE 49°
g/L=1
0
2
4
6
8
10
æ [%]

Fig. 7