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Effectivity and Cost Optimization of Safeguards Systems

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Contents:

1. Introduction

In the present paper an effort has been made to quantify the term "effectivity" of a safeguards system and a method has been detailed which can be used to compare the effectivity of a number of competing safeguards systems.

It is quite evident that the actual costs incured by a safeguarding authority in implementing all the measures of a safeguards system, have to be associated very closely with the effectivity of that system. For example, a definite probability of detection for a given amount of fissile material can be set at the beginning as the objection of a safeguards system [1]. The costs of a specific safeguards system can then be optimized with respect to this probability by varying the cost parameters (measuring instruments, containment etc.) of the system. As has been shown later, such optimized costs may be used to determine the relative effectivity of comparable systems.

The question of effectivity has been studied with two different models of safeguards systems. The first one may be called the inspector's system. In this model the inspectors can detect a diversion by observation and surveillance and the model is based on the theory of games [2]. The other is a statistical model based on the flow measurements of fissile material at strategic points. The second model has been treated at the beginning in a simplified manner, when statements of the form 'something has been diverted' or 'nothing has been diverted', have been treated. At a later stage this model has been elaborated.

It may be noted that both these models do not represent the reality exactly and completely. Certain salient features of these two models have been chosen to bring out the important characteristics of the method, which has been developed in this paper for quantifying the effectivity of a safeguards system.

2. Model for the Inspector's System; Cost Optimization and Definition of Effectivity

The game theoretical model for the inspector's system is of the following form: In a nuclear facility the operator can divert a certain (not quantified) amount of fissile material at r places. There are k ($\leq$ r) inspectors who safeguard the plant in a certain interval of time $t_o \leq t \leq t_1$. If an inspector is at a place where a diversion takes place he detects it with the probability q. (This is an extension of the model given in [2]; it is assumed for example, that one inspector has a number of assisting personnel who safeguard a certain area of the plant according to a given strategy.) The plant operators and the inspectors have been assumed to choose, with the help of stochastic experiments, certain strategies that is their mode of operation. A strategy of the operator would be to choose in the interval $t_o \leq t \leq t_1$ the places where he would divert a certain amount of fissile material, similarly a strategy of the inspectors would be to choose the places where they would control. It is assumed that the operators and the inspectors behave in an optimal way that is both choose optimal strategies.

To complete the game theoretical model, the pay off matrix has to be defined, i.e. the gain or the loss which the operator and the inspectors will have in case the operator diverts fissile material and this diversion is detected or not detected. Similarly, the gain or the loss of the inspectors when they detect or do not detect the diversion of fissile material. It is assumed that in all intervals of time in question the operator diverts at m_o places and m_o = const. It is defined that

(i) If the operator does not divert any material he has the gain (and the loss) zero.

(ii) If the operator diverts material and this diversion is detected at least once, he has the loss c, the inspectors have the gain c.

(iii) If the operator diverts material and this diversion is not detected, he has the gain d, the inspectors have the loss d.

With these definitions the gain of the operator becomes the loss of the inspectors and this game becomes a so called two person zero sum game.

Now it can be shown /¯2_7 that the optimal strategies for the operator and the inspectors are those strategies in which all possible strategies occur with equal probabilities. In this case the gain(or the loss) of the inspectors is given by

$$W = c - (c + d) \left(1- \frac{qk}{r}\right)^{m_o} \qquad (1.1)$$

(definitions of all the symbols used are given at the end of this paper)

This gain is greater than zero, if the number k of the inspectors is less than k_o, where

$$k_o = r \left(1- \sqrt[m_o]{\frac{c}{c+d}}\right) \qquad (1.2)$$

The probability of detection $P(d,m_o)$ is defined as the probability that the inspectors detect at least at d places a diversion when the operator diverts at m_o places. In the following the simplified expression

$$p(m_o) = P(d = 1, m_o) \qquad (1.3)$$

is used which gives the probability that the inspectors detect a diversion at least at one place when the operator diverts at m_o places. If it is assumed that the operator diverts the same amount of material at all places, m_o is a measure for the diverted material. The calculation gives

$$p(m_o) = 1 - \left(1- \frac{qk}{r}\right)^{m_o} \qquad (1.4)$$

The error second kind $\beta(m_o)$, that is the probability that nothing will be detected although the amount m_o will be diverted, is given by

$$\beta(m_o) = 1 - p(m_o) = \left(1- \frac{qk}{r}\right)^{m_o} \qquad (1.5)$$

The error first kind, i.e. the probability that a diversion will be detected if nothing will be diverted, is zero (the inspectors indicate diversions only, if they see them directly).

In order to arrive at a definition of effectivity one could proceed in the following manner:

a) Some common property of all the safeguards systems is postulated, which relate the probability of detection with the diverted amount.

b) The costs for each of the systems are optimized with respect to the postulate.

As will be seen, the costs optimized in this manner, as a function of the diverted material, can be defined as a measure of the effectivity of a system.

The postulates used in this paper are:

(i) Postulate 1

There exists a relation between p (m_o) and the amount diverted m_o for example, of the following type:

$$p(m_o) = 1-e^{-\frac{m_o}{a}} \qquad (1.6)$$

This corresponds to the assumption that the probability of detection for smaller amounts should be smaller and should increase with larger amounts.

Eq. (1.4) gives together with (1.6) a condition for k and q

$$q \cdot k = r\,(1- e^{-\frac{1}{a}}) = \alpha_3 \qquad (1.7)$$

(ii) Postulate 2

The error of the second kind $\beta(m_o)$ is less than β_o for m_o greater than m_{oo}:

$$\beta(m_{oo}) \leq \beta_o \qquad (1.8)$$

This gives with (1.5)

$$q \cdot k \geq r\,(1- \beta_o^{\frac{1}{m_{oo}}}) \qquad (1.9)$$

Equ. (1.7) and (1.9) are compatible if

$$a \leq - \frac{1}{m_{oo}} \ln \beta_o \qquad (1.10)$$

(iii) Postulate 3

The error first kind α is less than α_o. This is always fulfilled in the framework of the inspectors system chosen in this paper.

Note: Further work to quantify these postulates is in progress. For example instead of (1.6) a step function which is zero for $m_o \leq m_{oo}$ might be more reasonable.

The total costs of the inspector's system as a function of k and q are assumed to be as follows:

$$C = \alpha_1 k + \alpha_2 q \qquad (1.11)$$

Optimization of C with respect to the boundary condition (1.7) (it is assumed that (1.10) is fulfilled) gives

$$C_{opt} = 2 \; (\alpha_1 \alpha_2 \alpha_3)^{\frac{1}{2}} \; ; \; k_{opt} = \left(\frac{\alpha_2 \alpha_3}{\alpha_1} \right)^{\frac{1}{2}} ; \; q_{opt} = \left(\frac{\alpha_1 \alpha_3}{\alpha_2} \right)^{\frac{1}{2}} \qquad (1.12)$$

It may be seen from (1.12) that C_{opt} is independent of m_o (Curve I in Fig. 1).

Let us assume that another safeguards system is represented by curve II in Fig. 1. It may then be defined that

(i) The effectivity of a safeguards system is given by the costs as a function of the amount m_o of diverted material optimized with respect to the postulates given above.

(ii) One safeguards system is more effective than another in a certain interval of m_o, if in that interval of m_o the optimized costs of the former system are less than those of the latter system.

Some further comments may be made on the last definition:

(i) According to this definition system I in Fig. 1 is more effective than system II for $m_o < m_o'$. For $m_o > m_o'$ system II is more effective than system I.

(ii) In reality a given amount of budget may be taken to be available for safeguards purposes. It is quite conceivable that the actual amount of the budget determines the effectivity of any two systems as shown in Fig. 2. For $C_{opt} = C'_{opt}$ system II is more effective as it enables one to detect a smaller amount of diverted material. For $C_{opt} = C_{opt}^2$, system I is more effective.

(iii) In case the two systems do not differ according to the definition given above, the probability of detection $P(d, m_o)$ instead of $p\ (m_o)$ may be used for fixing the effectivity of a system. In that case two sets of curves, instead of two curves as given in Fig. 1, have to be compared, and a more sophisticated comparison may be possible.

In the following pages it has been shown that the same definition of effectivity can be used for the second model which is based on the statistical measurement of fissile material throughputs. Only the statement "detection" has to be defined in an appropriate manner.

3. Models for the Flow Measurement System

The following models refer to those parts of the fuel cycle for which one can establish a material balance.

In the interval of time $t_o \leq t \leq t_1$ the inspector is continously measuring, n-times at the same time, the throughput of a plant (input J_{in} and output J_{out}) at strategic points by flow measurements. Because of measuring errors these measurements are not exact. The variance of one measurement is σ_J^2, it is independent of the amount of measured material. One exact measurement would give at $t = t_1$ the inventory $J = J_{in} - J_{out}$ of the plant. The result of n measurements is the average value $\bar{J}$ (in general not equal J). At time $t = t_1$ the inventory I of the plant is measured m-times. The variance of the measurement is σ_I^2, the average value of the m measurements is $\bar{I}$. The inspector compares the two values $\bar{J}$ and $\bar{I}$ and states that a diversion has/has not taken place. Two kinds of statements are possible:

a) The unaccounted losses l of the plant (MUF) are taken into consideration in form of a fixed fraction ε of J_{in}, that is $l = \ J_{in}$. The inspector states either, that something has been diverted or that nothing has been diverted; he does not state anything on the amount of diverted material.

b) The inspector states that an amount m greater than m' and smaller than m'' is missing. Only after this statement he takes into consideration the possible unaccountable losses (MUF) of the plant.

These models can be described with help of different statistical procedures namely:

(i) the classical Bayes procedure,

(ii) the method of confidential intervals,

(iii) the testing procedure.

In the first method it has to be assumed that the parameter of the stochastic variable which is to be estimated, is itself a stochastic variable. The distribution function of this variable has to be known or some suitable assumptions have to be made. In the second method the statement of likelihood has to be introduced. In this paper the last named method has been used as it allows for statements which can be used directly for calculating the effectivity defined in part 2.

3.1 Model A: Not Quantified Statements of the Inspector

3.1.1 Statements of the Inspector

The inspector's hypothesis is $J = I + l$, $l = \zeta J_{in}$. This means that he assumes that the fraction ζ of J_{in} has been lost in the plant (MUF). Naturally this fraction can vary for different campaigns. It is assumed here that the inspector can estimate from his experience the value of ζ for the next campaign.

Note: Because of measuring errors, the inspector does not know J_{in} exactly, therefore, he does not know 1 exactly even if he knows ζ . It is further assumed that he can estimate 1 in a satisfactory manner.

The inspector tests his hypothesis with the help of his n and m measurements by a test δ. This test δ is defined by a region of acceptance $A(\delta) = (-\infty, z)$ and a critical region $K(\delta) = \bar{A}(\delta)$ in the following way: If on the basis of the next n and m measurements the following relation holds

$$\bar{J} - \bar{I} - 1 \in K(\delta) \qquad (3.1)$$

the inspector rejects his hypothesis that is he states that there was a diversion. If, on the other hand, the following condition is fulfilled

$$\bar{J} - \bar{I} - 1 \in A(\delta) \qquad (3.2)$$

he does not reject his hypothesis, in actual practice this means that he states that there has not been any diversion.

The value of z and the error first kind, i.e. the probability that $\bar{J} - \bar{I} - 1 \in K(\delta)$ for $J - I - 1 = 0$, are intimately connected. In order to establish this connection and for later purposes a special case of the general law of transformations of distribution functions is introduced here (see for example [3]):

Let $f_{a_1}(x_1)$ and $f_{a_2}(x_2)$ be the frequency functions of a_1 and a_2. Then

$$f_b(z_1) = \int_{-\infty}^{\infty} dz_2 f_{a_1}(z_1+z_2) f_{a_2}(z_2) \qquad (3.3)$$

is the frequency function of $b = a_1 - a_2$.

Now the error first kind α is given by

$$\alpha = p(\bar{J} - \bar{I} - 1 \geq z \,/\, J - I - 1 = 0) \qquad (3.4)$$

If it is assumed that $\bar{J}$ and $\bar{I}$ are normally distributed with expectation values J and I and variances σ_J^2/n and σ_I^2/m, then with (3.3)

$$1-\alpha = \frac{\sqrt{nm}}{2\pi\sigma_J\sigma_I} \int_{-\infty}^{z+1} dz_1 \int_{-\infty}^{\infty} dz_2 \exp\left[-\frac{1}{2}\left(\frac{n}{\sigma_J^2}(z_1+z_2-J)^2 + \frac{m}{\sigma_I^2}(z_2-J+1)^2\right)\right]$$

This result in

$$\alpha = 1-\phi\left(\frac{z}{\sigma}\right);\ \sigma = \left(\frac{\sigma_J^2}{n} + \frac{\sigma_I^2}{m}\right)^{\frac{1}{2}} \tag{3.5}$$

Here, ϕ (x) is given by

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt \tag{3.6}$$

Eq. (3.5) gives the connection between α and z. Lateron the value of z and therefore that of α will be established with the help of the postulates given in part 2.

3.1.2 Probability of Detection and Error Second Kind

The probability of detection p (m_o) is defined by the probability that $\bar{J}-\bar{I}-1 > z$, if m_o is to be diverted. This means

$$p(m_o) = p(\bar{J}-\bar{I}-1 > z / J-I-1 = m_o) \tag{3.7}$$

Again with the help of (3.3.)

$$p(m_o) = 1-\phi\left(\frac{z-m_o}{\sigma}\right);\ \sigma = \left(\frac{\sigma_J^2}{n} + \frac{\sigma_I^2}{m}\right)^{\frac{1}{2}} \tag{3.8}$$

With the help of the probability of detection the error second kind β (m_o), i.e. the probability that $J-I-1 \varepsilon A(\delta)$ if m_o is to be diverted, can be calculated:

$$\beta(m_o) = p(\bar{J}-\bar{I}-1 < z / J-I-1 = m_o) = 1-p(m_o) \tag{3.9}$$

Therefore from (3.8)

$$\beta(m_o) = \phi\left(\frac{z-m_o}{\sigma}\right) \tag{3.10}$$

3.1.3 Costs and Optimization of Costs; Effectivity

The following form of the costs for the safeguards system based on flow measurement has been assumed:

$$C = \frac{\alpha_1}{\sigma_J^2} + \frac{\alpha_2}{\sigma_I^2} + \beta_1 n + \beta_2 m \tag{3.11}$$

With this form the capital and the operating costs of the system have been symbolized. The postulates of part 2 are now

(i) A function for the probability of detection $p(m_o)$ is postulated. Together with (3.8) this gives a condition for σ:

$$\sigma = \frac{z - m_o}{\phi^{-1}\ (1-p(m_o)} \tag{3.12}$$

Here, ϕ^{-1} is the inverse function for ϕ.

(ii) It is postulated that the error second kind is less than β_o for m_o greater than m_o . This gives with (3.8,10)

$$1 - p\ (m_{oo}) = \phi(\ \frac{z-m_{oo}}{\sigma}\ ) < \beta_o \tag{3.13}$$

In the case of eq. (1.6)

$$a \leq \frac{1}{m_{oo}}\ \ln\ \frac{1}{\beta_o} \tag{3.14}$$

is obtained.

(iii) It is postulated that the error first kind is less than α_o. This gives with (3.5)

$$1 - \phi(\frac{z}{\sigma}) = \alpha_o \tag{3.15}$$

or in the limiting case

$$1- \phi\ (\frac{z}{\sigma}) = \alpha_o \tag{3.16}$$

Eq. (3, 16), that is condition (iii) fixes the value of z.

Eq. (3.12) and eq. (3.16) give a new condition for σ:

$$\sigma^2 = S^2 = \frac{m_o^2}{(\phi^{-1}(1-\alpha_o) - \phi^{-1}(1-p(m_o)))^2} \tag{3.17}$$

Now again the problem is to optimize the costs in (3.11) with respect to the variables $\sigma_J 2$, $\sigma_I 2$, n, m. Besides the condition (3.17) there are natural conditions of the following form

$$n,\ m > 1;\ a_1 < \sigma_J 2 < a_2;\ b_1 < \sigma_I 2 < b_2 \tag{3.18}$$

Example

Let n=m=1 and assume that in the following the conditions (3.18) and (3.14) (or the equivalent for (3.14) for another form of the postulated $p(m_o)$) are fulfilled. Then from (3.11) and (3.17)

$$C\ (\sigma_I 2) = \frac{\alpha_1}{S^2 - \sigma_I 2} + \frac{\alpha_2}{\sigma_I 2} + \beta_I + \beta_2 \tag{3.19}$$

$C\ (\sigma_I 2)$ has the form given in Fig. 3.

The optimum costs C_{opt} and the optimum values of $\sigma_I 2$ and $\sigma_J 2$ are obtained in the following form:

$$C_{opt}\ (\sigma^2_{Io}) = \frac{1+\gamma^2}{S^2}\left(\frac{\alpha_1}{\gamma^2} + \alpha_2\right);\ \sigma^2_{Io} = \frac{S^2}{1+\gamma^2}\ ;\ \sigma^2_{Jo} = \frac{\gamma^2 S^2}{1+\gamma^2}\ ;$$

$$\gamma^2 = \left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{2}} \tag{3.20}$$

The optimum costs as a function of m_o can be expressed as follows

$$C_{opt}(m_o) = \chi \frac{(\phi^{-1}(1-\alpha_o) - \phi^{-1}(1-p(m_o))^2}{m_o^2};\ \chi = (1+\gamma^2)\left(\frac{\alpha_1}{\gamma 2} + \alpha_2\right) \tag{3.21}$$

Two numerical examples are given to illustrate the method:

(i) For $p(m_o)$ eq. (1.6) let a = 0.5; let α= 0.05. Furthermore, let m_o be chosen that $0 \leq m_o \leq 1$ in the sense that $m_o = 1$ means the unity of the effective mass [4]. The results are shown in Fig.4a and 4b: Fig. 4a gives the postulated probability of detection and Fig. 4b gives the optimum costs divided by $\varkappa$ as a function of m_o. Here, $\varkappa$ is defined by (3.21)

(ii) Let $p(m_o)$ be a step function as shown in Fig. 5a, and $\alpha_o = 0.05$. Fig. 5b gives the optimum costs divided by $\varkappa$ as a function of m_o.

Now, according to the definition given in part 2, fig. 4b or fig. 5b gives except for the factor $\varkappa$ the effectivity of the safeguards system based on flow measurement. It can be compared with that of the inspector's system once the cost factors are known.

Note: From (3.21) it is seen that the optimal value of σ_I^2 is a function of the amount of diverted material. This is not disturbing for the following reason. As already said in part 2 in practice there is a fixed budget C'_{opt}. This budget establishes with eq. (3.20) the smallest amount m_o' of fissile material which can be detected (see Fig. 4b). Since σ_J^{o2} is an essentially ascending function of m_o, it would mean that all diverted amounts m_o greater than m_o can be detected for this fixed budget. The same is true for the boundary of the critical region z.

3.1.4 Comparison of the Statements with those introduced earlier [1]

(1) The likelihood of diversion P_d was defined as a statement of the inspector, as the assessment that an amount has been diverted with likelihood P_d. The errors first and second kind in this paper are fixed before the measurements of the inspector take place. However, they can be interpreted as being used after the measurements in the following way:

(i) $\bar{J}-\bar{I}-1 \in A$: With likelihood $1-\beta$ no diversion has taken place.

(ii) $\bar{J}-\bar{I}-1 \in K$: With likelihood $1-\alpha$ a diversion has taken place.

In this sense, $1-\alpha$ and $1-\beta$ are the above mentioned likelihoods of diversion.

(2) The risk of detection was defined as a statement of the operator: If the operator intends to divert a certain quantity m_o, he is able to calculate before the beginning of the measurements the risk R_D that the inspector comes to a statement P_d. This is the probability of detection in this paper which has been calculated for fixed α, β. Therefore the probability of detection is - in the words of [1] - the risk for a fixed threshold of alarm.

(3) The probability P_p of proving was defined as a **statement** of the safeguards system designer; it is the probability that in the case of the diversion of the amount m_o of fissile material the inspector makes any statement P_d about the diversion of the fraction ξ of m_o. In our case P_d is fixed, namely $1-\alpha$ and $1-\beta$. Therefore, in our case the probability of detection corresponds to P_p, too.

3.2 Model B: Quantified Statements of the Inspector

3.2.1 Statements of the Inspector

As mentioned earlier, in this model the inspector first states the missing material. Thereafter he takes into consideration the losses.

The inspector divides his n flow measurements and his m inventory measurements in the following form:

$$n = n_1 + n_2; \; m = m_1 + m_2 \tag{3.22}$$

The n_1 and m_1 measurements are used by him for making a hypothesis on the missing material whereas he uses the n_2 and m_2 measurements for testing his hypothesis. The average values of the n_1 and m_1 measurements are $\bar{J}_{n_1}$ and $\bar{I}_{m_1}$.

Let

$$\Delta_1 = \bar{J}_{n_1} - \bar{I}_{m_1} \tag{3.23}$$

Now the hypothesis of the inspector is

$$\Delta_1 - \sigma_1 \leq J - I \leq \Delta_1 + \sigma_1; \quad \sigma_1 = \left(\frac{\sigma_J^2}{n_1} + \frac{\sigma_I^2}{m_1} \right)^{\frac{1}{2}} \tag{3.24}$$

or, in short

$$H = \lambda : \left\{ \Delta_1 - \sigma_1 \leq \lambda \leq \Delta_1 + \sigma_1 \right\} ; \quad \lambda = J-I \qquad (3.25)$$

(More generally $H = \left\{ \lambda : \Delta_1 - a_1 \leq \lambda \leq \Delta_1 + a_2 \right\}$).

With the help of the n_2 and m_2 measurements the inspector tests his hypothesis with a test δ which is given by the critical region $K(\delta)$ for $\Delta_2 = \bar{J}_{m_2} - \bar{I}_{m_2}$

$$K(\delta) = \left\{ \Delta_2 ; \ \Delta_2 \leq \Delta_1 - \sigma_1 ; \ \Delta_2 \geq \Delta_1 + \sigma_1 \right\} \qquad (3.26)$$

This means that the inspector rejects his hypothesis if $\Delta_2 \varepsilon K(\delta)$ and that he does not reject his hypothesis if $\Delta_2 \notin K(\delta)$.

<u>Note:</u> In the case that the hypothesis (3.25) is rejected, the inspector divides his n and m measurements anew, that is he takes another set of n_1 and m_1 measurements to form his hypothesis.

The test given above is characterized by an operation characteristic $P(\lambda)$ which is defined by

$$P(\lambda) = p(\Delta_2 \varepsilon K(\delta) / J-I = \lambda) = 1 - p(\Delta_1 - \sigma_1 \leq \Delta_2 \leq \Delta_1 + \sigma_1 / J-I = \lambda) \qquad (3.27)$$

The calculation gives

$$P(\lambda) = 1 + \phi\left(\frac{\Delta_1 - \sigma_1 - \lambda}{\sigma_2} \right) - \phi\left(\frac{\Delta_1 + \sigma_1 - \lambda}{\sigma_2} \right); \quad \sigma_2 = \left(\frac{\sigma_J^2}{n_2} + \frac{\sigma_I^2}{m_2} \right)^{\frac{1}{2}} \qquad (3.28)$$

The form of $P(\lambda)$ is given in Fig. 6.

For example, the upper boundaries for the errors first and second kind can be expressed in terms of the operation characteristic $P(\lambda)$:

(i) α_1 the upper boundary for the error first kind is given by

$$p(\Delta_2 \varepsilon K(\delta) / J-I \,\varepsilon\, H) \leq \alpha \qquad (3.29)$$

With (3.28) one obtaines

$$\alpha = P(\lambda = \Delta + \sigma_1) = \frac{3}{2} - \phi\left(2 \frac{\sigma_1}{\sigma_2} \right) \qquad (3.30)$$

(ii) β, the upper boundary for the error second kind, is given by

$$p\ (\Delta_2 \notin K(\delta)/J\text{-}I \notin H) \leq \beta \tag{3.31}$$

With (3.28) one obtains

$$\beta = 1-P(\lambda = \Delta_1 + \sigma_1) = \phi(\frac{2\sigma_1}{\sigma_2}) - \frac{1}{2} \tag{3.32}$$

From (3.30) and (3.32) one sees

$$\alpha + \beta = 1 \tag{3.33}$$

Thus α and β cannot be made as small as one would like to have. However, this would not matter much, as instead of postulating certain values of α and β, it may be postulated that the generalized errors are smaller than some given values. More exactly

(i) It is postulated that the generalized error first kind, that is the probability that $\Delta_2 > \Delta_1 + a\sigma_1$ or $\Delta_2 < \Delta_1 - a\ \sigma_1$ for $\lambda \varepsilon H$ and $a > 1$ is smaller than α_o:

$$p(\Delta_2 > \Delta_1 + a\sigma_1, \Delta_2 < \Delta_1 - a\sigma_1\ /\lambda\ \varepsilon\ H) \leq \alpha_o \tag{3.34}$$

or

$$\alpha_o = 2 - \phi(\frac{(a+1)\sigma_1}{\sigma_2}) - \phi(\frac{(a-1)\ \sigma_1}{\sigma_2}) \tag{3.35}$$

(ii) It is postulated that the generalized error second kind, that is the probability that $\Delta_1 - \sigma_1 \leq \Delta_2 \leq \Delta_1 + \sigma_1$ for $\lambda > \Delta_1 + b\sigma_1$ and $b > 1$ is smaller than β_o:

$$p(\ \Delta_2 \notin K(\delta)/\ \lambda < \Delta_1 - b\ \sigma_1, \lambda > \Delta_1 + b\sigma_1) \leq \beta_o \tag{3.36}$$

or

$$\beta_o = \phi((b+1)\ \frac{\sigma_1}{\sigma_2}) - \phi((b-1)\ \frac{\sigma_1}{\sigma_2}) \tag{3.37}$$

These postulates mean that the probability, that for $\lambda \in H$ the value of Δ_2 falls far from H ($>\Delta_1 + a\sigma_1$ or $<\Delta_1 - a\sigma_1$) respectively that for λ far from H ($>\Delta_1 + b\sigma_1$ or $<\Delta_1 - b\sigma_1$) the value of Δ_2 falls into the region of acceptance, is smaller than α_o respectively β_o.

3.2.2 Probability of Detection

It is assumed that in the course of the next campaign the amount m_o of fissile material disappears. The probability that from the n_1 and m_1 measurements the inspector will find a value of Δ_1 between Δ and $\Delta + d\Delta$ is given by

$$p(\Delta \leq \Delta_1 \leq \Delta + d\Delta \; / \; J = I + m_o) \tag{3.38}$$

The probability that from the n_2 or m_2 measurements the inspector will find a value of Δ_2 between $\Delta - \sigma_1$ and $\Delta + \sigma_1$, is given by

$$p(\Delta - \sigma_1 \leq \Delta_2 \leq \Delta + \sigma_1 / J = I + m_o) \tag{3.39}$$

In this case the inspector states that the amount m greater than $\Delta - \sigma_1$ and smaller than $\Delta + \sigma_1$ is missing.

As probability of detection $p(m_o; m', m'')$ the probability is defined that the inspector makes a statement of the form 'an amount m with $\hat{m} - \sigma_1 \leq m \leq \hat{m} + \sigma_1$ is missing' where $\hat{m} - \sigma_1 \geq m'$, $\hat{m} + \sigma_1 \leq m''$. Then

$$p(m_o; m', m'') = \int_{m'+\sigma_1}^{m''-\sigma_1} d\Delta \; p(\Delta - \sigma_1 \leq \Delta_2 \leq \Delta + \sigma_1, \Delta \leq \Delta_1 \leq \Delta + d\Delta \; / J = I + m_o) =$$

$$= \int_{m'+\sigma_1}^{m''-\sigma_1} d\Delta \; p(\Delta - \sigma_1 \leq \Delta_2 \leq \Delta + \sigma_1 \; / \; J = I + m_o) \cdot p(\Delta \leq \Delta_1 \leq \Delta + d\Delta \; / J = I + m_o)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{m' - m_o + \sigma_1}{\sigma_1}}^{\frac{m'' - m_o - \sigma_1}{\sigma_1}} dx \left[\phi\left(\frac{\sigma_1}{\sigma_2}(x+1)\right) - \phi\left(\frac{\sigma_1}{\sigma_2}(x-1)\right) \right] \exp\left(-\frac{x^2}{2}\right) \tag{3.40}$$

Example: Let

$$m' = m_o - 2\sigma_1 ; \; m'' = m_o + 2\sigma_1 \tag{3.41}$$

In this case from (3.40)

$$p(m_o) = 2\phi(1) - \sqrt{\frac{2}{\pi}} \int_{-1}^{+1} dx \; \phi\left(\frac{\sigma_1}{\sigma_2}(x-1)\right) \exp\left(-\frac{x^2}{2}\right) \tag{3.42}$$

Up to now the unaccountable losses (MUF) were not considered. They may be taken into consideration in the following way. The inspector who states that an amount m greater m' and smaller than m'' is missing, says: Because of my experience I assume that the fraction ζ of the total input is MUF. Therefore I state according to (3.24) that the amount λ with

$$\Delta_1 - \sigma_1 - \zeta J_{in} \leq \lambda \leq \Delta_1 + \sigma_1 - \zeta J_{in} \tag{3.43}$$

has been diverted. Accordingly the probability of detection $p(m_o, m', m'')$ is the probability that the inspector will state that the amount λ with

$$\hat{m} - \sigma_1 - \zeta J_{in} \leq \lambda \leq \hat{m} + \sigma_1 - \zeta J_{in} \tag{3.44}$$

where $\hat{m} - \sigma_1 \geq m'$, $\hat{m} + \sigma_1 \leq m''$, has been diverted if the amount m_o will be diverted.

3.2.3 Costs and Optimization of Costs; Effectivity

The costs of the safeguards system are as given by (3.11). Now the inspector makes quantitative statements, therefore the postulates of part 2 have to be modified as they were adjusted to the case of non-quantitative statements.

The following postulates are established to take care of the quantitative nature of the model B.

(i) In connection with the probability of detection it is postulated
   (1) m' and m'' are functions of m_o, that is $m'(m_o)$ and $m''(m_o)$.
   (2) $\hat{p}(m_o) := p(m_o; m'(m_o), m''(m_o))$

(ii) The generalized error first kind is less than α_o.

(iii) The generalized error second kind is less than β_o.

Example

Let according to (3.41)

$$m' = m_o - 2\sigma_1;\; m'' = m_o + 2\sigma_1;\; \sigma_1 \text{ fixed} \tag{3.45}$$

and as in (1.6)

$$\hat{p}\,(m_o) = 1 \cdot e^{-\frac{m_o}{a}}$$

Then from (3.42) and (3.45) the following conditions are obtained

$$\sigma_1^{\,2} = b = \text{const};\; \sigma_2^{\,2} = g\,(m_o) = :\, g \tag{3.46}$$

Furthermore it is assumed that the postulates (i, ii) are satisfied by (3.46).

Therefore, the problem is to optimize

$$C = \frac{\alpha_1}{\sigma_J 2} + \frac{\alpha_2}{\sigma_I 2} + \beta_1 n + \beta_2 m \tag{3.47}$$

with respect to the conditions (3.46)

$$\frac{\sigma_J 2}{n_1} + \frac{\sigma_I 2}{m_1} = b;\; \frac{\sigma_J 2}{n_2} + \frac{\sigma_I 2}{m_2} = g \tag{3.48}$$

and the natural conditions

$$n_1 + n_2 = n;\; m_1 + m_2 = m;\; n_\nu, m_\nu > 0;\; a_1 \le \sigma_J^2 \le a_2;\; b_1 \le \sigma_I^2 \le b_2 \tag{3.49}$$

The solution is sketched below:

Let

$$n_1 = y_1 n;\; m_1 = y_2 m \tag{3.50}$$

According to (3.49) the allowed regions for y_1 and y_2 are

$$\frac{1}{n} \leq y_1 \leq 1-\frac{1}{n}; \quad \frac{1}{m} \leq y_2 \leq 1-\frac{1}{m} \qquad (3.51)$$

From (3.48) one obtains

$$\sigma_J^2 = nS_1; \quad \sigma_I^2 = m\,S_2 \qquad (3.52)$$

where

$$S_1 = \frac{y_1(1-y_1)}{y_1-y_2}\,(-y_2(b+g)+b); \quad S_2 = \frac{y_2(1-y_2)}{y_1-y_2}\,(y_1(g+b)-b) \qquad (3.53)$$

The costs (3.47)

$$C = \frac{\alpha_1}{nS_1} + \frac{\alpha_2}{nS_2} + \beta_1 n + \beta_2 m \qquad (3.54)$$

are optimized so that the relation

$$C(n_o,m_o) = 2\left(\frac{\alpha_1\beta_1}{S_1}\right)^{\frac{1}{2}} + 2\left(\frac{\alpha_2\beta_2}{S_2}\right)^{\frac{1}{2}}; \quad n_o = \left(\frac{\alpha_1}{\beta_1 S_1}\right)^{\frac{1}{2}}; \quad m_o = \left(\frac{\alpha_2}{\beta_2 S_2}\right)^{\frac{1}{2}} \qquad (3.55)$$

is obtained.

To optimize $C(n_o, m_o)$ with respect to σ_J^2, σ_I^2, the eq. (3.52) is inserted into eq. (3.55) and the following is obtained:

$$C\,(n_o,m_o;\ \sigma_J^2,\ \sigma_I^2) = \frac{2\,\alpha_1}{\sigma_J^2} + \frac{2\,\alpha_2}{\sigma_I^2} \qquad (3.56)$$

The minimum is given by the boundary values of σ_J^2, σ_I^2:

$$C(n_o,m_o;\ \sigma_{Jo}^2,\ \sigma_{Io}^2) = \frac{2\alpha_1}{a_2} + \frac{2\,\alpha_2}{b_2} \qquad (3.57)$$

This is the absolute minimal value of the costs C. By fixing $\sigma^2_{J_o}$ and $\sigma^2_{I_o}$ the values of y_1 and y_2 are fixed and therefore, by eq. (3.50), n_v and m_μ for all v, μ with $1 \leq v \leq n - 1$, and $1 \leq \mu \leq m - 1$. This assumes that the boundary condition (3.51) for $y_{1,2}$ is fulfilled.

The equations determining the optimal values of y_1 and y_2 are

$$S_{10} = \frac{\sigma^2_{J_o}}{n_o} = \frac{\beta_1 a_2^2}{\alpha_1} =: A; \quad S_{20} = \frac{\sigma^2_{I_o}}{m_o} = \frac{\beta_2 b_2^2}{\alpha_2} =: B \tag{3.58}$$

therefore

$$\frac{y_1(1-y_1)}{y_1-y_2} \cdot (y_2(b+g)-b) = -B; \quad \frac{y_2(1-y_2)}{y_1-y_2} \cdot (y_1(g+b)-b) = A \tag{3.59}$$

Eq. (3.59) leads to an equation of third order in α:

$$y_1^3 + C_2 y_1^2 + C_1 y_1 + C_o = 0 \tag{3.60}$$

where

$$C_o = \frac{A}{A_1}\left(1+B-\frac{b}{A}\right)$$

$$C_1 = \frac{A}{bA_1}\left[\left(\frac{b}{A}-B\right)(A_1-A_2) - (g+b)\left(1+B-\frac{b}{A}\right) + \left(\frac{b}{A}-1\right)B\right]$$

$$C_2 = \frac{A}{bA_1}\left[2A_1\left(B-\frac{b}{A}\right) + \frac{bA_2}{A} + (g+b)\left(1-\frac{b}{A}\right)\right]$$

$$A_1 = (g+b)\left(\frac{b}{A}-1\right) - \frac{b^2}{A}; \quad A_2 = (g+b)\left(\frac{b}{A}-1\right)B + bB$$

As follows from (3.51) one has to choose that solution of (3.60) which satisfies the condition

$$\frac{\beta_1 a_1}{\alpha_1} \leq y_1 \leq 1 - \frac{\beta_1 a_2}{\alpha_1}; \quad \frac{\beta_2 a_2}{\alpha_2} \leq y_2 \leq 1 - \frac{\beta_2 b_2}{\alpha_2} \tag{3.61}$$

Eq. (3.57) gives the effectivity of the system under consideration - it is independent of m_o,

<u>Note:</u> In the framework of the inspector's system one had quantitative statements, too, but they were not used, for one wanted to compare the inspector's system with the system based on flow measurement in the form of the former model A. For example, in the framework of the inspector's system, the probability of detection p (m_o; m', m'') is given by

$$p(m_o;\, m',\, m'') = \sum_{v=m'}^{m''} \binom{m_o}{v} \left(\frac{qk}{n}\right)^{2} \left(1 - \frac{qk}{n}\right)^{m_o - v} \qquad (3.62)$$

Therefore, it would be possible to compare the inspector's system with the system based on flow measurement in the form of model B also.

4. Compilation of Main Formulas of Different Models

The more important expressions developed for the probability of detection, error first and second kind, cost functions, and effectivity, for the two safeguards systems, have been shown in Table I.

5. Conclusion

The analysis carried out in this paper shows that the effectivity of a safeguards system can be defined in a quantitative way. It also shows that the effectivity defined in this manner can be used to compare widely different safeguards systems.

Two models for a safeguards system with flow measurement, were studied on the basis of testing hypotheses for the amount of diverted fissile material. From the mathematical stand-point it is of particular interest, to note that these tests were adjusted only with respect to the costs which they themselves cause, and not, as is usually the case, the losses (positive or negative) which they cause. This slightly unusual way was chosen as in the case of safeguards systems, the losses are difficult to quantify.

The models are not complete and further work is in progress.

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TABLE I. COMPILATION OF MAIN FORMULAS

	Inspector's system	Instrumented System	
		Model A	Model B
Probability of Detection	$p(m_o)=1-\left(1-\frac{qk}{n}\right)^{m_o}$ (1.4)	$p(m_o)=1-\phi\left(\frac{z-m_o}{\sigma}\right)$ (3.8)	$p(m_o;m';m'')=\frac{1}{\sqrt{2\pi}}\int_{\frac{m'-n_o+\sigma_1}{\sigma_1}}^{\frac{m''-m_o-\sigma_1}{\sigma_1}} dx\left[\phi\left(\frac{\sigma_1}{\sigma_2}(x+1)\right)-\phi\left(\frac{\sigma_1}{\sigma_2}(x-1)\right)\right]\exp\left(-\frac{x^2}{2}\right)$ (3.40)
Error first kind	$\alpha=0$	$\alpha=1-\phi\left(\frac{z}{\sigma}\right)$ (3.5)	(Upper boundary) $\alpha=\frac{3}{2}-\phi\left(\frac{2\sigma_1}{\sigma_2}\right)$
Error second kind	$\beta(m_o)=\left(1-\frac{qk}{n}\right)^{m_o}$ (1.5)	$\beta(m_o)=\phi\left(\frac{z-m_o}{\sigma}\right)$ (3.10)	(Upper boundary) $\beta=\phi\left(\frac{2\sigma_1}{\sigma_2}\right)-\frac{1}{2}$
Cost function	$C=\alpha_1 k+\alpha_2 q$ (1.11)	$C=\frac{\alpha_1}{\sigma_J^2}+\frac{\alpha_2}{\sigma_I^2}+\beta_1 n+\beta_2 m$ (3.11)	
Effectivity (optimum costs as a function of diverted material)	$C_{opt}(m_o)=2(\alpha_1\alpha_2\alpha_3)^{\frac{1}{2}}$	In the case $n=m=1$ $C_{opt}(m_o)=\chi\frac{\left(\phi^{-1}(1-\alpha_o)-\phi^{-1}(1-p(m_o))\right)^2}{m_o^2}$ (3.21)	In the special case given in eq.(3.45) $C_{opt}(m_o)=\frac{2\alpha_1}{a_2}+\frac{2\alpha_2}{b_2}$ (3.57)

List of Symbols

$A(\delta)$	Region of acceptance of the test δ
C	Costs of a safeguards system
c,d	Losses of the operator in the framework of the inspector's system
I	True inventory at time t
$\bar{I}$	Measured inventory at time t
$J = J_{in} - J_{out}$	True difference between input and output
$\bar{J}$	Measured difference between input and output
$K(\delta)$	Critical region of the test δ
k	Number of inspectors in the inspector's system
$l = \zeta J_{in}$	Losses of fissile material, as a fraction of input
m_o	Amount of diverted fissile material
m	Number of inventory measurements
n	Number of input and output measurements
$P(\lambda)$	Operation characteristic as a function of $\lambda = J-I$
p	Probability
$p(m_o)$	Probability of detection
q	Probability of detection at a certain area in case of diversion of fissile material in the framework of the inspector's system
r	Number of areas at which fissile material can be diverted, in the case of the inspector's system
t	Time
W	Value of the game which describes the inspector's system
z	Boundary of the critical region of the test in model A

List of Symbols (continued)

α, β	Errors first and second kind
$\alpha_{1,2}$; $\beta_{1,2}$	Coefficients of costs of safeguards systems
δ	Test
ε	Mathematical symbol for 'element'
ϕ	Error function

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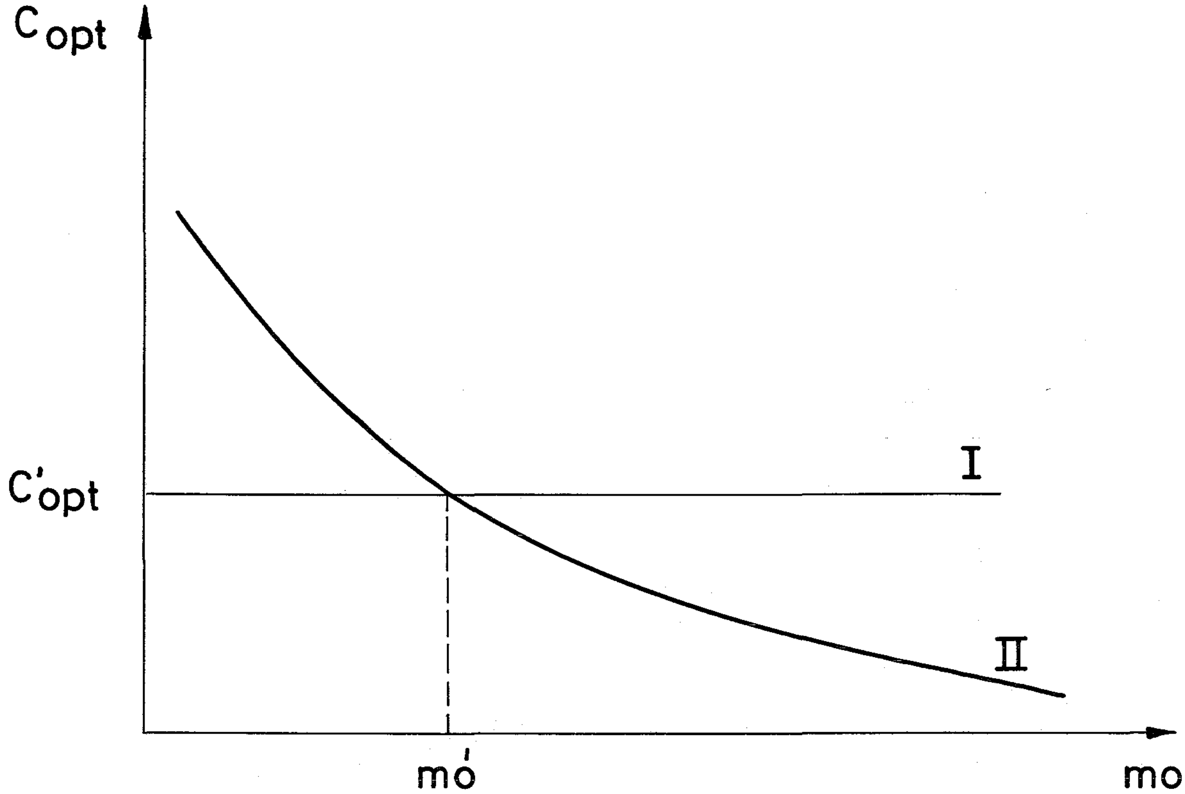

Fig 1: Effectivity of safeguards systems: Optimized costs as a function of material mo to be diverted.

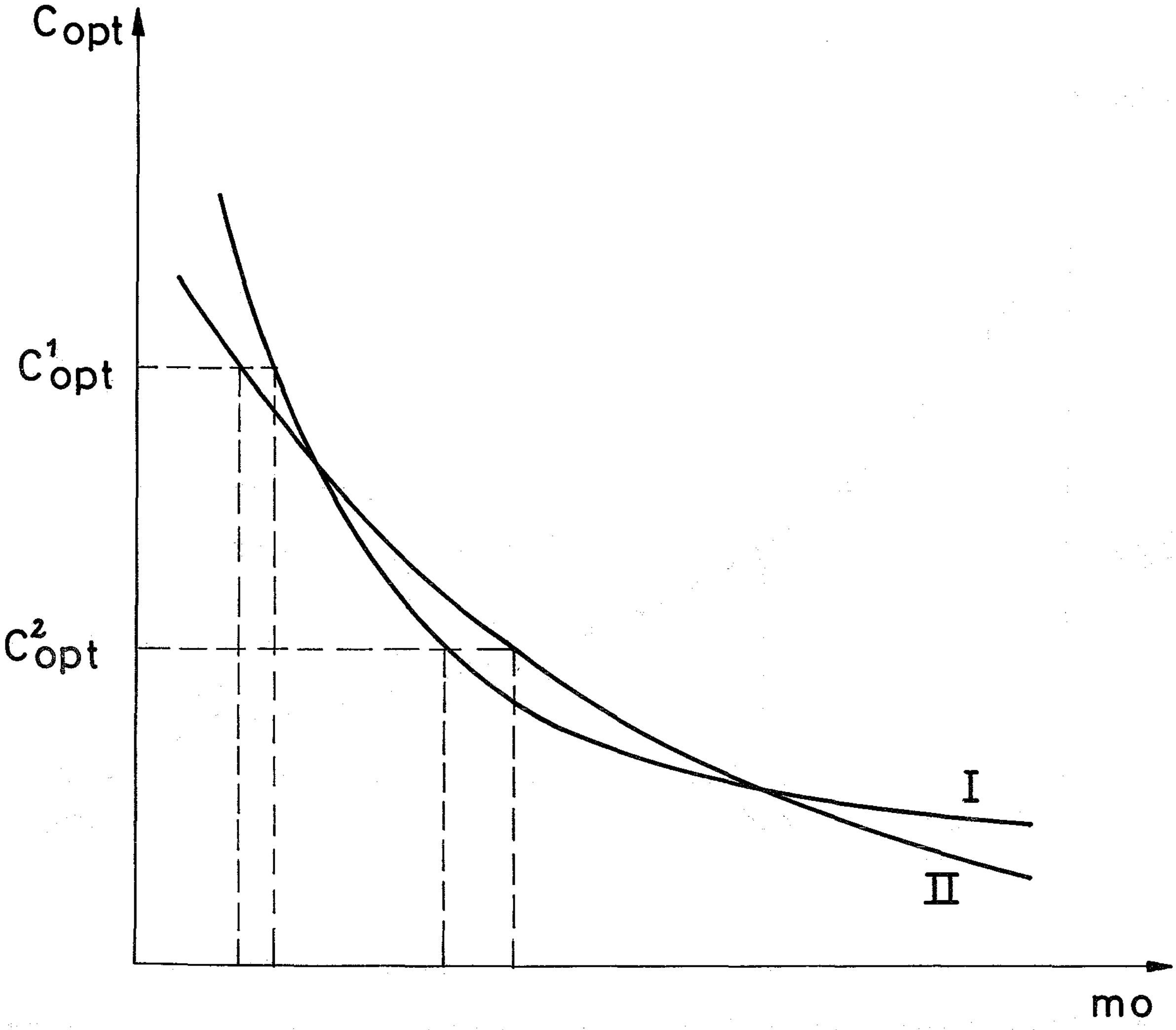

Fig.2: Effectivity of safeguards systems: For fixed budget C^1_{opt} system II is more effective, for fixed budget C^2_{opt} system I is more effective.

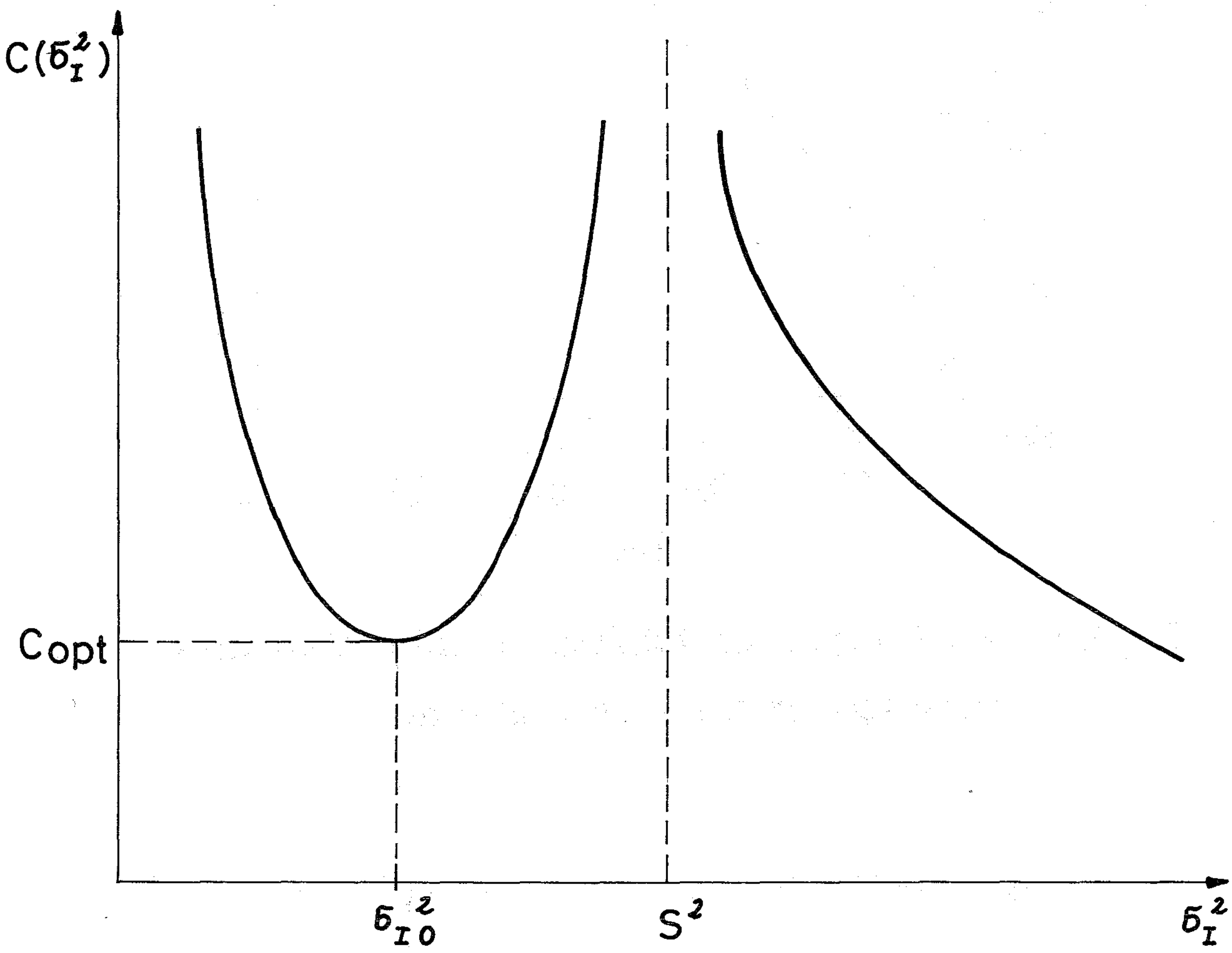

Fig. 3 Costs of a safeguards system as a function of σ_I^2.

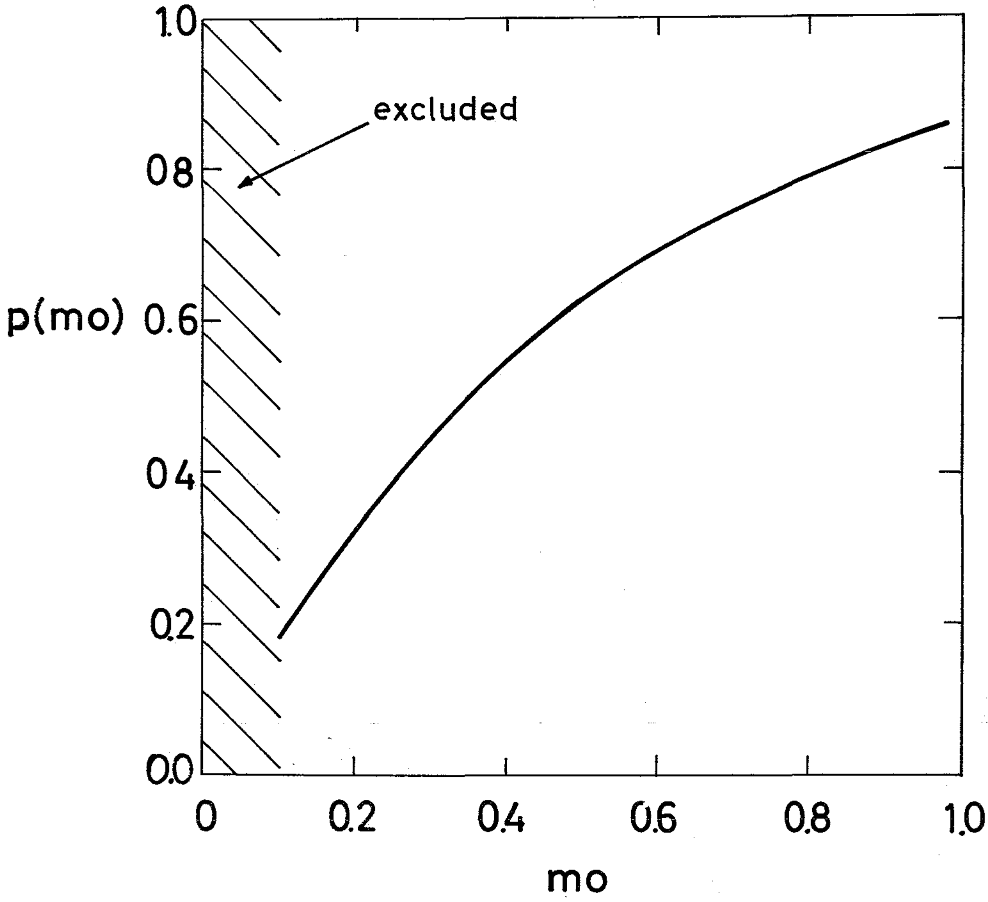

Fig. 4a: Postulated probability of detection as a function of diverted material.

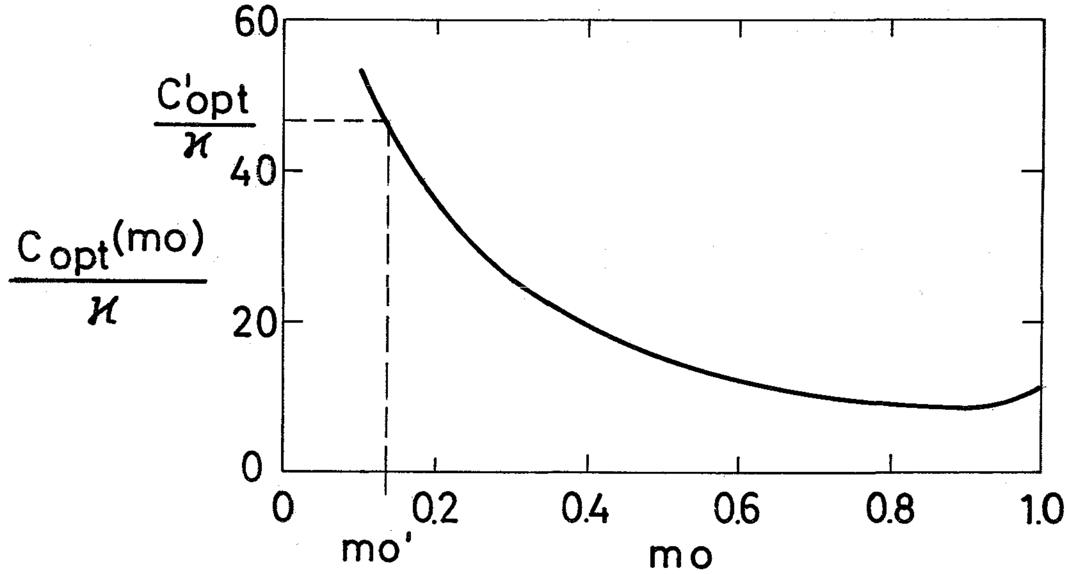

Fig. 4b: Effectivity of the system: Optimized safeguards costs as a function of diverted material. Meaning of $C_{opt}/\varkappa$ and mo' see text.

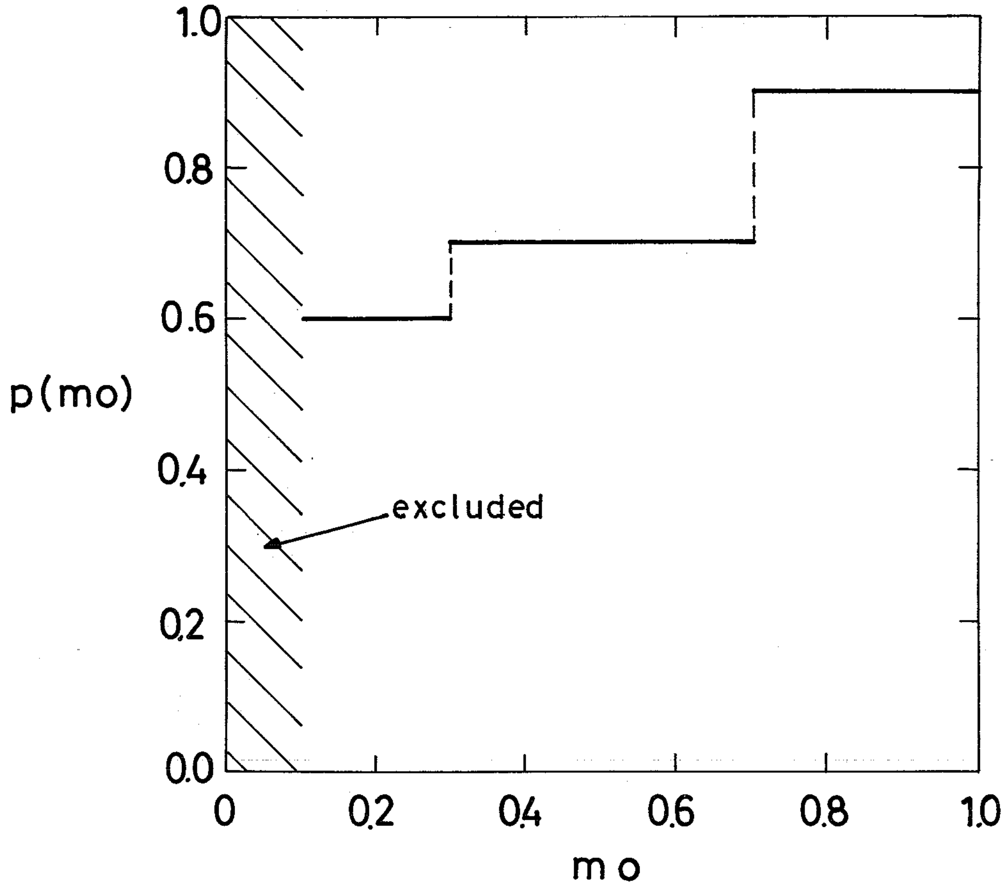

Fig.5a: Postulated probability of detection as a function of diverted material.

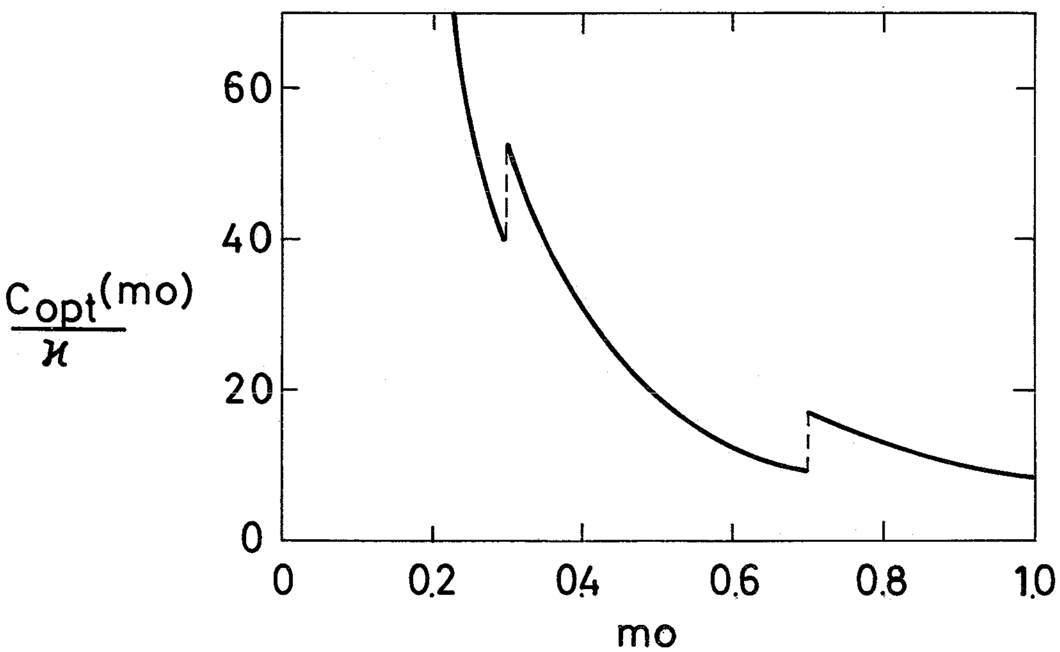

Fig. 5b: Effectivity of the system: Optimized safeguards costs as a function of diverted material.

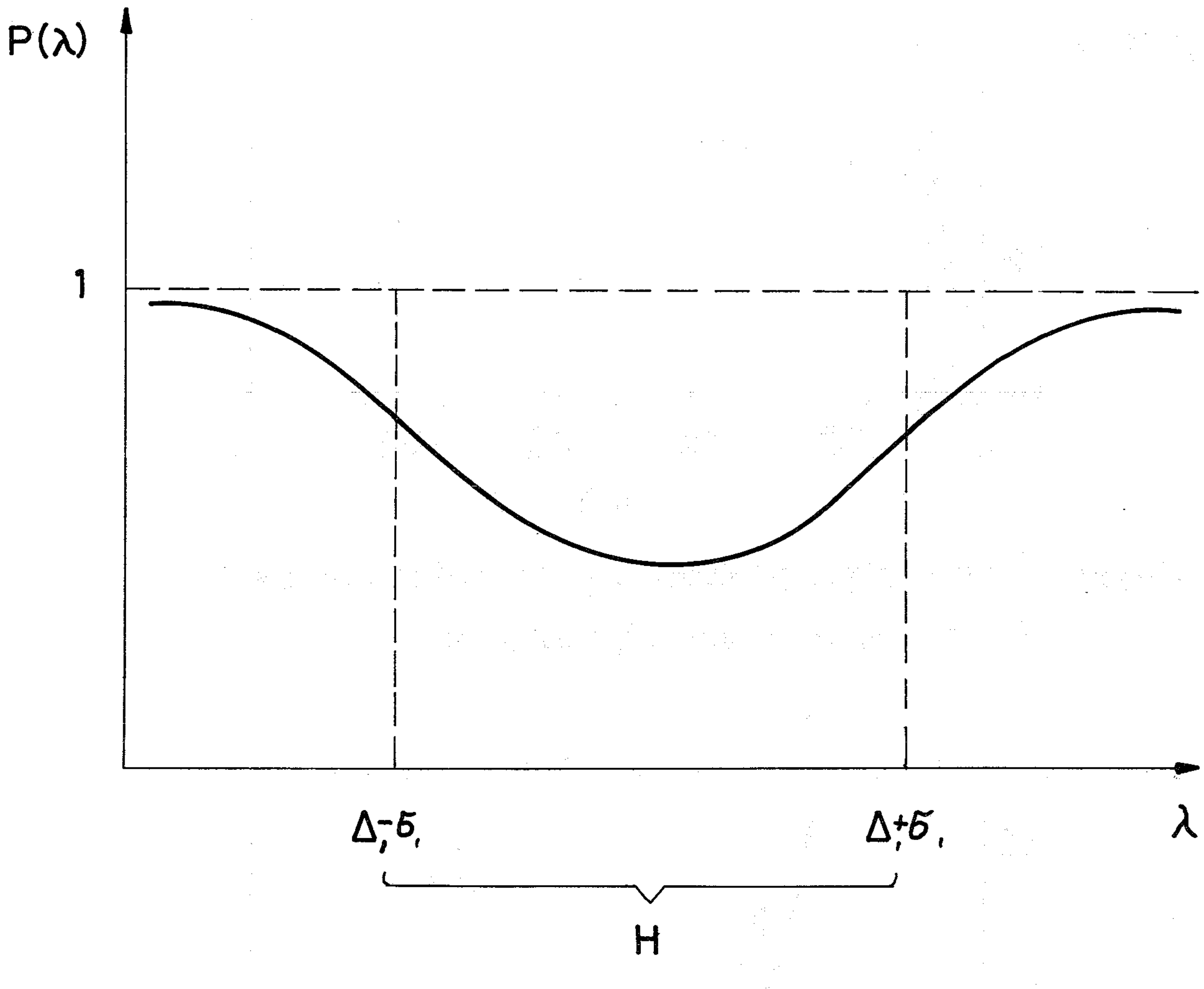

Fig. 6 Operation characteristic of the test given by (3.28)