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Criteria for Action Levels for Inspection Based on a Game Theoretical Model

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CRITERIA FOR ACTION LEVELS FOR INSPECTION BASED ON A GAME THEORETICAL MODEL [+)]

by

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+) Paper presented at the IAEA Working Panel on Systems Analysis, held on 25-29 August 1969, Vienna

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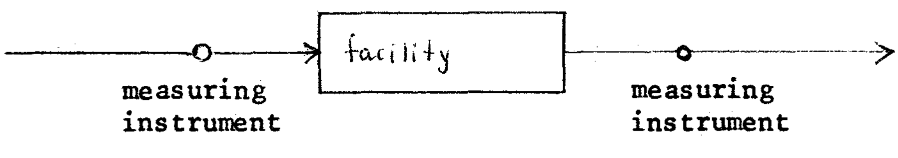
1. Introduction

Safeguards systems are installed in nuclear facilities to prevent or to detect diversion of fissile material. A condition on the systems considered is that they must have a "negligible" chance of declaring a false alarm under normal management practices.

In this paper a game theoretic model for a safeguards system based on flow measurements of fissile material is developed and analysed. The optimal behaviour of the facility's profit-maximizing operator and safeguards authority is defined and determined.

2. Description of the Safeguards Procedure

We conceive the fabricating, reprocessing or other facility (possibly part of a larger facility) as possessing one legal entrance and one legal exit for the fissile material.



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Between two inventory takings, a certain amount m of material is to be weighed out and processed in a number of prescribed portions. This amount m will later be taken as a unit of mass for the material. Since the weighing will be inexact due to stochastic measurement errors, the true input M is a random variable. Stochastic measurement errors are also made in determining the output.

These measurements, as well as m, are registered into a record book, on the basis of which the nominal value, i.e. the book inventory can be calculated.

When the facility is closed down - for instance during an inventory - the book inventory can be compared with the physical inventory. The physical inventory is treated as a portion of the output. We assume the measurements of the material flow are unbiased; and in particular that the expected value of M is m. This can be achieved by concentrating the measuring instruments in so-called "strategic points", where inspectors of the safeguards authority can control the measurements.

Let U be the difference between the portion m and the output-measurement.

U is the sum

$$U = e + v + m - M - \mu \qquad (2.1.)$$

where v is the net amount of retained material in the facility, e the actual amount of illegally diverted material, m - M the deviation of the real input M from the nominal m, and μ the output measurement error, that is the deviation of the measurement from the true value. We assume the input M to scatter so slightly that we can take v, m-M, μ stochastically independent. We define λ as

$$\lambda := m - M - \mu \qquad (2.2.)$$

and get

$$U = e + v + \lambda \qquad (2.3.)$$

The normal working process of the facility allows only a finite variation interval $[\underline{v},\overline{v}]$ for v.

As the measurement of the throughput is composed of several stochastically independent measurements of portions of the whole input and output, the distribution of λ resembles a normal distribution (by the central limit theorem). We take for this distribution a 'normal distribution without tails', which has a density

$$g(y) = \begin{cases} r \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}} & \text{for } y \in [-\Sigma, \Sigma] \\ o & \text{else} \end{cases} \qquad (2.4.)$$

$\Sigma (\geq 4\sigma)$ has still to be fixed, $r > 1$ is the normalising factor such that

$$\int_{-\Sigma}^{\Sigma} g(y)\, dy = 1$$

To facilitate some of the computations we shall later on sometimes use the normal density

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}} \quad \text{for } y \in \mathbb{R}$$

instead of $g(y)$

The operator and the safeguards authority are supposed to know the distribution function of v and λ. If the operator is interested in an illegal diversion of fissile material, he decides before the beginning of the working period, which rate $e \in [0,1]$ of the nominal input m he wants to divert (Yet we assume e to be less than the divertable amount $\hat{M} - \hat{v}$).

Just so, the safeguards authority has to choose (possibly in a random process) a number θ, before the beginning of the measurements or book checking. $\theta \in (0,1)$ is a bound such that in the case $\hat{U} \leq \theta$ legal processing is stated; in the case $\hat{U} > \theta$ a search in the facility is arranged (here and in the following, we denote the realisation of a random variable y by $\hat{y}$).

The search includes a thorough cleaning of the facility and yields the portion αv of the material v retained in the plant. α is a constant close to 1, known to operator and safeguards authority.

The measurement of αv is the random variable $W = \alpha v + k$ (2.7) where the random variable k is the deviation of the measurement of the washed out material αv from the true value αv. k is not necessarily independent of αv. In the case

$$\hat{U} - \frac{1}{\alpha}\hat{W} > \varepsilon$$

the inspector states: The amount of $\hat{U} - \frac{1}{\alpha}\hat{W} - \varepsilon$ has been diverted. In the case $\hat{U} - \frac{1}{\alpha}\hat{W} \leq \varepsilon$ legal processing is stated. The choice of the value of ε is a matter of judgement and should be so large, that legal processing is recognized with probability close to 1. For the game ε is a constant.

3. The game in normal form

Gains and losses of the safeguards authority and the operator depend on the outcome of the safeguards procedure. These pay-offs are partly fixed by political agreements. In the following disposition the pay-offs depend linearly (up to constants) on the diverted amount e respectively on the amount $\hat{U} - \frac{1}{\alpha}\hat{W} - \varepsilon$ stated as diverted. In general the pay-offs will depend on e or $\hat{U} - \frac{1}{\alpha}\hat{W} - \varepsilon$ monotonely in the same direction as in the linear disposition shown in table 3.1.

Considering the purpose of safeguarding, case I is a bad outcome for the safeguards authority (e>o), case II_1 is even worse, whereas case II_2 is a good outcome.

Therefore the pay-off or the utility of the inspection authority can be taken as the negative of the operator's pay-off. We assume (as usual) that the players i.e. operator and inspector guide their behaviour along the expected value of the pay-offs. Let $\sigma(e,\theta)$ be the expectation of the operator's pay-off under the strategies e and θ. The operator is player 1 and the safeguards authority player 2. Then the normal form of the two-person zero sum game is

$$(e \in [0,1]\ ,\ \ \theta \in [0,1]\ ,\ \ \sigma) \qquad (3.1.)$$

For the calculation of σ we use the auxiliary function

$$f(e,\theta,\eta) = p(U>\theta \wedge U - \frac{1}{\alpha} W-\varepsilon > \eta) \qquad (3.2.)$$

which is the probability that, under the choice of e and θ, an illegal diversion of an amount $>\eta$ is stated by the inspector. We call $f(e,\eta,o)$ the probability of detecting an illegal action.

Probabilities further needed can all be expressed by $f(e,\theta,\eta)$:

$$p(U\leq\theta) = 1 - f(e,\theta,-\infty) \qquad (3.3.)$$

$$p(U>\theta \wedge U - \frac{1}{\alpha} W-\varepsilon\leq 0) = f(e,\theta,-\infty) - f(e,\theta,o)$$

The pay-off function σ involves the expectation $\mathcal{E}X$ of the amount X declared as diverted:

$$X := \begin{Bmatrix} o & \text{for } U\leq\theta \\ o & \text{for } U>\theta,\ U - \frac{1}{\alpha} W - \varepsilon \leq 0 \\ U - \frac{1}{\alpha}W-\varepsilon & \text{for } \quad U - \frac{1}{\alpha} W - \varepsilon > 0 \end{Bmatrix} \qquad (3.4.)$$

Let f be piecewise smooth and

$$\lim_{\eta\to\infty} \eta \quad f(e,\theta,\eta) = o \qquad (3.5.)$$

which is true in the special cases treated in the end of this paper. By the mean value theorem of differential calculus the probability of stating the diversion of an amount

$\varepsilon(\eta,\eta+\Delta\eta)$ with $\eta\geq o$ is equal to

$$f(e,\theta,\eta) - f(e,\theta,\eta+\Delta\eta) = -\frac{\partial f}{\partial\eta}(e,\theta,\eta+ h\Delta\eta)\Delta\eta$$

with $h\in(0,1)$. It follows by partial integration (3.6.)

$$\mathcal{E}X = \int_o^\infty \eta(-\frac{\partial f}{\partial\eta}(e,\theta,\eta) \quad d\eta = \int_o^\infty f(e,\theta,\eta)d\eta$$

Using f, we can express σ as

$$\sigma(e,\theta) = de(1 - f(e,\theta,o)) + E\ (f(e,\theta,-\infty) - f(e,\theta,o)) - C\ f(e,\theta,o) - c\int_o^\infty f(e,\theta,\eta)d\eta \qquad (3.7.)$$

One can assume that f depends continuously on (e,θ,η) in all practically interesting cases. Then σ is continuous in (e,θ). The discrete extension (P,Q,σ) of the game is strictly determinate, and has a game value and optimal strategues, i.e. min - max strategies. Using his optimal strategy, player 1 gets at least the game value, whereas player 2 has to pay no more than the game value when using his optimal strategy x).

In the following, we make assumptions about the distribution of measuring deviations λ and k, such that already the nonextended game $(e \epsilon [0,1], \theta \epsilon [0,1], \sigma)$ is strictly determinate, and both players have optimal strategies. These are calculated for fixed parameters.

4. Costs of safeguards

Besides the costs of the control of measuring instruments, in this model there arise costs for the safeguards authority depending on the outcome of the safeguards procedure. In the case II_1 the safeguards authority has to pay the costs A of the search, the production losses during the search, lost orders and the amendment for false charge.

In the case II_2 the safeguards authority can be rewarded by getting the fines included in the penalties.

The expectation of the costs under the strategies (e,θ) is

$$K(e,\theta) = A\left[f(e,\theta,-\infty) - f(e,\theta,o)\right] - Bf(e,\theta,o) - b\int_{o}^{\infty} f(e,\theta,\eta)d\eta \qquad (4.1.)$$

where A and B are components of C $(A+B \leq C)$ and b is component of c.

If the safeguards authority has several min-max strategies, the optimal strategies can be restricted to the min-max-strategies with favourable expected costs.

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x) See, e.g. M. Dresher, Games of Strategy

5. Reduction of the sets of strategies

Before the calculation of min-max-strategies in 2 person zero sum games one eliminates the dominated strategies at first.

Def. X dominates X' respecting Y
if $a(x,y) \geq a(x'y)$ for all $y \in Y$ and
$a(x,y) > a(x',y)$ for one y

y dominates y' respecting X, if
$a(x,y) \leq a(x,y')$ for all $x \in X$ and $a(x,y) < a(x,y')$
for one $x \in X$

The elimination of dominated strategies is called reduction.

We assume that the random variables v, λ, k are essentially bounded. The bounds of their domain then yield statements about dominated strategies and a reasonable determination of the quantity ε.

The random variables take values in the following intervals with probability 1 (we take the smallest intervals with this property).

$$\begin{array}{lcl} [\underline{v}, \overline{v}] & & \ni \hat{v} \\ [-\Sigma, \Sigma] & & \ni \hat{\lambda} \\ \left\{ \begin{array}{l} [-T, T] \\ [-\alpha\hat{v}, T] \end{array} \right. & \text{for} & \left. \begin{array}{l} \alpha\hat{v} - T \geq 0 \\ \alpha\hat{v} - T < 0 \end{array} \right\} \ni \hat{k} \end{array} \tag{5.1.}$$

The condition that no false alarms should be possible, yields the requirement

$$\varepsilon \geq \varepsilon_o \tag{5.2.}$$

where

$$\varepsilon_o = \max \left\{ U - \frac{1}{\alpha} W / e = o \right\} = \left\{ \begin{array}{ll} \Sigma + \frac{T}{\alpha} & \text{for } \alpha\overline{v} - T \geq 0 \\ \Sigma + \overline{v} & \text{for } \alpha\overline{v} - T < 0 \end{array} \right\} \tag{5.3.}$$

We define $\overline{\theta}$ as the greatest lower bound of all θ with the property: $\hat{U} > \theta$ implicates $\hat{U} - \frac{1}{\alpha}\hat{W} > \varepsilon$ independent of the size of $\hat{W}$. Then $\overline{\theta}$ dominates all $\theta > \overline{\theta}$. The calculation yields

$$\bar{\theta} = \varepsilon - \varepsilon_o + \begin{cases} \bar{v} + \Sigma + \frac{2T}{\alpha} & \text{for} \quad \alpha\bar{v} - T \geq o \\ \bar{v} + \Sigma + \bar{v} + \frac{T}{\alpha} & \quad\quad \alpha\bar{v} - T < o \end{cases} \quad (5.4.)$$

Accordingly we define $\underline{\theta}$ as the least upper bound of all θ with the property: $\hat{U} \leqq \theta$ implicates $\hat{U} - \frac{1}{\alpha} \cdot \hat{W} \leqq \varepsilon$ independent of $\hat{W}$. Then $\underline{\theta}$ dominates all strategies $\theta < \underline{\theta}$. The inspector will therefore choose strategies only from the interval $[\underline{\theta}, \bar{\theta}]$

One gets for $\underline{\theta}$

$$\underline{\theta} = \varepsilon - \varepsilon_o + \begin{cases} \underline{v} + \Sigma & \bar{v} - \frac{T}{\alpha} \geq \underline{v} - \frac{T}{\alpha} \geq o \\ \frac{T}{\alpha} + \Sigma & \text{for} \quad \bar{v} - \frac{T}{\alpha} \geq o \geq \underline{v} - \frac{T}{\alpha} \\ \bar{v} + \Sigma & \bar{v} - \frac{T}{\alpha} < o \end{cases} \quad (5.5.)$$

regarding the assumption $W \geq o$.

For the values $\bar{v} = 0{,}02$; $\Sigma = 0{,}05$; $2T = 0{,}02$; $\alpha = 1$; $\underline{v} = 0{,}01$, $\varepsilon = \varepsilon_o$ we get

$$\bar{\theta} = 0{,}09; \ \underline{\theta} = 0{,}06$$

The range $[\underline{\theta}, \bar{\theta}]$ is therefore far smaller than $[0,1]$.

Just as the inspector, the operator can take into account that the inspector will use strategies only out of $[\underline{\theta}, \bar{\theta}]$. He will therefore take into account only strategies e which are favourable against the inspector's strategies $\in [\underline{\theta}, \bar{\theta}]$, i.e. he reduces his set of strategies, respecting $[\underline{\theta}, \bar{\theta}]$. For this purpose we define $\bar{e}$ by

$$\bar{e} := \inf \{ e \mid U > \bar{\theta} \} \quad (5.6.)$$

The calculation yields

$$\bar{e} = \bar{\theta} - \underline{v} + \Sigma = \varepsilon - \varepsilon_o + \begin{cases} \bar{v} - \underline{v} + 2\Sigma + \frac{2T}{\alpha} & \text{for} \quad \alpha\bar{v} - T \geq o \\ \bar{v} - \underline{v} + 2\Sigma + \bar{v} + \frac{T}{\alpha} & \quad\quad \alpha\bar{v} - T < o \end{cases} \quad (5.7.)$$

Strategies $e > \bar{e}$ are then no longer interesting for the operator.

Remark: For $\Sigma = T = o$, $\varepsilon = \varepsilon_o$ and $\underline{v} = \overline{v}$ we get $\overline{e} = o$ in accordance with reality.

The set of strategies to be considered by the operator can as well be bounded by $\underline{e}$ from beneath:

$$\underline{e} = \text{g. l. b.} \left\{ e / \ U - \frac{1}{\alpha} W \leqq \varepsilon \right\} \qquad (5.8.)$$

A little calculation gives

$$\underline{e} = \varepsilon - \varepsilon_o$$

It is therefore convenient, to choose $\varepsilon = \varepsilon_o$ (ξ_o defined above). We fix $\varepsilon = \varepsilon_o$ for the remainder of this paper.

Remark: For the reduction of the sets of strategies we have only used the fact that the cases I, II_1 are unfavourable for the inspector , whereas the case II_2 is favourable for him. The special form of the pay-offs has not been used.

For the reduction of the strategy sets it was essential that the random variables are essentially bounded. To get these bounds for approximately normally distributed random variables we cut off the tails of the respective normal densities symmetrically to the expected value such that there is to be expected only a slight operator's loss at false alarm. Therefore we get the condition that the terms of the pay off-function σ are close to zero for $e = o$ and every θ.

In the numerical example considered later on this condition yields bounds $\Sigma \varepsilon [4\sigma, 6\sigma]$ for λ. The domains $[\underline{\theta}, \overline{\theta}]$ and $[\underline{e}, \overline{e}]$ fixed above are then domains over which optimal strategies can be conjectured to randomize.

Remark: In this model an illegal diversion is stated only after a thorough search. If the random variables ν and λ are bounded as assumed above, then one can state a diversion already after the first measurement, namely in the case $\hat{U} > \Sigma + \overline{v}$. As the stated "diverted amount" one can then take $\max(\hat{U} - \Sigma - \overline{v}, \hat{U} - \frac{1}{\alpha}\hat{W} - \varepsilon)$. This additional possibility of detection is essential only if the measurement error k takes large values with a great probability; we exclude this case.

6. Discussion of parameters

The optiaml strategies of safeguards authority and operator are functions of the parameters c, d, C, , ... As the pay-off-function σ is very complicated these functions do not have a simple analytical form. Thetefore we reflect, in which regions the parameters can variate, we then compute function values for combinations of some arguments out of these regions and analyse the influence of the parameters on the function values.

The parameter regions listed in the following table 6-1 refer to a reprocessing plant with a half-year period between two inventory takings and a throughput of approximately 100 kg Pu per period. The price of Pu is assumed to be 40 \$/g. The control refers to Pu alone.

The mean value of C in the table has the order of magnitude of the loss caused by a production standstill of 8 days i.e. 200 000 \$.

7. Calculation of the optimal strategies

At first we assume that $v = \underline{v} = \overline{v}$ is constant and that the deviation k from the true value $\alpha\overline{v}$ is zero. The considerations on the reduction of the sets of strategies yield $\overline{\theta} = \overline{v} + \Sigma = \underline{\theta}$, i.e. there is exactly one undominated strategy of the inspector. Therefore $\overline{\theta} = \underline{\theta}$ is his unique optimal strategy. ε is equal to Σ. For this $\varepsilon = \Sigma$ $\theta := \overline{v} + \Sigma$ dominates all other θ, even if λ is assumed to be normally distributed with variance σ^2 and expectation 0.

For the auxiliary function (3.2.)

we get

$$\begin{aligned} f(e,\theta,\eta) &= p(e+\underline{v}+\lambda>\theta \wedge e + \lambda-\Sigma>\eta) \\ &= p(\lambda > \max\ (\theta-v,\ \eta+\ \Sigma) - e) \\ &= 1 - \phi\left(\frac{\max(\theta-\underline{v},\eta+\ \Sigma) - e}{\sigma}\right) \end{aligned} \quad (7.1)$$

The probability to detect an illegal action is then

$$f(e,\underline{\theta},0) = 1-\phi\left(\frac{\Sigma - e}{\sigma}\right) \qquad (7.2)$$

As the expectation of the amount to be declared as diverted we have in the case $\theta = \underline{\theta}$:

$$\mathcal{E}X = \int_0^\infty \left(1-\phi\left(\frac{\eta+\Sigma-e}{\sigma}\right)\right)d\eta$$

$$= \frac{\sigma}{\sqrt{2\pi}}\exp\left(-\frac{(\Sigma-e)^2}{2\sigma^2}\right)+(-\Sigma+e)\left(1-\phi\left(\frac{\Sigma-e}{\sigma}\right)\right) \qquad (7.3)$$

The safeguards costs are

$$K(e,\underline{\theta}) = -B\left(1-\phi\left(\frac{\Sigma-e}{\sigma}\right)\right) \qquad (7.4)$$
$$-b\left(\frac{\sigma}{\sqrt{2\pi}}\exp\left(-\frac{(\Sigma-e)^2}{2\sigma^2}\right)+(-\Sigma+e)\left(1-\phi\left(\frac{\Sigma-e}{\sigma}\right)\right)\right)$$

That means, the safeguards authority has no costs.

The pay-off-function for the operators is

$$\sigma(e,\overline{\theta}) = de\phi\left(\frac{\Sigma-e}{\sigma}\right)-C\left(1-\phi\left(\frac{\Sigma-e}{\sigma}\right)\right)$$
$$-c\left(\frac{\sigma}{\sqrt{2\pi}}\exp\left(-\frac{(\Sigma-e)^2}{2\sigma^2}\right)+(-\Sigma+e)\left(1-\phi\left(\frac{\Sigma-e}{\sigma}\right)\right)\right)$$

Instead of the unprecise demand:
$\lambda\varepsilon[-\Sigma, \Sigma]$ with probability close to 1, we now require the terms
$C\phi\left(\frac{\Sigma}{\sigma}\right)$, $c\phi\left(\frac{\Sigma}{\sigma}\right)$, $c\frac{\sigma}{\sqrt{2\pi}}\exp\left(-\frac{(3\Sigma)^2}{2\sigma^2}\right)$ of the pay-off function σ to be less than a little number, e.g. 1 \$. Then $\Sigma\varepsilon[4\sigma, 6\sigma]$, and the operator has to expect a loss of 3 \$.

By simple analytical arguments it can be shown that $\sigma(e,\underline{\theta})$ takes its maximum at a sole argument e_σ, which is in the interval $(0,\Sigma)$. Therefore e_σ is the unique optimal strategy of the operator.

In the tables 7-1 the strategies e_σ are listed for specific combinations of parameters. From the table 7-1, 1 and 2 one can perceive the relatively small influence of the penalties C and c on the optimal strategy e_σ and on the pay-off $\sigma(e_\sigma,\underline{\theta})$. From table 4 one can perceive that the pay-off $\sigma(e_\sigma,\underline{\theta})$ to the operator increases almost linearly with d, whereas the amount e_σ remains almost unchanged. On the contrary, the technical parameters σ and Σ have a strong influence on the optimal strategies of both players. From table 1 we perceive that e_σ has approximately the size of 0.8 Σ, and that the pay-off increases almost linearly with Σ.

The following graph shows the probability $f(e,\underline{\theta},0)$ of detection and the expectation $\mathcal{E}X$ of the amount X declared as diverted (Fig. 1).

In the following we leave the incisive assumption $\underline{v} = \overline{v}$. It seems to be reasonable to take v as rectangular distributed over the interval $[\underline{v}, \overline{v}]$. We assume the measurement of washed-out material αv to be exact enough, so that the error k can be subdued. Then we have

$$\varepsilon = \varepsilon_o = \Sigma, \quad \underline{\theta} = \underline{v} + \Sigma, \quad \overline{\theta} = \overline{v} + \Sigma$$

Let

$$d_v(x) = \begin{cases} 0 & x \notin [\underline{v}, \overline{v}], \\ \frac{1}{\overline{v}-\underline{v}} & x \in [\underline{v}, \overline{v}] \end{cases} \tag{7.6}$$

be the density of the random variable v. Then we get

$$f(\theta, e, \eta) = p\,(e+v+\lambda>\theta \wedge e+\lambda-\Sigma>\eta) \tag{7.7}$$

$$= \int_{-\infty}^{\infty} d_v(x)\,\left[1-\phi\left(\frac{\max(-\theta-x-e,\ \eta+\Sigma-e)}{\sigma}\right)\right]dx$$

Now it is easily shown that $\overline{\theta}$ dominates all $\theta>\overline{\theta}$ and that $\underline{\theta}$ dominates all $\theta<\underline{\theta}$, if λ is distributed according to $\phi(\frac{y}{\sigma})$. The probability of detecting a diversion for $\theta\in[\underline{\theta}, \overline{\theta}]$ is equal to

$$f(e,\theta,0) = 1 + \frac{1}{\bar{v}-\underline{v}} \Big[(\theta-\bar{v}-e)\phi\left(\frac{\Sigma-e}{\sigma}\right) - (\theta-\underline{v}-e)\phi\left(\frac{\theta-\underline{v}-e}{\sigma}\right)$$

$$+ \frac{\sigma}{\sqrt{2\pi}}\left(\exp\left(-\frac{(\Sigma-e)^2}{2\sigma^2}\right) - \exp\left(-\frac{(\theta-\underline{v}-e)^2}{2\sigma^2}\right)\right)\Big] \qquad (7.8)$$

For $\theta \varepsilon [\underline{\theta}, \bar{\theta}]$ the amount X to be declared as diverted has the expectation

$$\mathcal{E}X = -(\Sigma-e)\left(1-\phi\left(\frac{\Sigma-e}{\sigma}\right)\right)$$

$$+ \frac{1}{\bar{v}-\underline{v}} \Big[\left\{(\theta-\underline{v}-e)(\Sigma-e)+\sigma^2\right\} \left(\phi\left(\frac{\theta-\underline{v}-e}{\sigma}\right) - \phi\left(\frac{\varepsilon-e}{\sigma}\right)\right)$$

$$+ \frac{\sigma}{\sqrt{2\pi}} \left\{(-\theta+\bar{v}+e)\exp\left(-\frac{(\Sigma-e)^2}{2\sigma^2}\right) + (\Sigma-e)\exp\left(-\frac{(\theta-\underline{v}-e)^2}{2\sigma^2}\right)\right\}\Big] \qquad (7.9)$$

The second partial derivative $\frac{\delta^2 \sigma}{\delta\theta^2}$ is positive for all $\theta\varepsilon[\underline{\theta}, \bar{\theta}]$ and $e\varepsilon[0,1]$, i.e. σ is strictly convex in θ. Therefore a pure strategy $\theta\varepsilon[\underline{\theta}, \bar{\theta}]$ is the only optimal strategy of the inspector.

By a game theoretic theorem an optimal strategy for the operator is to randomize over at most two values e_1, $e_2 \varepsilon [0,1]$.

The coefficients of d,E,C,c are functions of e and Θ. Every function $h(e,\theta)$ of these coefficient functions has the property

$h(\Sigma+\bar{v}-\underline{v},\theta) > h(e,\theta)$ for $e > \Sigma+\bar{v}-\underline{v}$ simultaneously in all $\theta\varepsilon[\underline{\theta},\bar{\theta}]$.

Therefore the pure strategies e_1, e_2 over which an optimal strategy randomizes are $\varepsilon[0, \Sigma+\bar{v}-\underline{v}]$.

We did not succeed in proving that these coefficient functions are concave in $e\varepsilon[0,\Sigma+\bar{v}-\underline{v}]$. But it can be hoped that σ itself is strictly concave in e for the interesting parameter combinations of d,c,C,Σ,σ: then there exists exactly one optimal strategy; this is a pure strategy.

In the extreme case E=0 we have

$$\sigma(\underline{\theta},e) < \sigma(\theta,e) \text{ for } \theta\varepsilon(\underline{\theta}, \bar{\theta}]$$

simultaneously for all e. Therefore $\underline{\theta}$ is the optimal strategy of the operator. As $\underline{\theta}$ is a bound of the interval $[\underline{\theta}, \bar{\theta}]$ there is an optimal pure strategy of the operator which is an argument maximizing the function $\sigma(\underline{\theta}, e)$

Similarly, for very large E corresponding to d = c = 0, C = 0 we get $\overline{\theta}$ as the optimal strategy of the inspector, whereas the operator has again a pure optimal strategy.

8. Conclusion

This paper has shown that the conflict inherent in safeguarding fissile material can be handled in formal game theoretic terms. For several examples of realistic data, calculations have been given of the inspector's optimal behaviour according to the model developed.

It turns out the political parameters, whose values are hard to obtain, have only a small influence. The technical parameters, for instance measurement variance, influence the optimal strategies to a far greater extent.

The practical conclusions are two. The facility should be designed so that the technical parameters governing the uncertainty in the amount of fissile materials involved should be small.

In order that optimal inspection behaviour be calculable, the technical parameters must also be accurately known.

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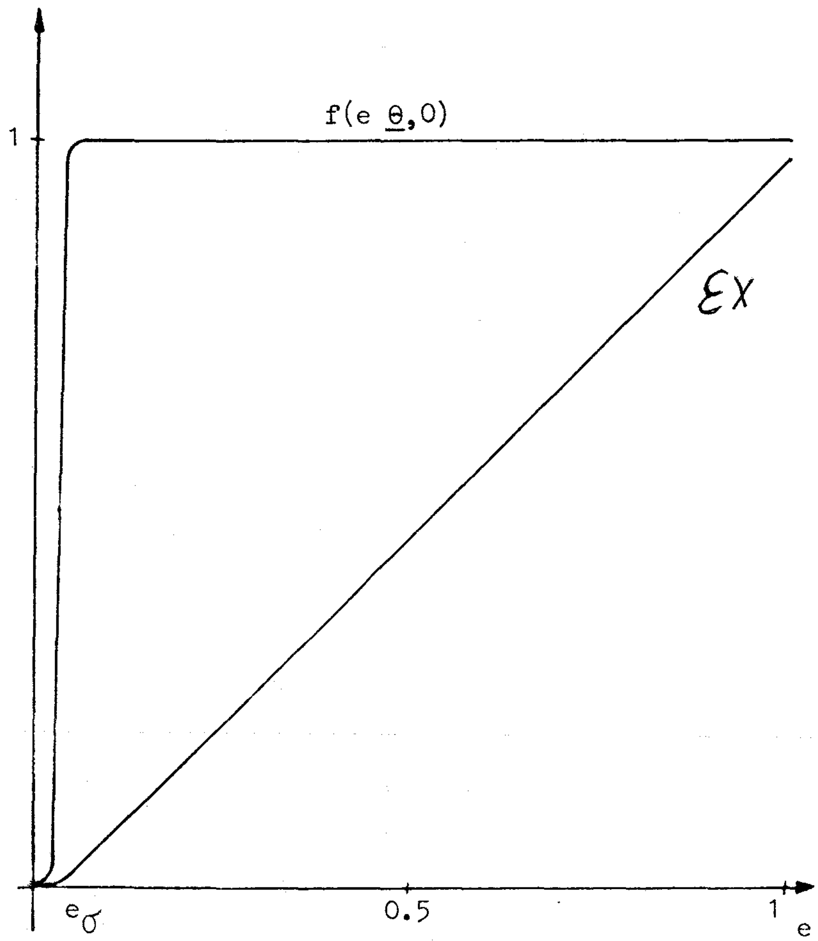

Fig. 1: Probability of detection $f(e,\underline{\theta},0)$ and expectation value $\mathcal{E}X$ of the amount X declared as diverted as a function of the amount e to be diverted ($\Sigma=\varepsilon=0.03$; $\sigma=0.005$)

Table 3-1: Outcomes and Pay-offs Possible for a Play

Operator chooses $e \epsilon [0,1]$	safeguards authority chooses $\theta \epsilon [0,1]$
Outcome of the safeguards procedure (play)	Pay-off to the operator (utility of the operator)
case I $\hat{U} \leqq \theta$ statement; legal processing, continuation of work	de, $d > o$ is the (subjective) utility of the operator per material unit (nominal input)
case II $\hat{U} > \theta$, search case II_1, $\hat{U} - \frac{1}{\alpha}\hat{W} \leq \epsilon$ statement: no deviation has taken place	de + E, $E > 0$ is composed of the political gain and an amendment for false charge
case II_2, $\hat{U} - \frac{1}{\alpha}\hat{W} > \epsilon$ statement: the amount of $\hat{U} - \frac{1}{\alpha}\hat{W} - \epsilon$ was diverted	$-C - c\,(\hat{U} - \frac{1}{\alpha}\hat{W} - \epsilon)$ $C > o$ is composed of the costs of the search and the production losses during the search, the losses by lost orders and the penalty for illegal diversion. $c(\hat{U} - \frac{1}{\alpha}\hat{W} - \epsilon)$ is an additional penalty for the declared amount of diverted material. The penalties can take into account the time of the plant's standstill

Table 6-1 : Cost parameters and technical parameters

value	minimal	mean	maximal	
C	$2\ 10^4$	2.10^5	$2\ 10^6$	$
E	$2\ 10^3$	$2\ 10^4$	$2\ 10^5$	$
c	$4\ 10^6$	$4\ 10^7$	$4\ 10^8$	$
d	$4\ 10^6$	$4\ 10^7$	$4\ 10^8$	$
σ	0.0015	0.005	0.015	
Σ	0.009	0.03	0.06	+)
$\bar{v}$	0.003	0.01	0.03	
$\underline{v}$	0	0.002	0.01	+)
T	$0.006\ \bar{v}$	$0.02\ \bar{v}$	$0.06\ \bar{v}$	
A	$\frac{C}{25}$	$\frac{C}{5}$	C	
B	0	$\frac{C}{10}$	$\frac{C}{2}$	
b	0	$\frac{C}{5}$	C	

+) marginal conditions

$4\sigma \leqq \Sigma$

$A+B \leqq C$

$\underline{v} \leqq \bar{v}$

Tables 7-1 : Optimal strategies

1. $C = 2.10^5$, $c = d = 4 \cdot 10^7$

σ	Σ	e_σ	$\sigma(e_\sigma, \underline{\theta})$ in $
0.0015	0.009	$6.6563\ 10^{-3}$	$2.3718\ 10^5$
0.005	0.03	$2.3125\ 10^{-2}$	$8.2212\ 10^5$
0.01	0.06	$4.625\ 10^{-2}$	$1.6611\ 10^6$
0.015	0.06	$4.312\ 10^{-2}$	$1.4350\ 10^6$

2. $c = d = 4.10^7$ $\Sigma = 0.06; \sigma = 0.01$

C	e_σ	$\sigma(e_\sigma, \underline{\theta})$
$2\ 10^4$	$4.6875\ 10^{-2}$	$1.6779\ 10^6$
$2\ 10^5$	$4.625\ 10^{-2}$	$1.6611\ 10^6$
$2\ 10^6$	$4.3125\ 10^{-2}$	$1.5470\ 10^6$

3. $C = 2\ 10^5$, $d = 4\ 10^7$, $\Sigma = 0.06$, $\sigma = 0.01$

c	e_o	$O(e_o, \underline{\theta})$
$4\ 10^6$	$4.6875\ 10^{-2}$	$1.6768\ 10^6$
$4\ 10^7$	$4.625\ 10^{-2}$	$1.6611\ 10^6$
$4\ 10^8$	$4.3125\ 10^{-2}$	$1.5615\ 10^6$

4. $C = 2\ 10^5$, $c = 4\ 10^7$, $\Sigma = 0.06$, $\sigma = 0.01$

d	e_o	$O(e_o, \underline{\theta})$
$4\ 10^6$	$4.1250\ 10^{-2}$	$1.4919\ 10^5$
$4\ 10^7$	$4.625\ 10^{-2}$	$1.6611\ 10^6$
$4\ 10^8$	$4.6875\ 10^{-2}$	$1.6938\ 10^7$