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Effectivity and Cost Optimization of Safeguards Systems
Part II

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EFFECTIVITY AND COST OPTIMIZATION OF SAFEGUARDS SYSTEMS

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by

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Effectivity and Cost Optimization of Safeguards Systems [1)]

Part II

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Introduction

Any systems analytical effort on optimization of safeguards measures, has to aim ultimately at a system which is economic and effective. These two factors are closely interrelated and cost optimization can hardly be undertaken without having some quantifiable basis for the effectiveness of safeguards systems.
In a previous paper with the same title [1] an effort was undertaken to define and analyse the effectivity of safeguards systems. It was shown that if one could make the postulate that there exists a relation between the probability of detection p (m_o) (to be attained by an inspection system), and the amount m_o assumed to be diverted then the costs of a safeguards system as a function of m_o, optimized with respect to this p (m_o) could be defined as the effectivity of a safeguards system. This effectivity could then be taken as a yard stick for the comparison of safeguards systems based on completely different methods. It was noted that the postulation of a relation between $p(m_o)$ and m_o might be associated with some difficulties and that further work on the effectivity was continued.

In the present paper a different method has been developed for the definition and analysis of the effectivity of safeguards systems. One of the main features of this method is the fact that it does not require the existence of a relation between $p(m_o)$ and m_o.

[1)] Paper presented at the IAEA Working Panel on Systems Analysis, held on 25-29 August 1969, Vienna

[2)] Institut für Angewandte Reaktorphysik, Kernforschungszentrum Karlsruhe

The basic idea underlying the new method may be stated in the following manner.

All technical parameters, which influence the probability of detection in the case of a diversion, in a realistically designed safeguards system, can be expressed as equivalent random measuring variances. The sum of all these variances after establishing the material balance in a nuclear facility, over a period of time, determines almost uniquely the quality of statement on the probability of detection. If one assumes as a first approximation that all these variances are normally distributed, one can easily visualize the fact that the quality of the probability statement improves when the sum of the variances decreases.

The generation of all these variances (while establishing the material balance), is associated with costs. These costs may be taken to be the safeguards budget available to a safeguards authority over a time period for executing various safeguards measures. The main objective of these measures is to make statement on a detection in case of a diversion, which can only be done if a material balance has been established. It can therefore be said that the more accurate these statements are, the more effective is the safeguards system. However, accuracy of statements can not be improved indefinitely without increasing the costs of safeguards also indefinitely. Because of this restriction, it would be reasonable to utilize a given safeguards budget in such a way that the combinations of all the variances in establishing material balance, give the highest possible probability of detection $p(m_o)$ for a given m_o, assumed to be diverted. This optimized $p(m_o)$ can be assertained as a function of different inspection budgets and has been defined as the effectivity of the inspection system in this paper.

It is to be noted that such a definition is bi-parametric in nature, i.e. it depends both on the costs of safeguards and on the amount assumed to be diverted. In App. I an effort has been made to eliminate the explicit dependence of effectivity on the amount assumed to be diverted.

In the first chapter of this paper all the relevant variances which go in, in the establishment of a material balance, have been considered. The second chapter deals with the cost functions in connection with the generation of these variances. In the third chapter the statement which an inspection authority can make on the basis of the measured values, has been developed. On the basis of these three considerations, the concept of effectivity has been developed in the fourth chapter. A simplified numerical example has been given in chapter five to indicate the influence of various cost functions on the minimization of the sum of the variances for a given budget. Some generalized conclusions complete the paper.

1. Variances in Establishing Material Balance in a Nuclear Facility

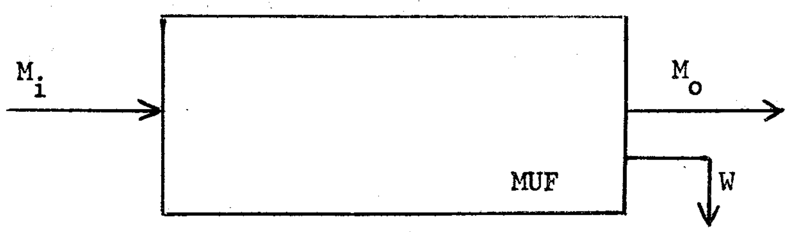


Fig. 1

Let the period under consideration be one year. During this period, a measured amount of fissile material M_i flows into a typical nuclear facility as shown in Fig. 1. During the same time, corresponding measured amounts of product M_o and waste W leave the facility. Besides this, a certain amount of material V is assumed to remain in the plant (Material Unaccounted For, MUF).

Process inventory taking takes places n times in a year so that the interval between two consecutive inventory takings is $t = \frac{1}{n}$ [yr]. This time is taken as the inventory period. This inventory taking ensures that at the time of taking the inventory and establishing a material balance the fissile material amount I is actually present in the plant.

For developing the subsequent model it is assumed that once in a year the operator plans to divert m_o amount of fissile material. The time required for diverting this amount is small compared to the interval between two inventories.

(Note: it can also be assumed that the diversion of m_o takes place spread over the whole year. This case is more complicated and will be treated later.)

If it is further assumed that at the beginning of an inventory period, the inventory is known exactly (which is only an approximation; this problem has been analysed in [2]), then the true values of the total material balance at the end of an inventory period are given by :

$$M_i - M_o - W - V - I - m_o = 0 \qquad (1.1)$$

The measured values of M_i, M_o, I and W are all considered to be random in nature.

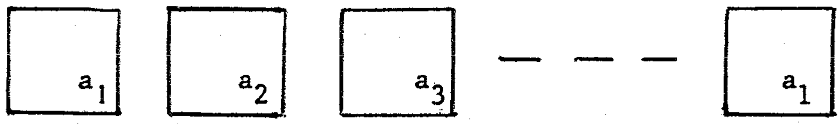
1.1 <u>Input and Output Measurement, M_i, M_o</u>

These measurements are assumed to be normal distributed with the expectation values EM_i and EM_o with the variances $\sigma^2_{M_i}$ and $\sigma^2_{M_o}$. As is shown below, these variances increase linearly with time:

$$\sigma^2_{M_i} = \delta^2_{M_i} \cdot \frac{dE^2 M_i}{ds} \cdot s \; ; \; \sigma^2_{M_o} = \delta^2_{M_o} \cdot \frac{dE^2 M_o}{ds} \cdot s \tag{1.2}$$

where s is the running time
and $\delta^2_{M_i}$ = const; $\delta^2_{M_o}$ = const.

Let



be the batches each masured with the variance δ^2_a :

$$Ea_\nu = Ea; \; \sigma^2_{a_\nu} = \sigma^2_a \; ; \; \sigma^2(a) = \delta^2_a \, E^2 a; \delta^2_a = \text{const.}$$

Then

$$\sigma^2(a_\nu) = 1 \cdot \delta^2_a \; E^2 a$$

Since $1 = \frac{s}{k}$ where k is the time between the measurement of the two batches, going over to the limit for small time intervals between batches

$$k \to ds; \; E^2 a \to dE^2 a$$

$$\sigma^2(\Sigma a_\nu) = \delta_a{}^2 \frac{dE^2 a}{ds} \cdot s$$

(This proves equ. 1.2.)

The input and output measurements consist normally of a measurement giving the amount of the material (volume or rate) and another measurement giving the concentration of fissile material in this material so that:

$$M = c \cdot V \tag{1.3}$$

The variances are then given by:

$$\sigma_M{}^2 = \sigma_c{}^2 \cdot E^2 V + \sigma_V{}^2 E^2 c \tag{1.4}$$

The variance $\sigma_c^{\,2}$ consists of the variance σ_{cz}^{o2} the random error and σ_{cs}^{2}, the systematic error, which is a characteristic of the analytical laboratory in which the measurements are carried out. Since the systematic errors may vary randomly from time to time in the same laboratory, they can also be expressed as a random distribution. Therefore:

$$\sigma_c^{\,2} = \sigma_{cz}^{o2} + \sigma_{cs}^{\,2} \tag{1.5}$$

With q repetitions the value of σ_{cz}^{o2} is reduced by a factor of q:

$$\sigma_{cz}^{\,2} = \frac{\sigma_{cz}^{o2}}{q} \tag{1.6}$$

The whole relative variance is then givenby:

$$\delta_J^2 = \frac{\sigma_J^2}{E^2 J} = \delta_z^{\,2} + \delta_R^{\,2};\ \delta_z^{\,2} = \frac{\sigma_{cz}^{\,2}}{E^2 c};\ \delta_R^2 = \frac{\sigma_{cs}^{\,2}}{E^2 c} + \frac{\sigma_V^2}{E^2 V} \tag{1.7}$$

1.2 Inventory I

The measurements for the inventory are also assumed to be normally distributed with the expectation value EI and the variance σ_I^2. The following relation is valid

$$\sigma_I^2 = \delta_I^2 \cdot E^2 I \tag{1.8}$$

where again δ_I^2 = const and is the relative standard deviation for a single measurement.

1.3 Material Unaccounted For, V

In the context of the present paper, the material unaccounted for (MUF) has been taken to be equivalent to the losses V which occur inside the plant. These losses V have been assumed to be related to the input in the following manner:

$$V = v \left(\frac{dM_i}{ds}\right)^{\frac{1}{2}} \tag{1.9}$$

This means that the losses V (MUF) have been taken to be proportional to the size of the plant, which, as a first approximation, may be assumed to increase with the square-root of the throughput.

Under normal operating conditions the MUF may be caused by a number of factors /‾3‾7 namely, a) systematic errors in measurements, b) unmeasured wastes leaving the plant, c) mal operations, d) fissile material plating out on plant component surface and similar phenomena. The first three components can either be eliminated or with certain efforts accounted for. Therefore, only the fourth component has been taken to be the cause of MUF in this paper and denoted as the internal losses, V.

It has been assumed that these losses V attain a maximum value within a short time after startup of the plant and then vary statistically around this value. This means that V has also been considered to be normally distributed varying randomly with the expectation value EV = 0 and the variance σ_V^2. It is further assumed that σ_V^2 is known both to the inspector and the operator.

1.4 Wastes, W

In this paper it has been assumed that

$$W = w \cdot M_i \tag{1.10}$$

and that w is a constant. The measured values for the wastes are normally distributed with the expectation value EW and the variance σ_W^2. For this variance the same type of relation as in equ. (1.2) can be postulated.

$$\sigma_W^2 = \delta_W^2 \cdot w^2 \frac{dE^2M_i}{ds} \cdot s \; ; \; \delta_W^2 = \text{const.} \tag{1.11}$$

where δ_W^2 is the relative variance for a single measurement for the waste stream.

It is to be noted that the assumption W = const is not exactly consistent with the condition that V is random. It has been implicitely assumed here that all the random variations in V are reflected in M_o. This is justified by the fact that W normally consists of a small fraction of M_o.

2. Cost Factors in Connection with the Variances

The generation of the measuring variances is associated with costs. It is assumed that an inspection authority has a fixed budget to spend over a given period of time (i.e. one year), in establishing a complete fissile material balance around a nuclear facility over a given period of time. This budget can be spent on the following two categories of measures:

2.1 Flow measurements
2.2 Inventory measurements

2.1 Flow Measurements

2.1.1 Input and Output: Let there be $r_1(r_2)$ input (output) batches for measurement in a year. Every input (output) batch is measured $q_1(q_2)$ times. The unit costs of measurement are $\gamma_1(\gamma_2)$ for each input (output) batch. The total costs for flow measurements K_F/yr are then given by

$$K_F = r_1 q_1 \cdot \gamma_1 + \gamma_2 \cdot q_2 \cdot \gamma_2 \qquad (2.1)$$

if the relative variance for a single measurement be denoted by δ^{o2}_{zi} (δ^{o2}_{zo}) for the input (output) batch, then equ. (2.1) in conjunction with equ. (1.6) is reduced to the form

$$K_F = \frac{r_1 \Gamma_1}{\delta^2_{M_i} - \delta^2_{R_i}} + \frac{r_2 \Gamma_2}{\delta^2_{M_o} - \delta^2_{Ro}} \; ; \; \Gamma_1 = \gamma_1 \delta^{o2}_{zi} ; \; \Gamma_2 = \gamma_2 \delta^{o2}_{zo} \qquad (2.2)$$

2.2.2 Waste Measurements: Similar considerations as in the case of input and output measurements (equs. 1.4 to 1.7 and 2.1 to 2.2) lead to the cost relation of the following form:

$$K_W = \frac{r_3 \Gamma_3}{\delta^2_W - \delta^2_{RW}} \; ; \; \Gamma_3 = \gamma_3 \delta^{o2}_{zW} \qquad (2.3)$$

where r_3 gives the number of waste measurements per year, γ_3 the unit cost for analysis and δ^{o2}_{zW}, the relative variance for a single analysis.

2.2 Inventory Taking

The costs for inventory taking have been assumed to be proportional to

a) the frequency. One inventory taking costs γ_4 units

b) repetition of measurements. The costs for each measurement is γ_2 units, i.e. same as that for the output measurement

As mentioned in the beginning, n inventories are taken per year and the measurements are repeated m times. $t = \frac{1}{n}$ is the interval for an inventory period. With σ_I^2 defined similarly as in eqs.(1.3) and (1.7), the total inventory costs/year are given by

$$K_I = n\gamma_4 + n \cdot m \cdot \gamma_2 \tag{2.4}$$

or

$$K_I = \frac{1}{t} \left(\gamma_4 + \frac{\Gamma}{\sigma_I^2 - \sigma_R^2}\right); \Gamma = \gamma_2 \cdot \sigma_{zo}^2 \tag{2.5}$$

The interval t can be limited on both sides with the help of the following consideration:

Inventory period cannot be less than t_u corresponding to the residence time of fissile material in a nuclear facility. It should not be greater than a critical time t_c, before which the material balance has to be completed to reduce the consequences of a diversion. This means

$$t_u \leq t \leq t_c$$

2.3 Summary

The total costs K for establishing the material balance in a nuclear facility comprises therefore of:

$$K = K_F + K_W + K_I \tag{2.6}$$

As indicated in the introduction, for a given cost, the variables t, δ_{zi}^2, δ_{zo}^2, δ_{zW}^2 and σ_I^2 have to be minimized in such a way that the probability of detection p (m_o) becomes maximum for an amount of fissile material m_o assumed to be diverted. It should be noted that for the three throughput measurements the relative variance δ^2, but for the inventory measurement the absolute variance σ^2 have been considered for optimization. This is because of the fact, that the inventory period t has been explicitely taken out for optimization and the σ^2s for the throughput measurements are a function of this time t.

3. Statement of the Inspector and Some Related Parameters

3.1 Expectation Values

Let the nominal value of the book inventory J be given by

$$J = M_i - M_o - W \tag{3.1}$$

Then under the conditions that in the time interval (0, t) the MUF losses $\hat{V}$ will be realized and that the amount m_o will be diverted:

$$p(a \leq J-I \leq b/\hat{V};\ m_o) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_a^b \exp\left[- \frac{(x-(\hat{V}+m_o))^2}{2\sigma^2}\right] dx \tag{3.2}$$

$$\sigma^2 = \sigma_{M_i}^2 + \sigma_{M_o}^2 + \sigma_W^2 + \sigma_I^2$$

Besides this the following relation is valid

$$p\ (y \leq V \leq y+dy) = \frac{1}{\sqrt{2\pi}\,\sigma_V} \exp\left[- \frac{y^2}{2\sigma_V^2}\right] dy \tag{3.3}$$

so that

$$p(a \leq J-I \leq b/m_o) = \frac{1}{\sqrt{2\pi}\,\Sigma} \int_a^b dx \exp\left[- \frac{(x-m_o)^2}{2\Sigma^2}\right]$$

$$\Sigma^2 = \sigma_{M_i}^2 + \sigma_{M_o}^2 + \sigma_W^2 + \sigma_I^2 + \sigma_V^2 \tag{3.4}$$

It is seen from equ. (3.4) that

$$E(J-I) = m_o; \qquad \sigma^2(J-I) = \Sigma^2 \tag{3.5}$$

Note: The expectation value E(J-I) should not be confused with the "conditioned expectation value " $EM_i - EM_o - EW - EI$ (which correspond to the expectation values of the measurements M_i M_o W and I with the realization of a particular value of $\hat{V}$.)

Equ. 3.4 gives

$$p(-\infty \leq J-I-m_o \leq c/m_o) = \frac{1}{\sqrt{2\pi}\,\Sigma} \int_{-\infty}^{c} dx \exp\left(-\frac{x^2}{2\Sigma^2}\right)$$

or

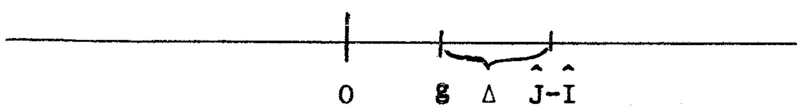
$$p(-\infty \leq J-I-m_o \leq c/m_o) = \phi\left(\frac{c}{\Sigma}\right) \qquad (3.6)$$

3.2 Inspectors Statement

In principle, an inspector can make several types of statements in case of a diversion of m_o, after comparing the nominal book values with the measured values of the inventory obtained from the material balance. For example he can say that an amount > 0 has been diverted or that an amount $\geq x$ has been diverted or he can say that an amount in the interval $c_1 \leq x \leq c_2$ has been diverted or he can also say that an amount x has been diverted. The different methods used to arrive at such statements and the relative merits of such statements will be analysed in a later publication. In the present paper, the last type of inspector's statement (i.e. to state that a definite amount x has been diverted in case the operator diverts an amount m_o) has been used as an example. The inspector can proceed in the following manner to come to this type of statement:

Under the condition that nothing will be diverted, the measured values J-I will lie in the interval $(-\infty, g)$ with the probability $1-\alpha$. Corresponding to the equ.(3.6), the following relation exists between α and g:

$$1-\alpha = \phi\left(\frac{g}{\Sigma}\right) \qquad (3.7)$$



The fixation of α, the probability of error, is a matter of judgement on the part of the inspector. It may be fixed (normally around 5 %) on the basis of experience, economic and other considerations. Once the value of α has been fixed, the inspector can make the following statement after obtaining the measured values of $\hat{J}+\hat{I}$ as shown in the above sketch (Fig. 2). "On the basis of my measurements I declare that the amount

$$\Delta = \hat{J}-\hat{I}-g$$

has been diverted.

If the measured values of $\hat{J}-\hat{I}$ fell in the interval (0,g), the inspector will make the statement that

"nothing has been diverted".

In this connection, the meaning of g can be illustrated with some simplifying assumptions. In the case of a diversion of the amount m_o, if by chance the true values of J+I would be measured and the unaccountable losses V be = 0, than g indicates the difference between the amount diverted m_o and the amount declared as diverted. That means

$$g = m_o - \Delta \tag{3.9}$$

This is done to keep the error second type (i.e. accusing the operator for diversion although no diversion has taken place) within reasonable limits (see below).

3.3 Probability of Detection; Inspector's Errors

3.3.1 Probability of Detection: A detection takes place if $\Delta > 0$, i.e. when according to equ. (3.8)

$$\hat{J}-\hat{I}-g > 0 \tag{3.10}$$

The probability that $J-I > g$ when the amount m_o has been diverted, has been defined in this paper as the probability of detection $p(E/m_o)$

$$p(E/m_o)=p(J-I > g/m_o)=1-p(J-I \leq g/m_o) \tag{3.11}$$

According to equ. (3.4)

$$p(-\infty \leq J-I \leq g/m_o)= \frac{1}{\sqrt{2\pi}\,\Sigma} \int_{-\infty}^{g-m_o} dx \exp\left[-\frac{x^2}{2\Sigma^2}\right] \tag{3.12}$$

so that

$$p\ (E/m_o) = \phi\left(\frac{m_o - g}{\Sigma}\right) \tag{3.13}$$

3.3.2 Inspector's Errors: The probability that a diversion will be stated if nothing is diverted is according to p. 12 equal to α. The probability that an amount Δ will be declared as diverted which is greater than the amount m_o which will be diverted in reality, is also α.

There is a second possibility for the inspector to err.: The inspector states 'no diversion' even if the operator has diverted the amount m_o. The probability for this case is given by

$$\beta(m_o) = p\ (J-I-g \leq 0/m_o) = 1-p\ (E/m_o) = 1-\phi\ (\frac{m_o-g}{\Sigma}) \qquad (3.14)$$

This probability decreases with increasing m_o.

4. Effectivity

It was indicated earlier that choice of the values of α(probability of error) and similarly that of β is a matter of judgement. However, it may be assumed that they can be established. Then the probability of detection $p(E/m_o)$ may be regarded as the criterion according to which the quality of safeguards systems can be assessed. And the effectivity of such a system can be defined as the highest probability of detection as a function of costs which can be achieved by the optimum utilization of a given budget:

$$Eff(K) = p_{opt}(K) \qquad (4.1)$$

The optimum utilization of the costs can be developed in the following manner:

According to equ. (3.13)

$$p(E/m_o) = \phi\ (\frac{m_o}{\Sigma} - \frac{g}{\Sigma})$$

or after fixing α

$$p(E/m_o) = \phi\ (\frac{m_o}{\Sigma} - \phi^{-1}(1-\alpha)) \qquad (4.2)$$

Equ. (4.2) shows that $p(E/m_o)$ increases monotoneously with decreasing Σ . This means that the optimum utilization of safeguards costs will be obtained when the sum of all the variances Σ (in establishing the material balance) becomes a minimum for these costs. Since this is valid irrespective of the value of m_o, and $p(E/m_o)$ is always positive, minimization of the value of Σ means simultaneously maximization of the value of $p(E/m_o)$.

Accordingly, the optimization problem is given by the following:
The relation

$$\Sigma^2=(\delta_{M_i}^2 \frac{dE^2M_i}{ds} + \delta_{M_o}^2 \frac{dE^2M_o}{ds} + \delta_W^2 w^2 \frac{dE^2M_i}{ds})\, t + \sigma_V^2 + \sigma_I^2$$

is to be minimized with respect to

$$\delta_{zi}^2 = \delta_{M_i}^2 - \delta_{R_i}^2;\ \delta_{zo}^2 = \delta_{M_o}^2 - \delta_{R_o}^2;\ \delta_{zw}^2 = \delta_w^2 - \delta_{Rw}^2;\ t; \sigma_I^2$$

with the following boundary conditions

$$K = \frac{r_1\Gamma_1}{\delta_{Mi}^2-\delta_{Ri}^2} + \frac{r_2\,\Gamma_2}{\delta_{Mo}^2-\delta_{Ro}^2} + \frac{r_3\Gamma_3}{\delta_w^2-\delta_{Rw}^2} + \frac{1}{t}\,(\gamma_4 + \frac{\Gamma}{\sigma_I^2-\sigma_R^2})$$

$$t_u \leq t \leq t_c$$

$$\delta_{Ri}^2 \leq \delta_{zi}^2 \leq \delta_{zi}^{o2}$$

$$\delta_{Ro}^2 \leq \delta_{zo}^2 \leq \delta_{zo}^{o2}$$

$$\delta_{Rw}^2 \leq \delta_{zw}^2 \leq \delta_{zw}^{o2}$$

$$\sigma_R^2 \leq \sigma_I^2 \leq \chi^2 \qquad (4.3)$$

χ^2 is the variance obtained by measuring the inventory only once.
For different K's different minimim Σ's are obtained

$$\Sigma_{opt}^2 = \frac{1}{f\,(K)} \qquad (4.4)$$

where f(K) is an ever increasing function of K.

The effectivity of a safeguards system is then given together with equ.(4.2)

$$Eff(K) = \phi(m_o f(K) - \phi^{-1}(1-\alpha)) \quad (4.5)$$

In App. I it has been shown that under certain conditions the optimum values of probability of detection cannot be obtained by minimizing the total variance Σ, but the probability itself has to be optimized. A method for tackling such optimization problems with these conditions has been sketched there.

5. Numerical Example

5.1 Technical and Cost Data

A numerical example for a hypothetical reprocessing plant has been worked out in this chapter. The example indicates how the sum of all the variances can be minimized to obtain the highest possible values of the probability of detection $p(E/m_o)$ (for m_o assumed to be diverted) for a given safeguards budget used to establish a material balance. It is needless to mention that the absolute values used in this example will necessarily change from plant to plant.

The technical and the cost data used for the reprocessing plant, are summarized in Table 5-1. Following comments on Table 5-1 might be useful:

a) The waste streams are assumed to contain 0.5 % of Pu present in the input stream.

b) Two types of inventory taking have been assumed; one is based on the difference in isotope composition of two consecutive batches [4] denoted as the tracer method; the other is the washout method in which the process inventory is washed out after a certain time.

c) The MUF(V) has been assumed to reach its maximum value during the start up of the plant and not during the equilibrium operation under consideration. Therefore, only the variance σ_V for V has been considered with the expectation value EV = 0.

d) The number of inventories for the upper and lower limit may appear to be on the high side. They should be considered only as an illustration.

5.2 Optimization

To keep the complexity of the numerical solution within limits and to maintain the transparancy, only t (inventory period) and σ_I^2 have been varied and the rest of the variables have been kept constant in this example.

The simplified optimization relation takes then the following form:

To optimize

$$\Sigma^2 = \sigma_V^2 + At + \sigma_I^2 \tag{5.1}$$

with respect to the variables t, σ_I^2 with the following boundary conditions

$$K = \frac{1}{t}\left(\gamma_4 + \frac{\Gamma}{\sigma_I^2 - \sigma_R^2}\right) \quad 5.2)$$

$$t_u \leq t \leq t_c$$

$$\sigma_R^2 < \sigma_I^2 \leq \chi^2 \tag{5.3}$$

The boundary condition (5.3) with respect to σ_I^2 means that even with repeated analyses, the value of σ_I^2 cannot be reduced below that of σ_R^2. σ_R^2 consists of the error in estimating the amount (in the case of a reprocessing plant it is the measurement of volume in a tank which cannot be reproduced normally), and the systematic error specific for the laboratory in which the analyses have been carried out. The upper limit is given by χ^2 which is the value of σ_I^2 for a single measurement. Higher values of σ_I^2 are not possible.

The minimum cost K_{min} is given when $t = t_c$ and $\sigma_I^2 = \chi^2$. This is given by

$$K_{min} = \frac{1}{t_c}\left(\gamma_4 + \frac{\Gamma}{\chi^2 - \sigma_R^2}\right) = \frac{\gamma_4 + \gamma_2}{t_c} \tag{5.4}$$

According to equ. (5.4) the actual values of both K_{min} for the two inventory variants are

$$K_{min} = \begin{cases} 43{,}200 \text{ DM for tracer} \\ 644{,}000 \text{ DM for washout} \end{cases} \tag{5.5}$$

The solution of the simplified optimization problem is obtained by first getting the analytical minimimum of the function (5.1) under the condition (5.2), and then eliminating σ_I^2 with the help of eqs. (5.2) and (5.1).

$$\sigma_I^2 = \sigma_R^2 + \frac{\Gamma}{Kt-\gamma_4} \qquad (5.6)$$

$$\Sigma^2 = \sigma_V^2 + \sigma_R^2 + At + \frac{\Gamma}{Kt-\gamma_4} \qquad (5.7)$$

The optimum t is then given by

$$t_o = \frac{1}{K}\left(\gamma_4 + \left(\frac{\Gamma K}{t\ A}\right)^{\frac{1}{2}}\right) \qquad (5.8)$$

the minimum σ_I^2 is given by

$$\sigma_{Io}^2 = \sigma_R^2 + \left(\frac{A\Gamma}{K}\right)^{\frac{1}{2}} \qquad (5.9)$$

and the minimum Σ^2 is given by

$$\Sigma_o^2 = \sigma_V^2 + \sigma_R^2 + 2\left(\frac{A\Gamma}{K}\right)^{\frac{1}{2}} + \frac{A\gamma}{K} \qquad (5.10)$$

The K can be eliminated from eqns. (5.8) and (5.9) so that

$$t_o(\sigma_{Io}^2) = \frac{\gamma_4}{A\Gamma}(\sigma_{Io}^2 - \sigma_R^2)^2 + \frac{1}{A}(\sigma_{Io}^2 - \sigma_R^2) \qquad (5.11)$$

Figs. 3 and 6 have been constructed to indicate the cost limits within which the optimization can be carried out. In both these Figs., constant cost lines have been plotted in the t, σ_I^2 plane, along with the curve for optimized $t_o(\sigma_{Io}^2)$ according to equ. (5.11). Fig. 3 is for the tracer technique and Fig. 6 for the washout method. Within the permissible limits of t and σ_I^2 as given by equ.(5.3), the points at which the curve $t_o(\sigma_{Io}^2)$ cuts the constant cost lines, the analytical minimum of Σ^2 gives also the required minimum of Σ^2 for the cost optimization.

For the tracer method (Fig. 3) the range of costs [in DM/yr] for optimization is given by

$$82.1 \cdot 10^4 \leq K \leq 508 \cdot 10^4 \qquad (5.12)$$

and that for the washout method (Fig. 6)

$$7.4 \cdot 10^5 \leq K \leq 1.6 \cdot 10^6 \qquad (5.13)$$

For all other costs outside these intervals the required optimum values lie at the boundary of these intervals i.e. between the points λ_1, λ_2 and λ_3 and λ_4 in Figs. 3 and 6.

The solutions for the optimization problems will then have the following forms.

a) Tracer method (Figs. 4a, 4b, 4c, 5)

$$t_{opt} = \begin{cases} t_c & \text{for } K_{min} \leq K \leq 82.1 \cdot 10^4 \\ \frac{1}{K}\left(\gamma_4 + \left(\frac{\Gamma K}{A}\right)^{\frac{1}{2}}\right) & \text{for } 82.1 \cdot 10^4 \leq K \leq 508 \cdot 10^4 \\ t_n & \text{for } 508 \cdot 10^4 \leq K < \infty \end{cases} \qquad (5.14)$$

$$\sigma^2_{Iopt} = \begin{cases} \sigma^2_R + \frac{\Gamma}{Kt_c - \gamma_4} & \text{for } K_{min} \leq K \leq 82.1 \cdot 10^4 \\ \sigma^2_R + \left(\frac{A\Gamma}{K}\right)^{\frac{1}{2}} & \text{for } 82.1 \cdot 10^4 \leq K \leq 508 \cdot 10^4 \\ \sigma^2_R + \frac{\Gamma}{Kt_n - \gamma_4} & \text{for } 508 \cdot 10^4 \leq K < \infty \end{cases} \qquad (5.15)$$

$$\Sigma^2_{opt} = \begin{cases} \sigma^2_V + \sigma^2_R + At_c + \frac{\Gamma}{Kt_c - \gamma_4} & \text{for } K_{min} \leq K \leq 82.1 \cdot 10^4 \\ \sigma^2_V + \sigma^2_R + 2\left(\frac{A\Gamma}{K}\right)^{\frac{1}{2}} + \frac{A\gamma_4}{K} & \text{for } 82.1 \cdot 10^4 \leq K \leq 508 \cdot 10^4 \\ \sigma^2_V + \sigma^2_R + At_n + \frac{\Gamma}{Kt_u - \gamma_4} & \text{for } 508 \cdot 10^4 \leq K < \infty \end{cases} \qquad (5.16)$$

It is to be seen that the cost optimization can be done in the following manner. Starting from the minimum costs $K_{min} = 4.32 \cdot 10^4$ one has to keep the number of inventories/yr corresponding to t_c constant till the value of $K = 82.1 \cdot 10^4$ has been attained (Fig. 3). Then one has to reduce both t (larger number of inventories/yr) and σ^2_I (repeated measurements) until

$K = 508 \cdot 10^6$ corresponding to t_u has been achieved. After that one has to keep t_u constant (largest number of inventories/yr possible) and decrease σ_I^2. With infinite costs the following asymptotic value of Σ^2 is obtained

$$\Sigma_{as}^2 = \sigma_V^2 + \sigma_R^2 + A \cdot t_u = 0.414 \qquad (5.17)$$

Fig. 5 gives the optimum probability of detection $p(E/m_o)$ i.e. effectivity of the safeguards system for different costs as well as the asymptotic values for infinite costs. It may be noted that the values for lower costs converge fairly rapidly to the asymptotic values. This means that beyond about 10^5 DM/yr no further improvement can be obtained in effectivity of the safeguards system under the given conditions.

b) Washout method (Figs. 7a, 7b, 7c, 8)

$$t_{opt}^2 = \begin{cases} t_c & \text{for } K_{min} \leq K \leq 7.4 \cdot 10^5 \\ \frac{1}{K}\left(\gamma_4 + \left(\frac{\Gamma K}{A}\right)^{\frac{1}{2}}\right) & \text{for } 7.4 \cdot 10^5 \leq K \leq 16 \cdot 10^5 \\ t_u & \text{for } 16.1 \cdot 10^5 \leq K < \infty \end{cases} \qquad (5.18)$$

$$\sigma_{Iopt}^2 = \begin{cases} \sigma_R^2 + \frac{\Gamma}{Kt_c - \gamma_4} & \text{for } K_{min} \leq K \leq 7.4 \cdot 10^5 \\ \sigma_R^2 + \left(\frac{A\Gamma}{K}\right)^{\frac{1}{2}} & \text{for } 7.4 \cdot 10^5 \leq K \leq 16 \cdot 10^5 \\ \sigma_R^2 + \frac{\Gamma}{Kt_u - \gamma_4} & \text{for } 16 \cdot 10^5 \leq K < \infty \end{cases} \qquad (5.19)$$

$$\Sigma_{opt}^2 = \begin{cases} \sigma_V^2 + \sigma_R^2 + At_c + \frac{\Gamma}{Kt_c - \gamma_4} & \text{for } K_{min} \leq K \leq 7.4 \cdot 10^5 \\ \sigma_V^2 + \sigma_R^2 + 2\left(\frac{A\Gamma}{K}\right)^{\frac{1}{2}} + \frac{A\gamma_4}{K} & \text{for } 7.4 \cdot 10^5 \leq K \leq 16 \cdot 10^5 \\ \sigma_V^2 + \sigma_R^2 + At_u + \frac{\Gamma}{Kt_u - \gamma_4} & \text{for } 16 \cdot 10^5 \leq K < \infty \end{cases} \qquad (5.20)$$

In this case again the optimization consists of keeping the number of inventories/yr corresponding to t_c constant till the value of $K=7.4 \cdot 10^5$ has been attained (Fig. 6). Then one has to reduce both t (larger number of inventories/yr) and σ_I^2 (repeated measurements) until $K = 16 \cdot 10^5$ corresponding to t_u has been achieved. After that one has to keep t_u constant (largest number of inventories/yr possible) and decrease σ_I^2. With infinite costs the following asymptotic value of Σ^2 is obtained

$$\Sigma_{as}^2 = \sigma_V^2 + \sigma_R^2 + At_u = 0.258 \qquad (5.21)$$

In Fig. 8 the optimum probability of detection $p(E/m_o)$ i.e. effectivity is plotted against m_o, the amount assumed to be diverted, with optimized costs as parameter. The asymptotic values of $p(E/m_o)$ for infinite costs have been shown, too. It may be seen here that beyond the minimum value K_{min} of the budget, further increase will not bring any improvement in the effectivity of the safeguards system.

Conclusions

The method developed in this paper to define the effectivity of safeguards systems, does not require any prior knowledge of the relation between probability of detection and the amount assumed to be diverted. It requires a prior knowledge only of the measuring errors and other technical conditions, which may influence a diversion. This method also gives a criterion according to which the utilization of safeguards budget can be optimized. A number of other conclusions can be drawn based on the analysis given in this paper.

a) The effectivity defined in this paper is for a single nuclear facility. For developing the concept of effectivity for a safeguards system covering a number of facilities or a fuel cycle, the fact that fissile material will have different values in different facilities (effective kgs), will have to be taken into consideration.

b) The definition of effectivity is based on the "detection" in case of a diversion. However, the question of "prevention" has also been touched by introducing the concept of critical time t_c, which compels an inspection authority to take an inventory and complete a material balance before a certain time has lapsed. This point can be further emphasized by optimizing the relation p/t. In that case optimization of Σ alone, as has been done here, will not help.

c) In a number of other cases also (for example, if the variance for MUF losses V were not normally distributed, or in case the diversion was assumed to be spread over the whole year etc.), minimization of Σ alone will not give the optimum probability of detection. In such cases the probability of detection which is independent of m_o might be a better value to optimize (see App. I).

d) Considerable amount of further work is required to solve still open problems in connection with the effectivity of safeguards system.

Acknowledgement

The authors would like to thank W. Häfele for his interest in this work.

Appendix I

It was indicated in the paper that under certain conditions the minimization of the value of Σ^2 might not give the highest probability of detection $p(E/m_o)$. Such conditions arise when for example, one of the random events is not normally distributed, or when it is assumed that diversion takes place spread over the whole year. Optimization of $p(E/m_o)$ under such conditions might lead to inspection procedures which are strongly dependent on the absolutevalues of m_o assumed to be diverted. In general the explicit dependence of the probability of detection on m_o can be eliminated if it s defined in the following manner·

$$P = \int p(E/m_o)\, dp\, (m_o) \tag{I.1}$$

Here dp (m_o) is the probability with which the operator plans to divert the amount m_o. It can be regarded as the "operator's strategy". One has to have some idea on this strategy before one can optimize P. One way of fixing the strategy is discussed below:

1. Estimate of the operator's strategy $(dp(m_o))$

It may be assumed that the amount $m_o = m_{oo}$ assumed to be diverted, where

$m_{oo} = m_{in}$ (e.g. 0.01 M_i, 6 kg Pu)

has to be detected with a probability $> p_o$.

This means, it is required that

$$p(E/m_o) \geq p_o \text{ for } m_o \geq m_{oo} \tag{I.2}$$

In the case of the model used in the paper it would mean

$$\phi\left(\frac{m_{oo}}{\Sigma} - \phi^{-1}(1-\alpha)\right) \geq p_o$$

or

$$\Sigma \leq \frac{m_{oo}}{\phi^{-1}(p_o + \phi^{-1}(1-\alpha))} \tag{I.3}$$

The requirement I.3 means an additional boundary condition for the optimization problem formulated in the text under (4.4). This can be argumented in the following way:

Because of the fact that the probability of detection p_o is extremely high (it can be chosen to be 99 %) for a $m_o > m_{oo}$, the operator will not divert an amount $\geq m_{oo}$. Below this value if the operator intended to divert an amount, the probability of diversion would be the same for all the amounts. With

$$dp\ (m_o) = f\ (m_o)\ dm_o \qquad (I.4)$$

the following type of equal distribution can be considered:

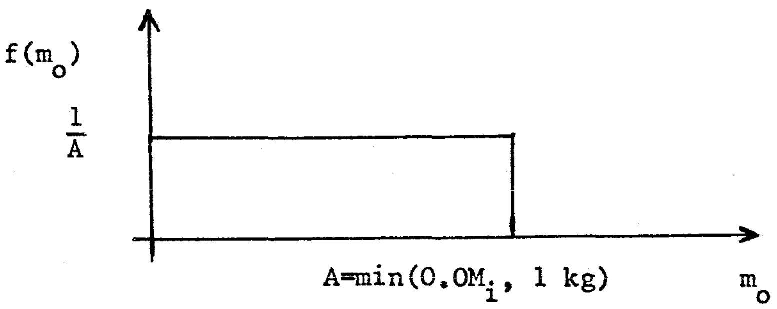


Fig. 2

The probability of detection according to eqn. I.1

$$P = \frac{1}{A} \int_0^A p(E/m_o)\,dm_o \qquad (I.5)$$

or"the effectivity" as defined in the text is then given by

$$Eff(K) = \frac{1}{A} \int_0^A \phi(m_o f(K) - \phi^{-1}(1-\alpha))\,dm_o \qquad (I.6)$$

With the help of the eqn.

$$\int_{-\infty}^{x} \phi(\frac{t+\alpha}{\beta})\,dt = (x+\alpha)\phi(\frac{x+\alpha}{\beta}) + \frac{\beta}{\sqrt{2\pi}} \exp\left[- \frac{(x+\alpha)^2}{2\beta^2}\right] \qquad (I.7)$$

the integration of I-6 can be carried out with the result:

$$Eff(K) = \frac{1}{A}\left[(A - \frac{\phi^{-1}(1-\alpha)}{f(K)})\phi(\frac{A-\phi^{-1}(1-\alpha)}{f(K)}) + \frac{\phi^{-1}(1-\alpha)}{f(K)} \cdot \phi(-\frac{\phi^{-1}(1-\alpha)}{f(K)}) + \right.$$

$$\left. + \frac{1}{f(K)\sqrt{2\pi}} (\exp(-\frac{(Af(K)-\phi^{-1}(1-\alpha))^2}{2}) - \exp(-\frac{(\phi^{-1}(1-\alpha))^2}{2})\right]$$

(I.8)

A further solution for $dp(m_o)$ can be obtained with the theory of games. The choice of $dp(m_o)$ would be the operator's strategy and the choice of α (probability of error) would be the inspector's strategy. The pay-off functions would be the expectations values of the gains or the losses in case of a detection or a non-detection or a false detection of a diverted amount.

List of symbols

Ea	Expectation value of the random variable a
Eff(K)	Effectivity as a function of the costs
g	Threshold of alarming
I	Physical inventory
J	Book inventory
K	Costs of the safeguards measurements
M_i, M_o	Input, Output
m_o	Amount of fissile material, assumed to be diverted
n	Number of inventory takings per year
$p(E/m_o)$	Probability of detection as a function of the material m_o assumed to be diverted
t	Time interval between two inventory taking
t_c, t_u	Upper and lower boundary for t
V	Material unaccounted for (MUF)
W	Waste
α, β	Error probabilities
γ, Γ	Cost factors
δ_a^2	Relative variance of the random variable a
Δ	Amount of material, declared as diverted
$\phi(x)$	Error function
$\phi^{-1}(x)$	Inverse of the error function
σ_a^2	Variance of the random variable a

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Determination of In-Process Inventory in a Reprocessing Plant by means of Isotope Analysis.
KFK 904

Table 5-1: Technical and cost data for a hypothetical reprocessing plant, used to illustrate effectivity of safeguards systems

1.	Type of plant		Reprocessing plant operating batchwise at the input and output end
2.	Number of working days/year		200
3.	Throughput		
	a) Input	Kg Pu/d	3
		Number of batches/d	3
		Relative variance/analysis [%]	1.0
		Costs [DM/analysis]	400
	b) Output	Kg Pu/day	2.985
		Number of batches/d	7.5
		Relative variance/analysis [%]	1
		Costs [DM/analysis]	100
	c) Waste	Kg Pu/d	0.015
		Number of analysis/d	1
		Relative variance/analysis [%]	10
		Costs [DM/analysis]	25
4.	Inventory		
	a) Tracer	Relative variance/measurement	
		Volumetric+Systematic (δ_R) [%]	2
		Concentration (δ_{zo}) [%]	8
		Costs(γ_4) [DM/inventory]	5000
		Costs(γ_2) [DM/analysis]	400
	b) Washout	Relative variance/measurement	
		Volumetric+Systematic (δ_R) [%]	0.3
		Concentration (δ_{zo}) [%]	0.7
		Costs(γ_4) [DM/inventory]	80,000
		Costs(γ_2) [DM/analysis]	400
5.	Limits for inventory period		
		Lower limit [d]	10
		Number of inventories/yr	20
		Upper limit [d]	25
		Number of inventories/yr	8
6.	MUF(V) Variance (σ_V^2) [kg^2 Pu]		0.25
7.	Mean Hold-up [kg]		20

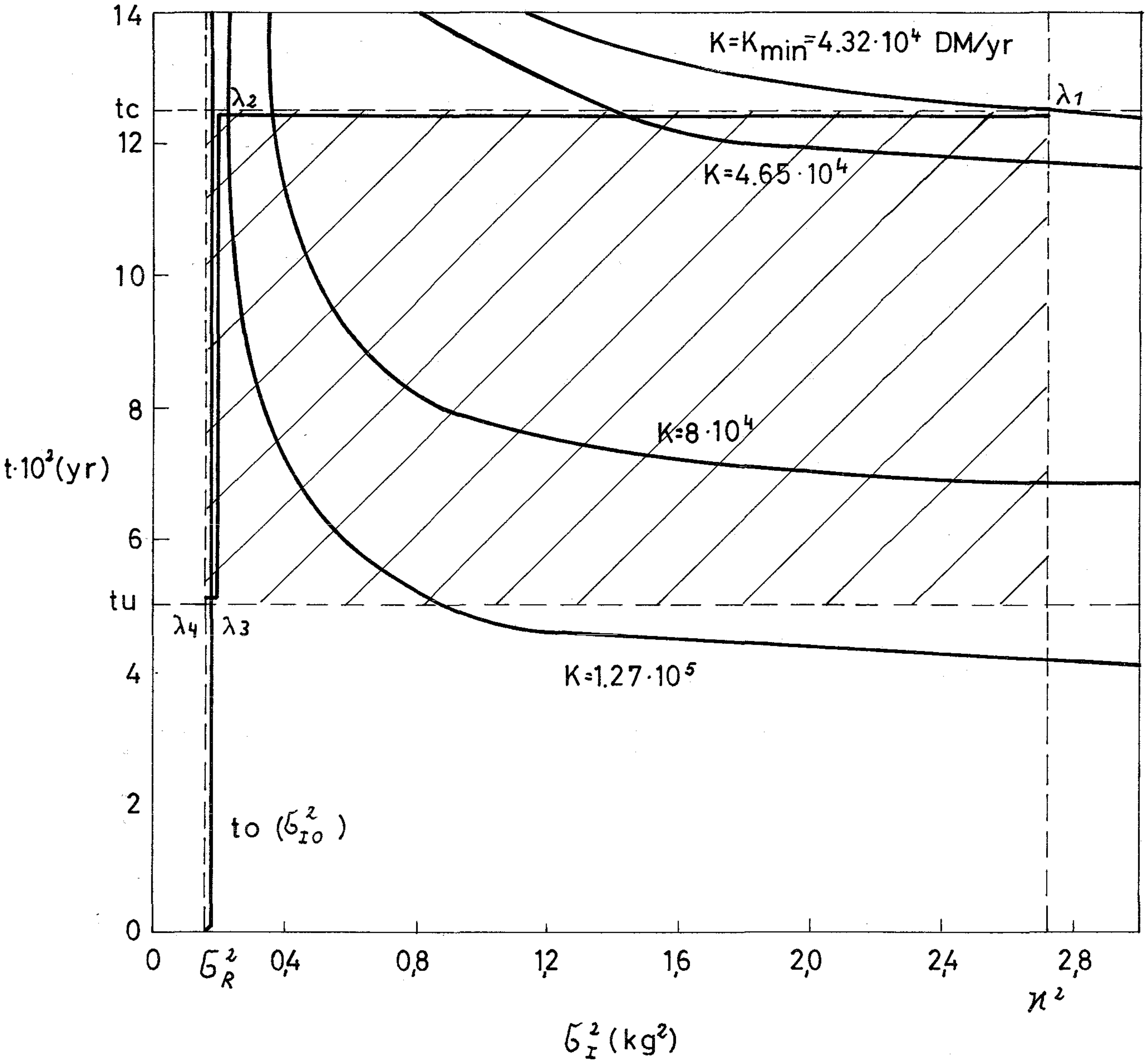

Fig. 3: Limits of cost optimization with t and σ_I^2 as variables for tracer method.

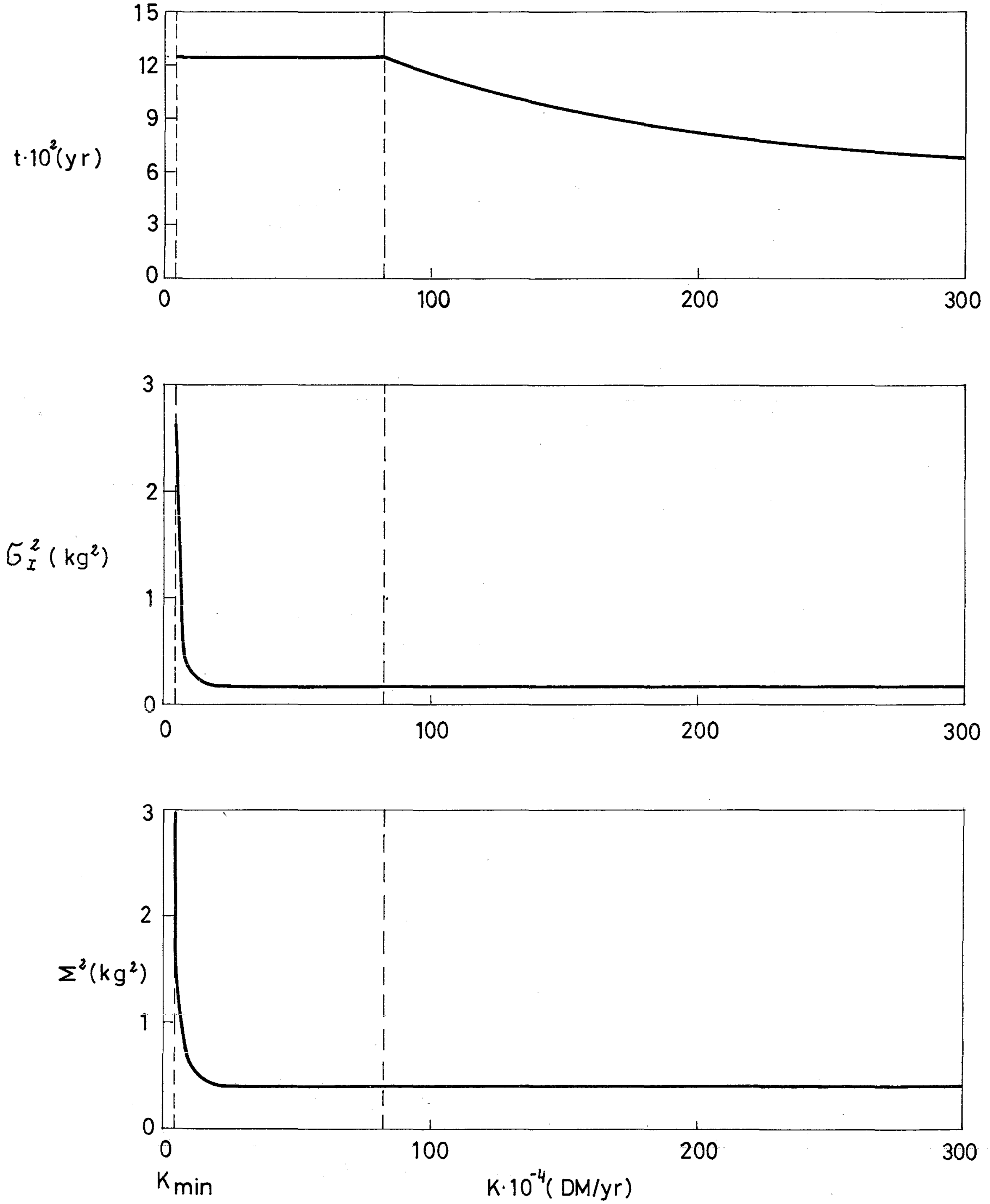

Fig. 4a,b,c: Optimum values of inventory periods $t = \frac{1}{n}$ (4a), variances for inventory measurement (4b) and total variance (4c) as a function of safeguards costs; tracer methods.

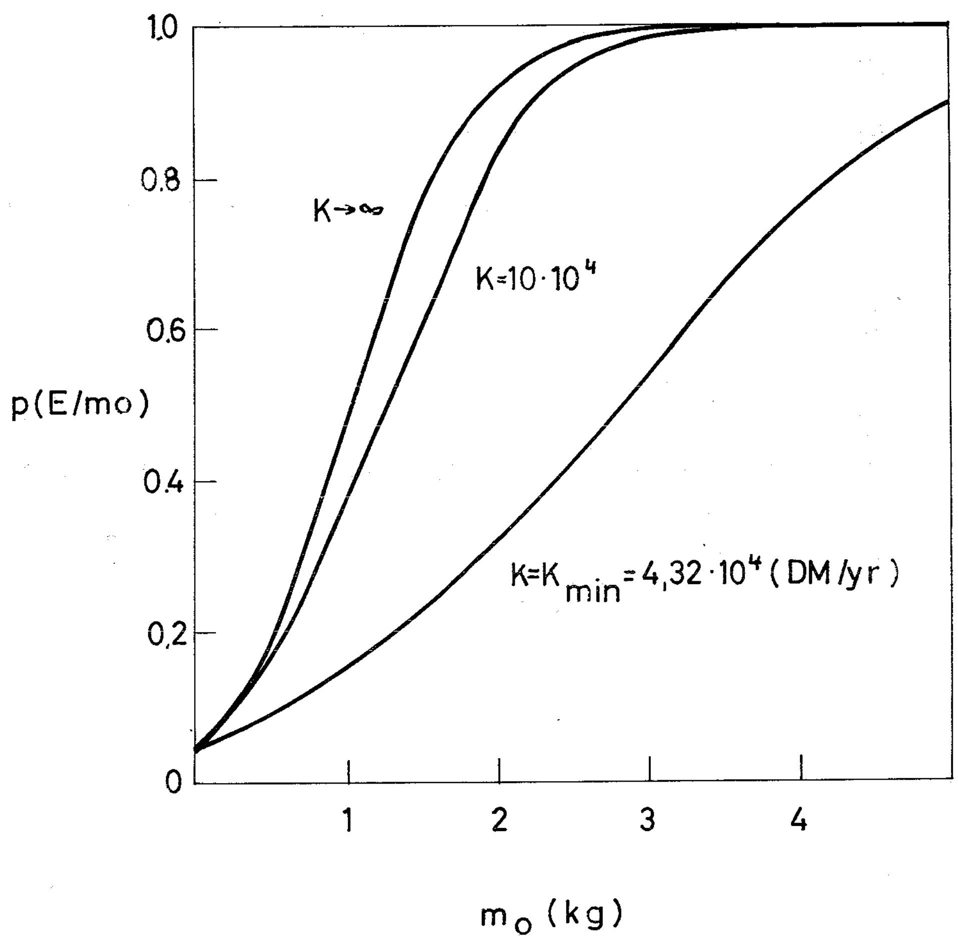

Fig. 5: Optimum probability of detection (effectivity) vs amounts assumed to be diverted, with optimized safeguards costs as parameter; for tracer method

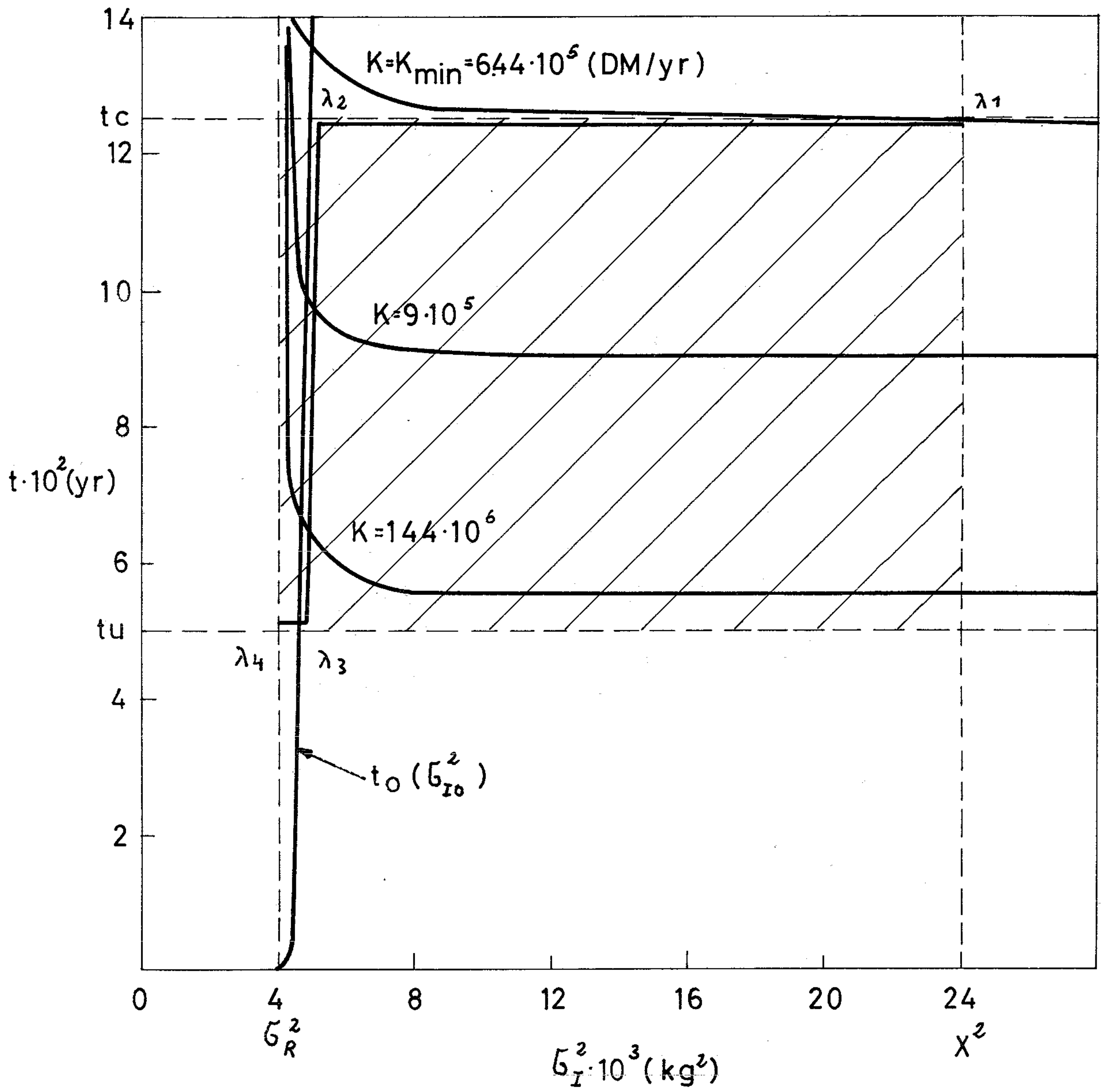

Fig. 6: Limits of costs optimization with t and σ_I^2 as variables, for washout method.

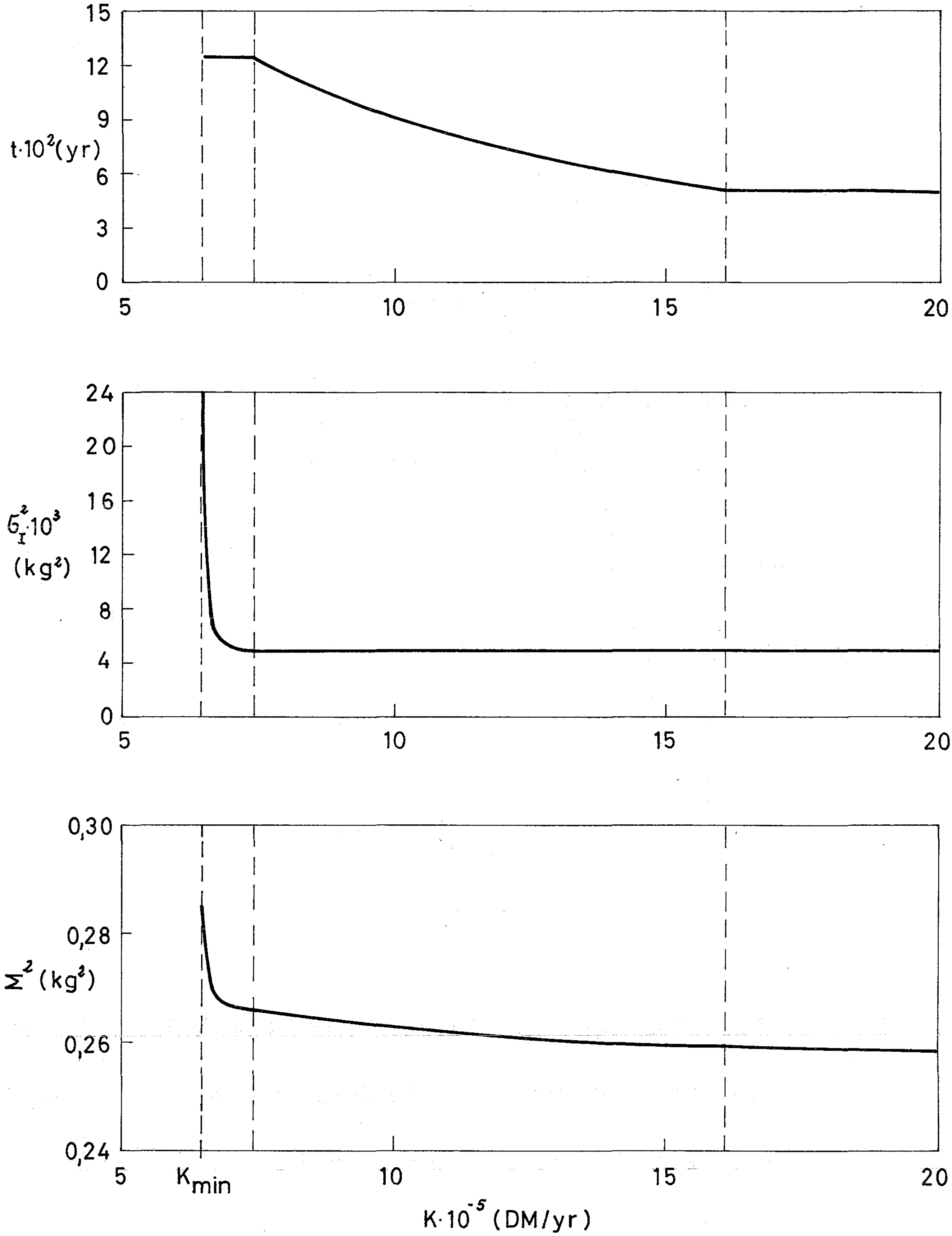

Fig. 7a,b,c: Optimum values of inventory periods $t = \frac{1}{n}$ (4a), variances for inventory measurement (4b) and total variance (4c) as a function of safeguards costs; washout method.

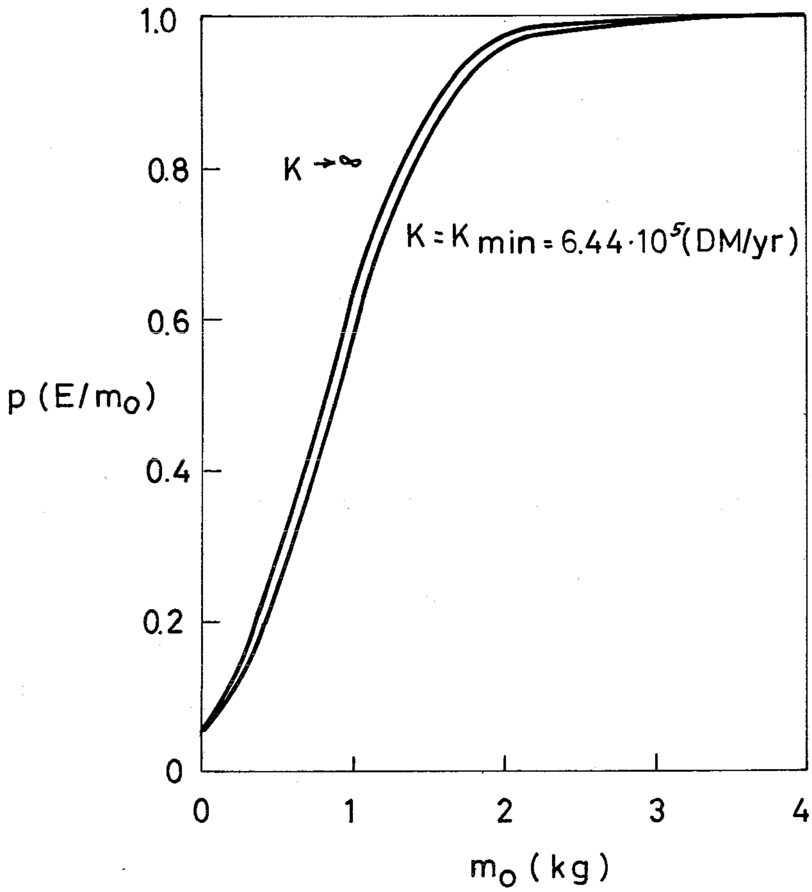

Fig. 8: Optimum probabilities of detection (effectivity) vs amounts assumed to be diverted, with optimized safeguards costs as parameter; for washout method.