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# The Solution of Heat Convection Problems by the Monte Carlo Method

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## Abstract

The application of the Monte Carlo Method (MCM) to heat convection problems is discussed. It is shown that the restriction $0 \leq p_i \leq 1$ for the transition probabilities $p_i$ does not limit this application, if a suitable differencing technique is used.

Thus the MCM can be a useful tool for the calculation of temperatures in heat convection problems, provided that the fluid properties may be assumed constant and the velocity field is known. An example is given for laminar, steady-state cylindrical pipe flow.

In the consideration of heat convection problems with temperature dependent fluid properties, one is led to the result that the application of the MCM in this case has no advantage compared to other solution methods.



## Zusammenfassung

Es wird die Anwendung der Monte-Carlo-Methode (MCM) auf Wärmeübergangsprobleme diskutiert. Durch die Restriktion $0 \leq p_i \leq 1$ für die Übergangswahrscheinlichkeiten $p_i$ wird diese Anwendung nicht begrenzt, wenn ein dem Problem angemessenes Differenzenschema verwendet wird.

Die MCM kann also ein nützliches Werkzeug für die Berechnung von Temperaturen bei Wärmeübergangsproblemen sein, wenn die Stoffeigenschaften des Fluids konstant angenommen werden können und das Geschwindigkeitsfeld bekannt ist. Ein Beispiel wird angegeben für eine laminare, stationäre Kreisrohrströmung.

Bei der Untersuchung von Wärmeübergangsproblemen mit temperaturabhängigen Stoffeigenschaften ergibt sich, daß die Anwendung der MCM in diesem Fall keinen Vorteil gegenüber anderen Lösungsmethoden hat.

# The Solution of Heat Convection Problems by the Monte Carlo Method

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## 1. Introduction

Elliptic and parabolic partial differential equations can be solved in many cases by the Monte Carlo Method (MCM), if the boundary conditions and/or initial values are given [1]. So the MCM has proven useful e.g. in solving the energy equation for the calculation of temperature distributions, if only heat conduction is considered [2,3]. For heat convection problems, that means heat transfer from a solid to a flowing fluid (or vice versa), the energy equation is extended by the convection terms, e.g. for an incompressible fluid without heat sources (i.e. without dissipation, too)

$$c_p \rho \frac{D\vartheta}{Dt} = \nabla(\lambda \nabla \vartheta) \tag{1}$$

or written in Cartesian coordinates for two dimensions

$$c_p \rho \left(\frac{\partial \vartheta}{\partial t} + u \frac{\partial \vartheta}{\partial x} + v \frac{\partial \vartheta}{\partial y}\right) = \frac{\partial}{\partial x}\left(\lambda \frac{\partial \vartheta}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial \vartheta}{\partial y}\right) \tag{2}$$

If the velocities u(t,x,y) and v(t,x,y) are known, equation (2) describes the temperature field in the fluid. For the solution of equation (2) the following boundary conditions and initial values are necessary:

- the temperature distribution or the distribution of the heat flux at the surface of the solid;
- the temperature distribution at the "inlet" of the fluid;
- the temperature distribution in the fluid at a time $t = t_o$, if a transient problem is considered ($\frac{\partial \vartheta}{\partial t} \neq 0$).

This paper is concerned with the question, whether the solution of equation (2) by the MCM is possible and useful. For convenience only two-dimensional problems are considered. However there is no fundamental difference to the treatment of three-dimensional problems.

Solving differential equations by the MCM has an important advantage compared to other numerical methods (in addition to the fact, that only rather simple computer programs are necessary): values of the solution can be calculated at single points without calculating the values at all other points at the same time. But for technical problems in most cases only a few values of the solution are needed. For instance for the thermal design of nuclear reactors important design criteria are the maximum surface temperatures of the fuel pins and the outlet temperatures of the coolant, whereas all other temperatures are of minor interest.

## 2. Some Remarks to Numerical Solutions Using the MCM

Methods for the solution of partial differential equations of the second order by the MCM are described else-where [1,2,3,4,5,6]. In the case of equation (2), using its finite difference representation, it should be mentioned that for the calculation of the transition probabilities $p_i$ for the stepping of the random walkers from one grid point to the neighbouring points, the velocities u and v and the fluid properties $c_p$, $\rho$, $\lambda$ at the point in question must be known. Since the values of the probabilities are restricted,

$$0 \leq p_i \leq 1 \tag{3},$$

Emery and Carson [4] concluded, that the value of the grid mesh size has an upper limit, determined by the value of the velocity u or v respectively, and that this upper limit has to be very small in regions of high velocities. This conclusion of Emery and Carson was caused by the fact, that they used a differencing technique, which was unsuitable for their problem, though mathematically correct, as already mentioned by McMordie and Batton [7].

And more than that it can be shown, that the restriction (3) can be fulfilled with constant mesh size, the value of which is independent from the value of u and v.

Replacing the convection terms $u\,\partial\vartheta/\partial x$ and $v\,\partial\vartheta/\partial y$ in equation (2) by finite differences one has to account for the fact that heat is transported by convection only in the direction of the velocity. So one has to take backward differences, if $u \geq 0$ or $v \geq 0$ respectively holds, and forward differences otherwise. To obtain the required differences in any case, the following finite difference representation of equation (2) should be used:

$$\frac{\frac{u+|u|}{2}\vartheta(x,y) - \frac{u+|u|}{2}\vartheta(x-h,y)}{h} + \frac{\frac{u-|u|}{2}\vartheta(x+h,y) - \frac{u-|u|}{2}\vartheta(x,y)}{h} +$$

$$+ \frac{\frac{v+|v|}{2}\vartheta(x,y) - \frac{v+|v|}{2}\vartheta(x,y-h)}{h} + \frac{\frac{v-|v|}{2}\vartheta(x,y+h) - \frac{v-|v|}{2}\vartheta(x,y)}{h} =$$

$$= \frac{\lambda}{\rho c_p} \Big( \frac{\vartheta(x+h,y) - 2\vartheta(x,y) + \vartheta(x-h,y)}{h^2} + \frac{\vartheta(x,y+h) - 2\vartheta(x,y) + \vartheta(x,y-h)}{h^2} \Big) \qquad (2a).$$

(shown for the case $\partial\vartheta/\partial t = 0$ only; a similar proof can be given for the case $\partial\vartheta/\partial t \neq 0$, if an implicit procedure -relaxation factor = 1- is used). For $\vartheta(x,y)$ it follows from equation (2a)

$$\vartheta(x,y) = \vartheta(x+h,y)\cdot\frac{1+\frac{h\varrho c_p}{\lambda}\cdot\frac{|u|-u}{2}}{4+\frac{h\varrho c_p}{\lambda}\cdot(|u|+|v|)}+$$

$$+\vartheta(x-h,y)\cdot\frac{1+\frac{h\varrho c_p}{\lambda}\cdot\frac{|u|+u}{2}}{4+\frac{h\varrho c_p}{\lambda}\cdot(|u|+|v|)}+$$

$$+\vartheta(x,y+h)\cdot\frac{1+\frac{h\varrho c_p}{\lambda}\cdot\frac{|v|-v}{2}}{4+\frac{h\varrho c_p}{\lambda}\cdot(|u|+|v|)}+$$

$$+\vartheta(x,y-h)\cdot\frac{1+\frac{h\varrho c_p}{\lambda}\cdot\frac{|v|+v}{2}}{4+\frac{h\varrho c_p}{\lambda}\cdot(|u|+|v|)}=$$

$$=\vartheta(x+h,y)\cdot p_1+\vartheta(x-h,y)\cdot p_2+$$

$$+\vartheta(x,y+h)\cdot p_3+\vartheta(x,y-h)\cdot p_4 \qquad (2b).$$

The coefficients of the temperatures on the right-hand side of equation (2b) are the transition probabilities $p_i$, which in this form fulfill the restriction (3) in any case. Thus this restriction does not limit the value of the MCM in heat convection problems.

In the following, the application of the MCM will be discussed first for the solution of special cases of equation (2) and then for the solution of equation (2) without additional assumptions.

## 3. Temperature Fields in Laminar, Steady-State Flow with Constant Fluid Properties

With the assumption of laminar, steady-state flow and constant fluid properties, equation (2) can be written as

$$u \frac{\partial \vartheta}{\partial x} + v \frac{\partial \vartheta}{\partial y} = \frac{\lambda}{\rho c_p} \left( \frac{\partial^2 \vartheta}{\partial x^2} + \frac{\partial^2 \vartheta}{\partial y^2} \right) \tag{4}$$

For fluid flow of this kind in many cases the velocities u (x,y) and v (x,y) can be calculated from the momentum equation (Navier-Stokes-equation) and are known for the solution of equation (4). Hence there is no restriction for the application of the MCM in this case.

Temperature fields in fluid flow of this kind have been calculated with the MCM successfully by Chandler et al. [8], who treated the heat transfer to a laminar, steady-state flow between two infinitely extended parallel plates. The writer has investigated the heat transfer to laminar, steady-state cylindrical pipe flow, which is described by the differential equation

$$u \frac{\partial \vartheta}{\partial x} = \frac{\lambda}{\rho c_p} \left( \frac{\partial^2 \vartheta}{\partial r^2} + \frac{1}{r} \frac{\partial \vartheta}{\partial r} \right) \tag{4a}$$

(conduction in axial direction neglected). The boundary conditions are (e.g.)

$$x \leq 0 : \quad \vartheta(r) = \vartheta_0$$

$$x > 0 : \quad \vartheta(r = R) = \vartheta_w > \vartheta_0$$

The velocity u is given by

$$u = u(r) = 2 u_m \left( 1 - \frac{r^2}{R^2} \right) \tag{5}$$

as the solution of the momentum equation for this special case. Combining equation (4a) and equation (5) results in a differential equation for $\vartheta(r,x)$, which has already been solved by Nusselt /‾9_7 using a series expansion. For a certain case (fluid: water, $\vartheta_o = 20\ ^oC$, $\vartheta_w = 40\ ^oC$, $u_m = 0.5\ cm.s^{-1}$) in fig. 1 Nusselt's solution of the differential equation is compared to the solution of the finite difference equation by the MCM. The agreement is quite good.

## 4. Temperature Fields in Turbulent, Quasi-Stationary Flow with Constant Fluid Properties

Without the assumption of laminar and steady-state flow, but retaining all other assumptions of equation (4), the temperature field is described by the equation

$$\frac{\partial \vartheta}{\partial t} + u\,\frac{\partial \vartheta}{\partial x} + v\,\frac{\partial \vartheta}{\partial y} = \frac{\lambda}{\rho c_p}\left(\frac{\partial^2 \vartheta}{\partial x^2} + \frac{\partial^2 \vartheta}{\partial y^2}\right) \qquad (6).$$

Since the time-dependent velocity field u(t,x,y) and v(t,x,y) of a turbulent flow cannot be calculated, for quasi-stationary flow with steady-state time averaged values $\bar{\vartheta}$, $\bar{u}$, $\bar{v}$ one defines

$$\vartheta(t,x,y) = \bar{\vartheta}(x,y) + \vartheta'(t,x,y),$$
$$u(t,x,y) = \bar{u}(x,y) + u'(t,x,y),$$
$$v(t,x,y) = \bar{v}(x,y) + v'(t,x,y).$$

$\vartheta'$, $u'$, $v'$ are the turbulent fluctuations around the respective time averaged values. Inserting these definitions in equation (6) and taking the time average of all expressions results in the differential equation

$$\bar{u}\frac{\partial\bar{\vartheta}}{\partial x}+\bar{v}\frac{\partial\bar{\vartheta}}{\partial x}=\frac{\lambda}{\rho c_p}\left(\frac{\partial^2\bar{\vartheta}}{\partial x^2}+\frac{\partial^2\bar{\vartheta}}{\partial y^2}\right)-\left(\frac{\partial\overline{u'\vartheta'}}{\partial x}+\frac{\partial\overline{v'\vartheta'}}{\partial y}\right) \quad (7)$$

The correlations $\overline{u'\vartheta'}$ and $\overline{v'\vartheta'}$ may be expressed by approximate functions representing measurement results [e.g. 10], or replaced by one of the known formulas for eddy diffusivities. Then the solution of equation (7) is similar to the solution of equation (4), provided that for the turbulent flow in question the velocity field $\bar{u}(x,y)$ and $\bar{v}(x,y)$ can be given, which is possible in some cases.

## 5. Temperature Fields in Fluid Flow with Temperature-Dependent Fluid Properties

The assumption of constant fluid properties does not hold for nearly all non-isothermal fluid flows of technical interest. But if temperature dependent fluid properties are accounted for, the following two consequences have to be considered:

- Since the fluid properties are needed for the calculation of the transition probabilities, as mentioned above, these probabilities are temperature-dependent, too. In all other known applications of the MCM for the solution of partial differential equations the transition probabilities are independent from the values of the solution.
- The differential equation for the velocity field (momentum equation) and the one for the temperature-field (energy equation, equation (2)) now have to be solved simultaneously; the velocities are needed for the calculation of the transition probabilities, too.

The transition probabilities must be known in all grid points before the computing procedure using the MCM starts. Therefore the main advantage of the MCM is lost: the temperatures of all grid points have to be calculated. Furthermore an iteration procedure becomes necessary: before the first step, the transition probabilities may be calculated with estimated values of temperatures and velocities, and after each step they have to be calculated again, until the difference of the results between two subsequent steps will be sufficiently small.

Not considering the problem of solving the momentum equation,it can be concluded that the MCM is unsuitable for the solution of heat convection problems, if temperature-dependent fluid properties have to be accounted for.

The time needed for the iteration procedure described above may roughly be estimated using the following information: the simulation of $10^4$ random walks (all starting from one grid point) in a one-dimensional grid of 14 points was completed after about 30 sec on an IBM/360-65 computer, and it has to be mentioned, that for sufficient accuracy in general more than $10^4$ random walks will be necessary for the calculation of one value. In a two-dimensional grid of say $10^3$ points the averaged time needed for one random walk is longer, so the time needed for one iteration step (i.e. calculation of the values for all grid points) will be more than $10^3 \cdot 30$ sec, or about 8 hours. Of course for certain problems less than $10^3$ grid points will be sufficient and a faster computer may be used. Further two improved methods using Monte Carlo techniques have been suggested, the EXODUS-method [4] and the MCM with information storing [5,6], which both will reduce the computation time. But the MCM has to be compared to other solution methods: For the problems considered in this section even a reduction of the computation time of some orders of magnitude cannot give a significant advantage to the MCM compared to other numerical solution methods.

## 6. Summary

The MCM has proven useful for the solution of pure heat conduction problems, especially, if only a few temperatures have to be calculated in the temperature field in question. In the same way the MCM can be used for the calculation of temperature fields in heat convection problems, provided that the fluid properties may assumed to be constant and the velocity field is known. For the solution of heat convection problems with temperature dependent fluid properties the application of the MCM has no advantage compared to other solution methods.

## Nomenclature

| | | |
|---|---|---|
| $c_p$ | = | Specific heat of fluid at constant pressure |
| $h$ | = | Grid mesh size |
| $p$ | = | Transition probability |
| $r$ | = | Radial coordinate |
| $R$ | = | Pipe radius |
| $t$ | = | Time coordinate |
| $u$ | = | Velocity component in x-direction |
| $u_m$ | = | Mean velocity in x-direction |
| $\bar{u}$ | = | Time averaged velocity in x-direction |
| $u'$ | = | Fluctuation of the velocity in x-direction, $u = \bar{u} + u'$ |
| $v$ | = | Velocity component in y-direction |
| $\bar{v}$ | = | Time averaged velocity in y-direction |
| $v'$ | = | Fluctuation of the velocity in y-direction, $v = \bar{v} + v'$ |
| $x, r$ | = | Cylindrical coordinates (two dimensions) |
| $x, y$ | = | Cartesian coordinates (two dimensions) |
| $\vartheta$ | = | Temperature |
| $\bar{\vartheta}$ | = | Time averaged temperature |
| $\vartheta'$ | = | Temperature fluctuation, $\vartheta = \bar{\vartheta} + \vartheta'$ |
| $\lambda$ | = | Thermal conductivity |
| $\rho$ | = | Fluid density |

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Fig. 1: Temperature Distribution in a Laminar, Steady-State Cylindrical Pipe Flow Downstream of a Stepwise Change in Wall Temperature

+ + Nusselt's solution [9]

o o Solution using the MCM