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KARLSRUHE

November 1976 KFK 2362

Institut für Reaktorbauelemente

Contributions to Multidimensional Quadrature Formulas

C. Günther

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Gesellschaft für Kernforschung, mbH, Karlsruhe

"Beiträge zu mehrdimensionalen Quadraturformeln"

ZUSAMMENFASSUNG

Die allgemeine Zielrichtung der vorliegenden Arbeit liegt darin, mehrdimensionale Quadraturformeln, die den Gaußschen Quadraturformeln im Eindimensionalen entsprechen, zu konstruieren und für diese Formeln Zusammenhänge mit orthogonalen und nichtnegativen Polynomen herzustellen, wie das im Eindimensionalen schon lange bekannt ist. Es handelt sich dabei zum einen um die Konstruktion von mehrdimensionalen Quadraturformeln allein mit Hilfsmitteln der Algebraischen Geometrie, zum anderen wird versucht, unter Einschluß der algebraischen Mittel Aussagen über Quadraturformeln zu erhalten, die auf jeden Fall reelle Stützstellen besitzen und unter bestimmten Umständen auch positive Gewichte haben.
Die Ergebnisse dieser Untersuchungen umfassen sowohl den Nachweis der Existenz bestimmter Quadraturformeln, Aussagen über die vom Polynomgenauigkeitsgrad abhängige Anzahl bzw. die maximal mögliche Anzahl von Stützstellen dieser Formeln als auch deren Konstruktion.

SUMMARY

The general objective of this paper is to construct multidimensional quadrature formulas similar to the Gaussian Quadrature Formulas in one dimension. The correspondence between these formulas and orthogonal and nonnegative polynomials is established. One part of the paper considers the construction of multidimensional quadrature formulas using only methods of algebraic geometry, on the other part it is tried to obtain results on quadrature formulas with real nodes and, if possible, with positive weights.
The results include the existence of quadrature formulas, information on the number resp. on the maximum possible number of points in the formulas for given polynomial degree N and the construction of formulas.

Contributions to

Multidimensional Quadrature Formulas [x]

(Manuscript, Version 1.6.1976)

C. Günther

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[x] This report is a preliminary version of a paper which was originated during the authors investigations of the numerical integration over the angular domain in the neutron transport equation while he was working at the Institute of Neutron Physics and Reactor Technology (INR). As at this time there seems to be no chance to bring this manuscript into a final form, the author tries to publish this paper in the present form. Parts of section 5 of the paper have been presented at the Intern. Congress of Math., Vancouver, Canada, 1974 (See the Appendix).

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0. General introduction

0.1 Introduction

This paper is concerned with the construction of multidimensional quadrature formulas (= q.f.'s). All investigations are restricted to formulas which integrate exactly polynomials up to a certain given degree N that means, for a given integral I(f), written as

$$I(f) = \iint_{D} f(x,y, \ldots)\, dG$$

with a nonnegative set function G,

$$S(f) = \sum_{i=1}^{n} A_i f(x_i, y_i, \ldots.)$$

is called a quadrature formulas of degree N or of order N, if $I(f) = S(f)$ whenever f is a polynomial in x, y, ... of degree not exceeding N. This latter conditions are usually named "moment conditions".

Generally we restrict our attention to two dimensional problems. The main interest is devoted to the question how to get q.f.'s with <u>real</u> nodes (x_i, y_i) without claiming that all weights (or coefficients) A_i must be positive. It is tried in some sense to minimize the number n of points used in S(f). The points of the involved q.f.'s are located on $p = 0$ where p is an orthogonal polynomial of degree ℓ if N has been $N = 2\ell - 1$.

The results developed here must be considered to ly between some purely algebraic theorems of MYSOVSKIKH and STROUD and others at the one side and a well known theorem of TCHAKALOFF on the other side, where the A_i have to be positive and the nodes must be situated in D.

0.2 General objectives

The well known results concerning the so-called "Gaussian Quadrature Formulas" may be summarized as follows, see e.g. KRYLOW /28/.

a) For given I(f) and given degree $N = 2\ell - 1$ there is a q.f. with a minimal number n of points where $n = \ell$ and there is exactly one q.f. with ℓ points.

b) The points of this formula are inside of the interval under consideration, the weights respective are positive.

c) The points of this formula are the zeros of the corresponding orthogonal polynomial of degree ℓ. This allows to calculate fairly well these points.

d) These statements hold for all $\ell > 0$.

Besides the need of having available a similar procedure for multidimensional problems, there have been also other reasons for mathematicians to adress their attention to this field:

a) One assumed to find similar connections between q.f.'s and orthogonal polynomials as in one dimension.

b) There is and has been a provocation (as usually for problems with enumerating features), to find q.f.'s for a given N with ever lower, perhaps least possible number of points.

As is well known, the task to construct q.f.'s S(f) for a given (two dimensional) I(f) and given degree N with a minimum possible number of points which is to integrate exactly $1, x, y, x^2, \ldots\ldots\ldots, x^N, x^{N-1}y, \ldots\ldots\ldots, y^N$, cannot be achieved as in one dimension by equating the number 3n of parameters available in S(f) and the number of conditions $\frac{1}{2}\cdot(N+1)\cdot(N+2)$. This even fails for $N = 2$ where six polynomials have to be integrated exactly, yet $n = 2$ is impossible as APPELL has already pointed out in 1890, /2/. -

The concept introduced in section 5 is the following: Let p be an orthogonal polynomial of degree ℓ; we then construct an S(f) with points only on $p = 0$ which integrates exactly all polynomials up to degree $N = 2\ell-1$ and which in addition minimizes or maximizes some polynomial q of degree $2\ell = N + 1$.

This procedure allows

1. to get q.f.'s with relatively low n,
2. to get q.f.'s with real nodes,
3. to balance in a certain sense the number of conditions and the number of free parameters.

Until now the so-called identification problem has not been solved: To give a set of polynomials p_1, p_2, ... the common zeros of which are the nodes of S(f) obtained in the outlined way. As a consequence of this, the q.f.'s of the presented type must be calculated numerically by solving the formulated extremal problem.

It must be emphasized that this paper does not intend to get special favorable q.f.'s. We rather develop a theory which holds for arbitrary positive integrals given on some two dimensional region.

The reasons which suggest the use of q.f.'s with positive weights or with nodes situated in D, have been several times sketched out e.g. in GÜNTHER /15/. Therefore we do not explain further why such formulas are preferred.

0.3 State of the art

The development of multidimensional quadrature until today is summarized in the following section with main emphasis on two dimensions.

Many q.f.'s have been found by proceeding "straight forward". This method has proved successful for symmetrical and other regular regions with symmetrical mass distributions of the weight function. In these cases assumptions have been made on the number, on the size and the distribution of the nodes and the weights of the formulas; this yields a nonlinear system of equations with a relatively low number of unknowns. As representatives for a large number of contributions stressing this approach, we mention papers of HAMMER and STROUD /24/ and RABINOWITZ and RICHTER /41/.

Recent papers of this kind are PIESSENS and HAEGEMANS /39/ and for the triangle COWPER /4/ and LYNESS and JESPERSEN /30/.

Without disparing the success of this procedure, it may not be overlooked that this is a "try and error" procedure, where it remains open why the method fails or is successful.

Beyond this, there have been attempts to get general results using algebraic geometry. The first step in this direction made RADON /42/ who constructed seven point q.f.'s of degree five the points of which are the common zeros of three orthogonal polynomials of degree three. Further results of more general character have been derived by MYSOVSKIKH /34/, /35/, /37/ and /38/, STROUD /47/, /48/ and /49/, FRANKE /9/, GÜNTHER /14/ and /16/ and MÖLLER /32/. The more constructive direction of Radon's ideas was followed by FRANKE /10/, GÜNTHER /18/ and MÖLLER /31/.

Another direction is represented by contributions considering q.f.'s with points lying inside D ("self-contained q.f.'s") and with positive weights.

A very general result of great importance has been obtained by TCHAKALOFF /53/, this result has been proved otherwise e.g. by DAVIS /5/. More recent results are given by GÜNTHER /15/ and /16/ connecting algebraic methods and functional analysis.

There are also some investigations of FRITSCH /11/, the methods of his paper do not seem to allow to attack more general problems. -

Reviewing papers on multidimensional q.f.'s are due to P. HAMMER /23/ comprising the time before 1959, the development until 1965 is contained in STROUD /46/. A newer summary is contained in a survey article of HABER /22/, some sections are treated in detail in STROUD's book /50/ which was published in 1971.

0.4 General review of the methods

Some of the concepts of this paper are based on the following idea:

For a given integral $I(f) = \iint f(x,y)dG$, we are constructing q.f.'s $S(f) = \sum_i A_i f(x_i,y_i)$ which may be considered as special Lebesgue-Stieltjes-integrales $S(f) = \iint f(x,y)dG_s$ with mass only in discrete points (x_i,y_i), the point (x_i,y_i) contains mass A_i.

As all investigations e.g. the consideration of the conjugate spaces, are restricted to finite-dimensional subspaces $L(T)$ of $C(T)$ - the linear space of functions continuous on T - - here we have $L(T) = P^2_N(T)$ (= vector space of polynomials in x and y of degree $\leq N$ with range T), $I(f)$ and $S(f)$ are only different representations of the same element of $L^*(T)$. With this convention a q.f. is only a special representation of an element $\in [P^2_N(T)]^*$ with discrete mass distribution.

This point of view becomes important if we are considering supporting polynomials, that means nonnegative polynomials $\emptyset(x,y) \in P^2_k(T)$ with $k = N$ or $k = N+1$ such that the points (x_i,y_i) of $S(f)$ may be only situated where $\emptyset$ vanishes.

This is accomplished by embedding all in the space $P^2_{N+1}(T)$ and satisfying all moment conditions up to degree N, whereas $S(f)$ attains a certain extremal value if f is a special polynomial of degree $N+1$. This method, based on a support-concept (support: mass where $\emptyset$ vanishes), introduced by KREIN /27/ may be carried over to multidimensional problems, as this method does not make use of the factorization of polynomials in linear factors.

In this method, in one dimension, only the maximal number of zeros of an element of the space in consideration plays an essential role and thereby is applicable for Tchebycheff systems.

In two dimensions, a somewhat sophisticated procedure must be used because either the zeros of a supporting polynomial or the zeros of an orthogonal polynomial may be one dimensional. Finally, the maximum possible number of common zeros of two members of our polynomial space (Bézout's theorem) plays the same role as n, the maximal number of zeros in one dimension.

An approach using also a support-concept, has already been proposed by AALTO /1/; it seems not to yield better results as has been attained by algebraic means only. -

The investigations, based on algebraic geometry considerations only, are stressing the idea to construct a canonical basis $(p_1, p_2, \ldots\ldots)$ of an ideal of polynomials i_s which has the points of a q.f. $S(f) = \sum A_j f(x_j, y_j)$ as zeros. That means that each polynomial $p(x, y)$ of degree N_1 from i_s (: $p(x_j, y_j) = 0$ for $j = 1,2,\ldots\ldots, n$) may be written as

$$p(x,y) = \sum_k a_k(x,y)p_k(x,y)$$

with polynomials $a_k(x, y)$ of degree $\leq \mathrm{Max}(N_1 - \text{degree } p_k, 0)$. This statement is a degree-dependent version of the very famous fundamental theorem of algebraic geometry of MAX NOETHER see e.g. /55/. The most general variant of this theorem has been recently given by MÖLLER /32/.

Continuing the algebraic ideas in a more constructive sense, it is frequently used that the p_k are not independent as the basis elements of a vector space but are satisfying relations of the form

$$\sum_k b_k(x,y)p_k(x,y) = 0$$

with polynomials $b_k(x,y)$. These relations are called "syzygies" e.g. GRÖBNER /12/. The first systematic use of this fact appears in a famous paper of RADON /42/.

0.5 Assessment of new results

Some concepts of this paper have been presented at the International Congress of Mathematicians at Vancouver in

1974 /17/. They are incorporated in section 5 and are working with the idea to interprete q.f.'s as special set functions with mass only on curves p = 0 where p is an orthogonal polynomial. In the same way the content of section 7 is new in which algebraic conditions are given being fulfilled by the weights and the points of the q.f.'s containing as subset the moment conditions.

The results cited in section 6 have been mostly given in /15/. It shall be only mentioned here that they may by proved using similar methods as in section 5.

Section 4 is an attempt to give a more unified look to the results found using only algebraic geometry.

It must be added that some of the preliminary results (section 1 - 3) contain new statements e.g. theorems 1.3.4, 1.3.5, 1.4.1 or 1.4.2.

1. Prerequisites

1.1 The representation of linear functionals on C(T) and on subspaces of C(T)

Let C(T) be the linear vector space of real valued continuous functions on T, T a normal compact topological space, and L(T) subspaces of C(T) of finite dimension and $C^*(T)$ and $L^*(T)$ the corresponding conjugate spaces.

Introducing the concept of partially ordered Banach spaces we are showing that every nonnegative linear functional on C(T) and on L(T) can be written as Lebesgue-Stieltjes-integral with nonnegative regular, bounded and additive set function.

Let E be a partially ordered Banach space, E^* the conjugate to E, $E^{\oplus}$ is the cone of nonnegative linear functionals on E. For the definitions given here and in the following see DUNFORD-SCHWARTZ / 7/.

Let T be, as initially stated, a nonempty, normal topological space. C(T) is the vector space of real continuous functions defined on T. The norm in C(T) is given by

$$\| f \| = \max_{X \in T} |f(X)|$$

for $f(X) \in C(T)$.

$f \in C(T)$ is said to be nonnegative if $f(X) \geqslant 0$ for all $X \in T$. By this, C(T) is a partially ordered Banach space.

$L(T) \subset C(T)$ is of finite dimension and contains at least one positive element. Is T' a compact subset of T, L(T') is the restriction of L(T) on T'.

rba(T) is the linear space of regular bounded additive set functions defined on the Borel field $\sigma(T)$ of sets on T. The norm of $G(V) \in \mathrm{rba}(T)$ is given by

$$\| G \| = \sup \sum_{j=1}^{r} |G(V_j)| ,$$

the total variation of G(V) where the supremum is to be taken over all subdivisions of T in a finite number of disjoint subsets V_j, $V_j \in \sigma(T)$. The definition of a regular set function and of rba(T) is given in DUNFORD-SCHWARTZ /7/, p.137 and p.261.

rba(T) is also partially ordered, $G_1 \geqslant G_2$ if $G_1(V) \geqslant G_2(V)$ for all $V \in \sigma(T)$. In $C^*(T)$ we have $l_1 \geqslant l_2$ if for $l_1, l_2 \in C^*(T)$ $l_1(f) \geqslant l_2(f)$ for all nonnegative $f \in C(T)$. The norm of $l \in C^*(T)$ is induced by the norm of C(T)

$$\| l \| = \sup_{f \in C(T),\ \|f\| \leqslant 1.} |l(f)|$$

For every $f \in C(T)$ the integral

$$\int_T f(s) G(ds)$$

with $G(V) \in \text{rba}(T)$ exists, DUNFORD-SCHWARTZ /7/, p.261, is called Lebesgue-Stieltjes-integral of f with respect to $G(V)$ on T.

The elements of $C^*(T)$ are related by

THEOREM 1.1.1:
oooooooooooooo

If T is normal, there is a isometric isomorphism between $C^*(T)$ and $\text{rba}(T)$ such that corresponding elements $l(f) \in C^*(T)$ and $G(V) \in \text{rba}(T)$ satisfy the identity

$$l(f) = \int_T f(s)G(ds), \text{ for all } f \in C(T).$$

Furthermore, this isomorphism preserves order, DUNFORD-SCHWARTZ, p.262.

Theorem 1.1.1 ascertains that elements of $C^{\oplus}(T)$ can be written as Lebesgue-Stieltjes-integrals with nonnegative $G \in \text{rba}(T)$. An immediate consequence is the following

COROLLARY 1.1.1:
oooooooooooooooo

For normal T, there is a homomorphism between $L^*(T)$ and $\text{rba}(T)$ such that corresponding elements $l(f) \in L^*(T)$ and $G(V) \in \text{rba}(T)$ are related by

$$l(f) = \int_T f(s)G(ds) \text{ for all } f \in L(T).$$

Following statements can be given as to the orderings in $L^*(T)$ and $\text{rba}(T)$. If $l_j(f) \in L^*(T)$ and $G_j(V) \in \text{rba}(T)$, $j = 1,2$,

$$l_j(f) = \int_T f(s)G_j(ds),$$

from $G_1 \geq G_2$ follows $l_1 \geq l_2$, the converse must not be true. A well known theorem of TCHAKALOFF /53/ ascertains that for each $l(f) \in L^{\oplus}(T)$, $l \geq 0$, there is at least one $G(V) \in \text{rba}(T)$,

$G \geqslant 0$.

THEOREM 1.1.2:
oooooooooooooo

Let T be compact and L(T) be the span of d continuous, linearly independent, real valued functions defined on T, containing an element g, $g > 0$ on T. Each $l(f) \in L^{\oplus}(T)$ can be written as

$$l(f) = \sum_{j=1}^{d'} A_j f(X_j) \text{ for all } f \in L(T),$$

where the $A_j > 0$, $X_j \in T$ and $d' \leq d$.

The sum is equivalent to an element $G \in rba\ (T)$, $G \geqslant 0$, having mass in all X_j, X_j containing the mass A_j for $j = 1, \ldots\ldots, d'$. The correspondence of the elements of $C^{\oplus}(T)$ resp. $L^{\oplus}(T)$ with the nonnegative elements of rba(T) justifies the terminology, the elements of $C^{\oplus}(T)$ ($L^{\oplus}(T)$) are induced by a mass distribution on T.

A basis $f_1, f_2, \ldots\ldots\ldots, f_d$ of L(T) induces a basis of $L^*(T)$ by setting $c_j = l(f_j)$, $j = 1, \ldots\ldots, d$, for all $l \in L^*(T)$. The representation as point $\vec{c}$ in some $\mathbb{R}^d$ with components c_j is equivalent to the original l(f).

For $L^{\oplus}(T)$ we have

THEOREM 1.1.3:
oooooooooooooo

1. $L^{\oplus}(T)$ contains all point functionals.
2. $L^{\oplus}(T)$ is a cone in $L^*(T)$ with vertex in the origin.
3. $L^{\oplus}(T)$ is a closed convex cone.

See e.g. WILSON /56/, p. 243.

If elements $l(f) \in L^*(T)$ which are not $\in L^{\oplus}(T)$ are written as

$$l(f) = \int_T f(s)\ G\ (ds) = \sum_{j=1}^{d'} A_j f(X_j),$$

not all coefficients A_j are positive that means $G \in rba(T)$ is not definite. In these cases it is useful to search for representations of $l_1(f)$,

$$l_1(f) = \int_T f(s)\left|G\ (ds)\right| = \sum_{j=1}^{d'} |A_j|\, f(X_j).$$

The form of $|G|$ finally permits to get knowledge of the structure of G.

1.2 Supporting polynomials

A statement is given which states the following: To each $l(f) \in L^{\oplus}(T)$ which is not a positive linear functional ($l \in \partial L^{\oplus}(T)$), there is a nonnegative function $\emptyset \in L(T)$, $\emptyset \not\equiv 0$, with $l(\emptyset) = 0$ and with (positive) mass only where $\emptyset$ vanishes in T.

We assume first that $L^{\oplus}(T)$ has inner points. If $\vec{c} \in \partial L^{\oplus}(T)$, $l(f)$ corresponding to $\vec{c}$, a separation theorem, stamming from the theory of convex bodies, is used to show the existence of a supporting hyperplane to $L^{\oplus}(T)$ in $\vec{c}$. This theorem states

THEOREM 1.2.1:
oooooooooooooo

Let B_1 and B_2 be two convex sets of a vector space L with B_1 and B_2 not empty, $B_1 \cap \text{int } B_2 = \emptyset$. Then there is a hyperplane H in L separating B_1 and B_2.

H can be written as $\emptyset(c) = a$ with real a, $\emptyset \in L^*$, $\emptyset \not\equiv 0$, for all $c \in H$. If H separates B_1 and B_2, we can assume that $\emptyset(c_1) \leqslant a$ for $c_1 \in B_1$, $\emptyset(c_2) \geqslant a$ for $c_2 \in B_2$, VALENTINE /54/, pp. 31-34.

With $L = L^*(T)$, $B_1 = \vec{c}$, $B_2 = L^{\oplus}(T)$, from Theorem 1.2.1 follows that the supporting hyperplane $\emptyset$ fulfills $\emptyset(c) = a$ with real a and $\emptyset \in [L^*(T)]^* = L(T)$, $\emptyset \not\equiv 0$. It can easily be shown, VALENTINE /54/, p. 37, that a = 0. By this, the equation of H is $\emptyset = 0$. As $l_X(\emptyset) = \emptyset(X) \geqslant 0$ for all point functionals (- they are all $\in L^{\oplus}(T)$ -) $l_X(f) = f(X) \in L^{\oplus}(T)$, $\emptyset$ is nonnegative on T. We now have established that for every

$\vec{c} \in \partial L^{\oplus}(T)$ there is an element $\emptyset \in L(T)$, $\emptyset \not\equiv 0$, $\emptyset \geqslant 0$ in T and $l(\emptyset) = 0$.

Recalling that L in many cases consists of a space of polynomials, $\emptyset$ is called "supporting polynomial" of $\vec{c}$ resp. $l(f)$ on T, KARLIN a. STUDDEN /26/, p. 43.

A nonnegative set function $G(V) \in rba(T)$ corresponding to $l(f) \in \partial L^{\oplus}(T)$ has the following property. Let $T_{\emptyset}$ be defined as $T_{\emptyset} := \{X \in T / \ \emptyset(X) = 0\}$. Then $G(V) = 0$ if $V \subseteq T - T_{\emptyset}$, $V \in \sigma(T)$. This is shown in detail in SMIRNOW /44/, p.119. We summarize

THEOREM 1.2.2:

Let $l(f)$ be from $\partial L^{\oplus}(T)$. Then there is a not identically vanishing function $\emptyset(X) \in L(T)$, $\emptyset(X) \geqslant 0$ for $X \in T$, with $l(\emptyset)$ with following characterictic; if $G(V) \in rba(T)$, nonnegative, corresponds to $l(f)$, $G(V)$ has mass only where $\emptyset$ vanishes in T.

The statement of Theorem 1.2.2 is even correct if $l(f)$ is the zero element $l_0(f)$ of $L^*(T)$, $l_0(f) = 0$ for all $f \in L(T)$. In this case, each nonnegative $\emptyset$ from $L(T)$ is supporting polynomial of $l_0(f)$.

If $L^{\oplus}(T)$ has no inner points, $L^{\oplus}(T)$ is contained in a hyperplane in $L^*(T)$ and Theorem 1.2.2 in this case trivially also holds.-

Following consequence of Theorem 1.2.2 can be derived admitting a similar statement for elements $l(f) \in \operatorname{Int} L^{\oplus}(T)$, if T satisfies some additional assumptions. The main idea is the following one: If $l(f) \in \operatorname{Int} L^{\oplus}(T)$ and T is in some "continuous" manner decreased (this must be precisely defined), we find for some $T' \subset T$ that $l(f) \in \partial L^{\oplus}(T')$. The exact statement is taken from GÜNTHER /15/, where this theorem has been proved.

Let $F(\lambda)$ be a set of subsets of $\mathbb{R}^s$, depending on the real parameter λ, $0 \leqslant \lambda \leqslant 1$. The $F(\lambda)$ are supposed to be nonempty for $0 \leqslant \lambda \leqslant 1$.

DEFINITION:
The $F(\lambda)$ are continuous functions of λ for $\lambda \in [0, 1]$ if for arbitrary $\varepsilon > 0$ there is a $\delta > 0$ such that for each point $X \in F(\lambda)$ and each $\mu \in [0, 1]$, $|\mu - \lambda| < \delta$ there is at least one point $Y \in F(\mu)$ with

$$\| X - Y \| \leq \varepsilon .$$

THEOREM 1.2.3:
oooooooooooooo

Let T and T' be given, T and T' compact and $l(f) \in L^{\oplus}(T)$. Assume there is a continuous set of sets $F(\lambda)$, $0 \leq \lambda \leq 1$ with $F(0) = T$, $F(1) = T'$, $F(\lambda) \subset F(\lambda')$ for $0 \leq \lambda' \leq \lambda \leq 1$, $F(\lambda)$ compact for $\lambda \in [0, 1]$. Let $l(f) \notin \operatorname{Int} L^{\oplus}(T')$.

Then we have: There is a uniquely determined interval $\lambda_1 \leq \lambda \leq \lambda_2$ with $0 \leq \lambda_1 \leq \lambda_2 \leq 1$ such that

$$l(f) \in \partial L^{\oplus}(F(\lambda)) \text{ for all } \lambda \in [\lambda_1, \lambda_2]$$

For each $\lambda \in [\lambda_1, \lambda_2]$ there is a function $\emptyset \in L(T)$, $\emptyset \not\equiv 0$, $\emptyset \geq 0$ on $F(\lambda)$ and $l(\emptyset) = 0$. There is a $G(V) \in rba(T)$, $G \geq 0$, $l(f) = \int_T f(s)G(ds)$ for all $f \in L(T)$, $G(V) = 0$ for $V \in \sigma(T)$, $V \in T - \{(\emptyset = 0) \cap F(\lambda)\}$.
In addition, if $l(f) \in \operatorname{Int} L^{\oplus}(T)$, we have $\lambda_1 > 0$,

$$\text{if } l(f) \notin \partial L^{\oplus}(T'), \ \lambda_2 < 1.$$

1.3 Algebraic geometry

This section contains definitions and theorems about polynomials in two variables primarily due to algebraic geometry.-

A polynomial $p(x,y)$ with coefficients from the field of real (complex) numbers $\mathbb{R}$ ($\mathbb{C}$) is written as

$$p(x,y) = \sum a_{jk} x^j y^k, \quad j = 0,1, \ldots\ldots , \quad k = 0,1,\ldots$$

For $a_{jk} = 0$ if $j + k > N$ and one $a_{jk} \neq 0$, $j + k = N$,

$p(x,y)$ is said to be of degree N or of order N.

In $K[x,y]$, the ring of polynomials in x and y with coefficient field K where K either $\mathbb{R}$ or $K = \mathbb{C}$, the theorem on unique factorization holds.

$X = (\xi, \eta)$ is a point of multiplicity v of p, $v \geq 0$ if

$$p(x,y) = \sum b_{jk}(x-\xi)^j(y-\eta)^k, \; j \geq 0, \; k \geq 0,$$

with all $b_{jk} = 0$ for $j + k < v$ and at least one b_{jk} with $j + k = v$ different from zero. X is a root of p if X is a point multiplicity ≥ 1 of p. A point of multiplicity $v > 1$ of p is called a singular point of p. An algebraic curve of order N without multiple components has only a finite number of singular points, WALKER /55/, p.65. X is a common zero of p and q if X is a zero of p and a zero of q. The definition of multiple zeros shall not be given here, this may be seen from WALKER, p. 108.

A well known result on the number of common zeros of polynomials is Bēzouts theorem

THEOREM 1.3.1:
oooooooooooooo

Two polynomials P_1 and P_2 of degrees n_1 and n_2 without common component have exactly $n_1 \cdot n_2$ common zeros.

We are first concerned in some detail with multiple common zeros of two polynomials, thereafter with the common zeros of polynomials at infinity. For this reason we define the tangents of an algebraic curve $p = 0$ in a point X of this curve. Let X be a v-fold point of $p = 0$, then if $(\lambda_1, \mu_1), (\lambda_2, \mu_2), \ldots\ldots\ldots$ $\ldots\ldots\ldots (\lambda_v, \mu_v)$ are the roots of

$$b_{v,o}\lambda^v + b_{v,1} \cdot \lambda^{v-1}\mu + \ldots\ldots + b_{o,v}\mu^v = 0$$

with coefficients b_{jk} with $j + k = v$ of p, the lines

$$\lambda_r(x-\xi) + \mu_r(y-\eta) = 0, \; r = 1,\ldots\ldots,v$$

are called the tangents of p = 0 in X, WALKER /55/, p.54. With these definitions we formulate

THEOREM 1.3.2:
oooooooooooooo

If X is a point of multiplicity r of p and of multiplicity s of q, X is a common zero of p and q at least of multiplicity r·s. The multiplicity is exactly r·s, if no tangent of p in X is tangent of q in X, WALKER /55/, p. 114.

An immediate consequence is

COROLLARY 1.3.1:
oooooooooooooooo

X is at least a double common zero of p and of q if any (the) tangent of p in X is also tangent of q in X.

The following remarks are related with common zeros of two polynomials at infinity. Let P_k defined as

$$P_k(x,y) = \sum_{i+j \leq n_k} a_{kij} x^i y^j, \quad k = 1,2.$$

We introduce projective coordinates and define

$$\tilde{P}_k(\tilde{x},\tilde{y},\tilde{z}) = \tilde{z}^{n_k} \cdot P_k(\tilde{x}/\tilde{z}, \tilde{y}/\tilde{z}), k = 1,2,$$

and as "companion polynomial" of P_k

$$\hat{P}_k(t) = \sum_{i+j=n_k} a_{kij} \cdot t^{n_k - i}, \quad k = 1,2,$$

A point X "at infinity" has coordinates $(\tilde{\xi},\tilde{\eta},0)$. It is a common point of $P_1 = 0$ and $P_2 = 0$ if $\tilde{P}_k(\tilde{\xi},\tilde{\eta},0) = 0$, k = 1,2. From this immediately follows: If there are v different projective common zeros $(\tilde{\xi}_r,\tilde{\eta}_r,0)$, r = 1,2,.....,v of $P_1 = 0$ and $P_2 = 0$, P_1 and P_2 have at least v common zeros at inifinity and there are polynomials Q_1 of degree $n_2 - v$ and Q_2 of degree $n_1 - v$

such that

$$P_3(x,y) \quad = \quad Q_1(x,y) \cdot P_1(x,y) \quad + Q_2(x,y) \cdot P_2(x,y)$$

is of degree $< n_1 + n_2 - v$.

As long as the $(\tilde{\xi}_r, \tilde{\eta}_r, 0)$ must be different, $v \leq \min_k n_k$; it should be noted that there are polynomials which have more than $\min n_k$ common zeros at infinity including some multiple common zeros.

Many papers studying multivariate quadrature problems are looking for polynomials vanishing in the points of a q.f. This suggests to consider the ideal i_s of polynomials which are vanishing in the points X_i of a q.f. $S(f) = \sum A_i f(X_i)$.

We assume for simplicity that all X_i are distinct, that means that S(f) contains no terms with derivatives. It is known from algebraic geometry that i_S has a basis with a finite number of elements, $(p_1, p_2, \ldots\ldots\ldots, p_r)$. By this each polynomial $q \in i_s$ ($q(X_i) = 0$ for all i) can be written as

$$q \quad = \quad \alpha_1 \cdot p_1 + \alpha_2 \cdot p_2 + \ldots\ldots\ldots\ldots + \alpha_r \cdot p_r$$

which polynomials $\alpha_1, \alpha_2, \ldots\ldots\ldots\ldots, \alpha_r$.

The strongest and best possible form of this fact usually named Max Noethers theorem has been given by MÖLLER /32/.

THEOREM 1.3.3:
oooooooooooooo

For each i_s exists a "canonical basis" that means that there is a basis $\{p_1, p_2, \ldots\ldots\ldots, p_r\}$ such that for any $q \in i_s$ of degree m we have

$$q \quad = \alpha_1 \cdot p_1 + \alpha_2 \cdot p_2 + \ldots\ldots + \alpha_r \cdot p_r$$

where degree $\alpha_j \leq$ m - degree p_j if $\alpha_j \not\equiv 0$ for $j = 1,2,\ldots., r$.

Apparently the identification of a canonical basis rises no difficulties. This basis contains a maximal set of linearly independent polynomials $\beta_{g,k}$ of lowest possible degree g with $\beta_{g,k}(X_i) = 0$; if there are polynomials $\beta_{g,k} \in i_s$ of degrees g' = g+1, g+2,....., which are not of the form

$$\gamma_1 \cdot \beta_{g,1} + \dots\dots\dots \gamma_s \cdot \beta_{g,s}$$

degree γ_i = g' - degree $\beta_{g,i}$, with previously found $\beta_{g,i}$, this polynomials also belong to this canonical basis.

We include here two simple results on real polynomials p.

THEOREM 1.3.4:
oooooooooooooo

Let p be $p(x,y) = \sum a_{ij} x^i y^i$ with real a_{ij}. The real part of p = 0 is bounded if the corresponding companion polynomial $\hat{p}(t) = \sum_{i+j=n} a_{ij} t^{n-j}$ is definite.

Of similar type is

THEOREM 1.3.5:
oooooooooooooo

p with real a_{ij} has 2v real common zeros with an arbitrary large circle $x^2 + y^2 - R^2 = 0$ if $\hat{p}(t)$ has v real zeros.

The variant with multiple roots of p is more complicated. These theorems have some importance concerning the reducibility of polynomials over $\mathbb{R}$.

1.4 Nonnegative polynomials in one dimension

Results on nonnegative polynomials on the interval $[0,1]$ are found e.g. in POLYA a. SZEGÖ /40/ or in KARLIN a. STUDDEN /26/. The latter book also contains results for $[0,\infty)$ and for $(-\infty,\infty)$.

The problem in question in this section is more general. Let a real polynomial p in x and y of degree ℓ be given. We investigate how many common zeros - without counting multiplicities - has p with another real polynomial q of degree m, relatively prime to p, q nonnegative where p = 0. A classification of points where p and q may simultaneously vanish is as follows

LEMMA 1.4.1:
oooooooooooo

Let X be a common zero of p and q in D; if $X \in$ Int D, X is at least a double common zero of p and of q.

Proof: X either is a regular (simple) or a singular point of p (or q). In the first case q = 0 must be tangent to p = 0, otherwise q changes sign by passing through X along p = 0. In the second case, according to theorem 1.3.2, X is an at least double common zero of p and q.

We conclude using Bézouts theorem (theorem 1.3.1)

THEOREM 1.4.1:
oooooooooooooo

Is p = 0 an irreducible algebraic curve of degree m having v common points with ∂D and is q a polynomial of degree m, relatively prime to p which is nonnegative where p = 0 in D. Then p and q have - without taking into account multiplicities - at most $\frac{m \cdot \ell + v}{2}$ points where they vanish together in D.

If p and q have μ, $1 \leq \mu$ common zeros at infinity, subsequent modification of theorem 1.4.1 holds

THEOREM 1.4.2:
oooooooooooooo

If in addition to the assumptions of theorem 1.4.1 p and q have μ common zeros at infinity, p and q vanish in $\frac{m \cdot \ell + v - \mu}{2}$ points simultaneously in D.

1.5 Multivariate nonnegative polynomials

Theorems concerning the form of polynomials in two variables nonnegative on some elliptical region $D \subset \mathbb{R}^2$ are given in GÜNTHER /19/. There are two details essential for the subsequent considerations.

First a polynomial $p = p(x,y) \in \mathbb{R}[x,y]$, nonnegative on D, may have factors or powers of factors which either have one dimensional sets of real zeros or zeros of dimension zero (isolated real zero). One of the results of /19/ is

THEOREM 1.5.1:
oooooooooooooo

If q is a factor of a polynomial $p(x,y) \in \mathbb{R}[x,y]$, nonnegative on D with the real part of q = 0 of dimension one, q is a factor of even multiplicity of p.

Of special interest is a statement giving for a fixed degree N the maximum possible number of isolated zeros of a nonnegative polynomial of degree N.

THEOREM 1.5.2:
oooooooooooooo

Let $p(x,y) \in \mathbb{R}[x,y]$ of degree $N^{\geq 2}$ be nonnegative on D. Then if $p \geq 0$ in $\mathbb{R}^2$, p has at most $\frac{1}{2} \cdot (N-1) \cdot (N-2) + 1$ isolated zeros. If p may be < 0 outside of D, p has at most $\frac{1}{2} \cdot (N-1) \cdot (N-2)$ isolated zeros in Int D and at most N zeros on ∂D.-

The representation of nonnegative polynomials plays an important role not only in the problems of this paper. It is equally connected with questions of statistics (approximation of densities, Tchebychew inequalities), with problems of approximation theory (e.g. if g is a minimax approximation to f, $\| f - g \| = E_n$, f and g polynomials with degree f $>$ degree g, the polynomials $f - g + E_n$ and $g - f + E_n$ are nonnegative). A similar argument holds for one side approximations.

2. Orthogonal polynomials

2.1 Orthogonal polynomials, one dimensional

There is a well established theory connecting orthogonal polynomials and quadrature formulas see e.g. SZEGÖ /52/, DAVIS a. RABINOWITZ /6/ or STROUD a. SECREST /51/.

The most essential parts of this theory may be described as follows: To calculate approximately

$$I(f) = \int_0^1 f(x)w(x)dx, \quad w(x) > 0 \text{ in } [0,1],$$

a weighted sum of values of f is taken,

$$S(f) = \sum_{j=1}^{n'} A_j f(x_j)$$

with S(f) of degree $N = 2\cdot\ell - 1$, that means $S(p) = I(p)$ whenever p is a polynomial of degree $\leq N$. Following statements hold for ℓ fixed:

E0: For all S(f) we have $n' \geq \ell$.
E1: There is one uniquely determined $\hat{S}(f)$ with $n' = \ell$.
E2: The weights A_j, $j = 1,2,\ldots\ldots,\ell$ of this formula are positive.
E3: The points or nodes of $\hat{S}(f)$ are lying in (0,1).
E4: The x_j are the zeros of the polynomial $p_\ell(x)$ of degree ℓ which is orthogonal with respect to w(x) to all polynomials of degree $\leq \ell - 1$,

$$p_\ell(x) = x^\ell + a_1 \cdot x^{\ell-1} + \ldots\ldots + a_\ell \text{ and}$$

$$\int_0^1 p_\ell(x)x^v w(x)dx = 0 \quad \text{for } v = 0,1,\ldots\ldots,\ell-1.$$

The quadrature formulas specified by E1 - E4 are called "Gaussian Quadrature Formulas". As pointed out in E1 these formulas are those with fewest number of points for given degree $N = 2\cdot\ell - 1$. By E4 is given a method to calculate the x_j.

Besides these'favourable' properties, q.f.'s like the Gaussian Q.F.-s with weights A_j positive and points x_j from the interval $[0,1]$ behave well with regard to convergence and stability, HABER /22/, p. 495, KRYLOW /28/.

Orthogonal polynomials satisfy several extremal principles one of which is briefly touched. Consider the problem to calculate

$$\underset{G \geq 0}{\text{Min}} \int_0^1 x^{2\ell} dG, \qquad (2.1.1)$$

the minimum to be taken over all nonnegative $G \in rba(0,1)$, where $\int_0^1 x^j dG = I(x^j)$ for $j = 0,1,\ldots\ldots\ldots, 2\cdot\ell - 1$. The solution of this problem can be regarded as element of $\partial[P^1_{2\ell}\ [0,1]]^{\oplus}$; this can easily be seen: let $c_i = I(x^i)$, $i = 0,1,\ldots\ldots\ldots,2\ell$ be the coordinates of G_1 in $[P^1_{2\ell}\ [0,1]]^*$, then $c_{2\ell}$ cannot be decreased without the corresponding element being removed from $[P^1_{2\ell}\ [0,1]]^{\oplus}$. The corresponding supporting polynomial, the existence of which is shown by theorem 1.2.2, is $\emptyset = (p_\ell(x))^2$ and the mass of G_1 is contained in the zeros of $p_\ell(x)$, furthermore, G_1 is the set function corresponding to the Gaussian Q.F. of degree ℓ with respect to $I(f) = \int_0^1 f(x)w(x)dx$, KARLIN a. STUDDEN /26/.

It should be noted that the analoguous maximum problem in the same way is related to the q.f.'s of Radau.

A problem dual to the one treated here is the following: Try to find a nonnegative polynomial $q_{2\ell}$ with leading term $x^{2\ell}$ such that $I(q_{2\ell})$ is minimal; the solution of this problem is $q_{2\ell} = (p_\ell(x))^2$, LOCHER /29/.

2.2 Orthogonal polynomials, two dimensional

The knowledge of properties of orthogonal polynomials in more than one dimension is not nearly as well developed as in one dimension. The following comments deal with two dimensional results, some crude results can be taken from ERDÉLYI /8/ e.g.:

2.2.1 Orthogonal polynomials exist with respect to positive $I(f) = \iint_D f(x,y)dG$, that means $I(f) > 0$ for all f, $f(x,y) \geq 0$ in D, $f \not\equiv 0$ where the scalar product is defined as $(f,g) = I(f \cdot g)$.
If this statement only holds for polynomials p up to a certain degree 2ℓ, the existence of orthogonal polynomials of degree $\leq \ell$ can be shown.
2.2.2 For a simply connected region $D \subset \mathbb{R}^2$ each factor q of an orthogonal polynomial is not a multiple factor and q has zeros in Int D, APPELL /2/, H8.
2.2.3 The $\ell + 1$ polynomials $p_{ij}(x,y)$for given $l = 0,1,\ldots..$

$$p_{ij}(x,y) = x^i y^j + q_{ij}(x,y)$$

with $i + j = \ell$, $i = \ell, \ell - 1, \ldots\ldots 0$ where degree $q_{ij} \leq \ell - 1$, which are orthogonal to all polynomials of degree $\leq \ell - 1$, are called basic orthogonal polynomials of degree ℓ.
2.2.4 Examples of orthogonal polynomials for several regions D and weight functions are given in ERDÉLYI /8/, GRÖBNER /13/ and STROUD /50/.
2.2.5 Beyond the results mentioned in 2.2.2 some details are known for low ℓ:

a) For given I(f) all linear orthogonal polynomials vanish in the center of mass $X = (\xi,\eta)$, $\xi = I(x)$, $\eta = I(y)$, APPELL /2/, H14.
b) Following sufficient criterion is known concerning the existence of four real common zeros of two orthogonal polynomials of degree two:
Let $\hat{C}(I(f))$ be $I(p_{20}p_{02} - p_{11}^2)$,

$$P_1 = a \cdot p_{20} + b \cdot p_{11} + c \cdot p_{02}$$
$$P_2 = A \cdot p_{20} + B \cdot p_{11} + C \cdot p_{02}$$

and $\delta = (A \cdot c - a \cdot C)^2 + (a \cdot B - A \cdot b) \cdot (c \cdot B - b \cdot C)$.

P_1 and P_2 have four real common zeros if $\hat{C}(I(f)) \neq 0$ and sign $\delta \neq$ sign $\hat{C}(I)$, GÜNTHER /16/. If $\hat{C}(I) = 0$, p_{20}, p_{11} and p_{02} have three real common zeros, MYSOVSKIKH /34/.
Some more advanced results are given in section 5.

3. Preliminaries about the number of points in quadrature formulas

The following theorem gives a relation between q.f.'s of certain degree d and orthogonal polynomials for multidimensional problems which is well known as it is a simple generalization from one dimension.

THEOREM 3.1:
oooooooooooo

If a formula S(f) is of degree d and its nodes belong to the hypersurface p = 0 of order v ($\leq$ d), p is orthogonal to all polynomials of degree $\leq$ d-v.

There are more general results, see HIRSCH /25/ and STROUD /50/.

The efforts to find q.f.'s of given degree d of some special kind with the fewest number of points has led to some results: In 1960 STROUD /45/ showed that a formula of degree d (in two dimension) must contain at least $n_{min}(d) = \frac{1}{2}([\frac{d}{2}] + 1)\cdot([\frac{d}{2}] + 2)$ points. A converse of this is due to MYSOVSKIKH /36/ who pointed out the equivalence of three facts:

1. The existence of a formula of degree $2\cdot\ell - 1$ with $n_{min}(2\ell - 1)$ points.
2. All orthogonal polynomials of degree ℓ have $n_{min}(2\cdot\ell-1)$ common zeros.
3. Some characteristics H_{ij}, i, j = 1,2,......., ℓ-1 vanish,

$$H_{ij} = \frac{1}{2} I(p_{\ell-i-1,i+1}p_{\ell-j+1,j-1} - 2p_{\ell-i,i}p_{\ell-j,j} + p_{\ell-i+1,i-1}\,p_{\ell-j-1,j+1})$$

These characteristics have already been defined by RADON /42/ for ℓ = 3. For ℓ = 2 we have (see 2.2.5) that there is exactly one three-point-formula of degree three iff $H_{11} = I(p_{20}p_{02} - p_{11}^2) = 0$.

The matrix of the characteristics H_{ij} also plays an important role for the calculation (constr.) of q.f.'s with more than n_{min} $(2\ell-1)$ points, RADON /42/, MÖLLER /31/. Also other matrices built up by the H_{ij} may enter when q.f.'s are constructed, GÜNTHER /18/.

Upper bounds for the minimum number of points in q.f.'s of given degree under different conditions on the size of the points and the sign of the weights are given by several authors which are cited in the subsequent sections.
A converse of the results of HIRSCH /25/ which ensures an upper limit on the number of linearly independent polynomials of degree ℓ vanishing in the nodes of a q.f. of degree $2 \cdot \ell - 1$ is given below; we restrict ourselves to the case of $\ell = 4$, the q.f. S(f) having $n = 14$ nodes. It can be shown that there are at most $v = 2$ linearly independent polynomials of degree four vanishing in the X_i using a theorem of algebraic geometry which is called "Cayley-Bacharach theorem" in SEMPLE a. ROTH /43/.

This result may be extended to lower n. It can be similarly shown that $v = 3$ for $n = 13$ and $n = 12$. An analogue for general n reads as follows: For given $\ell \geqslant 4$ and $n \geqslant \ell^2 - \frac{1}{2}(\ell-1)(\ell-2)+1$ we have $v \leqslant 2$.

4. Methods using algebraic geometry

4.1 Using a polynomial basis of the nodes

It is generally assumed in this section that all considerations are made in $\mathcal{C}[x,y]$. This does not matter the fact that one wants to have q.f.'s with real points and real (and rather positive) weights. Starting point of our investigations as in section 2. is a positive integral I(f) on $P^2_{2\ell-1}(D)$ for which representations are determined in the sense of 1.1 (that means q.f.'s of degree $2 \cdot \ell - 1$); the orthogonal polynomials are understood to be orthogonal with respect to the scalar product $(f,g) = I(f \cdot g)$. If S(f) is such a representation of I(f), $S(f) = \sum A_j f(X_j)$, let i_s be the ideal of polynomials $\in \mathcal{C}[x,y]$ vanishing in all points X_j, $j = 1,2,\ldots\ldots,n$. It is known that i_s has a finite basis, using the theorems of 1.3 we conclude that i_s has a canonical basis.

We now consider the problem how to calculate in some cases a canonical basis of a q.f. of degree $2 \cdot \ell - 1$ with a relatively small number of points.

An essential fact is the following: if $(p_1, p_2, \ldots\ldots p_r)$ is a (non necessary canonical) basis of i_s, the basis elements $p_1, p_2, \ldots\ldots, p_r$ are non independent in the sense that from

$$a_1 \cdot p_1 + a_2 \cdot p_2 + \ldots\ldots + a_r \cdot p_r = 0 \quad (4.1)$$

with $a_j \in \mathbb{C}[x,y]$, $j = 1,2,\ldots\ldots,r$ follows $a_j \equiv 0$ for all j. On the contrary there is a module of such relations as (4.1), called "syzygy" in the classical textbooks on algebraic geometry, see SEMPLE and ROTH /43/ or GRÖBNER /12/, p. .

In some cases it is possible to use syzygies or at least one syzygy to calculate a canonical basis of i_s for some S(f).

The first to make use of this fact has been RADON /42/. His conclusion was as follows: assume there is a q.f. of degree five with seven points X_j. Then there are three linearly independent polynomials P_1, P_2 and P_3 of degree three vanishing in the X_j and being orthogonal, see theorem 3.1. These polynomials satisfy

$$L_1 \cdot P_1 + L_2 \cdot P_2 + L_3 \cdot P_3 = 0$$

with linear L_j, j = 1,2,3. Another partitioning leads to

$$x \cdot K_1 + y \cdot K_2 = K_3 \quad (4.2)$$

where the K_j are linear combinations of the P_j ·(4.2) allows to calculate K_1, K_2 and K_3. By equating the coefficients of the power of order four we have

$$K_1 = a \cdot p_{21} + b \cdot p_{12} + c \cdot p_{03}$$

$$-K_2 = a \cdot p_{30} + b \cdot p_{21} + c \cdot p_{12}.$$

K_3 by definition is orthogonal to all polynomials of order ≤ 1. Assuming very general conditions on I(f) RADON showed that a, b and c may be determined such that K_3 is also orthogonal to x^2, $x \cdot y$ and y^2.

If $K_3 \neq 0$, (4.1) permits to calculate the nodes X_j and thereafter S(f). If $K_3 = 0$ or $K_3 = c_1 \cdot K_1 + c_2 \cdot K_2$ with constants c_1 and c_2, there is a common factor Q of degree two of K_1 and K_2,

$$K_1 = y \cdot Q, \quad K_2 = -x \cdot Q.$$

In this case any orthogonal polynomial K_3^* of degree three linearly independent from K_1 and K_2 is vanishing in the origin that means, using Noether's Theorem in its simplest form, $K_3^* = x \cdot A + y \cdot B$ with quadratic A and B, this may be interpreted as

$$Q \cdot K_3^* = Q \cdot (A \cdot x + B \cdot y) = A \cdot K_1 + B \cdot K_2,$$

a second syzygy, independent from (4.2).

A similar procedure has used MÖLLER /33/ in a recent paper to get q.f.'s of degree 7 with 12 points and formulas of degree 9 with 17 (real) points for some functionals. In the second case there are used two syzygies to calculate S(f) resp. the four basis elements of i_s, one relation with linear coefficients, the second with quadratic coefficient polynomials. In addition MÖLLER /33/ has given an extensive part of a theory which considers formulas where all basis elements are of same degree.

The examples quoted until here have basis polynomials of same degree. An interpretation of formula (4.1.2) has given rise to an investigation leading to formulas with basis elements of distinct degrees. (4.2) means there are two polynomials K_1 and K_2 of degrees three with 9 common zeros, two of them being at infinity while K_3 has only the finite (seven) common zeros together with K_1 and K_2. In /18/ the author has used a similar idea to construct special 14-point formulas of degree seven. The basis of i_s consists of two orthogonal polynomials P_1 and P_2 of degree four and of a third polynomial P_3 of degree five orthogonal to all polynomials of degree ≤ 2. These basis elements are related by

$$Q_1 \cdot P_1 + Q_2 \cdot P_2 + L_3 \cdot P_3 = 0$$

with quadratic Q_1 and Q_2 and constant L_3. This means P_1 and P_2 have two common zeros at infinity, $P_3 = Q_1 \cdot P_1 + Q_2 \cdot P_2$ and S(f) has as nodes the finite common zeros of P_1 and P_2. By definition P_3 is orthogonal to at maximum linear polynomials; by requiring that P_3 is orthogonal to x^2, xy and y^2 also, P_1 and P_2 are determined. A more detailed exposition of this idea is given in GÜNTHER /18/.

No use of syzygies makes following theorem in 1969 independently given by MYSOVSKIKH /34/ and STROUD /48/.

THEOREM 4.1:
oooooooooooo

Assume $P_1(x,y)$ and $P_2(x,y)$ are two orthogonal polynomials of degree ℓ with exactly ℓ^2 common zeros (x_i,y_i), $i = 1, 2, \ldots\ldots, \ell^2$, all of which are distinct and none of which are at infinity.
Then there ecists a q.f. of degree $2 \cdot \ell - 1$ with the (x_i,y_i) as points.

Several generalizations of this theorem are known admitting multiple common zeros of P_1 and P_2, MYSOVSKIKH /35/ and /37/, GÜNTHER /14/. Extensions to higher dimensions have been given by FRANKE /9/ and MYSOVSKIKH /38/. The special case where P_1 and P_2 have common factors is treated in more detail later on.

4.2 Orthogonal polynomials with common factors

The preceding sections have been concerned with q.f.'s with nodes being situated an irreducible algebraic curves.
One variant not yet treated has to look at reducible orthogonal polynomials, the other deals with orthogonal polynomials with common factors.

This latter aspect has been treated in GÜNTHER /18/, a preliminary result is published in /16/.

Let P_1 and P_2 be defined as

$$\begin{aligned} P_1 &= T_1 \cdot Q \\ P_2 &= T_2 \cdot Q, \end{aligned}$$

with degree $P_i = \ell$, degree $T_i = s < \ell$, degree $Q = \ell - s$.

We now describe how to get q.f.'s of degree $2\cdot\ell - 1$, using the finite common zeros of T_1 and T_2 and some points on $Q = 0$.

Case a): The discrete common zeros of T_1 and T_2 are all of finite type. We then search for a third polynomial P_3 of degree m, m specified later on, vanishing in the common points of $T_1 = 0$ and $T_2 = 0$ and having $m\cdot(\ell - s)$ distinct common zeros with Q. From theorem 1.3.4 follows that $P_3 = U_1\cdot T_1 + U_2\cdot T_2$ degree $U_i = m - s$ for $i = 1,2$. If degree $U_i <$ degree T_1, all parameters of U_1 and U_2 are linearly independent, if this inequality does not hold, it must be clarified how many parameters of the U_i are linearly independent. The available parameters are chosen so that P_3 is orthogonal to all polynomials of degree $\leq 2\cdot\ell - m - 1$. By taking m sufficiently large, it can be arranged to have free parameters enough to find solutions different from the solutions $U_1 = Q$, $U_2 = 0$ or from $U_1 = 0$, $U_2 = Q$.

Case b): There must be made slight modifications if T_1 and T_2 have common zeros at infinity. Here P_3 is of degree $<$ degree U_i + degree T_i, such that the common zeros of P_1 and P_2 at infinity must not be zeros of P_3 (they may be).

An essential feature of the orthogonality relations is that P_3 by definition satisfies some conditions of orthogonality; let degree q be one, degree $P_3 = \ell$. Then we have that each $P_3 = a\cdot T_1 + b\cdot T_2$, degree a = degree b = 1, is orthogonal to all polynomials P of the form $P = Q\cdot c$ with degree $c \leq \ell - 2$ because

$$(P_3,P) = (a\cdot T_1 + b\cdot T_2,\ Q\ c) = (\underbrace{T_1\cdot Q}_{P_1}, a\cdot c) + (T_2\cdot Q,\ b\cdot c) = 0$$

In case b) it must be guaranteed that P_3 has no common zeros with P_1 and P_2 at infinity.

We remind ourselves that the syzygies are here

$$T_2 \cdot P_1 - T_1 \cdot P_2 = 0 \text{ and}$$
$$a \cdot P_1 + b \cdot P_2 - Q \cdot P_3 = 0$$

Calculating q.f.'s in the case of orthogonal polynomials with common factors, a simplification is possible. One first calculates the common zeros X_j, $j = 1, \ldots\ldots$ of T_1 and T_2. then by selecting appropriate Lagrangian polynomials the weights A_j, $j = 1, \ldots\ldots$. Thereafter a representation of $I(f) - \sum A_j f(X_j)$ as element $[P^2_{2\ell-1}(D) \mod Q]^*$ is determined.

The following table contains a list of all possibilities entering in case a) for $\ell = 6$. The column "method A" contains values attained using a method similar to that in section 5, to get discrete mass distributions on $Q = 0$.

degree Q	degree P_3	maximal number of points	method A
1	7	32	32
2	7	30	29
3	6	27	28
4	6	28	29
5	6	31	30

A corresponding table for two orthogonal polynomials P_1 and P_2 of degree $\ell = 6$ with four common zeros at infinity, v = degree Q of them being on $Q = 0$, looks as follows,

degree Q	degree P_3	maximal number of points
1	7	29
2	7	28
3	7	29

5. Quadrature formulas with real nodes

5.1 A basic result

Starting point of the investigations of this section is a theorem serving as a basis to concentrate the mass distribution of special set functions G on the zero-curve of orthogonal polynomials.

THEOREM 5.1.1:
oooooooooooooo

If $I(f) \in [P^2_{2\ell-1}(D)]^{\oplus}$ and if q is orthogonal of degree ℓ, we also have $I(f) \in [P^2_{2\ell-1}(D) \bmod q]^{\oplus}$ that means I(f) can be written as

$$I(f) = \iint\limits_{D} f\, dG$$

with $G \in \mathrm{rba}(D)$, $G(\Delta) = 0$ for $\Delta \in \sigma(D)$, $\Delta \subseteq \{D - (q = 0)\}$.

Proof: a) (Algebraic part) As for given orthogonal q and any p_1 and p_2 from $P^2_{2\ell-1}(D)$ which obey $p_1 = a \cdot q + p_2$ with degree $a \leq \ell - 1$, we have $I(p_1) = I(p_2)$, the definition of I(f) on $P^2_{2\ell-1}(D)$ is equivalent to the prescription of an orthogonal q and a definition of the functional for elements of $P^2_{2\ell-1}(D)$ mod q.

b) (Topological part) $L_1 = P^2_{2\ell-1}(D)$ mod q may be regarded as L_2 where $L_2 = P^2_{2\ell-1}(D')$, $D' = \{D \cap (q = 0)\}$. If p_1 and $p_2 \in P^2_{2\ell-1}(D)$ have the same values for $(x,y) \in D'$, we conclude that $p_1 - p_2 = a \cdot q$, degree $a \leq \ell - 1$ or $p_1 = p_2 \bmod q$ and by the orthogonality of q $I(p_1) = I(p_2)$. //

We point out that this proof would be incorrect if a statement as 2.2.2 did not hold.

Remark: Theorem 5.1.1 states that $I(f) \in L^{*}_2$, not $I \in L^{\oplus}_2$. If in any case were $I \in L^{\oplus}_2$, for each $p \in L_2$, $p \geq 0$, there must be a polynomial $p' \in P^2_{2\ell-1}(D)$, $p' = p + a \cdot q$, degree $a \leq \ell - 1$, $p' \geq 0$ in D. It can be seen by examples that this is impossible.

5.2 Points from $\partial\left[P^2_{2\ell-1}(D) \bmod q\right]^{\bullet}$

In theorem 5.1.1 was shown that for every given q orthogonal of degree ℓ, representations of I(f) exist with mass only on q = 0 in D. Another problem is now to find representations of I(f) with mass on q = 0 where the mass is located in single points. Moreover the number of points shall be small. This is established by repeating the preceding analysis on $P^2_{2\ell-1}(D) \bmod q$.

Let q be irreducible in $\mathbb{R}$ and let the number of common zeros of q = 0 with ∂D be equal $\alpha' = 2\alpha''$. We know from theorem 1.2.2 that there is a supporting polynomial $\emptyset \in L_1 = P^2_{2\ell-1}(D) \bmod q$ which is nonnegative on $\{q = 0 \cap D\}$.

$\emptyset = 0$ has at most $(2\ell - 1)\cdot\ell$ common points with q = 0, only α' of them may be simple common points. From this we conclude that we have not more than $n = \frac{1}{2}((2\ell - 1)\cdot\ell - 2\alpha'' + 1) + 2\alpha'' = \ell^2 - \left[\frac{\ell+1}{2}\right] + \alpha''$ points which may contain mass. For ℓ even α' may be zero, for ℓ odd $\alpha' \geqslant 2$, as has been pointed out in theorem 1.3.4. We have shown the following

THEOREM 5.2.1:
ooooooooooooooo

If q is an orthogonal polynomial of degree ℓ with respect to I(f), q irreducible in $\mathbb{R}$ and $I(f) \in \partial\left[P^2_{2\ell-1}(D) \bmod q\right]^{\bullet}$, I(f) can be written as weighted sum of point functionals involving at most $n = \ell^2 - \left[\frac{\ell + 1}{2}\right] + \alpha''$ points on q = 0 in D.

If q is reducible in $\mathbb{R}$ e.g. $q = q_1 \cdot q_2$, q_1 and q_2 in $\mathbb{R}$ irreducible, $\emptyset$ may contain q_1^2 and/or q_2^2 as factor. In this case a more detailed analysis has to be made considering separately the contributions of $\left[P^2_{2\ell-1}(D) \bmod q_1\right]^{\bullet}$ and of $\left[P^2_{2\ell-1}(D) \bmod q_2\right]^{\bullet}$. It can be shown that the result on the number n of points equally holds in this case.

5.3 Elements from Int$\left[P^2_{2\ell-1}(D) \bmod q\right]^{\oplus}$

If $I(f) \in \mathrm{Int}\left[P^2_{2\ell-1}(D) \bmod q\right]^{\oplus}$, theorem 1.2.3 is used instead of theorem 1.2.2 to get a similar result as theorem 5.2.1. We remind ourselves that theorem 1.2.3 states the following: there are subsets $D' \subset D$ such that $I(f) \in \partial\left[P^2_{2\ell-1}(D') \bmod q\right]^{\oplus}$. Using the same arguments as in 5.2, a minor modification has to be made if $\alpha' = 0$ due the fact the curve $q = 0$ is 'cut off' and two simple common zeros of $\emptyset$ and q are introduced. For ℓ odd, the situation remains unchanged. The general result with arbitrary $\alpha' = 2 \cdot \alpha''$ is

THEOREM 5.3.1:
oooooooooooooo

For $I(f) \in \mathrm{Int}\left[P^2_{2\ell-1}(D) \bmod q\right]^{\oplus}$, there is a representation of $I(f)$ with points on $q = 0$ in D with at most $n = n(\ell, \alpha'')$ points where

$$n(\ell, \alpha'') = \begin{cases} \ell^2 - \frac{\ell + 1}{2} + \alpha'' & \text{for } \ell \text{ odd}, = \ell^2 - \frac{\ell - 1}{2} \text{ for } \alpha'' = 1. \\ \ell^2 - \frac{\ell}{2} + \alpha'' & \text{for } \ell \text{ even}, \ \alpha'' > 0, \\ \ell^2 - \frac{\ell}{2} + 1 & \text{for } \ell \text{ even}, \ \alpha'' = 0. \end{cases}$$

Inserting for α'' the minimal possible values we find for moderate ℓ

ℓ	2	3	4	5	6	7
n	4	8	15	23	34	46

5.4 A second approach

Upper bounds on the number of special q.f.'s have been given in 5.3. This results together with their proof are nonconstructive. For this reason we give a second derivation of the same statements, following the original ideas of KREIN in /27/, which selects special q.f.'s, the parameters of which satisfy special conditions including the moment conditions. This is accomplished by the fact that each formula is the solution of an extremum problem. By the (necessary) conditions for such an extremum, a q.f. is uniquely determined. These conditions are treated in more detail in section 7.

We begin again by assuming an orthogonal polynomial q of degree ℓ is given. We then select a second arbitrary real polynomial p of degree ℓ, which is to have at least one zero at infinity not together with q. Let b = b(p,q) be the number of common zeros of p and q at infinity; b is not smaller than the number c of common zeros of the companion polynomials of p and q. Assume b = c, then for ℓ odd, we have $0 \leqslant b \leqslant \ell - 1$, the same holds for ℓ even, excepted the case where a (q) = 0, here $0 \leqslant b \leqslant \ell - 2$, as in this case c must be $\leqslant \ell - 2$, otherwise p could not be real (!).

Let I(f) be from $\left[P^2_{2\ell-1}(D) \bmod q\right]^{\bullet}$ with q irreducible in $\mathbb{R}$. Then let us calculate

$$\operatorname{Min} \iint\limits_{D} \emptyset^2 dG$$

subject to the constraints on G

1. $G(\Delta) = 0$ for $\Delta \in \sigma(D)$, $\Delta \subseteq \{D - (q=0)\}$,
2. $G \geqslant 0$,
3. $\iint\limits_{D} x^i y^j dG = I(x^i y^j)$ for $i,j \geqslant 0$, $i + j \leqslant 2 \cdot \ell - 1$ and modulo q.

From theorem 1.1.2 follows that the set of possible G is not empty, from theorems of Helly type is deduced that this problem has a solution. This solution $G \in \mathrm{rba}\,(D)$ may be considered as element $\vec{c}$ of the conjugate L_3^* to L_3 defined as $L_3 = \mathrm{lin}\,(P^2_{2\ell-1}(D),\ p^2)$. $\vec{c}$ with coordinates $(c_{oo}, c_{1o}, c_{o1} \ldots\ldots, c_{o,2\ell-1}, c_*)$ with $c_{ij} = I(x^i y^j)$ for $i,j \geq 0$, $i + j \leq 2\cdot\ell - 1$, and $c_* = \iint p^2 dG$, is lying on $\partial L_3^{\oplus}$, as its coordinate c_* cannot be decreased without removing $\vec{c}$ from $L_3^{\oplus}$.

As consequence of theorem 1.2.2 there is a supporting polynomial $\emptyset \in L_3$, $\emptyset \geq 0$ in $D' = \{(x,y) \in D \,/\, q(x,y) = 0\}$, $\iint \emptyset\, dG = 0$ and G has mass only where q and $\emptyset$ simultaneously vanish in D.

The maximum possible number of points where G may have mass is estimated using theorem 1.4.2.

THEOREM 5.4.1:

oooooooooooooo

Assume 1. q is a real, orthogonal polynomial of degree ℓ,
2. $I(f) \in [P^2_{2\ell-1}(D) \bmod q]^{\oplus}$ irreducible in $\mathbb{R}$.
3. p is an arbitrary real polynomial of degree ℓ having at least one zero at infinity not together with q.
4. q has $a' = 2 \cdot a''$ common points with ∂D.
5. $0 \leq b = b(p,q)$.

Then there is a q.f. of degree $2 \cdot \ell - 1$ with at most $n(\ell)$ nodes situated on q = 0 in D,

$$n(1) = \ell^2 + a'' - \frac{b}{2} \text{ for b even,}$$

$$n(1) = \ell^2 + a'' - \frac{b+1}{2} \text{ for b odd.}$$

Minimizing with respect to a" and b yields the following theorem 5.4.2. We remind the fact that for ℓ even q may be chosen as to have no real zeros with the line at infinity. If D is replaced by a sufficiently large $D_1 \supseteq D$, we have a" = 0. In this case, b can be chosen to be $\ell - 2$.

For odd ℓ, a" = 1 and b = $\ell - 1$ is possible.

THEOREM 5.4.2:

ooooooooooooooo

For $I(f) \in \text{Int} \left[P^2_{2\ell-1}(D) \bmod q\right]^{\oplus}$, there is a representation of I(f) with points on q = 0 in D with at most n points,

$$n = \ell^2 - \frac{\ell}{2} + 1 \qquad \text{for } \ell \text{ even}$$

$$n = \ell^2 - \frac{\ell-1}{2} + 1 \qquad \text{for } \ell \text{ odd.}$$

Comparing theorem 5.4.2 with theorem 5.3.1 with minimal a", there is found a slight difference for ℓ odd; we have n(theorem 5.4.2) = n(th. 5.3.1) + 1. This inconsistency can be removed by showing that the mass points of the solutuons G_{min} and G_{max} for the problems min $\iint p^2 dG$ and max $\iint p^2 dG$ strictly interlace. By this can be demonstrated that the solution G_{min} contains no boundary points (where q = 0 intersects ∂D_1). This point shall not be outlined here.

5.5 Points not $\in \left[P^2_{2\ell-1}(D) \bmod q\right]^{\oplus}$

If the assumption $I(f) \in \left[P^2_{2\ell-1}(D) \bmod q\right]^{\oplus}$ is not fulfilled, the statements of 5.4 contain in several cases one additional point. This has been briefly touched at the end of section 1.1 and is now discussed in more detail. In this case to have a well posed problem, we search for

$$\mathrm{Min} \iint\limits_{D} p^2 \,/dG/ \;,$$

subject to the previously given constraints imposed on G excepted the condition $G \geqslant 0$. Here we find that the solution G of this problem may be such that $/G/ \in \partial L_3^{\oplus}$ or from Int $L_3^{\oplus}$. If $/G/ \in \partial L_3^{\oplus}$, the theorems of 5.4 also hold. For $/G/ \in \mathrm{Int}\, L_3^{\oplus}$, following well known ideas, see e.g. KARLIN a. STUDDEN /26/ /G/ may be written as positive weighted sum of two elements of $\partial L_3^{\oplus}$ as

$$/G/ = \lambda \cdot f(X_1) \;+\; (1-\lambda)\cdot l(f) \;, \quad 0 < \lambda < 1,$$

where $f(X_1)$ is an arbitrary chosen point functional with $X_1 \in \{q = 0 \wedge \partial D\}$ and $l(f)$ is the element of $\partial L_3^{\oplus}$ being the intersection point of $\partial L_3^{\oplus}$ and the straight line joining $f(X_1)$ and /G/ in L_3^{*}. As G has the same mass points as /G/, we arrive at the main result of this section

THEOREM 5.5.1:
oooooooooooooo

To each integral $I(f) \in \left[P^2_{2\ell-1}(D)\right]^{\oplus}$ there is a q.f. of degree $2\cdot\ell - 1$ with at maximum $m = \ell^2 - \frac{\ell}{2} + 1$ real points for ℓ even, with at most $n = \ell^2 - \frac{\ell+1}{2} + 1$ for ℓ odd.

Such formulas exist for each orthogonal polynomial q of degree ℓ with minimal $a'(q)$, the nodes of the formula being situated on $q = 0$.

6. Quadrature formulas with real nodes and positive weights

Results concerning the existence of q.f.'s with nodes and positive weights are due to TCHAKALOFF /53/ and GÜNTHER /15/. The first cited paper is considering a linear topological space of continuous functions L(T) which are given on a compact topological space T. It is shown (theorem 1.1.2) that each element $\in L^{\oplus}$ can be written as weighted sum of at most n point functionals $f(X_j)$, $j = 1,2,\ldots\ldots, n$ with positive weights and points $X_j \in T$ where $n = \dim L(T)$.

Assuming the elements of L(T) to be polynomials of degree $\leq 2\ell-1$ and T to be an ellipse D, in /15/ is demonstrated that there are formulas containing at most $2\cdot\ell^2 - 3\cdot\ell + 2$ points from D and positive weights (Tchakaloffs theorem: $2\cdot\ell^2 + \ell$ points). The method used in /15/ can be replaced by another one involving also an extreme value formulation.

Let $I(f) \in \left[P^2_{2\ell-1}(D)\right]^{\oplus}$ and $\emptyset$ a real polynomial of degree ℓ. Then we search for the minimum of

$$\iint\limits_{D} \emptyset^2 dG$$

among all nonnegative $G \in rba(D)$ with $\iint_D x^i y^j dG = I(x^i y^j)$ for $i,j \geq 0$, $i + j \leq 2\cdot\ell - 1$. The solution of this problem is connected with the existence of a supporting polynomial (see theorem 1.2.2) $\psi = \psi_{min} = \emptyset^2 + g_{2\ell-1}$ with $g_{2\ell-1}$ of degree $\leq 2\cdot\ell - 1$. The set of points $\{(x,y) \in D / \psi(x,y) = 0\}$ may be of dimension zero or one. If this set or a subset of this set is of dimension one the mass is once more deduced to be in isolated points of D, see /15/.

REMARKS:

1) For spaces $P^2_{2\ell}(D)$ with even maximal degree $2\cdot\ell$, $\emptyset$ must be chosen to be of degree $\ell + 1$ to get similar results.
2) A similar results also holds for the corresponding maximum problem. In this case $\psi = \psi_{max} = -\emptyset^2 + g_{2\ell-1}$; this generalizes the theorem in one dimension, for the interval $[0,1]$,

which states

$$\psi = \psi_{\min} = \prod_{j=1} (x - x_j)^2 = x^{2\ell} + \ldots\ldots ,$$

$$\psi = \psi_{\max} = x \cdot (1 - x) \prod_{k=1}^{\ell-1} (x - x_k)^2 = - x^{2\ell} + \ldots ,$$

see KARLIN and STUDDEN/26/, p.111.

The following problem to date not has been investigated: Is it possible to get better upper bounds for the maximum number of points in q.f.'s of a certain degree with real nodes by allowing to have points with negative weights?

7. The calculation of quadrature formulas

7.1 Interpolation on q = 0 and moment equations on q = 0 showing the purely algebraic point of view

Theorem 5.1.1 admits an interpretation of the following type: Q.f.'s may be constructed as reduced interpolating q.f.'s, if suitably chosen N points ($N = \dim L$, $L = P^2_{2\ell-1}(D) \bmod q$) on $q = 0$ or generally spoken N linearly independent elements from L are selected as nodes of an interpolating polynomial from L. Integrating this interpolating polynomial yields a result, weaker than theorem 5.2.4.

THEOREM 7.1.1:

oooooooooooooo

To each orthogonal polynomial q of degree ℓ there is a reduced interpolating q.f. of degree $2\cdot\ell - 1$ with at most $N = \ell\,(\frac{3\cdot\ell\ -1}{2})$ (real) points on $q = 0$.

To reduce significantly the number n of nodes for q.f.'s of given degree $2\cdot\ell - 1$, it could be tried to equate the number of conditions to be fulfielled and the number of free parameters for L. If this could be achieved, this seems not to be possible, $n = \frac{\ell}{2}\,\frac{(3\cdot\ell\ + 1)}{2}$ points points were necessary, $\approx \frac{3}{4}\,\ell^2$ points.

7.2 Algebraic conditions for the solution of the extremum problem

We assume the following:

a) $I \in \left[P^{2}_{2l-1}(D) \bmod q \right]^{\oplus}$

b) q irreducible and $a(q) = 0$, this means q has no real zeros at infinity, this includes ℓ is even.

Under these assumptions, the solution $\hat{G}$ of the minimum problem

$$\min \iint_{D} p(x,y)\,dG$$

with $p(x,y)$ of degree $2\cdot\ell$ and q not a factor of p ($p = 0$ real not of dimension zero !), consists of discrete points $X_i = (x_i, y_i)$ containing mass A_i, that means

$$\hat{G} \equiv S(f) = \sum_{i=1}^{n} A_i f(X_i). \qquad (7.2.1)$$

The constraints imposed on all admissable G are

$$\sum_{i=1}^{n} A_i x_i^k y_i^m = I(x^k y^m) \text{modulo } q \text{ for } k,m \geq 0,\; k + m \leq 2\cdot\ell - 1 \qquad (7.2.2)$$

and

$$q(x_i, y_i) = 0 \quad \text{for } i = 1,2,\ldots\ldots, n \qquad (7.2.3)$$

modulo q in (7.2.2) means: there are no indices (k, m) and (k', m') in this set, such that $x^k y^m - x^{k'} y^{m'} = \alpha\cdot q$ with polynomic α.

As necessary conditions for the minimum problem with constraints (7.2.2) and (7.2.3) we find

$$p(x_i, y_i) + \sum_{k,m} t_{k,m} x_i^k y_i^m = 0 \text{ for } i = 1,2,\ldots\ldots,n \qquad (7.2.4)$$

$$A_i \cdot \left[\frac{\partial p}{\partial x} (x_i , y_i) + \sum_{k,m} k \cdot t_{k,m} x_i^{k-1} y_i^m \right] + s_i \frac{\partial q(x_i , y_i)}{\partial x} = 0 \tag{7.2.5}$$

$$A_i \cdot \left[\frac{\partial p}{\partial y} (x_i y_i) + \sum_{k,m} m \cdot t_{k,m} x_i^k y_i^{m-1} \right] + s_i \frac{\partial q(x_i , y_i)}{\partial y} = 0 \tag{7.2.6}$$

i varies from 1 to n in (7.2.5) and (7.2.6).

The sums in (7.2.4) - (7.2.6) over k and m contain all terms entering in (7.2.2); $\ell \cdot (2 \cdot \ell + 1) - \ell \cdot (\ell + 1)/2 = \frac{1}{2} \cdot \ell \cdot (3 \cdot \ell + 1)$ terms. There are $w = 4 \cdot n + \frac{1}{2} \cdot \ell \cdot (3 \cdot \ell + 1)$ equations for the same number of unknowns, the A_i, x_i, y_i, $t_{k,m}$ and s_i.

Some remarks are necessary:

1. The variables $t_{k,m}$ and s_i are the Lagrangian multipliers of the necessay conditions for the solution of the extremum problem.
2. Assume that n has any value. Generally, the number of original moment equations ($= \ell \cdot (2 \cdot \ell + 1)$) is not consistent with the number $3 \cdot n$ of available parameters. The conditions (7.2.2)-(7.2.6) represent a system of $w (\approx 5{,}5 \cdot n)$ equations for w unknowns. We add a table containing representative values

ℓ	$2 \cdot \ell - 1$	$(\ell^2 - \ell + 2) = n$	$4n$	$\frac{1}{2}(3 \cdot \ell + 1)$	w
2	3	4	16	7	23
4	7	14	56	26	82
6	11	32	128	57	185
8	15	58	232	1oo	332
2o	39	382	1528	61o	1757

3. It has been pointed out that the solution of the minimum problem is of the form (7.2.1). A numerical solution of such a system seems only to be possible if n is known; this is a partial justification of the results given in section 5.

Equations (7.2.4) - (7.2.6) may be slightly modified, setting

$$\emptyset(x,y) = P(x,y) + \sum_{k,m} t_{k,m} x^k y^m,$$

as

$$\emptyset(x_i,y_i) = 0 \qquad (7.2.4')$$

$$A_i \frac{\partial\emptyset}{\partial x}(x_i,y_i) + s_i \frac{\partial q}{\partial x}(x_i,y_i) = 0 \qquad (7.2.5')$$

$$A_i \frac{\partial\emptyset}{\partial y}(x_i,y_i) + s_i \frac{\partial q}{\partial y}(x_i,y_i) = 0 \qquad (7.2.6')$$

These equations are exhibiting the fact that the curves $q = 0$ and $\emptyset = 0$ have a common tangent (for $s_i \neq 0$) in (x_i,y_i) or another at least double common zero in this point (for $s_i = 0$). $\emptyset$ is immediately identified to be the supporting polynomial equally named $\emptyset$ in sections 1.2 and 5.2 and points out the dual way in which the relations (7.2.4) - (7.2.6) might have been developped. At this point the connection between optimization problems, the necessary conditions of which are (7.2.4) - (7.2.6), and the separation theorem (theorem 1.2.2) can immediately be seen. This connection plays on important role in the proof of conditions for the existence of extrema in general optimization problems.

If in this section the assumption $a(q) = 0$ is dropped, we must include intersection points $Z_j = (\tilde{x}_j,\tilde{y}_j)$ of ∂D and $q = 0$ with fixed coordinates. By each point ot this kind, the number of conditions in (7.2.4) is augmented by one.

7.3 Algebraic equations for the extremum problem with free nodes

This section is concerned with q.f.'s which are the solutions of the e.g. minimization problem

$$\text{Min} \iint_D p(x,y)dG,$$

with $p(x,y)$ of degree $= 2\cdot\ell$, $\iint_D x^i y^i dG = I(x^i y^i)$ for $i,j \geq 0$, $i + j \leq 2\cdot\ell - 1$, $G \geq 0$. Here the mass of the solution $\hat{G}$ must mot be contained in discrete points. This may be achieved by a suitable choice of p; if we have a positive definite companion polynomial of p, $\hat{G}$ has mass only in discrete points. We first have as moment conditions

$$\sum_{i=1}^{n} A_i x_i^k y_i^m = I(x^k y^m) \text{ for } k + m \leq 2\cdot\ell - 1, \quad k,m \geq 0 \qquad (7.3.1)$$

and as Lagrangian conditions for the minimum problem with constraints

$$\phi(x_i,y_i) = p(x_i,y_i) + \sum_{k,m} t_{k,m} x_i^k y_i^m = 0 \qquad (7.3.2)$$

for $i = 1,2,\ldots,n$

$$\frac{\partial\phi}{\partial x}(x_i,y_i) = \frac{\partial p}{\partial x}(x_i,y_i) + \sum_{k,m} t_{k,m}\cdot k\cdot x_i^{k-1} y_i^m = 0 \qquad (7.3.3)$$

$$\frac{\partial\phi}{\partial y}(x_i,y_i) = \frac{\partial p}{\partial y}(x_i,y_i) + \sum_{k,m} t_{k,m}\cdot m\cdot x_i^k y_i^{m-1} = 0 \qquad (7.3.4)$$

for $i = 1,2,\ldots,n$

These are $w = 3\cdot n + \ell\cdot(2\cdot\ell + 1)$ equations for the same number of unknowns. As $n \leq 2\cdot\ell^2 - 3\cdot\ell + 2$, we have $w \leq 2\cdot\ell^2 + \ell + 6\cdot\ell^2 - 9\cdot\ell + 6 = 8\cdot\ell^2 - 8\cdot\ell + 6$.

7.4 A numerical procedure

This section gives an idea how to calculate a solution of the systems of equations given by (7.2.2) - (7.2.6) or by (7.3.1) - (7.3.4). For simplicity we restrict ourselves to the second set of equations. The fundamental idea of the proposed procedure has been introduced for onedimensional problems by GUSTAFSON /21/.

We assume that p is definite - then only discrete points enter as solutions - and that n, the number of points, is known. The equations to be solved are

$$\sum_{i=1}^{n} A_i \, x_i^k \, y_i^m \quad - \quad I_{k,m} \quad = 0 \quad \begin{matrix} k,m \geq 0 \\ k+m \leq 2\cdot\ell -1 \end{matrix} \qquad (7.4.1)$$

$$\phi(x_i,y_i) = p(x_i,y_i) + \sum_{k,m} t_{k,m} x_i^k y_i^m = 0, \qquad (7.4.2)$$

$$i = 1,\ldots,n$$

$$\frac{\partial \phi}{\partial x}(x_i,y_i) = \frac{\partial p}{\partial x}(x_i,y_i) + \sum_{k,m} t_{k,m} K x_i^{k-1} y_i^m = 0, \qquad (7.4.3)$$

$$i = 1,\ldots,n$$

$$\frac{\partial \phi}{\partial y}(x_i,y_i) = \frac{\partial p}{\partial y}(x_i,y_i) + \sum_{k,m} t_{k,m} m x_i^k y_i^{m-1} = 0, \qquad (7.4.4)$$

$$i = 1,\ldots,n$$

We introduce a real parameter λ, $0 \leq \lambda \leq 1$, and let x_i, y_i, A_i and $t_{k,m}$ depend on λ . For $\lambda = 1$, $x_i(1)$,..... are the solution of (7.4.1) - (7.4.4). The values of x_i (0), y_i (0), A_i (0) and $t_{k,m}$ (0) are some reasonable estimations of the corresponding values for $\lambda = 1$. This may be achieved in the following manner: A 2ℓ - th degree polynomial $\Psi(x,y)$, $\Psi(x,y) \geq 0$ in D, $\Psi \not\equiv 0$ is introduced which has n zeros $(x_i(0), y_i(0))$, i = 1,2,....,n in D. By the coefficients of $\Psi(x,y)$, the $t_{k,m}$ (0) are defined. The $A_i(0)$

are either chosen arbitrary positive or are to satisfy n moment conditions. By this, $I_{k,m}(0)$ is defined.

The one-parameter family of (nonlinear) problems with solution $\mathfrak{u}(\lambda) = \left\{ x_i(\lambda), y_i(\lambda), A_i(\lambda), t_{k,m}(\lambda) \right\}$ by (7.4.1) - (7.4.4) with

$$I_{k,m}(\lambda) = \lambda \cdot I_{k,m} + (1-\lambda) \cdot I_{k,m}(0)$$

instead of $I_{k,m}$ in (7.4.1).

Differentiating of (7.4.1) - (7.4.4) with respect to λ gives

$$\sum_{i=1}^{n} \left\{ \frac{dA_i}{d\lambda} x_i^k x_i^m + A_i \frac{dx_i}{d\lambda} k x_i^{k-1} y_i^m + A_i x_i^k m y_i^{m-1} \cdot \frac{dy_i}{d\lambda} \right\} = I_{k,m} - I_{k,m}(0)$$

and

$$\frac{\partial f}{\partial x_i} \cdot \frac{dx_i}{d\lambda} + \frac{\partial f}{\partial y_i} \cdot \frac{dy_i}{d\lambda} + \frac{\partial f}{\partial t_{k,m}} \cdot \frac{dt_{k,m}}{d\lambda} = 0 \tag{7.4.5}$$

for $f = \emptyset$, $\frac{\partial \emptyset}{\partial x}$ and $\frac{\partial \emptyset}{\partial y}$, and for $i = 1,2,\ldots,n$.

Our solution $\mathfrak{u}(1)$ is obtained by solving the initial value problem given by (7.4.5) and $\mathfrak{u}(0)$ as initial value, (7.4.5) is of implicit form

$$\mathfrak{M} \cdot \frac{d\mathfrak{u}}{d\lambda} = \mathfrak{r} \quad , \mathfrak{u}(0) \text{ given.} \tag{7.4.6}$$

$\mathfrak{r}, \mathfrak{u}(\lambda)$ are vectors of dimension $d = 3n + x$, $\mathfrak{M}$ is a $d \ast d$-matrix, x is the number of admissable tupels (k,m) in (7.4.1).

(7.4.6) may be solved using the Euler-Cauchy-method.

For each integration step, the matrix $\mathfrak{M}$, depending on all dependent and independent variables, must be inverted. The procedure has an additional characteristic feature. One may subdivide the interval $0 \leq \lambda \leq 1$, in equally or unequally spaced subintervals I_j ; $\lambda_{j-1} \leq \lambda \leq \lambda_j$, $j = 1,\ldots\ldots,r$, $\lambda_0 = 0$, $\lambda_r = 1$, and after integrating (7.4.6) from λ_{j-1} to λ_j, the nonlinear system (7.4.1) - (7.4.4) may be solved iteratively by NEWTON-RHAPSON- iteration for $\lambda = \lambda_j$, with the result $\mathfrak{u}(\lambda)$ of the preceding integration step as starting value.-

For some simple examples, this method has proved successful.

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Appendix

(MANUSKRIPT)

Quadrature formulas with real points

by

Claus Günther

The problem of constructing quadrature formulas (= QFs) for multidimensional problems which are exact for polynomials up to a certain degree has initiated a number of investigations.

1. People have generalized one-dimensional algebraic methods to get analoguous formulas of Gaussian type for multidimensional problems; an example is the theorem of MYSOVSKIKH |8| and STROUD |11|, published in 1969.

2. People have searched for the existence of certain types of formulas, e. g. self-contained QFs or QFs whose weights are positive (TCHAKALOFF |13|, FRITSCH |1|, GÜNTHER |2|).

There have been few contributions to the problem of finding QFs

$$S(f) = \sum_{i=1}^{n} A_i f(X_i)$$

with arbitrary (real) weights, but which must have real points X_i.
Let us recall what is known on this subject. We restrict ourselves to two dimensions. There is a well known result of STROUD |10|, a little bit modified:

THEOREM 1:

If S(f) is of degree N with real nodes X_i, we have at least $n(N) = \frac{1}{2} \cdot ([\frac{N}{2}] + 1)([\frac{N}{2}] + 2)$ weights A_i positive.

For N = 3, MYSOVSKIKH |8| showed in 1969 that there is always a QF of third degree with four real nodes.
Let us return to the above cited result of MYSOVSKIKH and STROUD which can be stated as follows:

THEOREM 2:

Assume two orthogonal polynomials P1 and P2 of degree 1 have exactly 1^2 distinct common zeros X_i none of which is at infinity. Then we can construct a 1^2-point-QF of degree 21 - 1 with the X_i as points.

A result of myself |3| specifies all pairs of orthogonal polynomials of second degree for a given integral which have four real common zeros and the weights of the corresponding QF suiting Theorem 2 are positive. I add for completeness, that 1 = 1, N = 1, is trivial.
This talk will contribute to the existence od QFs with real nodes. The method has partially been used by other authors (KREIN|6|, GÜNTHER |2|).
We analyze in the following the case 1 = 3. Let D be a circle and

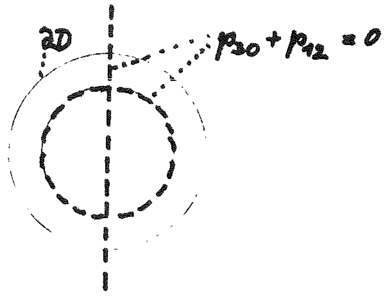
$$I(f) = \iint_D f(x,y)dG, \quad D: x^2 + y^2 = 1, \ x,y \text{ real},$$

with G an arbitrary nonnegative (, regular and bounded) set function on D. We remember the following fact:
1. Each factor of an orthogonal polynomial vanishes also in Int D, STROUD |12|.
We make the following definition:
2. Let $P_N^2(D)$ be the linear topological space of polynomials in v variables of degree $\leqslant$ N with range D. Then we have, if P1 is a real orthogonal polynomial of third degree and Q1 and Q2 of degrees $\leqslant$ 5 satisfy

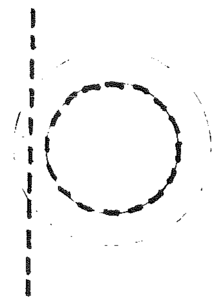
$$Q1 - Q2 = a \cdot P1, \text{ degree } a \leqslant 2,$$

and by this I(Q1) = I(Q2), that means $I \in [P_5^2(D) \bmod P1]^*$, not $I \in [\ldots\ldots]^{\circledast}$. Therefore I can be represented by Stieltjes-integrals with a set function which has mass only on $(P1 = 0) \cap D$.
3. We select a P1 of third degree, P1 orthogonal with respect to I, which has at most 4 common zeros with ∂D. As follows from Bézout's Theorem, WALKER |14|, P1 and ∂D have six common zeros , if $x^2 + y^2 - 1$ is not a factor of P1. If we set $P1 = p_{30} + p_{12} = x \cdot (x^2 + y^2)$ + terms of order $\leqslant 2$, using the basic orthogonal polynomials $p_{i,3-i}$ of third degree, we have at most four real common zeros of P1 and ∂D, at least two common zeros of P1 and ∂D are at "at infinity".
Examples:

a)



b)



4. If we define for simplicity L to be $P_5^2(D)$ mod P1, we now are concerned with the problem to find representations of elements of L^* resp. of $L^\oplus$, the cone of nonnegative linear functionals on L.
If P1 is reducible, P1 = Q1•Q2, theorems on the form of the contributions from $P_5^2(D)$ mod Q_i could be used. We know that $P_5^2(D)$ mod Q_i, if degree Q_i is $\leqslant 2$, has a Tchebycheff system of functions as basis. These theorems may be found in the book of KARLIN and STUDDEN |5|. We remember that

$$P_N^2(D) \bmod Q_i = P_N^1(T), \text{ if } Q_i \text{ is linear,}$$

with T the intervall of $Q_i = 0$, lying in D and

$$P_N^2(D) \bmod Q_i = (1, \sin x, \cos x, \sin 2x, \ldots\ldots\ldots, \cos Nx),$$

if Q_i is an ellipse.
We do not make use of this fact and proceed in another way. For that purpose we generalize a method described in detail in KARLIN a. STUDDEN, which has originally been introduced by M.G.KREIN |6|.
Among all set functions resp. mass distributions $\hat{G}$ on P1 = 0 in D which satisfy

$$\iint_D x^i y^j \, d\hat{G} = I(x^i y^j), \quad i,j \geqslant 0, \quad i + j \leqslant 5,$$

we search for a special mass distribution (named G), which minimizes

$$\iint_D \{(x^2 + y^2)\, y\}^2 \, d|G| \; .$$

Also here we have selected the integrand $\phi = ((x^2 + y^2)y)^2$ in such a manner to have many common zeros with P1 at infinity.
We deal first with the case $I \in L^\oplus$.
If we introduce L1 to be $\mathrm{lin}(L, \phi)$, the solution G of our minimum problem can be regarded as element of $\partial L1^\oplus$, the boundary of the nonnegative cone $L1^\oplus$, that means that there is a $\psi \in L1$ with

$$\iint \psi \, dG = 0, \quad \psi \geqslant 0 \text{ on } ((P1 = 0) \wedge D).$$

This fact holds independently from the other one that representations of I may exist with v points, v > 6 with v - 6 points with negative weights.
The consequence of this is that G has mass only in points where $\psi = 0$ in D. For this purpose we must investigate where ψ can have value zero on P1 = 0 in D. From Bézout's theorem

from Algebraic Geometry again we conclude that P1 and Ψ have exactly degree P1·degree Ψ = 18 common zeros provided these polynomials have no common factor. Except the maximal four points where both P1 and Q_∂ ($Q_\partial = 0$ equation of ∂D) vanish, P1 and Ψ can have only (at least) double common zeros. We conclude that among the 14 finite common zeros of P1 and Ψ, we have at most four simple common zeros. By this there are at most nine points which may contain mass. These points can be identified to be the points of a QF. of degree 5 for I(f).

If $I \notin L^{\oplus}$, that means $I \in L^{*}$, the minimum solution of our problem is not $\in \partial L1^{\oplus}$ but $|G| \in \mathrm{Int}\ L1^{\oplus}$. To obtain an analoguous representation of G, we draw in $L1^{*}$ a straight line S through one of the point functionals $f(X_i)$, where X_i is a simple common zero of P1 and Q_∂ (in the terminology of KARLIN and STUDDEN: points X_i of index 1/2), and through $|G|$; the second intersection point of S with $\partial L1^{\oplus}$ is a point Y in $L1^{*}$ of index $\leqslant 7$. Y may be written as

$$Y = \sum_{i=1}^{9} A_i f(Y_i),$$

with point functionals $f(Y_i)$, where $Y_i \in (P1 = 0)$. Either this representation contains 9 points Y_i, where one is Y or in all other cases there are only 8 points in this formula. By this also $|G|$ and G can be assumed to be representable in this form.

If we take a sufficiently large circular region D', containing D (, D arbitrarily compact), instead of D, P1 and $\partial D'$ have only two real common zeros. Now repeating the same arguments, we see that at most two points with index 1/2 may occur. In this case $|G|$ and therefore G can be written involving at most 8 points.

Because there are always at least two l.i. polynomials in x and y of degree $\leqslant 3$ which are vanishing in the Y_i and which are orthogonal as is well known, we arrive at

THEOREM 3:

To each positive integral I(f) on a compact region D there is a QF. of degree 5 with at most 8 real points X_i. The X_i are among the common zeros of two orthogonal polynomials P1 and P2 of degree 3. One polynomial can be taken to be an orthogonal polynomial with the third order terms

$$P1 = (a \cdot x + b \cdot y)(x^2 + y^2) + \ldots .$$

where a and b are arbitrary real with $|a| + |b| > 0$.

There remains to ask how are corresponding more general results to be obtained with the same methods.

1. If I(f) is central-symmetric that means the mass of I(f)

is invariant with respect to translations around the center of symmetry. We then find that for l even, the orthogonal polynomial

$$P_l = (x^2 + y^2)^{\frac{l}{2}} + \ldots.$$

consists of $l/2$ circles and $1 \in [P^2_{2l-1}(D) \bmod P_l]^{\oplus}$. For l arbitrary even, it can be shown that there is always a QF. of degree $2l - 1$ with at most $l^2 - l + 2$ (real!) nodes on $P_l = 0$. We have for low degrees

l	4	6	8	10
$2l - 1$	7	11	15	19
n	14	32	58	92

An analoguous result holds for l odd; we always have for central symmetric region and weight function a QF. of degree $2l - 1$ with at most $l^2 - 2l + 4$ points.

l	1	3	5	7	9	11
$2l - 1$	1	5	9	13	17	21
n	1	7	19	39	67	103

The general case, arbitrary nonnegative weight function, large circular region as before, permits similar results:
As not always holds $1 \in [\ldots\ldots]^{\oplus}$, we generally need one point more as in the preceding considerations.
For l even, we find $n = l^2 - l + 3$, the same bound for the maximum number of nodes is found for l odd.
As to the construction, for l odd, we can use RADONs |9| procedure to find QFs of degree $2l - 1$, the points of which are the common zeros of at least three orthogonal polynomials of degree l. For l even, there are analoguous methods, which are described in detail in GÜNTHER |4| for $l = 4$.

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