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AN ANALYTICAL MODEL FOR COMPUTATION OF RELIABILITY OF WASTE MANAGEMENT FACILITIES WITH INTERMEDIATE STORAGES

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Abstract

A high reliability is called for waste management facilities within the fuel cycle of nuclear power stations which can be fulfilled by providing intermediate storage facilities and reserve capacities.

In this report a model based on the theory of Markov processes is described which allows computation of reliability characteristics of waste management facilities containing intermediate storage facilities.
The application of the model is demonstrated by an example.

Ein analytisches Modell zur Berechnung der Zuverlässigkeit von Entsorgungseinrichtungen mit Zwischenlagern

Kurzfassung

An Entsorgungseinrichtungen des nuklearen Brennstoffkreislaufs wird eine hohe Zuverlässigkeitsforderung gestellt, die durch die Einrichtung von Zwischenlagern und Reservekapazitäten erfüllt werden kann.

In dem Bericht wird ein auf der Theorie der Markovschen Prozesse basierendes Modell entwickelt und beschrieben. Mit dem Modell lassen sich Zuverlässigkeitskenngrößen von Entsorgungseinrichtungen mit Zwischenlagern berechnen. Die Anwendung des Modells wird anhand eines Beispiels gezeigt.

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1. Introduction

The basis of these considerations is the waste management for spent fuel elements from nuclear power stations, which consists of reprocessing and waste treatment.

Besides safety the most important design criterion in waste management is the reliability, which may be shown by the following numerical example: The planned German reprocessing facility for light water reactor fuel elements is capable of managing the output of about 50 GWe nuclear power stations /1/. The costs for the erection of such a reprocessing plant are approximately the same as the costs for the erection of 2 GWe nuclear power station /1/. This means that the failure of the reprocessing plant over a longer period of time may entail the shutdown of a plant park requiring a very large investment volume.

The requirements to the reliability of waste management can be fulfilled by the erection of intermediate storage facilities and by the provision of reserve capacities.

In this report an analytical model is presented which allows to compute the reliability characteristics of waste management facilities containing intermediate storage facilities and reserve capacities. These characteristics can then be used as a help when decisions are made on the design.

The model is based on the theory of Markov processes. The necessary input data are the distribution of failures of the processing units, the capacities of intermediate storage facilities, and the input and output rates. Subsequently, various reliability characteristics are computed such as the probability of an overflow of the storage facility, the average time until the first overflow of the storage facility, etc..

The practical design of redundant waste management systems with intermediate storage facilities relies on rules of thumb today as for example the worst case rule. Analytical models have so far been missing according to our knowledge.

So, in the reliability theory only few approaches exist to tackle the problem /2,3/, since it is a substantial assumption of this theory that the failure of a unit (e.g., an electronic unit) might directly entail a halt of the whole system while in systems provided with intermediate storage facilities part of the failures can be accommodated by the intermediate storage facilities and, in case that a unit is defective over a longer period, its failure results in the stopping of the whole system only after a certain time lag.

On account of the specific costs structure of waste management facilities, i.e., extremely high investment costs caused by high safety requirements (protection against plane crashes, earthquakes, sabotage) and the little capital bound in the goods to be processed (practically zero in case of waste), the models of inventory theory /4/ cannot be employed here.

A similarity with our problem is given under the dam theory /5/ which in turn is related to the queuing theory /6,7/. By these models the probability of an overflow and evacuation, respectively, of a dam is usually calculated for variable input flows and a constant rate of release, which is subsequently used in decision making on the design of dam capacities. However, the models can be transferred to waste management to a limit extent only since other restrictions are encountered such as varying release rates (cf. also Chapter 2).

Following a description of the basic system and its restrictions (Chapter 2) a model based on the theory of Markov processes will be presented which allows to calculate reliability characteristics of intermediate storage systems (Chapter 3). The application of the model to an example will be demonstrated in Chapter 4. Finally, numerical problems will be discussed.

2. Intermediate Storage and Processing Systems (ISP-Systems)

Fig. 1 shows an example of a general ISP-system. It consists of intermediate storage facilities and processing units which may be arranged in series or in parallel. The input into the system is the output to be handled from larger facilities such as the waste from the nuclear power stations or reprocessing plants. Following suitable treatment, this waste is brought into several final storage facilities. The connecting lines mark the flow of the material to be processed, which may consist of piece goods (e.g. fuel elements) or fluid goods (e.g. aqueous waste). The treatment can be either continuous or discontinuous. The basic structural elements of an ISP-system is represented in Fig. 2.

Part of them can be reduced to simpler elements. For instance intermediate storage facilities and processing units, respectively, arranged in series or in parallel as in Fig. 2c-f might be combined to form one intermediate storage facility and processing unit, respectively, if the capacities and failure distributions are suitably linked and if reliable connection lines and switching systems are assumed (cf. Chapter 3.3).

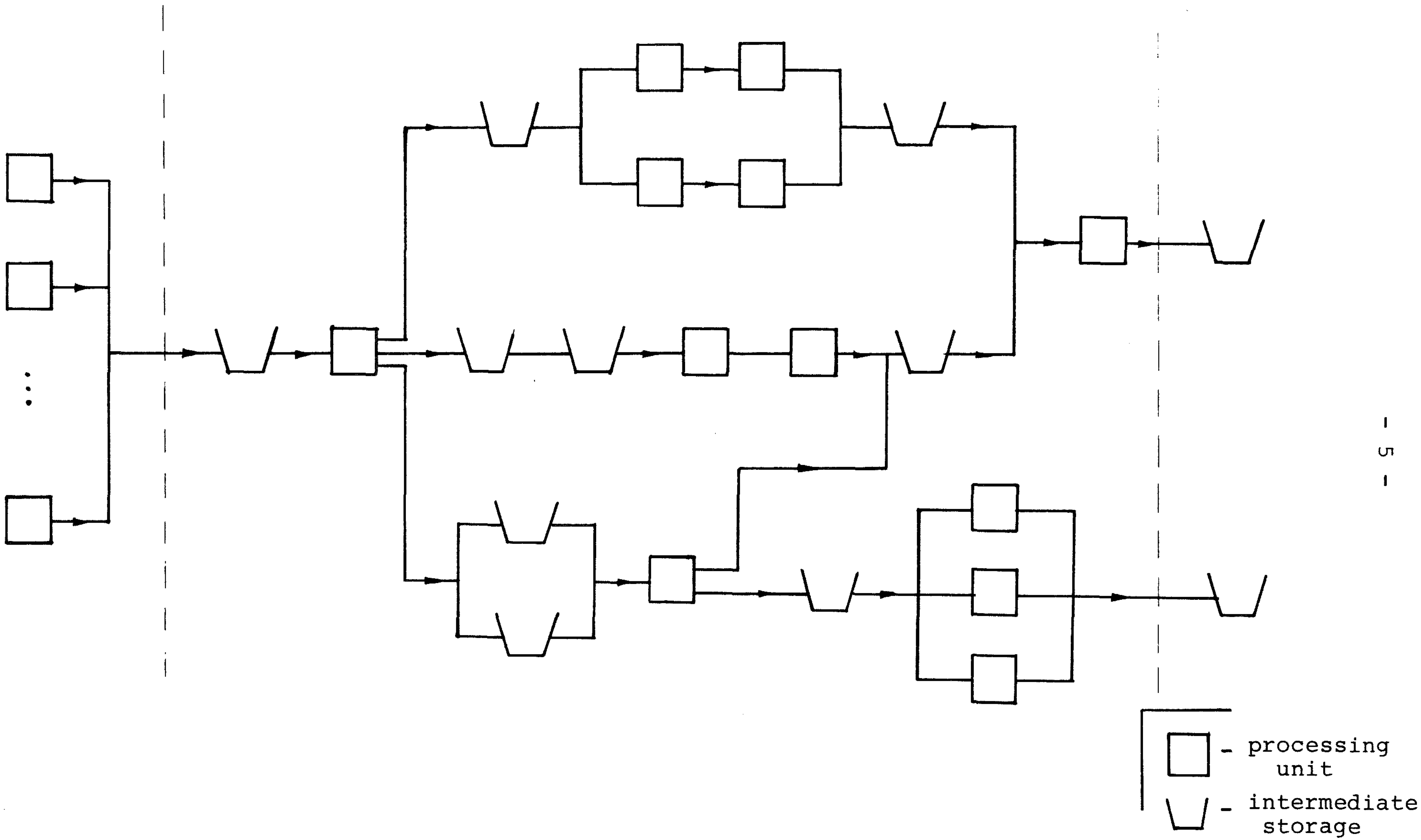


Fig. 1 Example of a waste management system with intermediate storage facilities.

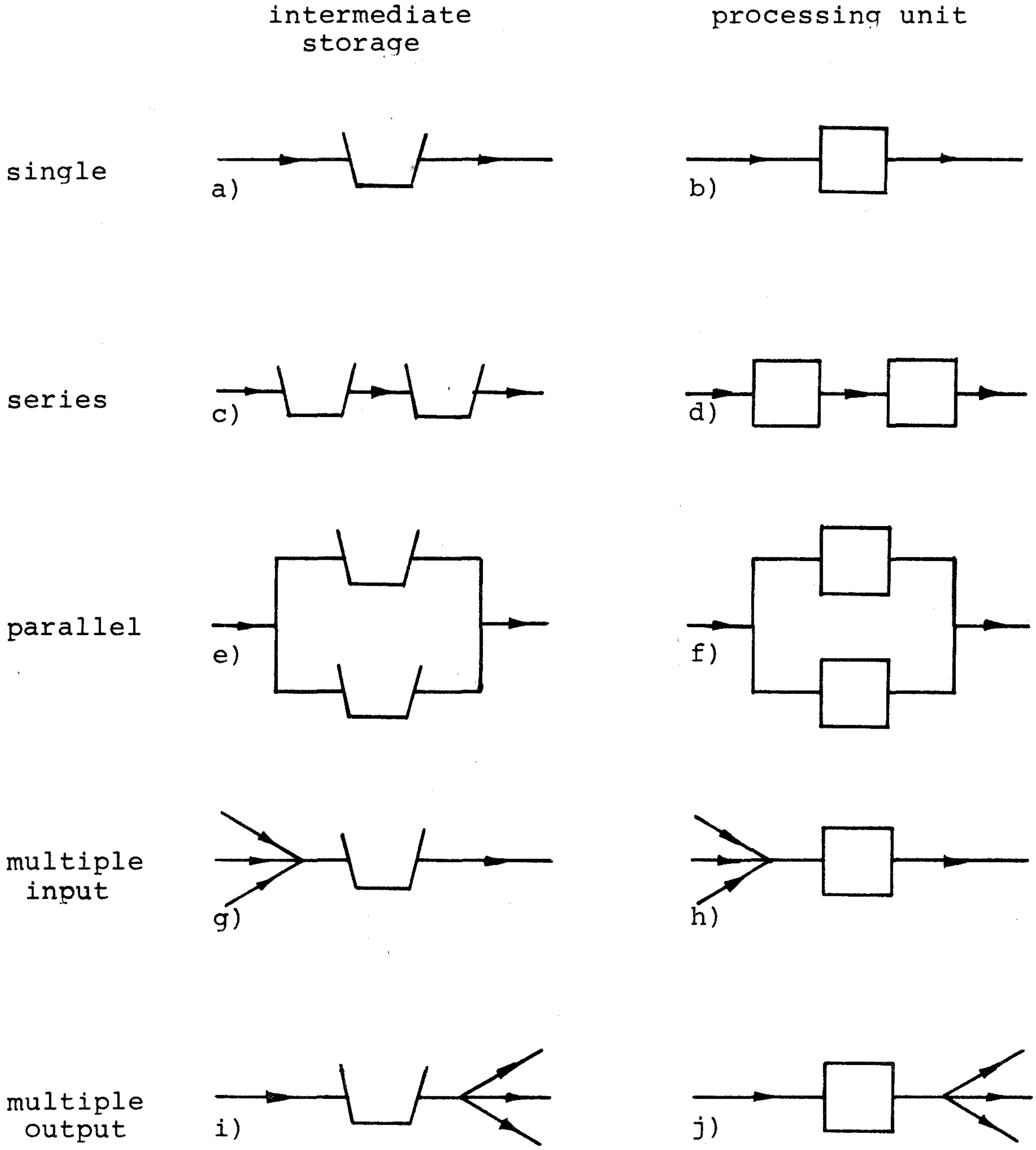


Fig. 2 Structural elements of an intermediate storage and processing system (ISP-system).

After such combination the structure represented in Fig.2h and i occurs very seldom in waste management, since in an intermediate storage facility only one type of material usually exists which is exclusively processed by a definite type of processing. If several materials are present in one intermediate storage facility, the storage capacity can generally be partitioned in waste management.

This means that, as a whole, a general intermediate storage system can be reduced to a system of basic components (cf. Fig.3). consisting of an intermediate storage facility and two processing units (cf. Fig. 4).

The first processing unit provides the material to be handled which is stored in the intermediate storage facility in case that the subsequent unit fails. The output stream from the second processing unit is either directed towards a final storage facility or to other subsequent basic components, which means that the first case is not relevant for the design of the basic component whilst feedbacks on the subsequent components must be taken into account.

The variable to be determined for such a basic component is the reliability which is identical with the probability of overflow of a storage facility caused by the failure of the second processing unit and implying standstill of the first.

To calculate this target variable the following assumptions are made:

- The intermediate storage facilities, connecting lines and switching systems are absolutely reliable.
- The intermediate storage facilities considered here do not act as operation buffers but only as holdup tanks in case of failures of processing units.

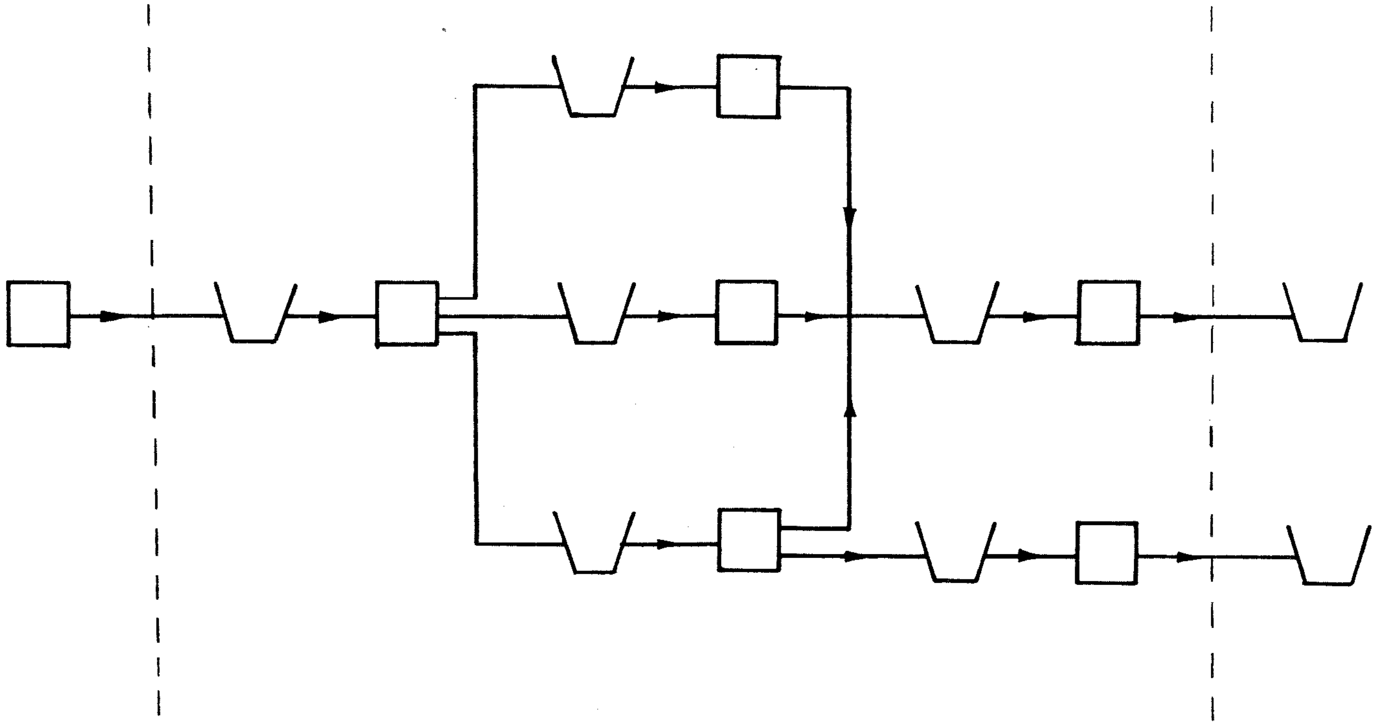
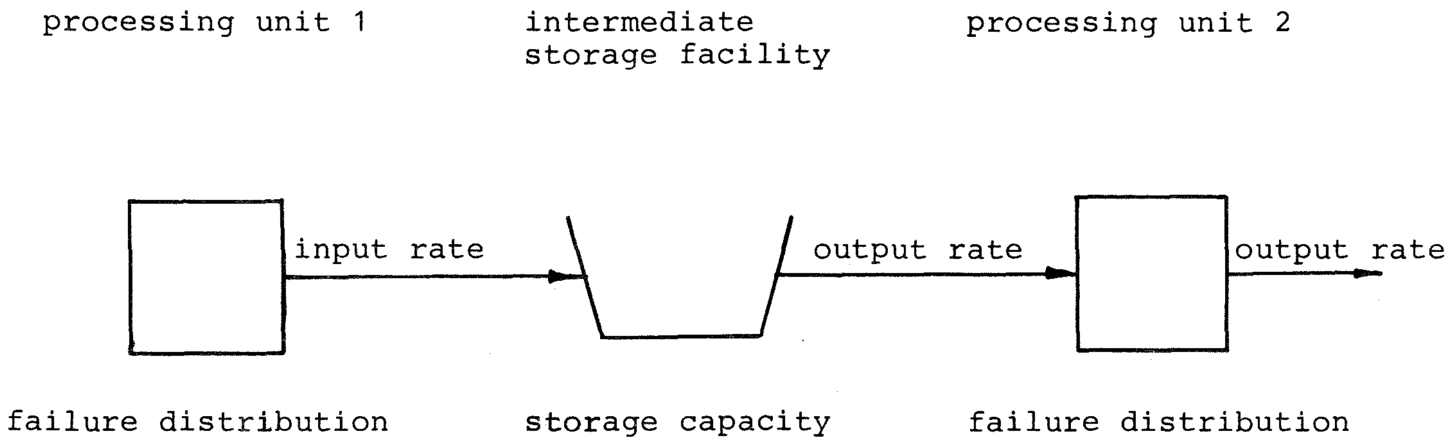


Fig. 3 Waste management system of Fig. 1 after reduction to basic components.



<u>Fig. 4</u> Basic components of an intermediate storage and processing system including system variables.

System variables are:

- input into an intermediate storage facility,
- output from an intermediate storage facility equal to input into a processing unit,
- output from a processing unit,
- storage capacity,
- inventory at the time t,
- distribution of failures of a processing unit.

Since in waste management processing units have been provided to precede each intermediate storage facility, the input equals the output of the preceding processing unit in connection with their failure distributions.

In practice, the output from the storage facility often depends on the inventory such as increase in shift operation from two to three shifts in cases where the inventory has surpassed a critical limit. So, the output is expressed as a function of the inventory.

Since, generally, a processing unit transforms its input flow into a different output flow, the output is equal to the input times a constant value.

3. Analytical Models of Intermediate Storage and Processing Systems

Based on a simple model comprising one storage facility, ISP-systems in series and in parallel are considered in this chapter.

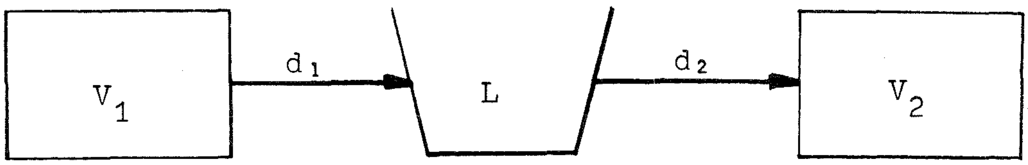
If the state i denotes that exactly i units of quantity are present in the storage facility, the so-called Markov property applies to the random process so defined on the assumptions made in Chapter 2.

As a matter of fact, the inventory at the time t depends only on the inventory at the time $t-\Delta t$ as well as on the intermediate additions and withdrawals, i.e., each state is only dependent on the state in only one preceding period. Given an initial distribution p(0) of the storage inventory (e.g. empty storage facility) and the transition rates of individual states obtained from the failure rates, a Markov process with a steady-state parameter range has thus been defined (cf. also Annex A).

Since the objective of storage facilities provided in waste management systems consists in a permanent capability of accepting waste from preceding processing units, the model aims at an examination of system failures which are attributable to storage overflows. By contrast, failures caused by evacuation of a storage facility are not as significant as in conventional storage systems where a permanent readiness to deliver to the succeeding units is to be ensured.

3.1 The Model Comprising One Storage Facility

We start from the basic system consisting of two processing units V_i and the storage facility L in between.



Subsequently a unit of quantity (UQ) and a unit of time (UT) are defined, e.g. one weekly charge, one day. Material is assumed to flow from V_1 to L at the rate d_1 $[\frac{UQ}{UT}]$ and from L to V_2 at the rate d_2. As already mentioned in Chapter 2, $d_1 = d_2$ under normal operation.

In the first model approach the assumption is made that both processing units perform independent of the storage inventory at the same rate d unless they are not operating. In 3.1.2 the case is examined in which d_2 is adapted to a high storage inventory such that a storage inventory exceeding a so-called critical inventory will result in an increase of d_2 to d_2*.

3.1.1 The Basic Model

Let a_i be the constant failure rates of V_i, a_i (j) indicating how often on an average the processing unit V_i can no longer process j units of quantity during a period of T units of time.

The capacity of the storage facility L is assumed to be K units of quantity and

$$\kappa := \max \{k : a_2(k) \neq 0\}.$$

Then exclusively for computations using the APL programm from Annex B, the assumption is made that κ is smaller than K. This means that also the maximum individual failure of V_2 (worst case) can be accommodated by L in case that the storage facility has been previously empty.

The transition rates from the state i into the state i+j, i.e., the rates applicable to an addition to the storage facility by j UQ will be defined by

$$b_{i+j\ i} := \frac{a_2(j)}{T} \quad \text{for} \quad 0 \leq i < K$$

and correspondingly the output by

$$b_{i-j\ i} := \frac{a_1(j)}{T} \quad \text{for} \quad 1 \leq i \leq K.$$

In order to be able to compute the probability of an overflow it is necessary to introduce the excessive capacities

$$K < i \leq N := K + \kappa \quad .$$

For the case of evacuation negative storage inventories

$$-\kappa \leq i < 0$$

must be introduced accordingly, which will be explained more detailed later.

So, if the storage inventory is greater than K, d_1 must be set equal to zero which yields

$$b_{i+j\ i} := 0 \quad \text{for } K < i < N,\ i+j \leq N \quad .$$

On the other hand the storage inventory decreases due to the absence of additions. However, since V_1 has to stand still only for the period until the storage inventory attaines again the scope of its capacity, the rates for an evacuation of the storage facility $b_{i-j\ i}$ caused by this standstill are different from zero only if i-j = K. Now within the period [O, T] withdrawal from the storage facility is

$$d_2 T - \sum_{l=1}^{\kappa} l a_2(l) \quad UQ.$$

This defines

$$b_{i-j\ i} := \begin{cases} 0 & \text{for } i-j \neq K \\ \dfrac{d_2T - \sum\limits_{l=1}^{K} l \cdot a_2(l)}{j \cdot T} & \text{for all other cases,} \end{cases}$$

when $K < i \leq N$.

Similar to the situation indicated above that the storage inventory is greater than K, a negative storage inventory will be treated now. It yields for $i < 0$

$$b_{i-j\ i} := 0 \quad \text{for } i-j \geq -\kappa$$

and

$$b_{i+j\ i} := \begin{cases} 0 & \text{for } i+j \neq 0 \\ \dfrac{d_1T - \sum\limits_{l=1}^{K} la_1(l)}{j \cdot T} & \text{for all other cases .} \end{cases}$$

Summarizing, the matrix $B := (b_{ij}) \in MAT(N+\kappa+1)$ of the transition rates are obtained with

$$b_{i+j\ i} := \begin{cases} \dfrac{d_1T - \sum\limits_{l=1}^{K} la_1(l)}{j \cdot T} & \text{for } i+j=0 \\ \dfrac{a_2(j)}{T} & \text{for } 0 \leq i \leq K \\ 0 & \text{for all other cases} \end{cases}$$

$$b_{i-j\ i} := \begin{cases} \dfrac{a_1(j)}{T} & \text{for } 0 \le i \le K \\[2ex] \dfrac{d_2 T - \sum\limits_{l=1}^{\kappa} l a_2(l)}{j \cdot T} & \text{for } i-j=K \\[2ex] 0 & \text{for all other cases} \end{cases}$$

for $j \ge 1$, $i+j \le N$, $i-j \ge -\kappa$. Finally, as in every transition matrix (cf. Annex A), the following relation must hold

$$b_{jj} := -\sum_{\substack{i=-\kappa \\ i \ne j}}^{N} b_{ji} \quad .$$

Using the formulas derived in Annex A, this produces as a result the distribution $p(t)$ of the inventory at the time t and the limit distribution p, that is

$$p(t) = \sum_{i \ge 0} \frac{(Bt)^i}{i!} \; p(0)$$

and $p \ge 0$ is the solution of the linear system of equations

$$B \cdot p = 0 \; , \; \sum_{i=-\kappa}^{N} p_i = 1 \quad .$$

The probability of failure of the storage facility due to overflow is

$$p_O(t) = \sum_{i=K+1}^{N} p_i(t)$$

while the probability of failure due to evacuation during failure of V_1 is calculated to be

$$p_E(t) = \sum_{i=-\kappa}^{-1} p_i(t) \quad .$$

This leads to the following corrections

$$p_K(t) := p_K(t) + p_O(t)$$

$$p_o(t) := p_o(t) + p_E(t)$$

and the mean storage inventory is thus obtained to be

$$E[SI](t) = UQ \cdot \sum_{i=1}^{K} i \cdot p_i(t)$$

with the variance

$$\sigma_{SI}(t) = UQ \sqrt{\sum_{i=1}^{K} i^2 p_i(t) - E[SI](t)^2}$$

Instead of a formalistic proof of the model the action of it will be demonstrated with the aid of a simple but expansible example.

We assume that there are only inputs to the storage facility, i.e. $a_1 \equiv \mathcal{O}$. Because of simplicity we consider only inputs of one kind, therefore

$$a_2(j) := \begin{cases} k & \text{for } j=i \\ 0 & \text{otherwise.} \end{cases}$$

With the suitable definition of the UQ it is i=1.

If the storage facility now first has an "infinite" capacity, then the matrix B looks like

$$B := \begin{bmatrix} 0 & & & & \mathcal{O} \\ \frac{k}{T} & 0 & & & \\ & \frac{k}{T} & 0 & & \\ & & \ddots & \ddots & \\ \mathcal{O} & & & \ddots & \ddots \end{bmatrix} + \begin{bmatrix} -\frac{k}{T} & & & & \mathcal{O} \\ & -\frac{k}{T} & & & \\ & & -\frac{k}{T} & & \\ & & & \ddots & \\ \mathcal{O} & & & & \ddots \end{bmatrix}$$

and therefore we have

$$\exp Bt_o := \sum_{i\geq 0} \frac{t_o^i}{i!} B^i$$

$$= e^{-ht} \cdot \begin{pmatrix} 1 & & & & & \\ ht & 1 & & & 0 & \\ \cdot & ht & 1 & & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \frac{(ht)^i}{(i-1)!} & \frac{(ht)^{i-1}}{(i-2)!} & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

with $h := \frac{k}{T}$.

If now the storage inventory at the time $t=0$ is 1 UQ , then the average inventory at time $t=t_o$ is

$$E[SI](t_0) = e^{-h\cdot t_0} \left(1 + \sum_{j\geq 1} (1+j) \frac{(h\cdot t_0)^j}{j!} \right)$$

$$= (1 + h\cdot t_0)\cdot e^{-h\cdot t_0} \sum_{j\geq 0} \frac{(h\cdot t_0)^j}{j!}$$

$$= 1 + \frac{k\cdot t_0}{T} \quad UQ .$$

Herewith it can be easyly shown, that a given deterministic development of inventory in the case of infinite capacity exactly finds its equivalent in the expectation of the stochastic interpreted process.

If we now consider a storage capacity of K UQ, the above exspected value changes in

$$E[SI](t_0) = e^{-h \cdot t_0} \left(1 + \sum_{j=1}^{K-1-1} (1 + j) \frac{(h \cdot t_0)^j}{j!} \right)$$

$$= 1 \cdot e^{-h \cdot t_0} \sum_{j=0}^{K-1-1} \frac{(h \cdot t_0)^j}{j!}$$

$$+ h \cdot t_0 \cdot e^{-h \cdot t_0} \sum_{j=0}^{K-1-2} \frac{(h \cdot t_0)^j}{j!}$$

$$+ K \cdot e^{-h \cdot t_0} \sum_{j \geq K} \frac{(h \cdot t_0)^j}{j!} \quad UQ.$$

In the same way the case of overflow, which is shown in Fig. 5, shall be discussed shortly.

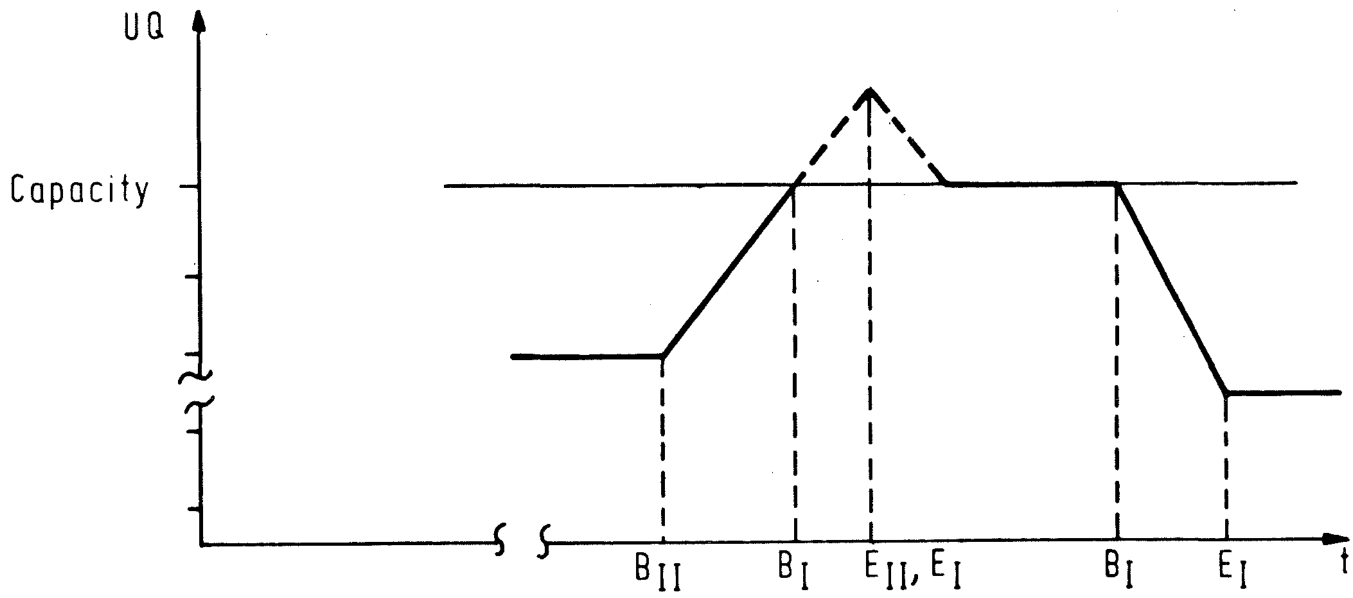


Fig. 5 Development of inventory in case of overflow of a storage facility

(B_I (B_{II}) denoting the beginning, E_I (E_{II}) the end of a failure of the processing unit $V_1(V_2)$)

At time B_{II} V_2 begins to fail, the inventory in the storage facility raises monotonously until the facility is completely filled at B_I. At this moment V_1 must be stopped since the failure of V_2 continues. This entails a standstill of the whole system until E_{II}, E_I. At this moment both processing units resume operation and the subsequent plot should be obvious.

During the period in which the storage facility is filled

($[B_I, B_I]$) a distinction must be made between the period during which V_2 fails leading to a shutdown of V_1 and consequently a standstill of the whole system $[B_I, E_{II}]$, on the one hand, and the period during which the two processing units are performing $[E_{II}, B_I]$. Only the first interval has to be recorded as the overflow interval.

This is in principle done by the introduction of so-called excessive capacities, which will be shown again with the above simple example.

Now we assume that there are K UQ at the storage at time t=O, the storage is filled. In order to compute the probality of an overflow of the storage, we first see that with

$$C := \begin{bmatrix} -b & a \\ b & -a \end{bmatrix}$$

it is

$$\exp C\cdot t = \frac{1}{(a+b)} \begin{bmatrix} a+be^{-(a+b)t} & a-ae^{-(a+b)t} \\ b-be^{-(a+b)t} & b+ae^{-(a+b)t} \end{bmatrix} .$$

If now

$$B := \begin{bmatrix} -\frac{k}{T} & & & & \mathcal{O} & \\ \frac{k}{T} & \ddots & & & & \\ & \ddots & \ddots & -\frac{k}{T} & & \\ & \mathcal{O} & & \frac{k}{T} & & C \end{bmatrix}$$

is again the transition matrix of this process with

$$C := \begin{bmatrix} -\frac{k}{T} & d_2-\frac{k}{T} \\ \frac{k}{T} & -d_2+\frac{k}{T} \end{bmatrix}$$

then it is

$$\exp B\cdot t_0 = \left[\begin{array}{ccc|c} \cdot & & & \\ & \cdot & & \emptyset \\ & & \cdot & \\ * & & & \cdot \\ \hline & & & \exp C\cdot t_0 \end{array}\right] .$$

Therefore we have the probability of an overflow at time $t=t_o$

$$p_{K+1}(t_0) = \frac{1}{d_2}\cdot\frac{k}{T}\cdot\left(1 - e^{-t_0\cdot d_2} \right) .$$

In case $t \rightarrow \infty$ this is again exactly equivalent to the value, computable in deterministic approach.

The same considerations must also apply in case of evacuation of a storage facility. Negative inventories must be introduced accordingly.

3.1.2 The Basic Model with Adaptation of Rate

To avoid premature overflow of the storage facility, an upper storage inventory limit, the so-called critical inventory, is introduced into the model. This term appears in a dualistic meaning in the inventory theory as the time of ordering as a lower inventory limit.

If the storage inventory exceeds this limit, the subsequent processing unit V_2 is to increase its rate from d_2 to d_2^*, e.g. by changing over from 2 to 3 shift operation in order to ensure a more rapid reduction of the storage inventory.

This will bring about a change of the growth rates $b_{i+j\ i}$ only for $M < i \leq K$

$$b_{i+j\ i} := \frac{a_2^*(j)}{T},$$

where a_2^* is the failure rate of V_2 in case that this processing unit is operated at the rate d_2^*.

Due to the lack of additional information, we can assume quite often that a higher utilization of the facility will imply a more frequent occurrence of outages of the same duration i.e.,

$$a_2^*\ (j) := C\ a_2(j)$$

where C is a constant dependent on $\frac{d_2^*}{d_2}$.

In this case the output rates will change for $M < i < N$ in such a way that for $i > K$

$$b_{i-j\ i} := \begin{cases} \dfrac{d_2^* T - \sum\limits_{l=1}^{K} l \cdot a_2^*(l)}{j \cdot T} & \text{for } i-j=K \\ 0 & \text{for all others,} \end{cases}$$

since for $i > M$ the processing unit V_2 operates with rate d_2^*.

Assuming that $M < i \leq K$, within the period of observation $[0,T]$

$$(d_2^* - d_2)\, T - \sum_{l=1}^{K} l(a_2^*(l) - a_2(l))$$

units of quantity will be additionally withdrawn from the storage facility. Because of this only for $i-j = M$, $a_1(j)$ is increased by $c(j)$, i.e.

$$b_{i-j\ i} := \frac{a_1(j) + c(j)}{T}$$

with

$$c(j) := \frac{(d_2^* - d_2)T - \sum l(a_2^*(l) - a_2(l))}{j}.$$

Summarizing we obtain

$$b_{i+j\ i} := \begin{cases} \dfrac{d_1 T - \sum\limits_{l=1}^{K} l \cdot a_1(l)}{j \cdot T} & \text{for } i+j=0 \\[2ex] \dfrac{a_2(j)}{T} & \text{for } 0 \le i \le M \\[2ex] \dfrac{a_2^*(j)}{T} & \text{for } M < i \le K \\[2ex] 0 & \text{for all others} \end{cases}$$

$$b_{i-j\ i} := \begin{cases} \dfrac{a_1(j)}{T} & \text{for } 0 \le i \le K\ ,\ i-j \ne M \\[2ex] \dfrac{a_1(j)}{T} + \dfrac{(d_2^* - d_2)T - \sum\limits_{l=1}^{K} l \cdot (a_2^*(l) - a_2(l))}{j \cdot T} & \text{for } i-j=M \\[2ex] \dfrac{d_2^* T - \sum\limits_{l=1}^{K} l \cdot a_2^*(l)}{j \cdot T} & \text{for } i-j=K \\[2ex] 0 & \text{for all other cases} \end{cases}$$

and with

$$b_{ii} := - \sum_{\substack{j=-K \\ j \ne i}}^{N} b_{ji}$$

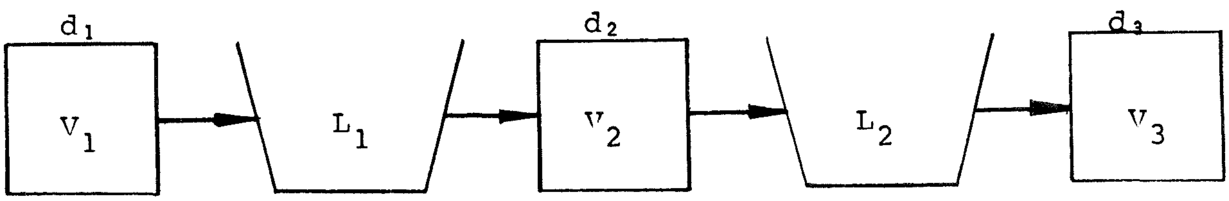
finally $B := (b_{ij}) \in MAT(N+K+1)$ is obtained as in the first case.

3.2 Models Comprising Two Storage Facilities

To extend the model containing one storage facility of Chapter 3.1 the simplest case of a many storage facilities model will be considered now, i.e., a system including two storage facilities and three processing units. The case will be studied of a series connection and of a parallel connection.

3.2.1 Series Connection

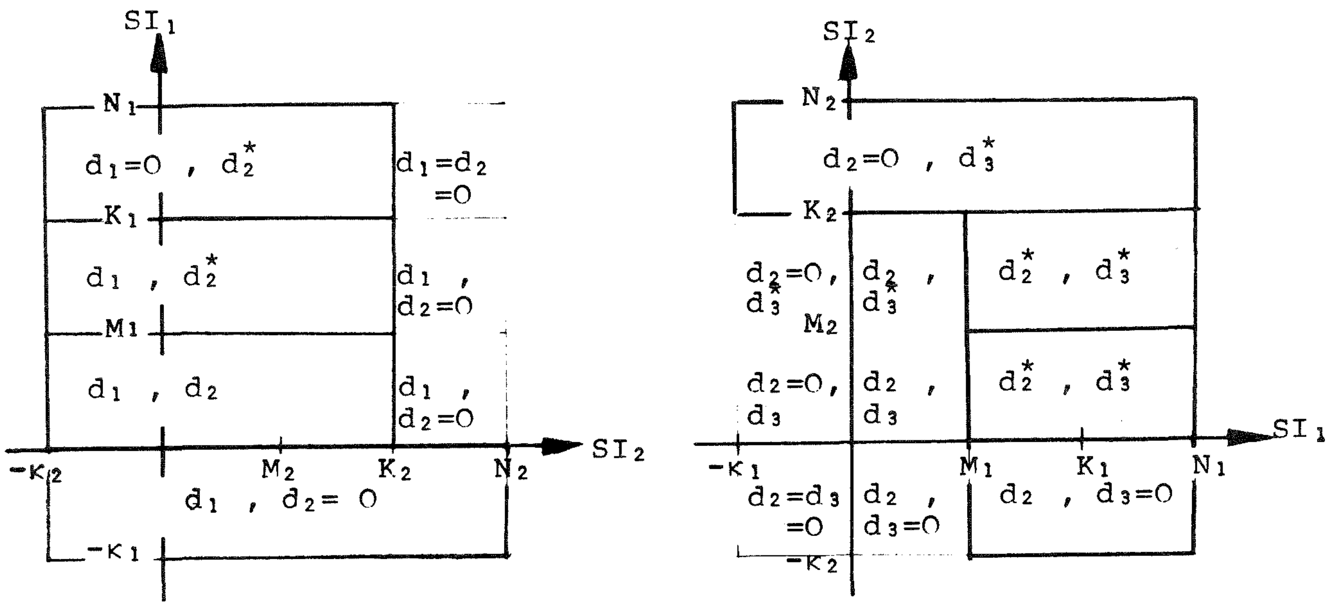
We consider a system of three processing units and two storage facilities connected in series.



According to the definition of a UQ and UT the processing units V_i perform again at the same rate d_i while in this case a conversion factor might have to be taken into account.

Again, the failure rates $a_i(j)$, $a_i^*(j)$ $(1 \leq i \leq 3,\ 1 \leq j \leq \kappa_i)$ of the processing units V_i examined separately shall be predetermined. It is assumed that the storage facilities L_i have a capacity K_i and again a critical inventory M_i shall be defined beyond which the succeeding processing unit is operated at the higher rate d_i^*.

In this way, it is defined at which rates the processing units V_i must be operated as a function of the storage inventory.



<u>Fig. 6</u> Rates of processing units as a function of the storage inventories SI_i of L_i ($N_i := K_i + \kappa_i + 1$)

It is assumed that the rate d_i is increased to d_i^* if and only if the inventory of the preceeding storage facility has surpassed the critical level M_i. The output flow of the succeeding storage facility is then assumed to get adapted to the higher input flow. Also the second condition is certainly reasonable because in practice such a high excessive capacity will be generally used in the course of rate adaptation, e.g., changeover from 2 to 3 shift operation, that this must also result in an appropriate increase of the output rate if the succeeding storage facility is not bound to flow over precociously.

Using these prerequisites the model can be conveniently established. The state (i,j) is to denote the state of L_i on the assumption that the other storage facility is under the state j, where

$$(1,j) := \begin{cases} (1,1), \text{ if } SI_2 \in [-\kappa_2, K_2] \\ (1,2), \text{ if } SI_2 \in [K_2, N_2] \end{cases}$$

$$(2,j) := \begin{cases} (2,1), \text{ if } SI_1 \in [-\kappa_1, 0] \\ (2,2), \text{ if } SI_1 \in [0, M_1] \\ (2,3), \text{ if } SI_1 \in (M_1, N_1] \end{cases} .$$

This means that by analogy with the procedure described in Chapter 3.1 the matrices B_i of the transition rates of L_i can be determined from the matrices B(i,j) to be defined.

In the following it is assumed that the expressions $j \geq 1$, $i-j \geq -\kappa$, $i+j \leq N$ are always observed and that with

$$\alpha_i := \sum_{i=1}^{\kappa_i} l \cdot a_i(l) \quad , \quad \alpha_i^* := \sum_{i=1}^{\kappa_i} l \cdot a_i^*(l) \quad \text{for } 1 \leq i \leq 3$$

the elements of B(i,j) differing from zero are defined as follows :

<u>B(1,1)</u>

$$b_{i+j\ i} := \begin{cases} \frac{1}{j} \left(d_1 - \frac{\alpha_1}{T} \right) & \text{for } i+j=0 \\ \frac{a_2(j)}{T} & \text{for } 0 \leq i \leq M_1 \\ \frac{a_2^*(j)}{T} & \text{for } M_1 < i \leq K_1 \end{cases}$$

$$b_{i-j\ i} := \begin{cases} \frac{a_1(j)}{T} & \text{for } 0 \leq i \leq K_1\ ,\ i-j \neq M_1 \\ \frac{a_1(j)}{T} + \frac{1}{j}\left(d_2^* - d_2 - \frac{\alpha_2^* - \alpha_2}{T} \right) & \text{for } i-j = M_1 \\ \frac{1}{j}\left(d_2^* - \frac{\alpha_2^*}{T} \right) & \text{for } i-j = K_1 \end{cases}$$

<u>B(1,2)</u>

$$b_{i+j\ i} := \begin{cases} \frac{1}{j}\left(d_1 - \frac{\alpha_1}{T} \right) & \text{for } i+j=0 \\ \frac{a_2(j)}{T} \ \frac{d_1 T - \alpha_1}{\alpha_2} & \text{for } 0 \leq i \leq K_1 \end{cases}$$

The hypothesis is put forward that the inputs to the storage facility resulting from $d_2 = 0$ can be distributed proportional to the other inputs expressed by a_2.

<u>B(2,1)</u>

$$b_{i-j\ i} := \begin{cases} \frac{a_3(j)}{T} \ \frac{d_3 T - \alpha_3}{\alpha_3} & \text{for } 0 \leq i \leq K_2\ ,\ i-j \neq M_2 \\ \frac{a_3(j)}{T} \ \frac{d_3 T - \alpha_3}{\alpha_3} + \frac{1}{j}\left(d_3^* - d_3 - \frac{\alpha_3^* - \alpha_3}{T} \right) & i-j = M_2 \\ \frac{1}{j}\left(d_3 - \frac{\alpha_3^*}{T} \right) & \text{for } i-j = K_2 \end{cases}$$

The same hypothesis applies as in B(1,2).

B(2,2)

$$b_{i+j\ i} := \begin{cases} \frac{1}{j}\left(d_2 - \frac{\alpha_2}{T} \right) & \text{for } i+j=0 \\ \frac{a_3(j)}{T} & \text{for } 0 \le i \le M_2 \\ \frac{a_3^*(j)}{T} & \text{for } M_2 < i \le K_2 \end{cases}$$

$$b_{i-j\ i} := \begin{cases} \frac{a_2(j)}{T} & \text{for } 0 \le i \le K_2\ ,\quad i-j \ne M_2 \\ \frac{a_2(j)}{T} + \frac{1}{j}\left(d_3^* - d_3 - \frac{\alpha_3^* - \alpha_3}{T} \right) & \text{for } \quad i-j=M_2 \\ \frac{1}{j}\left(d_3^* - \frac{\alpha_3^*}{T} \right) & \text{for } i-j=K_2 \end{cases}$$

B(2,3)

$$b_{i+j\ i} := \begin{cases} \frac{1}{j}\left(d_2^* - \frac{\alpha_2^*}{T} \right) & \text{for } i+j=0 \\ \frac{a_3^*(j)}{T} & \text{for } 0 \le i \le K_2 \end{cases}$$

$$b_{i-j\ i} := \begin{cases} \dfrac{a_2^*(j)}{T} & \text{for } 0 \le i \le K_2 \\ \dfrac{1}{j}\left(d_3^* - \dfrac{\alpha_3^*}{T} \right) & \text{for } i-j=K_2 \end{cases}$$

In this case the assumption was made that $d_2^* = d_3^*$. Otherwise, only the last matrix must be subjected to changes.

Finally, we must put for all matrices

$$b_{ii} := \sum_{\substack{j \neq i \\ j=-\kappa}}^{N} b_{ji} \qquad \text{for} \quad -\kappa \le i \le N \;.$$

Then the matrices $\mathbf{B}(i,j)$ must first be treated independent of each other by determination of

$$p_{j_i}(i,j)(t)$$

as the probability calculated by means of $B(i,j)$ that the storage facility L_i at the time t is in the state j_i provided that the other storage facility is in the state j.

If the following relations are defined

$$\Pi_1(t) := p(SI_1(t) \in [-\kappa_1, 0))$$

$$\rho_1(t) := p(SI_1(t) \in [0, M_1])$$

$$\Pi_2(t) := p(SI_2(t) \in [-\kappa_2, K_2)),$$

the theorem of total probability furnishes the equation

$$P(t)\cdot\begin{pmatrix}\Pi_1(t)\\ \rho_1(t)\\ \Pi_2(t)\end{pmatrix} = \begin{pmatrix}p_1(1,2)(t)\\ p_2(1,2)(t)\\ p_1(2,3)(t)\end{pmatrix},$$

where

$$P(t) := \begin{pmatrix}1 & 0 & p_1(1,2)(t)-p_1(1,1)(t)\\ 0 & 1 & p_2(1,2)(t)-p_2(1,1)(t)\\ p_1(2,3)(t)-p_1(2,1)(t) & p_1(2,3)(t)-p_1(2,2)(t) & 1\end{pmatrix}.$$

However, for simplification it must be considered that

$$p_1(2,1)(t)\xrightarrow[t\to\infty]{} 1$$

$$p_1(1,2)(t)\xrightarrow[t\to\infty]{} 0$$

$$p_2(1,2)(t)\xrightarrow[t\to\infty]{} 0$$

to become

$$\begin{pmatrix}1 & 0 & -p_1(1,1)\\ 0 & 1 & -p_2(1,1)\\ p_1(2,3)-1 & p_1(2,3)-p_1(2,2) & 1\end{pmatrix}\cdot\begin{pmatrix}\Pi_1\\ \rho_1\\ \Pi_2\end{pmatrix} = \begin{pmatrix}0\\ 0\\ p_1(2,3)\end{pmatrix}$$

reducing the system of equations in the case of $t \to \infty$.

It is easily noted that the matrix of coefficients p is a regular one.

In this way, the matrices of the transition rates of the coupled storage facilities are defined for $t \to \infty$ by

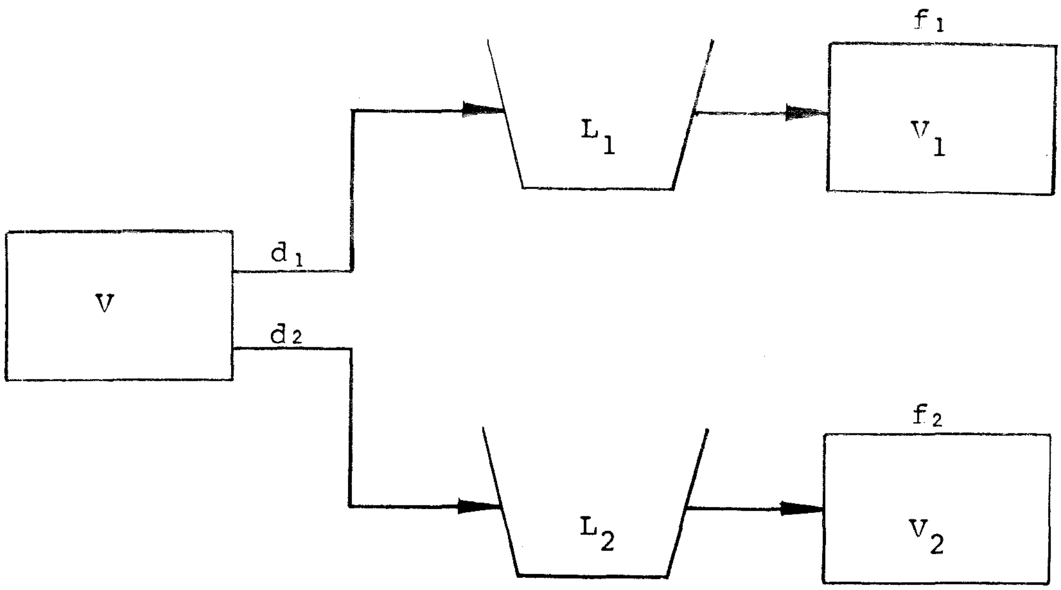
$$B_1 := \Pi_2 \, B(1,1) + (1-\Pi_2) \, B(1,2)$$

$$B_2 := \Pi_1 \, B(2,1) + \rho_1 \, B(2,2) + (1-\Pi_1-\rho_1) \, B(2,3)$$

and the reliability characteristics of L_i can be determined as in Chapter 3.1.

3.2.2 Parallel Connection

Two parallel connected storage facilities will now be studied as the second significant case of a system comprising two storage facilities.



Two different problems arise. On the one hand, the two flows of goods can be identical. This means that the system has been designed so as to be redundant, which means that in case of failure of one partial system the other system takes over the function of the first. As outlined in Chapter 2, this case can be conveniently reduced.

More practical importance is attributed to the case that the two partial systems are different, i.e., the failure of one partial system already gives rise to total failure. Only this situation is to be treated more thoroughly here.

Again one UQ, one UT and the rates d_i, f_i shall be given with d_i again equal to f_i, but not $d_1 = d_2$, $f_1 = f_2$. The failure rates a, a_1, a_2 of the processing units and the capacities of the storage facilities M_i, K_i are given.

It appears immediately that the completely symmetric storage facilities depend on each other only in case that one of these storage facilities flows over.

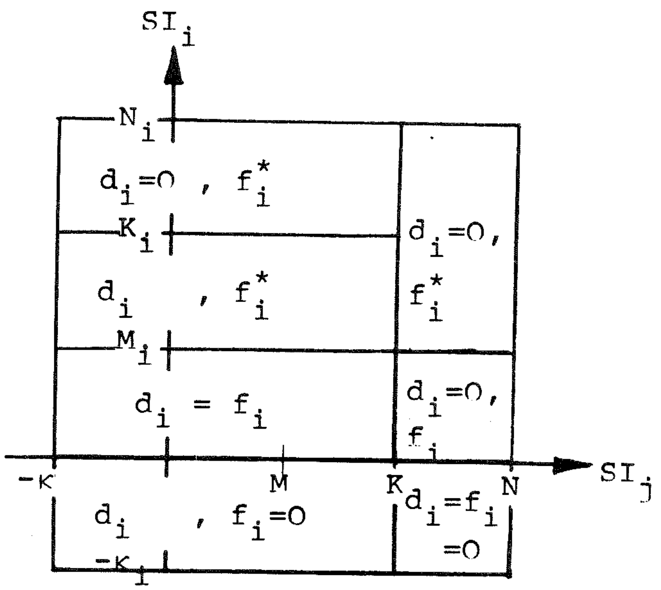


Fig. 7 Rate of the processing unit V_i as a function of the storage inventories SI_i, SI_j $(N_i := K_i + \kappa_i + 1)$

If $(k,j) \in \{1,2\}^2$ again designates the state k of the storage facility in case that the other storage facility is characterized by the state j, where

$$j := \begin{cases} 1 & \text{if storage inventory} \leq K \\ 2 & \text{for all other cases} \end{cases} .$$

B(k,1) ,exactly as in chapter 3.1.2, is defined by

$$b_{i+j\ i} := \begin{cases} \frac{1}{j}\left(d_k - \frac{\alpha_k}{T} \right) & \text{for } i+j=0 \\ \frac{a_k(j)}{T} & \text{for } 0 \le i \le M_k \\ \frac{a_k(j)}{T} & \text{for } M_k < i \le K_k \end{cases}$$

$$b_{i-j\ i} := \begin{cases} \frac{a(j)}{T} & \text{for } 0 \le i \le K_k\ ,\ i-j \ne M_k \\ \frac{a(j)}{T} + \frac{1}{j}\left(f_k - f_k - \frac{\alpha_2 - \alpha_2}{T} \right) & \text{for } i-j = M_k \\ \frac{1}{j}\left(f_k - \frac{\alpha_2}{T} \right) & \text{for } i-j = K_k \end{cases} ,$$

whereas B(k,2) is defined by

$$b_{i+j\ i} := 0 \qquad \text{for } -\kappa_k \le i \le K_k$$

$$b_{i-j\ i} := \begin{cases} \frac{a_k(j)}{T}\ \frac{f_k T - \alpha_k}{\alpha_k} & \text{for } 0 \le i \le K_k\ ,\ i-j \ne M_k \\ \frac{1}{j}\left(f_k - \frac{\alpha_k}{T} \right) & \text{for } i-j \in \{M_k, K_k\} \end{cases} .$$

Like in Chapter 3.2.1 the probabilities

$$p_{j_i}(i,j)(t)$$

are again calculated with the matrices

$$B\ (i,j),\quad (i,j)\in\{1,2\}^2$$

and with

$$\Pi_i(t) := p(SI_i(t)\in[-\kappa_i, K_i])$$

we obtain the equation

$$\begin{bmatrix} 1 & p_1(1,2)-p_1(1,1) \\ p_1(2,2)-p_1(2,1) & 1 \end{bmatrix} \cdot \begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix} = \begin{bmatrix} p_1(1,2) \\ p_1(2,2) \end{bmatrix}.$$

However, it must be considered again that

$$p_1(1,2)(t) \underset{t\to\infty}{\longmapsto} 1$$

$$p_1(2,2)(t) \underset{t\to\infty}{\longmapsto} 1$$

which means that in the steady-state case

$$\Pi_1 = \frac{p_1(1,1)}{p_1(1,1)+p_1(2,1) - p_1(1,1)p_1(2,1)}$$

$$\Pi_2 = \frac{p_1(2,1)}{p_1(1,1)+p_1(2,1) - p_1(1,1)p_1(2,1)} .$$

Finally, we have again the matrix of the transition rates of the coupled storage facilities

$$B_1 := \Pi_2 B(1,1) + (1-\Pi_2)B(1,2)$$

$$B_2 := \Pi_1 B(2,1) + (1-\Pi_1)B(2,2) ,$$

allowing to determine the reliability characteristics of the coupled storage facilities L_i, as it was done in Chapter 3.1.

3.3 Models Comprising N Storage Facilities

Models comprising N storage facilities with N > 2 are generally too complex for providing an analytical solution. A means of solution consists in a reduction to simpler systems as well as in computer aided simulation.

The first possibility was outlined in Chapter 2. The reduction of two temporary storage facilities according to Fig. 2e and f to one storage facility is quite simple assuming an absolute reliability of storage facilities and connection lines.

In parallel or series connected processing units the input and output rates and the failure distributions must be linked to each other when reduction to one unit shall be made. This is easily done for the inputs and outputs. In the case of distributions familiar rules of linking can be used /8,9/. This yields again relatively simple distributions unless an excessive number of distributions must be linked or the failure distributions obey a complex pattern. However, the prerequisites required are generally fulfilled.

After reduction to an intermediate storage facility or a processing unit the models developed in Chapters 3.1 and 3.2 can generally be applied.

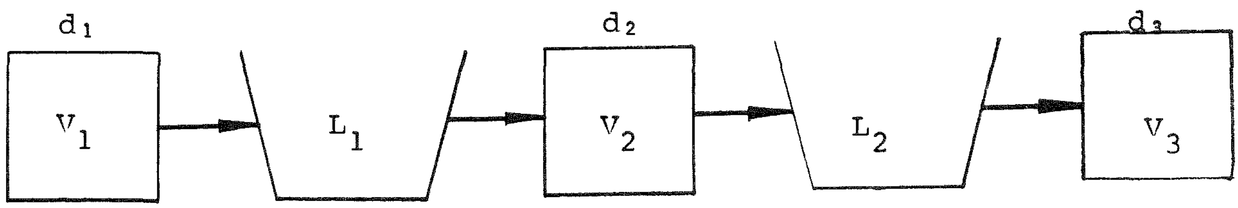
If the system is still too complex, computer aided simulation will offer a solution. However, it can be applied to a limited extent only since the events to be analyzed, such as the overflow of an intermediate storage facility, rarely occur and a great number of simulation runs are required to attain a given confidence level, thus entailing a long computation time.

4. Application of the Model

By a typical example from waste management the application of the model described in Chapter 3 will be presented in this chapter. Following a description of the example 4.1 some typical results will be discussed in 4.2. Finally, in Chapter 4.3, numerical problems of a general nature will be discussed which arise in applications.

4.1 An Example

The starting point is a system of two storage facilities in series.



The assumption shall be made that V_1 is preceded by a source (e.g. permanently filled storage facility) and that a sink (e.g. ultimate storage facility of any dimension) follows V_3.

Having defined a UQ - e.g. one weekly charge - and a UT - e.g. one day - the problem shall be to determine with the data of Table 1, selected from practical application, the reliability characteristics for the steady-state system in order to be able to evaluate the capacity required by the storage facility L_1.

	V_1	V_2	V_3	L_1	L_2
Rate d $\|\frac{UQ}{UT}\|$	0.2	0.2	0.2		
Adapted Rate d* $\|\frac{UQ}{UT}\|$	0.3	0.3	0.3		
Availability	90%	80%	80%		
Capacity K_i |UQ|				?	10
Critical inventory M_i |UQ|				$\frac{3}{4} \cdot K_1$	8

Table 1 Data applicable to the example

The only information about failure rates shall say that the failures occur in an order of magnitude of 5, 10, 15 and 20 UT with a failure of 5 UT duration in V_1 taking place 10 times more frequently and in V_2 and V_3 about 5 times more frequently than a failure of 20 UT duration.

It will be further assumed that the failure rates are distributed exponentially, an assumption which is nearly always confirmed in practice. By this assumption the failures within T = 200 UT (e.g., 1 year of operation) of normal operation are recorded as shown in Fig. 8.

If, finally, a processing unit adapts its rate, e.g., by changeover from 2 to 3 shift operation, the failure rates of this unit will increase by 10% due to the higher utilization.

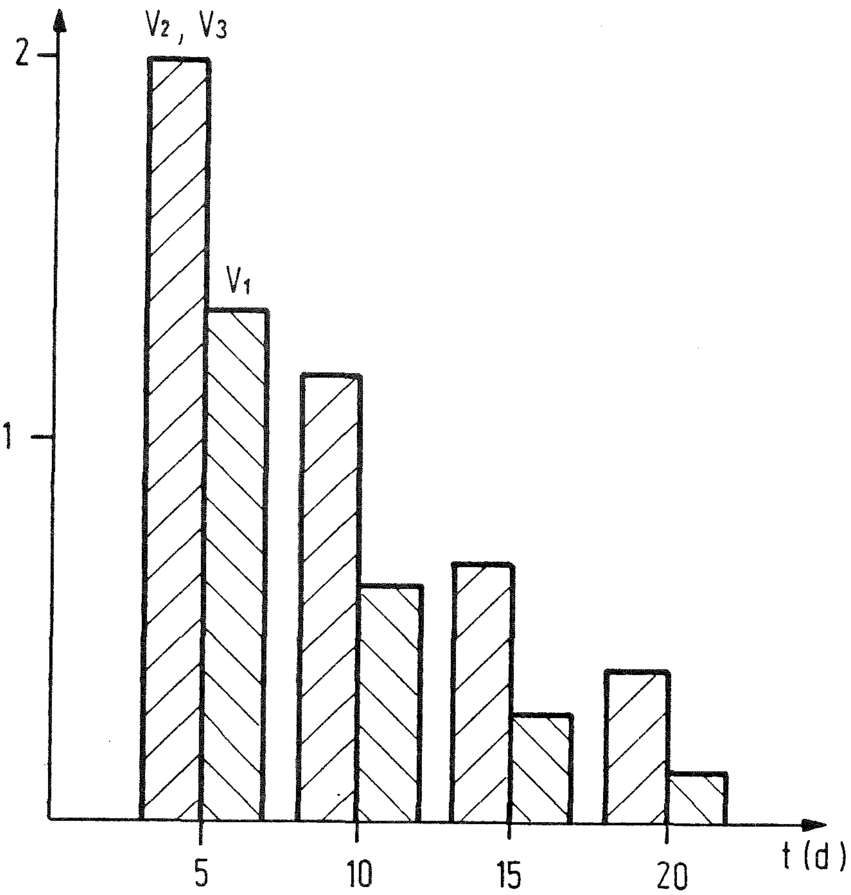


Fig. 8 Frequency of failures within T=200 UT

4.2 Results

The first question asked under this problem is to what extent the two storage facilities can be treated as independent units. This is answered by Table 2.

		$p_O \cdot 10^2$	$p_E \cdot 10^2$	E[T] |UQ|
L_1	coupled	2.09	0.32	810
	not coupled	2.54	0.25	712
L_2	not coupled	1.25	5.24	1130
	coupled	1.87	6.02	865

Table 2 Overflow probabilities p_O, evacuation probabilities p_E, and average time until the first overflow of the storage facility E [T] for a storage facility empty at the beginning under the steady-state condition $(t \to \infty)$ for the two storage facilities L_1 and L_2 and the capacity K_1 = 12 UQ.

It appears that the error of nearly 10% in the treatment of the first storage facility is relatively small. This is clear if one considers that backfeeding from the second storage facility occurs only in case that the latter flows over which, obviously, is a very rare event.

On the other hand, considerable errors result from the separate examination of the second storage facility.

K_1 |UQ|	$p_O \cdot 10^2$	$p_E \cdot 10^2$	E[T] |UT|	σ_T |UT|	E[SI] |UQ|	σ_{SI} |UQ|
8	1.74	6.25	910	792	4.06	3.26
16	1.93	5.95	846	731	4.18	3.28
24	1.99	5.93	830	716	4.20	3.29
32	2.01	5.93	823	711	4.21	3.29

Table 3 Reliability characteristics for the second storage facility and different capacities K_1.

Table 3 shows that the capacity of the first storage facility K_1 exerts a relatively low influence on the degree of coupling. It is rather decisive that a storage inventory of L_1 exceeding M_1 leads to an increase of also the rate of V_3 from $d = 0.2$ to $d^* = 0.3$. However, in this case failures of V_i imply additions to and withdrawals from the storage facility, which are by 50% higher than failures of the same duration at $d = 0.2$. This explains why the reliability of the decoupled second storage facility is much lower than that of the coupled storage facility.

Besides the very little overflow and evacuation probalities the typical case is recognized in Table 3 that the random variables of time until the first overflow of the storage facility and of storage inventory are associated with very high standard deviations σ_T and σ_{SI}, respectively. This means that averaging for an expected value is not very meaningful here.

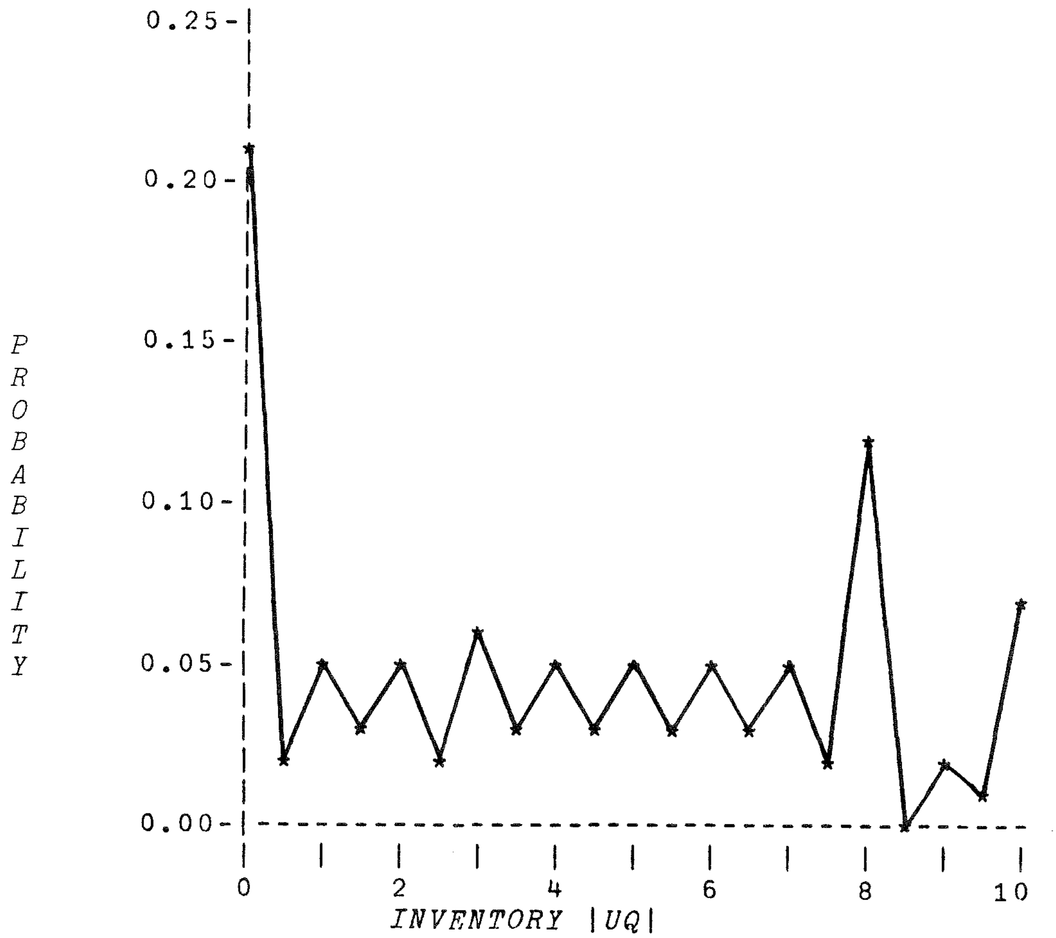


Fig. 9 Distribution of inventory of the second storage facility (for K_1 = 20 |UQ|).

Fig. 9 shows the three peaks O, M_2 = 8 and K_2 = 10 in the distribution of the storage inventory of L_2 while the storage inventory between O and M_2 is nearly evenly distributed, whereat the scattering in this region results from the adaption of the rate of V_3.

K_1 |UQ|	$p_O \cdot 10^2$	$p_E \cdot 10^2$	E[T] |UQ|	σ_T |UT|	E[SI] |UQ|	σ_{SI} |UQ|
8	3.64	o.725	428	313	5.1	2.1
12	2.54	o.246	712	493	8.0	2.7
16	1.88	o.o86	1o81	736	11.1	3.1
20	1.48	o.o31	1464	988	14.2	3.3
24	1.19	o.o11	1923	13o6	17.3	3.5
28	o.98	o.oo4	2461	17oo	2o.3	3.7
32	o.81	o.oo1	3o81	2174	23.4	3.8

Table 4 Reliability characteristics for the first storage facility L_1 and different capacities K_1.

In Table 4 the most significant reliability characteristics of the first storage facility have been listed. It strikes first that the average storage inventory of L_1 is always a bit lower than the critical inventory $M_1 = 3/4$ of K_1 with a relatively low standard deviation in this case. This gets obvious immediately: Since the input into the storage facility is higher than the output, the inventory on an average takes a high value. If, on the other side, it is higher than M_1, such a high excessive capacity is used that the storage inventory is directly reduced to M_1. This is also made clear by Fig. 1O.

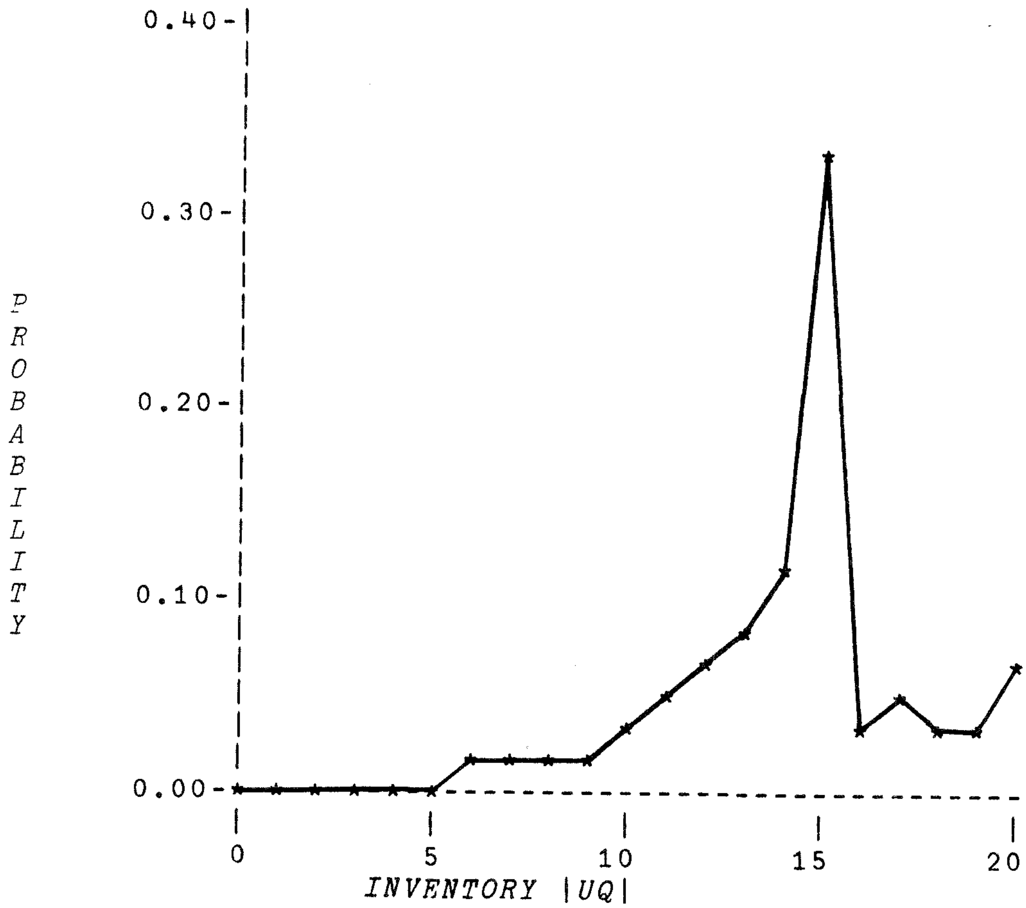


Fig. 10 Distribution of inventory of the first storage facility (for K_1 = 20 |UQ|).

Based on the reliability characteristics of Table 4 a decision must be made now on the capacity required for the storage facility L_1. In waste management this is usually done by ensuring a given failure-free period of operation of the entire system, beginning with startup, taking into account that the failure structure variables with the time, e.g., increase in availability of prototype facilities.

In the same way as an increase in K_1 results in a higher reliability of this system, the reduction of the critical inventory M_1 brings about the same effect. In this case, at K_1 = 20, an increase in K_1 by 4 UQ corresponds to a reduction of M_1 by 1 UQ with respect to the overflow probability, as demonstrated in Fig. 11 , but not with respect to E[T].

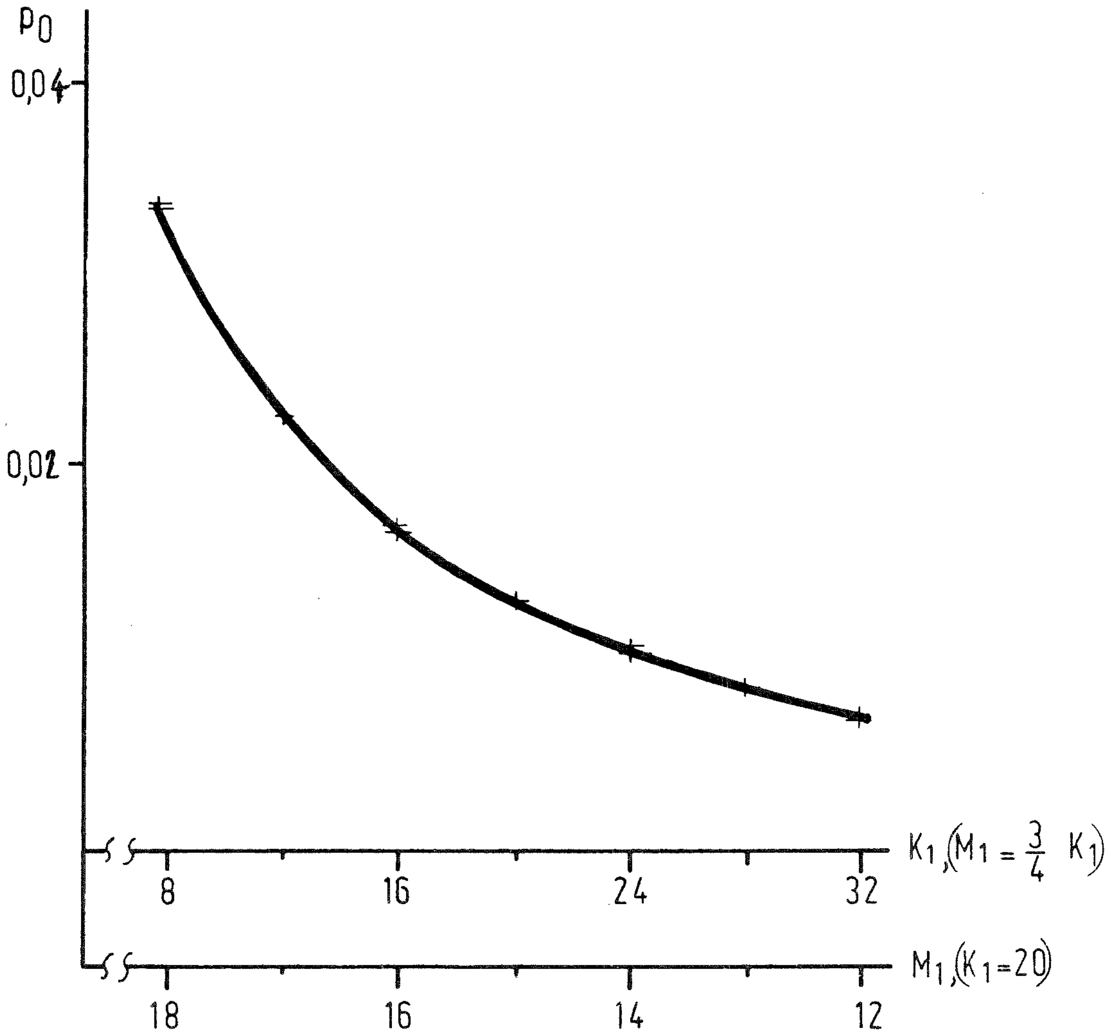


Fig. 11 Overflow probability of L_1.

Finally, the dependence of the reliability of this system on the failure structure of its components - not changing their availability - will be briefly dealt.

$\gamma=(\gamma_1,\gamma_2,\gamma_3)$	L_1		L_2	
	$p_0 \cdot 10^2$	E[T] |UT|	$p_0 \cdot 10^2$	E[T] |UT|
(10, 10, 10)	1.28	1581	1.69	917
(10, 10, 5)	1.31	1559	1.89	862
(5, 10, 10)	1.30	1568	1.69	918
(5, 10, 5)	1.33	1547	1.89	863
(10, 5, 10)	1.45	1484	1.76	887
(10, 5, 5)	1.48	1464	1.97	836
(5, 5, 10)	1.47	1.473	1.76	.888
(5, 5, 5)	1.50	1453	1.96	837

Table 5 Reliability characteristics of L_i for different exponentially distributed failure structures, K_1=20.

Table 5 contains the reliability characteristics of L_i for which the structural parameters

$$\gamma_i := \frac{a_i(1)}{a_i(4)} \qquad 1 \le i \le 3$$

have been varied with the other conditions remaining unchanged. It is easily recognized that the characteristics of L_1 behave in a very stable mode if γ_2 remains constant. On the other side, only the failure structure of the third processing unit exerts a major influence on the reliability of the second storage facility.

4.3 Numerical Problems

The results of Chapter 4.2 were calculated with an APL program explained in Annex B. This programing language was used because of the short time of implementation it requires and the possibility of realizing quickly changes of structure of the underlying system. On the other hand, the major computations consist in matrix operations which can be very easily programed with APL.

The problem immediately arises of the order of magnitude of the transition matrix B. Matrices of a size greater than approximately 50 - 55 could no longer be processed on the system available for lack of memory. However, since the matrix B has a very special shape, it is not necessary here to refer directly to the theory of sparse matrices /12,13/. The block-tri-diagonality of B can either be used for direct methods /10/ or for iteration methods /10/. They converge since on account of the diagonal dominance and the theorem of Perron-Frobenius /11,10/ the spectral radius of the matrix shortened by one line and one column is smaller than unity.

Obviously, processing of greater matrices calls for a different implementation.

Last but not least, considerable numerical problems are encountered as in each analytical model if the underlying system is too complex in its structure and not reducible; but the past applications have not been more complicated than described in Chapter 3.

5. Conclusions

Summarizing it may be said that the analytical Markov model described above is an efficient instrument with regard to CPU-time and accuracy for computation of reliability characteristics mainly for simply structured intermediate storage and processing systems. The model is also applicable to more complex systems if a reduction is possible (cf. Chapter 3.3).

The Markov model was first applied to determine the capacity of the drum storage facility of the Medium Level Waste Treatment Facility (MAVA) in Karlsruhe.

6. References

/1/ Issel W., Knoch W.: Zur Auslegung einer großen Wiederaufarbeitungsanlage; Atomwirtschaft July/August 1976

/2/ Reinschke K.: Zuverlässigkeit von Systemen; VEB Verlag Technik Berlin, 1973

/3/ Gnedenko B.W.et.al.: Mathematische Methoden der Zuverlässigkeitstheorie; Akademie-Verlag, Berlin, 1968

/4/ Popp W.: Einführung in die Theorie der Lagerhaltung; Springer-Verlag 1968

/5/ Moran P.: The Theory of Storage; Methuen , 1958

/6/ Ghosal A.: Some Aspects of Queuing and Storage Systems; Springer-Verlag, 1970

/7/ Cohen J.W.: The Single Server Queue; North-Holland Comp., London, 1969

/8/ Riordan J.: An Introduction to Combinational Analysis; Wiley New-York, 1967

/9/ Petrich G.: Simulation durch kombinatorische Manipulation von Häufigkeitsverteilungen; KFK-Ext. 13/72-2 , August 1972

/10/ Varga R.S.: Matrix Iterative Analysis; Prentice Hall, 1962

/11/ Koehler G.J.: Optimization over Leontief Substitution Systems; North-Holland Comp., London, 1975

/12/ Young D.M.: Interative Solution of Large Linear Systems; Academic-Press, 1971

/13/ Tewarson R.P.: Sparse Matrices; Academic-Press, 1973

7. Index of Notations

$a(j), a_i(j)$	failure rates of V, V_i
$a_1^*(j)$	failure rates of V_i at a higher rate
α_i, α_i^*	$\sum_{l=1}^{\kappa} l a_i(l)$, $\sum_{l=1}^{\kappa} l a_i^*(l)$
b_{ij}	transition rates
B	matrix of transition rates
$B_i, B(i,\cdot)$	matrix of transition rates from L_i
d_i, f_i	rate of V_i
d_i^*, f_i^*	adapted higher rate of V_i
$E[SI](t)$, $E[SI]$	expected value of storage inventory
$E[T]$	expected value during the period until the first overflow of the storage facility
K, K_i	storage capacity of L, L_i
κ, κ_i	maximum failure of V_2, V_i
L, L_i	storage facility
M, M_i	critical storage inventory of L, L_i
$p(t)$	distribution of storage inventory at the time t
p	distribution of the storage inventory for $t \to \infty$
$p_E(t)$	probability of storage evacuation
$p_O(t)$	overflow probability
$\sigma_{SI}(t), \sigma_{SI}$	standard deviation of storage inventory
σ_T	standard deviation of the time until first overflow of storage facility

SI_i	storage inventory of L_i
T	period of observation
UQ	unit of quantity
UT	unit of time
V, V_i	processing unit

Annex A Mathematical Fundamentals

In the above paragraphs a random process $Z(t)$, $t \in IR$ was always taken as the base.This process is called Markov process with a steady-state parameter range if the so-called Markov property applies , i.e.

$$P(Z(t) \le x \mid Z(t)=z_1 , \ldots , Z(t_n)=z_n) = P(Z(t) \le x \mid Z(t_n)=z_n)$$

for all values of $t_1 < \ldots < t_n < t$ and all possible values of the random variables Z /14/.In this context only the case is interesting when the process Z adopts only the finite number of states z_i , $1 \le i \le n$ .So,the transition rates

$$b_{ji}(t) := \lim_{\Delta t \to 0} \frac{P(Z(t+\Delta t)=z_j \mid Z(t)=z_i)}{\Delta t}$$

$$b_{ii}(t) := - \sum_{\substack{j=1 \\ j \ne i}}^{N} b_{ji}(t)$$

can be defined.If $p_i(t)$ denotes the probality that the process at the time t is in the state i , the Kolmogoroff differential equations can quickly be deviated for $p(t) = (p_1(t), \ldots, p_n(t))$ /2, 4/

$$\frac{d}{dt} p(t) = B(t) \cdot p(t) = p(t) \cdot B(t)$$

where $B(t) := (b_{ij}(t))_{1 \le i,j \le n}$.

In the homogeneous case of constant transition rates,which is the only case of interest here, we obviously obtain

$$p(\tau) = (\exp B \cdot (\tau - t)) \cdot p(t)$$

where

$$\exp B \cdot t := \sum_{i \ge 0} \frac{t^i}{i!} B^i .$$

Now exp Bt can be calculated numerically in an approximation with the help of the finite sum. However, the error with respect to the column sum norm

$$\| \exp(B\tau) - \sum_{i=0}^{K} \frac{\tau^i}{i!} B^i \| \leq \frac{2\tau \cdot \max |b_{ij}|}{(K+1)!} e^{2\tau \max |b_{ii}|}$$

is very high. Also the estimate

$$\| \exp(\mathbf{B} \cdot \mathbf{L}\tau) - \left(\sum_{i=0}^{K} \frac{\tau^i}{i!} B^i \right)^{\mathbf{L}} \| \leq \left(1 + \frac{2\tau \max |b_{ii}|}{(K+1)!} e^{2\tau \max |b_{ii}|}\right)^{\mathbf{L}} - 1$$

is only satisfactionary in part so that other familiar methods /10,12/ should be consulted to calculate $\exp B\tau$.

It is clear that in the case $t \to \infty$ for $p := \lim_{t \to \infty} p(t)$

$$B \cdot p = p \cdot B = 0$$

with $\sum_{i=1}^{N} p_i = 1$. If B is irreducible, we obtain

$$\tilde{B} := \begin{pmatrix} b_{11} & \cdots & b_{1\ n-1} \\ \vdots & & \vdots \\ b_{n-1\ 1} & \cdots & b_{n-1\ n-1} \end{pmatrix}$$

regular since for $\tilde{B} \cdot x = 0$ it follows

$$(E - \exp \tilde{B} \cdot t) \cdot x = 0 \quad \text{for all } t \in \mathbb{R} ,$$

whilst with

$$B^* := \begin{pmatrix} & \tilde{B} & & \begin{matrix} 0 \\ \vdots \end{matrix} \\ b_{n1} & \dots & b_{n\,n-1} & 0 \end{pmatrix}$$

$$\exp B^*\cdot t = \begin{pmatrix} & \exp \tilde{B}t & & \begin{matrix} 0 \\ \vdots \end{matrix} \\ c_1 & \dots & c_{n-1} & 1 \end{pmatrix}$$

with $c_i \neq 0$ and hence

$$\sum_{i=1}^{N-1} (\exp \tilde{B}t)_{ij} < 1 \quad \text{for all} \quad 1 \leq j \leq N-1.$$

This means

$$\| \exp \tilde{B}t \| < 1 \quad \text{and}$$

$$\sum_{i \geq 0} (\exp \tilde{B}t)^i = (E - \exp \tilde{B}t)^{-1},$$

with finally, $x=0$.

This means that p is unambiqously determined by $B.p = 0$, $\sum_{i=1}^{N} p_i = 1$ provided that B is irreducible.

Let us assume now that $T \subset \{1,2, \dots ,n\}$ and the process starts at time $t=0$ in the state $i_0 \varepsilon T$. With

$$\sigma(t) := \sum_{i \varepsilon T} p_i(t)$$

and $a_i := 1$ for $i \varepsilon T$ and $:= 0$ otherwise the expected value for the time until the process first leaves the states of T is

$$E[T] = \int_0^\infty \alpha(t)\, dt = \int_0^\infty a \cdot p(t)\, dt$$

$$= \int_0^\infty a \cdot (\exp Bt) \cdot p(0)\, dt = \int_{-\infty}^0 a \cdot (\exp -Bt) \cdot p(0)\, dt$$

$$= a \cdot B^{-1} \cdot p(0) = \sum_{i \varepsilon T} (B^{-1} \cdot p(0))_i \quad .$$

Using this formula the time until the first overflow of the storage facility is determined in Chapter 4.2.

Annex B The APL-Program

On the following pages the APL-program which was used to compute the example in Chapter 4 is listed. Hereby *COMPUTATION* is the main program and *I MATRIX J* generates B(I,I) with (J=0) or without (J=1) diagonal. Further *MATRIX*12 generates B(1,2), *MATRIX*21 B(2,1) and *MATRIX*23 B(2,3). *COMPUTE* computes the reliability characteristics. *INPUT*1 and *INPUT*2 are auxiliary programs and *OUTPUT* causes the output. On the last page a test run with the data of *INPUT* is printed.

The underlined variables within the functions are nearly adequate to the symbols used in the model description above.

```
     ∇ COMPUTATION;C;D;E;F;PI;B1;B2
[1]    INPUT
[2]    1 MATRIX 0
[3]    C←-PE
[4]    D←-PM
[5]    2 MATRIX 0
[6]    E←PO
[7]    CAP[2]←⌊((CP←CAP[2])÷1.5)+0.5
[8]    MATRIX23 0
[9]    PI←(0,0,1+F)⌹ 3 3 ρ(1,0,C,0,1,D,F,(E+F←-PQ),1)
[10]   1 MATRIX 1
[11]   B1←PI[3]×B
[12]   B←ι0
[13]   MATRIX12
[14]   B←B1+B×1-PI[3]
[15]   B1←ι0
[16]   B←B-(0,-ιN-1)⌽⍉(N,N)ρ((+/[1] B),(N×N-1)ρ0)
[17]   OUTPUT 1
[18]   INPUT1
[19]   MATRIX21
[20]   B2←PI[1]×B
[21]   B←ι0
[22]   2 MATRIX 1
[23]   B2←B2+PI[2]×B
[24]   B←ι0
[25]   INPUT2
[26]   MATRIX23 1
[27]   B←B2+B×1-PI[1]+PI[2]
[28]   B2←ι0
[29]   B←B-(0,-ιN-1)⌽⍉(N,N)ρ((+/[1] B),(N×N-1)ρ0)
[30]   OUTPUT 2
     ∇
```

```
     ∇ I MATRIX J;H;K;L;M;O;P;Q;R;S;T;U;V;W;X
[1]    W←(-3×G)-U←-1-V←P-1-T←1+S←(L←CAP[I])-O←-G-Q←-G+1-R←(M←CR[I
       ])+P←2×1+G←KAPA
[2]    B←((O,T)ρ0),[1](U,T)↑(P-ιU)⌽(U,V)ρ(0,(⌽AS[K←I+1;]÷T),0,(H←
       A[I;]÷T),Sρ0)
[3]    B[Q;]←B[Q;]+(÷ιT)×DS-D+(÷T)×ALS[K]-AL[K]
[4]    B←(((R,O)↑(P-ιR)⌽(R,R)ρ(0,(⌽A[K;]÷T),0,H,Mρ0)),[1]((S,O)ρ0
       )),B
[5]    B←(((G,G)ρ0),[1](⌽(÷ιG)×D-AL[I]÷T),[1]((X←L+G),G)ρ0),B
[6]    B←B,((X,G)ρ0),[1]((÷ιG)×DS-ALS[K]÷T),[1](G,G)ρ0
[7]    →(1+⌶26)×J=0
[8]    B←B-(0,-ιN-1)⌽⍉(N,N)ρ((+/[1] B),(N×(N←L+P-1)-1)ρ0)
[9]    I COMPUTE 1
     ∇
```

```
     ∇ MATRIX12;H;H1;V;W
[1]    H←⌽(÷⍳G)×D-AL[1]÷T
[2]    H1←⌽A[2;]×(÷AL[2])×D-AL[1]÷T
[3]    B←(H,[1]((V-1),G)⍴0),(V,W)↑(G+1-⍳V)⌽(V,V←G+W)⍴(H1,(W←CAP[1
       ]+1)⍴0)
[4]    B←((G,N←G+V)⍴0),[1] B,(V,G)⍴0
     ∇
```

```
     ∇ MATRIX21;H;R;S;T;U;V;W
[1]    H←A[3;]×(D÷AL[3])-÷T
[2]    B←((V,U)↑(G-⍳V)⌽(V,V←G+T)⍴(H,T⍴0)),[1]((S←1+G←KAPA),U←1+T←
       CAP[2])⍴0
[3]    B[M;]←B[M←S+R;]+((R+1)⍴0),(÷⍳T-R←CR[2])×DS-D+(÷T)×ALS[
       3]-AL[3]
[4]    B←(((S+T+G),G)⍴0),B,((V,G)⍴0),[1]((÷⍳G)×DS-ALS[3]÷T),[1](G
       ,G)⍴0
     ∇
```

```
     ∇ MATRIX23 J;R;S;T
[1]    B←(N,R+1)↑(S-⍳N)⌽(N,N←R+S←1+2×G)⍴((⌽AS[3;]÷T),0,(AS[
       2;]÷T),(R←CAP[2])⍴0)
[2]    B←(((G,G)⍴0),[1](⌽(÷⍳G)×DS-ALS[2]÷T),[1]((T←G+R),G)⍴0),B
[3]    B←B,((T,G)⍴0),[1]((÷⍳G)×DS-ALS[3]÷T),[1](G,G)⍴0
[4]    →(1+⌶26)×J=0
[5]    B←B-(0,-⍳N-1)⌽⍉(N,N)⍴((+/[1] B),(N×N-1)⍴0)
[6]    2 COMPUTE 1
     ∇
```

```
     ∇ I COMPUTE J;M;S;Z
[1]    P←P÷+/P←(-(S↑B[;N])⌹(S,S←N-1)↑B),1
[2]    PM←(+/P[G+⍳CR[I]+1])+PE←+/G↑P
[3]    P[G+1]←P[G+1]+PE
[4]    P[N-G]←P[N-G]+PO←+/(-G)↑P
[5]    ESI←(P←G↓(N-G)↑P)+.×S←(0,⍳CAP[I])
[6]    SSI←((P+.×S*2)-ESI*2)*0.5
[7]    →(J=1)/3+⌶26
[8]    ET←-+/Z←((G⍴0),1,(M-G+1)⍴0)⌹(M,(M←N-G))↑B
[9]    ST←((2×+/Z⌹(M,M)↑B)-ET*2)*0.5
[10]   B←⍳0
     ∇
```

```
     ∇ INPUT1;B
[1]    D←2×D
[2]    DS←2×DS
[3]    KAPA←3×KP←KAPA
[4]    CAP[2]←2×CP
[5]    CR←2×CR
[6]    B←(3,KAPA)⍴0
[7]    B[;2×⍳KP]←A[;⍳KP]
[8]    AL←(÷1.1)×ALS←(AS←1.1×A←B)+.×⍳KAPA
     ∇
```

```
     ∇ INPUT2;B
[1]    B←(3,KAPA)⍴0
[2]    B[;3×⍳KP]←A[;2×⍳KP]
[3]    AL←(÷1.1)×ALS←(AS←1.1×A←B)+.×⍳KAPA
     ∇
```

```
     ∇ OUTPUT I
[1]    I COMPUTE 0
[2]    →(I=1)/3+⌶26
[3]    ESI←0.5×ESI
[4]    SSI←0.5×SSI
[5]    'STORAGE ';I
[6]    '=========='
[7]    'PROBALITY FOR AN OVERFLOW   : ';PO
[8]    'PROBALITY FOR AN EVACUATION : ';PE
[9]    'OVERFLOW AFTER               ';ET;' UT'
[10]   'WITH STANDARD DEVIATION      ';ST;' UT'
[11]   'EXPECTED STORAGE INVENTORY  : ';ESI;' UQ'
[12]   'WITH STANDARD DEVIATION      ';SSI;' UQ'
[13]   ' '
     ∇
```

```
    ∇ INPUT
[1]   KAPA←4
[2]   CAP← 12 10
[3]   CR← 9 8
[4]   A← 3 4 ρ0
[5]   A[1;]← 1.34469684 0.6241529838 0.289706152
      0.134469684
[6]   A[2;]← 2.0022061 1.17089 0.68474485
      0.40044122
[7]   A[3;]←A[2;]
[8]   D←0.2
[9]   DS←0.3
[10]  T←200
[11]  AL←(÷1.1)×ALS←(AS←1.1×A)+.×⍳KAPA
    ∇
```

COMPUTATION

```
STORAGE 1
=========
PROBALITY FOR AN OVERFLOW    : 0.02535
PROBALITY FOR AN EVACUATION  : 0.002456
OVERFLOW AFTER                 711.9 UT
WITH STANDARD DEVIATION        492.8 UT
EXPECTED STORAGE INVENTORY   : 8.046 UQ
WITH STANDARD DEVIATION        2.682 UQ

STORAGE 2
=========
PROBALITY FOR AN OVERFLOW    : 0.01867
PROBALITY FOR AN EVACUATION  : 0.06021
OVERFLOW AFTER                 865.3 UT
WITH STANDARD DEVIATION        749.2 UT
EXPECTED STORAGE INVENTORY   : 4.148 UQ
WITH STANDARD DEVIATION        3.274 UQ
```