

KERNFORSCHUNGSZENTRUM

KARLSRUHE

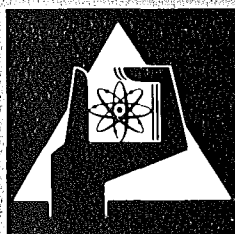
August 1977

KFK 2509

Institut für Neutronenphysik und Reaktortechnik

**Treatment of the Response of a Reactor to
Stochastic Reactivity Input**

N. K. Bansal



**GESELLSCHAFT
FÜR
KERNFORSCHUNG M.B.H.**

KARLSRUHE

Als Manuskript vervielfältigt

Für diesen Bericht behalten wir uns alle Rechte vor

GESELLSCHAFT FÜR KERNFORSCHUNG M. B. H.
KARLSRUHE

KERNFORSCHUNGSZENTRUM KARLSRUHE

KFK 2509

Institut für Neutronenphysik und Reaktortechnik

Treatment of the Response of a Reactor
to Stochastic Reactivity Input

N. K. Bansal*

*Permanent address: St. Stephen's College,
University of Delhi, Delhi 110007, INDIA

Gesellschaft für Kernforschung mbH., Karlsruhe

Contents

1. Introduction
 2. System's Equations
 3. Solution of the Exact System Equations
 4. A General Formulation for Calculating the Various
Correlation Functions of State Variables in a Reactor
System
 5. Critical Uncontrolled Reactor
 6. Conclusions
- References
- Appendix

Abstract

One of the important applications of reactor noise theory, which relies on the mathematical methods for treating stochastic processes, is to determine, either the confidence limits for the allowed deviations of the measured signals during normal reactor operation, or the statistical properties of their respective expectation values. In this report, we stress mainly on the general mathematical aspects for treating this problem.

A global description of a reactor system, perturbed by stochastic reactivity input, leads to a stochastic differential equation with parametric excitation. A discrepancy exists in literature about obtaining the correct solution of such an equation in its general frame. We discuss this discrepancy and review the work done for solving such an equation.

Some recent work indicates that linearisation of system's equations is justified in most cases of reactor operations. We develop a general scheme for calculating the various covariances and correlation functions in a stable and stationary system, which is perturbed by various noise sources and where linearisation of system's equations is justified. The formulation is easily extendable to an unstable, nonstationary system, like an uncontrolled critical reactor as demonstrated.

Untersuchung der Antwort eines Reaktors auf stochastische Reaktivitäts- anregung

Zusammenfassung

Eine der wichtigen Anwendungen der Reaktor-Rauschtheorie, die auf den mathematischen Methoden zur Behandlung von stochastischen Prozessen basiert, ist die Bestimmung der Vertrauensgrenzen für die bei ungestörtem Reaktorbetrieb zulässigen Abweichungen gemessener Reaktorsignale oder ihrer statistischen Kenngrößen vom jeweiligen Erwartungswert. In diesem Bericht betonen wir hauptsächlich die allgemeinen mathematischen Aspekte für die Behandlung dieses Problems.

Eine globale Beschreibung eines Reaktorsystems, das durch einen stochastischen Reaktivitäts-Input gestört ist, führt zu einer stochastischen Differentialgleichung mit parametrischer Anregung. In der Literatur gibt es eine Diskrepanz bezüglich einer korrekten Lösung einer solchen Gleichung in ihrer allgemeinen Form. Wir diskutieren diese Diskrepanz und geben eine Übersicht über die zur Lösung dieser Gleichung durchgeführten Arbeiten.

Einige neuere Arbeiten deuten darauf hin, daß für die meisten Fälle des Reaktorbetriebs eine Linearisierung der Zustandsgleichung gerechtfertigt ist. Wir entwickeln eine allgemeine Methode zur Berechnung verschiedener Kovarianzen und Korrelationsfunktionen für ein stabiles, stationäres System, das durch verschiedene Rauschquellen gestört ist, wobei eine Linearisierung der Zustandsgleichung erlaubt ist. Diese Formulierung kann, wie gezeigt, leicht auf instabile, nicht stationäre Systeme, wie z.B. einen unkontrollierten, kritischen Reaktor erweitert werden.

1. Introduction

In the theory of reactor noise, one should distinguish between the two categories of reactor noise, namely (a) zero power noise and (b) power noise.

The noise in reactors operating at low power, i.e. zero power noise, mainly arises due to the branching process in nuclear fission. In this case one deals with the behaviour of neutrons in an environment, which is fixed and stationary, or which changes with time in a deterministic manner. The behaviour of neutron noise source, in these cases, i.e. the branching process, is well known and may be considered to be arising from an external equivalent reactivity noise source, assumed to be white /1/. This additive term accounts for all correlated neutron pair terms resulting from the branching process within the neutron chains. For a system with well defined parameters and with a noise source of known statistical characteristics, it is not difficult to set up system's equations and treat the problem in all desired details. From various papers and reviews written on the subject, it is obvious that zeropower noise is fully understood and is finding applications in various laboratories /2/.

In a power reactor, the situation is, however, quite different. Zero power noise is still present in a power reactor, but in contrast to a fictitious noise equivalent source the true reactivity fluctuations due to sources such as thermohydraulics, the coupled processes of coolant flow and heat transfer, mechanical vibrations of fuel rods and control rods, completely masks the zeropower effects due to the branching processes. The physical reason for this is that the mean square amplitude of neutron fluctuations is proportional to reactor power for the branching process, while the reactivity induced noise is proportional to the square of reactor power in a power reactor /3/.

It is obvious that in contrast to a single noise source in a zero power reactor, there is a large number of noise sources in a power reactor. Thus in order to understand power noise completely, one should have a realistic

description of these various noise sources. An analysis of reactor noise can either be used (1) to obtain an early warning of the abnormal behaviour, or (2) to diagnose an existing malfunction. The use of noise analysis techniques to diagnose a malfunction can be made, if one has a priori knowledge about the type of noise expected for the type of malfunction being surveilled, i.e. a malfunction diagnosis would be possible if we could correlate various classes of defects with particular classes of reactivity noise spectra.

Another application of reactor noise theory comes, when we observe the response of the reactor system to a deterministic input, like a control rod movement to obtain an early information about its abnormal behaviour. Due to the omnipresence of fluctuations, the detector output will certainly deviate from the expected behaviour. It may then be our task to decide, whether the true behaviour of the detector signal is within the allowed deviations of the system or it is due to a malfunction. This problem could be termed as decision problem and for this we need assess the confidence limits for the allowed deviations of the measured signals. For such calculations it is not necessary to know exactly the detailed behaviour of all reactivity noise sources. Rather one could, in case of necessity, make physically acceptable, but not too conservative assumptions about them.

In a complete description of power reactor noise, space dependent effects should be included. Using simplified models, some work treating space dependence has been reported /4/ - /8/. Some experimental work on a boiling water reactor suggests that power noise may be treated by assuming it to be separable into two components, viz the local component and the global component /9/-/15/. The two components have different spatial correlations. The local component changes very rapidly along the axis, while the global component varies slowly in space. Using this model of local and global components, some authors have, recently, treated the power noise /16/ - /19/.

However, if one is interested in the mean square deviations of the system and if one wishes to take a decision about the allowed deviations in the signals of an out of core detector, it is sufficient to treat the reactor globally. Moreover, it should also be noted that an accurate description

of a reactor system perturbed by reactivity noise, even in the frame of the point reactor model treating global effects only, is complicated and leads to a stochastic differential equation with parametric excitation. The solution of such an equation has been of considerable interest and lead to some controversies. In Sec.3, we review the work done for the solution of such an equation and describe the limitations and the merits of this work. Earlier, in sec. 2 , we describe the state equations for a reactor system. A method for calculating the variances and covariances of a general reactor system perturbed by external noise sources is described in sec. 4 Finally, in sec. 5, we deal with some special aspects of a nonstationary system and show connections with the earlier reported work /52/-/53/.

2. The System's Equations: Parametric and Source Excitations

The kinetic equations of a general reactor system, in the point model approximation can be written in the following matrix equation form

$$\frac{dx(t)}{dt} = Ax(t) + A_1(t)x(t) \quad (2.1)$$

where $x(t)$ is the state vector, the components of which are various state variables, A is the system's matrix in the steady state and the matrix $A_1(t)$ contains the random parameters of the system. As an example, $x(t)$ may have the components

$$x(t) = \begin{bmatrix} N(t) \\ \vec{C}(t) \\ T(t) \end{bmatrix} \quad (2.2)$$

where N and \vec{C} are the neutron and delayed neutron precursor densities respectively and T is a representative temperature of the reactor.

For one group of delayed neutrons

$$A = \begin{bmatrix} -\frac{\beta}{\ell} & \lambda & \frac{\gamma N_0}{\ell} \\ \frac{\beta}{\ell} & -\lambda & 0 \\ \frac{T_d}{\tau} & 0 & -\frac{1}{\tau} \end{bmatrix} \quad (2.3)$$

and

$$A_1(t) = \begin{bmatrix} \rho(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (2.4)$$

where β is the delayed neutron fraction, λ the mean delayed neutron decay constant, ℓ is the mean neutron generation time and γ is the temperature-reactivity coefficient.

τ and T_d are the constants for power temperature coupling, τ being the time constant for temperature-changes in the core and T_d the difference in temperature resulting from the doubling of the neutron density. N_0 corresponds to the average neutron density.

For a system, which is critical on the average, $\rho(t)$ represents the fluctuations about the average zero value of the reactivity. In Eq. (2.1) $\rho(t)$ appears as a dependent term on the neutron density rather than as an inhomogeneous term like an external source. Such an equation is termed as a stochastic differential equation with parametric excitation.

The usual procedure in reactor kinetics is to linearise these system's equations (2.1), assuming that the fluctuations are small and the second order effects can be neglected. When linearised, the system's equations can be written as

$$\frac{dx(t)}{dt} = A(x(t)-x_0) + R(t) \quad (2.5)$$

where the noise source $R(t)$ is a functional of the random matrix $A_1(t)$ and average state vector x_0 , i.e.

$$R(t) = f(A_1(t), x_0) \quad (2.6)$$

It should be noted here, that after linearisation, the fluctuating term appears as an inhomogeneous term like an external source and this type of equation is termed a random differential equation with source excitation.

In the exact system's equations (2.1) as well as in the linearised system's equations (2.5), if $A_1(t)$ and $R(t)$ respectively are random, the corresponding solutions will also be random. The solution of such an equation then consists in finding the statistical properties of $x(t)$ given those of $A_1(t)$ or $R(t)$. This means that if the probability distribution of $A_1(t)$ or $R(t)$ is given, we have to either find the joint probability distribution of state variables or some moments of interest for the probability distribution.

The solution of an equation such as (2.1), has been a subject of considerable interest, for a long time, to mathematicians and scientists, because of the high importance of parametric excitations in technical fields like electronics and plasma physics. In reactor physics, we generally encounter a rather fortunate situation of dealing with a stable and stationary system, showing small fluctuations about its mean behaviour. Hence, one has more confidence in approximating the system with a linearized set of equations. However, as discussed briefly later, linearisation

in reactor physics is also not always valid and the higher order moments being more sensitive to parametric excitation, an extension of the linearised treatment to systems with many state variables for obtaining higher order moments should be done only after getting some results in the exact frame of parametric excitation. Without performing such calculations, the validity of linearisation can not be established. Because of this the solution of exact system's equations, rather recently, has drawn the attention also of reactor physicists.

3. Solution of the Exact System's Equations: Parametric Excitation

The function $A_1(t)$, in Eq. (2.1), is a random process, i.e. for each fixed t , the value of the function $A_1(t)$ is a random variable. A very commonly made assumption about $A_1(t)$ is that it is a stationary Gaussian white noise random process. This means that for each fixed t , the random variable has a Gaussian distribution with zero mean, i.e.

$$\langle A_1(t) \rangle \equiv 0, \quad (3.1a)$$

where the brackets $\langle \rangle$ denote the expectation, i.e. averaging over the statistical ensemble. Further, for any two times t_1, t_2 , with $t_1 \neq t_2$, the two random variables $A_1(t_1)$ and $A_1(t_2)$ are completely independent of each other. Mathematically

$$\langle A_1(t_1) A_1'(t_2) \rangle = c \delta(t_1 - t_2), \quad (3.1b)$$

where prime denotes the transpose of the matrix and the constant c is the white noise covariance matrix, which expresses how the components of $A_1(t_1)$ are correlated amongst themselves. The name white comes from the fact that the power spectral density function in this case is constant, independent of frequency, analogous to the spectrum of white light.

In the theory of random processes, white noise is the same kind of mathematical pathology, as the Dirac delta function is in the theory of deterministic functions. As long as one does only linear operations on a delta function, it is usually possible to interpret the result in a meaningful way. However, one runs into trouble in trying to do nonlinear things to a delta function. The square or the logarithmic of a delta function is meaningless. A similar situation exists in the case of white noise. For the linearised system equation (2.5), there is no difficulty in interpreting, what is meant by a solution of this differential equation. As a function of t , $x(t)$ turns out to be a Gaussian process, and there is no controversy about computing the means and covariances of this process.

However, when the exact system eq. (2.1) is considered, a problem of interpretation arises. A controversy exists in the literature about the correct interpretation of such an equation and gives rise to two equations for the joint probability distribution of state variables, which are different from each other. About the discussion of this discrepancy, we would refer the reader to a paper by Gray and Caughey /20/.

In reactor physics, the problem has been discussed rather recently by Akcasu and Karasulu /21/. The arguments are nearly the same as given in the papers by Gray and Caughey, but in the frame of reactor physics for the first time. The reasons for obtaining two different equations for the mean number of neutrons have also been discussed by Williams /22/, Karmeshu and Bansal /23/ and Kishida /24/.

Reading through these papers, however, one gets the impression that the subject is more bewildering to the reader than it was before he read the paper. One should note the point that the fundamental difficulty arises from the properties of the heuristic mathematical idealisation i.e. white noise or its rigorous counterpart Brownian motion, which is heuristically the time integral of white noise. The trouble arises when one attempts to apply the usual rules of differential and integral calculus to functions of time, which are actually sample functions of a stochastic process. Two possible ways of extending ordinary calculus to stochastic functions have been forwarded. These are known as the Ito calculus and Stratonovich calculus. From an engineering point of view, a very good review has been given by Mortensen /25/, who suggests an approach to the problems of

mathematical modeling analysis and computation, which seems to have the qualities of being both mathematically rigorous and consistent with physical reality.

The subject of solving stochastic differential equation with parametric excitation has been treated by a number of other workers /26-36/ and has now appeared in various books also on random differential equations /37,38/.

In reactor physics, the interest in the solution of such an equation started mainly from the work by Williams /39/.

He studied the point model reactor kinetic equations with one group of delayed neutrons. Assuming the reactivity to be a white noise function, the following Fokker Planck equation for the probability distribution of neutron density (without source), was formulated

$$\begin{aligned} \frac{\partial P}{\partial t} = & - \frac{\partial}{\partial N} \left\{ \left(\frac{1}{\ell} (\rho_0 - \beta) N + \lambda C \right) P \right\} \\ & - \frac{\partial}{\partial C} \left\{ \left(\frac{\beta}{\ell} N - \lambda C \right) P \right\} \\ & + \frac{\partial^2}{\partial N^2} \left\{ \left(\frac{\sigma_{11}}{\ell^2} N^2 \right) P \right\} \end{aligned} \quad (3.2)$$

where σ_{ij}^2 are a measure of the amplitude of fluctuations.

The equation (3.2) is obtained if one uses Ito calculus for evaluating the stochastic integral

$$\int_{t_0}^t N(s) dw(s) \quad (3.3)$$

However, if one uses Stratonovich calculus the Fokker Planck equation obtained is

$$\begin{aligned} \frac{\partial P}{\partial t} = & - \frac{\partial}{\partial N} \left\{ \left[\frac{\rho_0 - \beta}{\ell} - \frac{\sigma_{11}^2}{\ell^2} \right] N + \lambda C \right\} P \\ & - \frac{\partial}{\partial C} \left\{ \left(\frac{\beta}{\ell} N - \lambda C \right) P \right\} + \frac{\partial^2}{\partial N^2} \left\{ \left(\frac{\sigma_{11}^2}{\ell^2} N^2 \right) P \right\} \end{aligned} \quad (3.4)$$

The question now is, which one of the above two equations (3.2 and 3.4) is the correct one. Without going into the details of their derivation, we would like to point out that Ito calculus is a strictly formulated mathematical problem, whereas Stratonovich calculus is a mathematical approximation to a physical problem. The mathematician starts with the transition density for the process for discussing Markov process. He is able to associate a Fokker Planck equation in an unambiguous way with this transition density. When he finds that he has two possible ways of modelling the process as the solution to a stochastic differential equation, he will choose the way which has the most mathematical elegance in its internal structure, and which is capable of the greatest generalization. Considered from this point, the Ito calculus is the right choice.

However, an engineer can not resolve the issue on the basis of mathematical elegance alone. The engineer does not start with the transition density. He starts with a differential equation (point reactor kinetic equations) which has been obtained on the basis of known physical laws. He then adds a white noise forcing term to get a stochastic model. If the coefficient of the noise is itself random, than there are two possible ways of interpreting the equation, leading to two different Fokker Planck equations and two different processes. The question is, which process does one "really" get in the physical world.

Without reproducing the arguments given in the literature /41-43/ we point out what seems to be the conclusions of interest.

Let $\Delta(t)$ be a stationary delta correlated normal process with zero mean (i.e., a white noise):

$$\begin{aligned} \langle \Delta(t) \rangle &= 0 \\ \langle \Delta(t_1) \Delta(t_2) \rangle &= \sigma^2 \delta(t_2 - t_1) \end{aligned} \quad (3.5)$$

This process is not physically realizable because it has a constant spectrum (the Fourier transform of the correlation function) i.e., infinite energy. Let $\Delta_z(t)$ be a physical approximation to $\Delta(t)$, i.e., a stationary, zero mean normal process with a correlation function concentrated mostly around $t_1 = t_2$:

$$\begin{aligned} \langle \Delta_z(t) \rangle &= 0 \\ \langle \Delta_z(t_1) \Delta_z(t_2) \rangle &= \sigma^2 h_\tau(t_2 - t_1) \end{aligned} \tag{3.6}$$

with

$$\lim_{\tau \rightarrow 0} h_\tau(t_2 - t_1) = \delta(t_2 - t_1) \tag{3.7}$$

Consider now the stochastic equation:

$$\frac{dx_\tau}{dt} = f(x_\tau) + g(x_\tau) \Delta_z(t) \tag{3.8}$$

with f and g arbitrary (deterministic) functions. (3.8) defines a stochastic process $x_\tau(t)$ which is not a diffusion process (in general it is not even Markov). If $h_\tau(s)$ is well behaved as $s \rightarrow 0$, $x_\tau(t)$ has sample paths that possess a derivative (almost everywhere). Therefore Eq.(3.8) is meaningful and can be formally integrated with an initial condition $x_\tau(t_0) = x_\tau^0$ to yield the solution $x_\tau(t)$. Let us now take the limit of (3.8) as $\tau \rightarrow 0$:

$$\frac{dx}{dt} = f(x) + g(x) \Delta(t) \tag{3.9}$$

where $\Delta(t)$ satisfies (3.5). In this case $x(t)$ is a diffusion process. Using the symmetrized integral defined by Stratonovich, Eq. (3.9) can be handled like an ordinary differential equation. The result is that the transition

probability density function ψ of $x(t)$ satisfies the Fokker Planck equation:

$$\frac{\partial \psi}{\partial t} = - \frac{\partial}{\partial x} \left\{ \left[f(x) + g(x)g'(x)\frac{\sigma^2}{2} \right] \psi \right\} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} [g^2(x)\psi] \quad (3.10)$$

Therefore, the process described by this F.P. equation is the one to which the physical process $x_\tau(t)$ converges.

What about if the Brownian motion between $w(t)$ is used? It should be recalled that $dw(t)$ exists, but not dw/dt . Therefore, the identity

$$\Delta(t) \equiv \frac{dw(t)}{dt} \quad (3.11)$$

is misleading. However, one can show that Eq.(3.9) can be thrown into the equivalent form

$$dx = \left(f + \frac{\sigma^2}{2} gg' \right) dt + \sigma g dw \quad (3.12)$$

where now Ito's integral must be used. Rewriting (3.9) as:

$$dx = f dt + g dw \quad (3.13)$$

is equivalent to changing model. The physical process $x_\tau(t)$ does not converge to the solution of (3.13).

Since there is no physical process which is strictly delta-correlated, the safest procedure is to use equations such as (3.9) together with Stratonovich's recipe.

Assuming the reactivity to be a Gaussian, stationary random function, the problem of neutron density fluctuations was again considered by Williams /44/ in 1971, where three simplified cases were studied

- (1) no delayed neutrons
- (2) prompt jump approximation
- (3) infinite delay time model, i.e. assuming that the delayed neutrons concentration remains constant during a transient and equal to its value prior to reactivity insertion.

For each of the three simple cases, the probability distributions for the resulting neutron densities have been computed and were shown to be non Gaussian. However, it is not possible to extend this analysis for obtaining the basic probability distribution in case of second order systems. Hence, Williams investigated the possibility of obtaining the mean and variances directly from the basic equations i.e. without solving it first. However, even for these calculations, he had to introduce the approximation (see Appendix)

$$\langle A(t)\Delta(t')N(t') \rangle = \langle \Delta(t)\Delta(t') \rangle \langle N(t') \rangle \quad (3.14)$$

But even after using this approximation, the moments obtained by Williams were of first order only, when delayed neutrons (i.e. the second order effects) are included.

In 1975, Karmeshu and Bansal /45/ found the first and second order moments of neutrons and precursors by assuming the fluctuations in reactivity to be a D.M.P., which is defined as a two valued stepwise constant Markov process in which the time changes are exponentially correlated. With this choice for the random process, the artificially introduced approximation (3.14) becomes exactly valid. Hence the expressions for the moments obtained by these authors are exact /23/. But from these expressions it is very difficult to extract, directly, any usable information and one has no hope for extending the analysis to systems with state vectors having more than two components.

Next to mention in this field is the work of Akcasu and Karasulu /29/. Using the simple point kinetic model with one group of delayed neutrons and no feedback terms, they formulate the Fokker Planck equation for the probability density using Stratonovich calculus. As mentioned earlier, these authors, for the first time in reactor physics, discuss the discrepancy about the use of a particular type of calculus.

For the simple case of no delayed neutrons, an exact form of the Fokker Planck equation is derived for a delta correlated reactivity from the stochastic Liouville equation, for the first time in reactor physics by Karmeshu and Bansal /46/. It is possible to extend this treatment to get the complete probability balance equation for an extended system /36/ (i.e. with two or more state variables), but the solution of this equation can only be obtained in some special cases /47/. Moreover it should be noted that the assumption of white noise has been made for the noise sources and this limiting case, sometimes leads to the problem of identifying various parameters to real kinetic parameters because of its infinite total power.

In spite of the limitations of the work done in the frame of parametric excitation, such as the noncapability of extension to multicomponent state vector and the direct extraction of useful information, due credit should be given to this. Firstly they are a step ahead of the linearised approximation in giving a solution for the nonlinearised kinetic equations. Secondly, without obtaining at least some results in the nonlinear frame, one would never be sure about the applicability of linearisation. Worthy to note in this respect is the note of Kebabze and Adamovski /48/. This work concludes that though linearisation is acceptable in most cases of a reactor, it is not sufficient in the vicinity of, and for fixing stability thresholds. This seems plausible as the nonlinearities may lead to either stabilising or destabilising effects.

In a recent work, Dutré and Deboscher /49/ tabulated the cases where linearisation is justified. Employing a point reactor model, which applies to low frequency range, where the delayed neutron dynamics can be reduced to the use of an effective neutron life time and where the feedback processes can be considered as promptly responding to power variations, these

authors, in general, conclude that for large α , the linearisation of system equations leads to accurate results for the probability distribution of state variables, where

$$\alpha = \frac{\ell k_0}{G_\rho}$$

and ℓ - effective neutron life time partially accounting for delayed neutrons
 $= (1-\beta)\ell_p + \sum \beta_i(\ell_p + \frac{1}{\lambda_i})$

G_ρ - mean square amplitude of reactivity fluctuations

k_0 - amount of reactivity needed to keep the reactor at operating condition

We find that the condition of large α is completely fulfilled.

If our aim is to find expressions for the confidence limits to the mean square deviations observed in a detector placed outside the core of a reactor, where one concerns with a general reactor system, it is obvious that one has to deal with a state vector with many components. It is thus essential to have a formalism which is transparent and easily operable for any general system.

4. A General Formulation for Calculating the Various Correlation Functions of State Variables in a Reactor System: Linearised State Equations

If we envisage a practical situation of a critical power reactor operating at steady state and at high power level, where the power fluctuations are really small and if we are working in situations far away from stability margins, we can employ a linearised system of equations, i.e.

$$\frac{dx(t)}{dt} = A x(t) + R(t) \quad (4.1)$$

where $R(t)$ is a vector whose components represent various noise sources present in the system. As the most acceptable model for $R(t)$, we assume it to be generated by a linear system forced by a white noise vector i.e.

$$\frac{dR(t)}{dt} = B R(t) + W(t) \quad (4.2)$$

where B is a square matrix.

The physical system described by Eqs. (4.1) and (4.2) is drawn in Fig. (1)

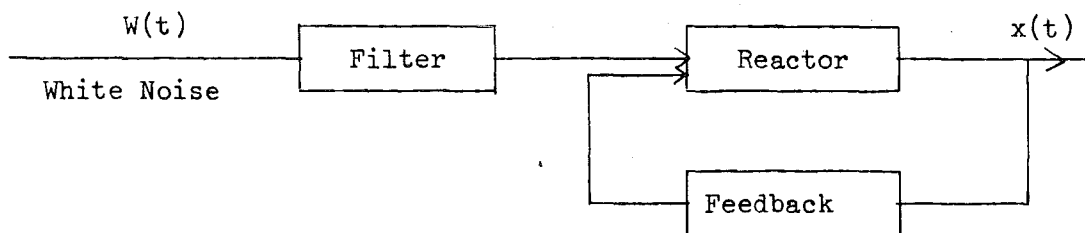


Fig. 1

The vector $W(t)$, contains various white noise sources and has the properties described by Eqs. (3.1) for $A_1(t)$, with a constant (or time independent) matrix c . The matrix B in Eq. (4.2) distributes various primary noise sources to the components of the state vector and is also necessary for introducing adequate corner frequencies.

Differential equations (4.1) and (4.2) can be combined to obtain another differential equation for an extended system, i.e.

$$\frac{dP(t)}{dt} = M P(t) + F(t) \quad (4.3)$$

where

$$P = \begin{pmatrix} x \\ R \end{pmatrix}, \quad (4.4a)$$

$$M = \begin{bmatrix} A & I \\ 0 & B \end{bmatrix}, \quad (4.4b)$$

$$F(t) = \begin{bmatrix} 0 \\ W(t) \end{bmatrix}. \quad (4.4c)$$

Let us now assume that the system is stable and stationary. This requires that all the eigenvalues of M have real negative parts, which means that A and B have eigenvalues with real negative parts. Then if A is a stable matrix, a stable B matrix should be used and the solution of Eq. (4.3) may be written as

$$P(t) = \int_{-\infty}^t e^{M(t-u)} F(u) du \quad (4.5)$$

where e^{Mu} is the exponential of a matrix defined by

$$e^{Mu} = \sum_{k=0}^{\infty} \frac{(Mu)^k}{k!} \quad (4.6)$$

Taking the expectation of Eq. (4.5) we get

$$\langle P(t) \rangle = 0$$

because of the fact that W(t) satisfies Eqs. (3.1) and because of the linearity of system's equations.

The covariance matrix is obtained by multiplying Eq. (4.5) by its transpose at some other time, say s, and taking the ensemble average; we have

$$\langle P(t) P'(s) \rangle = \int_{-\infty}^t du \int_{-\infty}^s dv e^{M(t-u)} \langle F(u) F'(v) \rangle e^{M'(s-u)} \quad (4.8)$$

From the definition (4.4c) of $F(u)$ and from the properties of $W(t)$ given by Eq. (3.1) we obtain

$$\langle P(t) P'(s) \rangle = \int_{-\infty}^t du \int_{-\infty}^s dv e^{M(t-u)} \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} \delta(u-v) e^{M'(s-v)}$$

Assuming arbitrarily, $t \geq s$, we obtain with some transformation

$$\langle P(t) P'(s) \rangle = e^{M(t-s)} \int_0^{\infty} e^{Mx} \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} e^{M'x} dx \quad (4.9)$$

The above expression gives the complete correlation matrix, the components of which are various auto and cross-correlation functions for the state variables. Expression (4.9) describes better than the expressions for the individual components, the dynamics of the system. It also ensures stationarity because of the dependence of various terms on the time difference $|t-s|$ only, because $t \geq s$ is only an arbitrary assumption. For treating the equation of the type such as (4.9), one can take recourse to established mathematical techniques.

The exponential of a matrix e^{Mt} , informally defined by (4.6), can be evaluated by using Sylvester's theorem, according to which

$$\exp(Mt) = \sum_{R=1}^n Z_R \exp(\lambda_R t) \quad (4.10)$$

where λ 's are the eigenvalues of the matrix M and

$$Z_R = \prod_{\substack{j=1 \\ j \neq k}}^n (M - \lambda_j I) / (\lambda_k - \lambda_j) \quad (4.11)$$

It is also possible and some times more transparent, specially to reactor physicists, to express the matrix in terms of its eigenvalues, eigenvectors and adjoint eigenvectors. In this representation, we have

$$e^{Mt} = U e^{\Lambda t} V \quad (4.12)$$

where

$$V = U^{-1} = \Lambda^{-1} U^{\dagger},$$

U being the matrix formed by the right hand eigenvectors and U^{\dagger} is the matrix formed by the adjoint left hand vecors. The diagonal matrix Λ is obtained from

$$U^{\dagger} U = \Lambda$$

For finding eigenvalues, one can use the standard computer programmes supplied as part of the system. However, it has been observed that even for simple systems, like described by Eq. (2.3), it is necessary to run the programme in double precision (on IBM and similar computers).

To find the constant term given by the integral

$$\int_0^{\infty} e^{Mx} S e^{M'x} dx = Y \quad (4.13)$$

where

$$S = \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} \quad (4.14)$$

we may multiply it by M from left

$$\int_0^{\infty} \frac{d}{dx} e^{Mx} S e^{M'x} dx = MY \quad (4.15)$$

and then from right by M' , i.e.

$$\int_0^{\infty} dx e^{Mx} S \frac{d}{dx} (e^{M'x}) = YM' \quad (4.16)$$

Addition of expressions (4.15) and (4.16) yields

$$\int_0^{\infty} \frac{d}{dx} (e^{Mx} S e^{M'x}) dx = MY + YM' \quad (4.17)$$

Since M has all eigenvalues with real negative parts, we have

$$0 = S + MY + YM' \quad (4.18)$$

The solution of Eq. (4.18) can be obtained following Dalfes /50/.

Y satisfies the symmetric property $Y = Y'$ and also S is symmetric. We can write

$$M = U \Lambda V \quad \text{and} \quad M' = V' \Lambda U' \quad (4.19)$$

where $V = U^{-1}$ and Λ is a diagonal matrix. Substituting (4.19) in (4.18), we get

$$\Lambda \Gamma + \Gamma \Lambda + VSU' = 0 \quad (4.20)$$

where

$$\Gamma = V Y U' \quad (4.21)$$

The solution of Eq. (4.20) can be written now as Λ is diagonal. One obtains for $k, l=0, \dots, m$

$$\Gamma_{kl} = - \frac{1}{\lambda_k - \lambda_l} \sum_{i,j=0}^N v_{ki} u'_{jl} s_{ij} \quad (4.23)$$

and from Eq. (4.22)

$$Y = U \Gamma V' \quad (4.24)$$

It should be pointed out that for the solution of Eq. (4.18), one may also rely on methods, proposed by Wiberg /51/ for solving a more general equation containing a nonlinear term, i.e. a matrix Riccati equation.

With the solution of Eq. (4.18) for Y and an expression (4.12) for the exponential of a matrix, we have a formulation (Eq. 4.9) to calculate the exact time development of all the correlation functions in any general reactor system perturbed by noise sources, which may be of a general type. One needs only to redefine the parameters characteristic of the system to obtain the desired results.

5. Critical Uncontrolled Reactor: One Group of Delayed Neutrons:

In this section we consider the special case of an uncontrolled reactor, at criticality, which is perturbed by stochastic reactivity input. The special feature of such an uncontrolled system is that it is nonstationary and hence the state of the system at any time would depend on its initial state. However, it is possible to extend the treatment developed in sec. 4 for a stationary system, to nonstationary systems by writing the solution (Eq. 4.5) as

$$P(t) = e^{Mt} P(0) + \int_D^t e^{M(t-s)} F(s) ds \quad (5.1)$$

In our earlier work /52,53 / we had considered the case of a critical reactor driven by band limited white reactivity noise with a corner frequency w_c . The first and second order moments were computed by employing two complementary assumptions, depending on the corner frequency w_c . Two limiting

cases of corner frequency were considered viz (1) $w_c \gg \beta/\lambda$ and (2) $w_c \ll \beta/\lambda$. The calculations were performed by linearising the equations in the first case, while for the second case the treatment was done in the prompt jump approximation, where it is possible to get the moments for the nonlinearised exact equations. The results of this analysis show /53/ an equivalence between linearised and nonlinearised treatments for most cases of practical interest.

As a special result of the above analysis, it was found that for an initial equilibrium between neutrons and precursors, all the normalised covariances have the same asymptotic development, i.e. $2Dt$, where D (we call it a diffusion coefficient) is expressed in terms of usual kinetic parameters. An estimate about the order of D gives a very low magnitude. This means that the reactor system is a damped system as a result of the precursors present in the system.

6. Conclusions

An important application of reactor noise theory is to assess the confidence limits for the allowed deviations of the measured signals during normal reactor operation. For taking a decision about the allowed deviations, from its expected behaviour, in the signals of a detector, placed outside the core of a reactor, it is sufficient to treat the reactor globally. For this one can use the point reactor model for the kinetic equations, where the reactivity appears as a dependent parameter on the neutron density. If the reactivity is fluctuating in a random manner, then we have to deal with the solution of a stochastic differential equation with parametric excitation. The solution of such an equation has led to a controversy, which we have discussed. A review of the work done for the solution of such equations has been presented.

In reactor kinetic problems, where one has to deal with a rather large system with a multicomponent state vector, the analysis, in an exact frame of parametric excitations, becomes very difficult. This is evident from the work done for the simple model of two components, i.e. the neutron and the precursor number, state vector. For a practical situation, where the

linearisation is a good approximation, a stable critical reactor operating at high power level with relatively small fluctuations of the state variables about their mean values, a complete matrix formulation is presented. From this formulation one could compute the exact time development of the various correlation functions. Input noise source to this system could rather be of general type.

A critical reactor without a control is an unstable, nonstationary system. The transient state of such a system depends on its initial state. It is, however, possible to extend the treatment, developed for the stationary system, to a nonstationary system if its initial state is known. For a single group of delayed neutrons and for a bandlimited white reactivity noise driving the system, we have calculated the exact time development of first and second order moments for the neutrons and precursors in /52/ and /53/. As a special result it is found that all the covariances have the common asymptotic time development, suggesting that the system behaves as a composite Brownian particle.

Acknowledgements

This report was written during authors visit, as a fellow of the German Academic Exchange Service, to the Institute for Neutronphysics and Reaktortechnik, Karlsruhe.

The author is thankful to the institute for the hospitality extended to him. The benefit obtained through unaccountable discussions with Dr. H. Borgwaldt is gratefully acknowledged. The author feels indebted to Dr. M. Edelmann for his untiring help during his entire stay in the institute.

Special thanks are due to Prof. Ricciardi of the University of Chicago, U.S.A., for some clarifications.

References

- /1/ C.E. Cohn
"A Simplified Theory of Pile Noise"
Nucl. Sci. Eng., 7, 472 (1960)
- /2/ W. Seifritz and D. Stegemann
"Reactor Noise Analysis"
Atomic Energy Review (IAEA), 9 (1), 129 (1971)
- /3/ H. Borgwaldt
"Neutron Noise in a Reactor with an External Control Loop"
Nukleonik, 11, 76 (1968).
- /4/ G. Kosaly, L. Maroti and L. Mesko
"A Simple Space Dependent Theory of the Neutron Noise in a
Boiling Water Reactor"
Annals of Nuclear Energy, 2, 315 (1975)
- /5/ G. Kosaly and L. Mesko
"Theory of the Autospectrum of the Local Component of
Power Reactor Noise"
Annals of Nuclear Energy, 3, 233 (1976)
- /6/ K. Behringer, G. Kosaly and Lj. Kostic
"Theoretical Investigations of the Local and the Global
Components of the Neutron Noise Field in a Boiling Water Reactor"
(To be published in Nucl. Sci. and Eng., 1977)
- /7/ G. Kosaly
"Investigation of the Local Component of Power Reactor Noise
via Diffusion Theory"
Report of the Central Research Institute for Physics,
Budapest KFKI-75-27 (1975)
- /8/ H. Dam
"Neutron Noise in Boiling Water Reactors"
Atomkernenergie, 27, 8 (1976)

- / 9/ W. Seifritz
"An Analysis of the Space Dependent Neutron Flux Density Fluctuations on the Lingeln Boiling Water Reactor (KWL) by Methods of Stochastic Processes"
Atomkernenergie, 19, 271 (1972)
- /10/ D. Wach and G. Kosaly
"Investigation of the Joint Effect of Local and Global Driving Sources in Incore Neutron Measurements"
Atomkernenergie, 23, 244 (1974)
- /11/ Y. Ando, N. Naito, A. Tababe and N. Kitamura
"Void Detection in BWR by Noise Analysis"
Journal of Nuclear Science and Technology (Japan), 12, 597 (1975)
- /12/ D. Stegemann, P. Gebureck, A.T. Mikulski and W. Seifritz
"Operating Characteristics of a Boiling Water Reactor"
Paper Presented at the Symposium on Power Plant Dynamics, Control and Testing, Knoxville, Tenn. (1973)
- /13/ T. Nomura
"BWR Noise Spectra and Application of Noise Analysis to FBR"
Annals of Nuclear Energy, 2, 379 (1975)
- /14/ P.E. Blomberg and F. Åkerhielm
"A Contribution to the Experience of Noise Measurements and Analysis in BWR Power Plant"
Annals of Nuclear Energy, 2, 323 (1975)
- /15/ D. Wach and G. Kosaly
"Investigation of the Joint Effect of Local and Global Driving Sources in Incore Neutron Noise Measurements"
Atomkernenergie, 23, 244 (1975)
- /16/ D. Wach
"Ein neues Messmodell der Neutronenflußkreuzleistungsdichte zur Identifizierung von Rauschquellen in Leistungsreaktoren"
Ph. D. Thesis, Technical University Munich (1976)

- /17/ W. Seifritz and F. Ciocli
"On-load Monitoring of Local Steam Velocity by Neutron
Noise Analysis"
Trans. Am. Nucl. Soc., 17, 451 (1973)
- /18/ D. Wach
"Ermittlung lokaler Dampfblasengeschwindigkeiten aus den Rausch-
signalen von Incore-Ionisationskammern"
Atomwirtschaft, 18, 580 (1973)
- /19/ Lj. Kostic
"Local Steam Bubble Velocity Measurements in a BWR Using
Reactor Noise Analysis Techniques"
Atomwirtschaft 21, 363 (1976)
- /20/ A.H. Gray and T.K. Caughey
"A Controversy in Problems Involving Random Parametric
Excitation"
Jr. Math. and Phys., Vol. XL IV, 288 (1965)
- /21/ Z.A. Akcasu and M. Karasulu
"Nonlinear Response of Point Reactors to Stochastic Inputs"
Annals of Nuclear Energy, 3, 11 (1976)
- /22/ M.M.R. Williams
"Comments on "Systems with Stochastic Parameters"
Nucl. Sci. Eng., 60, 447 (1976)
- /23/ Karmeshu and N.K. Bansal
"Rejoinder to "Comments on Systems with Stochastic Parameters"
Nucl. Sci. Eng., 60, 448 (1976)
- /24/ K. Kishida
"Theoretical Studies of Reactor Noise Analysis Based on Statistical Physics"
Ph. D. Thesis, Osaka University, Osaka, Japan (1977)
- /25/ R.E. Mortensen
"Mathematical Problems of Modeling Stochastic Nonlinear Dynamic Systems"
NASA CR-1168 (1968)

- /26/ J.C. Samuels
"On the Stability of Random Systems and the Stabilisation
of Deterministic Systems with Random Noise"
The Journal of the Acoustical Society of America, 32 (5), 594 (1960)
- /27/ S.T. Ariaratnam
"Random Vibrations of Nonlinear Suspensions"
The Jr. Mech. Eng. Sci, 2, 195 (1960)
- /28/ M.A. Leibowitz
"Statistical Behaviour of Linear Systems with Randomly
Varying Parameters"
J. Math. Phys., 4, 852 (1963)
- /29/ R. Kubo
"Stochastic Liouville Equations"
J. Math. Phys., 4, 174 (1963)
- /30/ S.T. Ariaratnam and P.W.U. Graefe
"Linear Systems with Stochastic Coefficients
Pt I, II, III"
International J. Control, Vol. I, 239 (1965);
Vol. II, 161 (1965), Vol. III, 205 (1965)
- /31/ A.H. Gray
"Behaviour of Linear Systems with Random Parametric Excitation"
The Journal of the Ac. Soc. of America, 37, 235 (1965)
- /32/ W.M. Wonham
"Random Differential Equations in Control Theory"
in Probabilistic Methods in Applied Mathematics, Vol. 2,
Ed. A.T. Bharacha Reid, Academic Press, New York (1970)
- /33/ J.A. Morrison and J. McKenna
"Analysis of Some Stochastic Ordinary Differential Equations"
Bell Labs., Murray Hill, N.J. (1972)

- /34/ "Stochastic Differential Equations", Ed. Joseph B. Keller and Henry P. McKean, Proc. of a Symposium in Applied Mathematics of the American Mathematical Society and the Society for Industrial and Applied Mathematics, held in New York City, March 29-30 (1972)
- /35/ J.A. Morrison
"Moments and Correlation Functions of Solutions of Some Stochastic Matrix Differential Equations"
J. Math. Phys., 13, 299 (1972)
- /36/ A. Brissaud and U. Frisch
"Solving Linear Stochastic Differential Equations"
J. Math. Phys., 15, 524 (1974)
- /37/ S.K. Srinivasan and R. Vasudevan
"Introduction to Random Differential Equations and Their Applications"
Elsevier, New York (1971)
- /38/ T. J. Soong
"Random Differential Equations in Science and Engineering"
Academic Press, New York (1973)
- /39/ M.M.R. Williams
"Probability Distributions in Simple Neutronic Systems via the Fokker Planck Equation"
Jr. Nucl. Energy, 23, 633 (1969)
- /40/ J.L. Doob
"Stochastic Processes"
John Wiley and Sons, Inc., New York (1953)
- /41/ R.L. Stratonovich,
"Conditional Markov Processes and their Applications to the Theory of Optimal Control"
pp. 42-55. Elsevier, 1968
- /42/ L. Arnold
"Stochastic Differential Equations: Theory and Applications"
John Wiley and Sons, Inc., New York (1976). Mainly Chapter 10.

- /43/ E. Wong
"Stochastic Processes in Information and Dynamical Systems"
Mainly Chapter 4. McGraw, 1971
- /44/ M.M.R. Williams
"The kinetic Behaviour of Simple Neutronic Systems with Randomly
Fluctuating Parameters"
Jr. Nucl. Energy, 25, 563 (1971)
- /45/ Karmeshu and N.K. Bansal
"Stability of Moments in a Simple Neutronic System with
Stochastic Parameters"
Nucl. Sci. and Eng., 58, 321 (1975)
- /46/ Karmeshu and N.K. Bansal
"Statistical Description of a Point Reactor in the Absence
of Delayed Neutrons"
Atomkernenergie, 28, 79 (1976)
- /47/ E. Wong
"The Construction of a Class of Stationary Markoff Processes"
In Proceedings of the Symposium on Applied Mathematics,
Ed. R. Bellman, American Mathematical Society, Philadelphia (1963)
- /48/ B.V. Keadze and L.A. Adamovski
"Non-linearity Consideration when Analyzing Reactor Noise
Statistical Characteristics"
Annals of Nuclear Energy, 2, 337 (1975)
- /49/ W.L. Dutré and A.F. Debosscher
"Exact Statistical Analysis of Non-linear Dynamic Nuclear Power
Reactor Models by the Fokker Planck Method - Part I: Reactor with
Direct Power Feedback"
Nucl. Sci. Eng., 62, 355 (1971)
- /50/ A. Dalfes
"The Correlation Function and Power Spectral Density of Nuclear
Reactors"
Nukleonik, 6, 5 (1964)

/51/ D.M. Wiberg

"Identification of the LMFBR Dynamic State for Detection of
Coolant Boiling"

Gesellschaft für Kernforschung mbH., Karlsruhe, KFK 1911 (1974)

/52/ N.K. Bansal and H. Borgwaldt

"The Asymptotic Behaviour of a Critical Point Reactor
in the Absence of a Controller"

Gesellschaft für Kernforschung mbH., Karlsruhe, KFK 2367 (1976)

/53/ N.K. Bansal and H. Borgwaldt

"Quasistatic Treatment of the Influence of Bandlimited White
Reactivity Noise on an Uncontrolled Critical Point Reactor"

Gesellschaft für Kernforschung mbH., Karlsruhe, KFK 2424 (1977)

Appendix

The necessity to introduce the approximation (3.14) arises due to the so called problem of closure, which is encountered at a number of places in mathematics. For example, in the solution of neutron transport equation, if one expands the flux in terms of a series of Legendre polynomials, then one gets an equation for the n^{th} order term in terms of the $(n+1)^{\text{th}}$ order term. One, then tries to express the $(n+1)^{\text{th}}$ term in terms of the lower order terms. Similarly, in the solution of random differential equations, one gets an expression for a moment in terms of one higher order moment. Let us look, for clarification, at a simple example of the solution of the equation

$$\frac{d}{dt} N(t) = \frac{\rho(t)}{\ell} N(t) \quad (\text{A.1})$$

where $\rho(t)$ is random. If

$$\rho(t) = \rho_0 + \Delta(t) \quad (\text{A.2})$$

where ρ_0 the average steady part and $\Delta(t)$ the fluctuating part, we have

$$\left(\frac{d}{dt} + \frac{\rho_0}{\ell}\right) N(t) = \frac{\Delta(t)}{\ell} N(t)$$

or

$$N(t) = N_0 G_0(t) + \int_0^t \frac{1}{\ell} \Delta(t') G_0(t-t') N(t') dt' \quad (\text{A.3})$$

where G_0 is the Green's function defined by

$$\left(\frac{d}{dt} + \frac{\rho_0}{\ell}\right) G_0(t) = \delta(t)$$

$$\text{i.e.} \quad G_0(t) = \begin{cases} e^{-\frac{\rho_0}{\ell} t} & t > 0 \\ 0 & t < 0 \end{cases}$$

Let us take the ensemble average of Eq. (A.3), we have

$$\langle N(t) \rangle = N_0 G_0(t) + \int_0^t \frac{1}{\ell} \langle \Delta(t') N(t') \rangle G_0(t-t') dt' \quad (\text{A.4})$$

Now $\langle \Delta N \rangle$ is not known. If one makes the assumption $\langle \Delta N \rangle \approx \langle \Delta \rangle \langle N \rangle$, this amounts to linearising Eq. (A.1). More useful results may be obtained by multiplying Eq. (A.3) by Δ and taking the ensemble average, i.e.,

$$\langle \Delta(t) N(t) \rangle = \frac{1}{\ell} \int_0^t \langle \Delta(t) \Delta(t') N(t') \rangle G_0(t-t') dt' \quad (\text{A.5})$$

At this stage one can make the less restrictive assumption (3.14), which expresses the unknown correlation in terms of the known autocorrelation of the stochastic input $\Delta(t)$.