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R. Avenhaus
Projekt Kernmaterialüberwachung

Kernforschungszentrum Karlsruhe

KERNFORSCHUNGSZENTRUM KARLSRUHE

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# Analysis of Decision Procedures for a Sequence of Inventory Periods

R. Avenhaus +)

+) Hochschule der Bundeswehr, München

Kernforschungszentrum Karlsruhe GmbH, Karlsruhe

Kernforschungszentrum Karlsruhe GmbH
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Abstract

Application of the material balance principle to nuclear material safeguards means comparison at the end of an inventory period of the book inventory, i.e. the amount of nuclear material which should be in the plant, with the real inventory, i.e. the amount actually found in the plant. By definition a statement about non-diversion or diversion can only be made at the end of the inventory period, which means that the detection time is determined by the length of that period. The question arises of the appropriate length of an inventory period. If one has in mind a fixed reference time, e.g. one year, this question is identical with the question of the appropriate number of inventories per reference time.

In this paper optimal test procedures for a sequence of inventory periods will be discussed. Starting with a game theoretical description of the conflict situation between the plant operator and the inspector, the objectives of the inspector as well as the general decision theoretical problem will be formulated. In the first part the objective of "secure" detection will be emphasized which means that only at the end of the reference time a decision is taken by the inspector. In the second part the objective of "timely" detection will be emphasized which will lead to sequential test procedures. At the end of the paper all procedures will be summarized, and in view of the multitude of procedures available at the moment some comments about future work will be given.

# Analyse von Entscheidungsprozeduren für eine Folge von Inventurperioden


## Zusammenfassung

Die Anwendung des Materialbilanzierungsprinzips auf Probleme der Überwachung des kerntechnischen Materials beinhaltet den Vergleich des Buchinventars, d.h. der Materialmenge, die in der Anlage sein sollte, mit dem realen Inventar, d.h. der Materialmenge, die am Ende einer Inventurperiode wirklich in der Anlage gefunden wird. Da per definitionem eine Aussage über mögliche Materialentwendung nur am Ende einer Inventurperiode gemacht werden kann, bedeutet dies, daß die Entdeckungszeit durch die Länge der Inventurperiode bestimmt wird, was die Frage nach der geeigneten Länge einer Inventurperiode aufwirft. Denkt man an eine bestimmte Referenzzeit, z.B. ein Jahr, so ist diese Frage gleichbedeutend mit der Frage nach der geeigneten Zahl von Inventuren in der Referenzzeit.

In dieser Arbeit werden optimale Testverfahren für eine Folge von Inventurperioden diskutiert. Ausgehend von einer spieltheoretischen Formulierung des Konflikts zwischen Anlagebetreiber und Inspektor werden die Ziele des Inspektors und das allgemeine entscheidungstheoretische Problem formuliert. Im ersten Teil wird das Ziel "sichere" Entdeckung betont, was bedeutet, daß vom Inspektor erst am Ende der Referenzzeit eine Entscheidung getroffen wird. Im zweiten Teil wird das Ziel "rechtzeitige" Entdeckung betont, was zu sequentiellen Testverfahren führt. Am Ende der Arbeit werden alle Verfahren zusammengestellt, und es werden angesichts der Vielfalt der gegenwärtig verfügbaren Verfahren einige Bemerkungen zu zukünftigen Arbeiten gemacht.

Table of Contents

# INTRODUCTION

The concept of material accountability as applied to nuclear material safeguards may be described as follows: At the beginning of an inventory period $[t_o, t_1]$, the real or physical inventory $I_o$ in the material balance area under consideration is measured. In the interval $[t_o, t_1]$ the net transfers $T_1$ are measured which give together with $I_o$ the so-called book inventory $B_1 = I_o + T_1$ at the time $t_1$, i.e. the amount of material which *should* be in the material balance area. This book inventory is compared with the real inventory $I_1$ at $t_1$, i.e. the inventory which is *actually* found in the material balance area. If no material has been lost or diverted (null hypothesis $H_o$), and if there have been no measurement errors, the difference

$$Z_1 := B_1 - I_1 = I_o + T_1 - I_1$$

should be zero; if, on the contrary, the amount $M_1$ has been lost or diverted, the difference should just be $M_1$ (alternative hypothesis $H_1$). Because of the random measurement errors the distributions of which are assumed to be known by long term experience, this will not exactly be true. Therefore, in order to decide whether or not a difference greater than zero can be explained by measurement errors, a *significance test* has to be performed with the two hypotheses that the expected value of the difference $Z_1$ is either zero or $M_1 > 0$, in formulae:

$$H_o\colon E(Z_1) = 0 \quad \text{and} \quad H_1\colon E(Z_1) = M_1 > 0.$$

In this paper a reference time $[t_o, t_n]$ is considered, at the beginning $t_o$ and at the end $t_n$ of which real inventories are taken, and during which the additional n-1 real inventories $I_i, i=1,\dots,n-1$ are taken. Thus, n book physical inventory differences $Z_i := I_{i+1} + T_i - I_i$, $i=1,\dots,n$ can be performed by which it shall be decided whether or not material has been lost or diverted in the interval of time $[t_o, t_n]$. It is the purpose of this paper to discuss *optimal test procedures*, one major issue being the question of the proper optimization criterion.

It should be emphasized that in the following paragraphs all test procedures are exclusively based on the source variables $Z_i, i=1,\dots,n$. This means that neither an information about single real inventories or net transfers is used, nor are these inventories or transfers estimated. This is in line with the goal of international nuclear material safeguards, i.e. the early detection of any diversion (or prevention by the risk of early detection, IAEO (1971)) which is our subject; for plant operations management it might be very reasonable to use the detailed information. We will come back to this point on several occasions.

If the operator wants to divert material - this has to be assumed for *technical analytical reasons* and should not be considered as a prejudice against any operator or state - it has to be assumed that he will do this in such a way that his "risk" is minimized so as to be on the safe side. On the other hand, the inspector should choose such a test procedure that his "success" of detecting a diversion is maximized. Thus, we are led to the analysis of a conflict situation with the *game theory* as its appropriate tool.

Game theoretical analyses of the nuclear material safeguards problem have been performed since 1968 (see e.g. Avenhaus (1978), Siri et al. (1978), Bennet et al. (1979)). These analyses have been criticized from various sides, essentially with the argument that they call for the knowledge of values of payoff parameters which will never be available, and that therefore traditional statistical tools have to be used, working simply with error probabilities of the first and second kinds. The analysts working in the nuclear material safeguards project of the Karlsruhe Nuclear Research Center took efforts and succeeded to show that the game theory represents the only way to appropriately formulate the objectives of safeguards and, furthermore, to show under which assumptions which results can be obtained that are *independent of the payoff parameter values*, thus establishing a link between the overall objective and traditional statistical procedures. This will be demonstrated in this paper, too, for the concrete problem of inventory sequences, even though today this basic issue is no longer discussed.

A "Review of the Application of Strategic Analysis to Material Accounting" by C.A. Bennett et al. (1979) was a "Consensus Report of the Peer

Review Group" established to answer five questions on the applicability of game theoretical methods to material accounting. Question 1 was "Does the use of game theory provide a viable analytical tool for the safeguards problems in general?", and question 5 "What are the disadvantages and benefits of using game theory?". The group members based their findings primarily on the paper by Siri et al. (1978) in which a payoff function was used which in our opinion does not adequately describe the problem of *international* nuclear material safeguards and which, by the way, made the results highly dependent on the values of the payoff parameters. Although two papers of the Karlsruhe group (Avenhaus and Frick (1974 a,b)) were mentioned in the literature review of the said group, they apparently did not play a major role in its deliberations. Therefore, the group as a whole expressed some reservation with respect to the practical and immediate applicability of game theoretical methods but nevertheless came to generally positive conclusions in the sense of our remarks above.

As already mentioned, the objective of international nuclear material safeguards is the early detection of the diversion of significant amounts of fissile material. This means, technically speaking, that the diversion of a given amount of material (the so-called *goal quantity*) should be detected as *early* as possible with as *high a probability of detection* as possible. As we will see, these two objectives may be conflicting such that a tradeoff between them has to be made. In addition, it has not been stated explicitly, but it has to be concluded from the boundary condition of the lowest possible plant disturbance by safeguards measures that the *false alarm probability* must not exceed a given value.

In the years after the formulation of safeguards objectives and procedures, i.e. in the years after 1971, the analyses centered around the problem of finding safeguards procedures and evaluation techniques which optimized the overall probability of detection for a reference time. Only in recent years the aspect of short detection time - which, as we will see, is even difficult to formulate as a technical optimization criterion - came to the forefront of investigations. According to this development and since this classification is natural, this paper is divided into two basic parts.

In the first part the *static approach* will be described: It is assumed that only at the end of the reference time $[t_o, t_n]$ a final decision will be taken whether or not material has been diverted.

> It should be noted that at the *first action level* it is only decided whether or not the null hypothesis is accepted. There will be several further action levels before a final decision about diversion will be taken. As we consider in this paper only the first level of action, we say instead of "accepting the alternative hypothesis" in a somewhat simplifying manner "stating a diversion".

This is in the sense of the objective of achieving a high overall probability of detection. In this approach the safeguards goal of "early detection" of a diversion plays the role of a *boundary condition* the weight of which has to be determined by practical arguments.

In the second part the *sequential approach* will be described: It is assumed that after each inventory taking during the reference time it will be decided whether or not material has been diverted, and that plant operations will be stopped immediately in the first case. This is in the sense of the objective of achieving a short detection time in case of diversion. This approach is much more difficult, both from a practical and a theoretical point of view: The practitioner has to say how much more important it is for the inspector to detect a diversion already after the i-th inventory rather than detecting it after the i+1st inventory only. The theoretician, on the other hand, is confronted with much more serious analytical problems.

In this connection the notion of the *abrupt diversion* of a large amount of nuclear material in one inventory period has to be introduced; in fact, the analysis of sequential test procedures has been emphasized for exactly this reason, namely to provide tools for the timely detection of abrupt diversion. On the contrary, static test procedures serve primarily the purpose of detecting *protracted diversion* of nuclear material. As the inspector never knows which diversion strategy will be used, if nuclear material will be diverted, in this paper we discuss - wherever possible - the efficiency of the test procedures with respect to both these diversion strategies.

It has already been said that the purpose of this paper is to give an overview of the test procedures developed in recent years, especially to precisely list their underlying assumptions and to determine their relative advantages and shortcomings with respect to the safeguards objectives. Naturally, there remain many questions to be solved in the future, both practical and theoretical ones. Except for these open questions, there are several aspects of the whole area which have *not* been tackled in this paper, some of which will be mentioned in the following paragraphs.

First, no attempt has been made to give a *complete list* of all test procedures considered so far in the literature. Some of them have been developed only for special loss strategies and cannot be generalized to cover more complicated ones, others are only slight modifications of procedures discussed here which have been adapted to computational requirements (e.g. the so-called V-mask techniques which represent only special variants of the CUMUF tests).

Second, this paper does not deal with the problem of *estimating* losses or the amount of diverted material. These problems have attracted considerable interest since the theory of *Kalman filters* (see e.g. Sage and Melsa (1971)) has been applied in this area in order to define "best solutions" and to find algorithms for their numerical calculation (Pike and Morrison (1977, 1979)). The reason for not considering problems of estimation in this paper is that the problem of detecting losses or diversion is basically a *test problem*, not an estimation problem. Estimation theory may provide test statistics in those cases where best tests cannot be found (see e.g. Stewart et al. (1979), Sellinschegg (1980)) - in fact we will consider such a situation in section 1.3. However, in this paper the general intention has been to formulate the basic problem with the help of first principles, and these are provided by the test theory.

Third, the very important problem of *verification* has not been discussed, although it is clear that one criterion for the practical application of procedures in *international safeguards* is their verifiability. One reason for this is that so far only the problem of verifying the data of *one* inventory period has been analyzed theoretically (Avenhaus and Beedgen (1980), Beedgen and Neu (1980)); another reason is that especially these problems cannot be solved without taking into account concrete plant and measurement conditions and, furthermore, that at the moment they are of a more constructive than analytical nature (requirement for tamper-proof instruments, data transmission lines, etc.).

A final word should be said about the mathematical-statistical tools which have been used in this paper and which were necessary to represent appropriately the problems and their solutions. Even though more complicated proofs are not given, the reader should be familiar with the elements of statistics and decision theory (e.g. at the level of Brownlee's book or, in view of nuclear material accountancy, at the level of the IAEA Technical Manual, Part F (1980)). The formal definitions, derivations and proofs, which have been given appear to be absolutely necessary for understanding the nature of the problem and of the corresponding mathematical models developed in the last ten years. To facilitate this understanding, emphasis has been laid on the use of standard statistical terminology; furthermore, at the end of this paper a list of the more important symbols used throughout this paper will be given.

# PART I

## The Static Approach

In this part it will be assumed that the operator decides at the beginning of the reference time whether or not he will divert material, and if yes, which amount he will divert in each inventory period. Furthermore, it will be assumed that the inspector decides at the beginning of the reference time which test procedure he will choose in order to make a statement *at the end of the reference time* whether or not the operator has diverted material.

The assumption that the inspector will not perform the single material balance tests at the moment at which the data are available to him, but only at the end of the reference time, is the most important assumption in this part; it expresses the fact that the main safeguards objective of the inspector is to detect a diversion with as high a "security" as possible. Moreover, it seems to be justified by the fact that plant operations will not be stopped as soon as the book physical differences for a single inventory period are significant.

Any game theoretical analysis of a conflict situation has to start with the definition of the *payoffs* to the players for the various outcomes of the game. Let the pair of payoffs to the inspector as player 1 and to the operator as player 2 be defined by

(-a, -b) in case of diversion and detection
(-c, d) in case of diversion and no detection
(-e, -f) in case of no diversion and "detection" (false alarm)
(0, 0) in case of no diversion and no detection,

where (a, b, c, d, e, f) > (0,...,0), where a < c, i.e. the inspector's loss in case of detected diversion is smaller than his loss in case of not detected diversion, and where b > f, i.e. the operator's loss in case of detected diversion is greater than his loss in case of a false alarm.

The true gains and losses of the inspector can hardly be quantified; we assume that the inspection effort is small compared to these quantities. Therefore, it is not part of the payoffs but is treated as a parameter of the set of inspector's strategies the value of which is determined a priori. Also it is assumed that the payoff of the operator is independent of the total amount of diverted material (which might be different in case of subnational diversion).

The notion "false alarm" needs an explanation. In the framework of the one-level safeguards procedure considered so far there is no possibility of correcting a false accusation. The inspector knows that according to the choice of the value of the false alarm probability a false alarm is raised with this probability; he can, however, not decide in the actual situation of an alarm whether or not this alarm is justified. Therefore, in this framework a "false alarm" does not exist, but only an "alarm". It is clear that especially in the case of international safeguards such a simple procedure will not be accepted in practice. In case of an alarm a "second action level" must follow which should permit to clarify whether or not data were falsified. There are no precise procedures; if they existed, a mathematical treatment of such a two-level procedure would have to take into account that the safeguards measures at the second level could have an impact on the behavior of the two players at the first level. In the following paragraphs, we will *not* analyze these possibilities in further detail but only assume that there are possibilities for clarifying false alarms and, furthermore, that these possibilities do not influence the player's behavior at the first level.

Let $1-\beta$ be the probability that a diversion is detected and let $\alpha$ be the probability that the inspector states a diversion if in fact the operator behaves legally. The *conditional* expected payoffs to the inspector and to the operator are

$(-a\cdot(1-\beta)-c\cdot\beta,\ -b\cdot(1-\beta)+d\cdot\beta)$ in case of diversion
$(-e\cdot\alpha,\ -f\cdot\alpha)$ in case of no diversion.

If the operator diverts material with the probability $p\in[0,1]$, the expected payoffs are defined by

$I := (-a+(a-c)\cdot\beta)\cdot p-e\cdot\alpha\cdot(1-p)$ for the inspector,
$B := (-b+(b+d)\cdot\beta)\cdot p-f\cdot\alpha\cdot(1-p)$ for the operator.

The operator who wants - if at all - to divert the amount of material $M > 0$, has the set of strategies

$$\{p\}\otimes Y_M := \{p: p\in[0,1]\}\otimes\{(M_1,\dots,M_n): \sum_i M_i=M\}.$$

The set of strategies of the inspector for the moment is simply called $X := \{x\}$. The dependence of the probability of detection $1-\beta$, the false alarm probability $\alpha$ and the expected payoff functions of the strategies are denoted by $\beta(x;y)$, $\alpha(x)$, $I(x;p,y)$ and $B(x;p,y)$, respectively.

As we assume that in international safeguards a cooperative behavior between the operator and the inspector with the goal of concealing diversion can be excluded, we are led to a non-cooperative, two-person game which is characterized by the sets of strategies and the function pair of payoff to the two players, which we therefore denote by the quadruple

$$(X, \{p\} \otimes Y_M, I, B).$$

The solution of this game is given by the so-called *equilibrium* point $(x^*; p^*, y^*)$ which is defined by

$$\begin{aligned} I(x^*;p^*,y^*) &\geq I(x;p^*,y^*) \\ B(x^*;p^*,y^*) &\geq B(x^*;p,y). \end{aligned}$$

It is important now that this equilibrium point can be determined by a *two step procedure* consisting of the solution of the following two games:

*Theorem 1*

Equilibrium points of the game $(X, \{p\} \otimes Y_M, I, B)$ with sets of strategies as defined above can be obtained by determining the equilibrium points of the following auxiliary games:

1. A two-person *zero sum* game $(X_\alpha, Y_M, 1-\beta)$ where $X_\alpha$ is defined by $\{x : x \varepsilon X,\ \alpha \text{ fixed}\}$. The optimal strategies of this game, $x^*(\alpha)$ and $y^*$, are the saddle points of $1-\beta$ on $X_\alpha \otimes Y_M$, defined by

$$1-\beta(x^*(\alpha),y,\alpha) \geq 1-\beta(x^*(\alpha),y^*,\alpha) \geq 1-\beta(x(\alpha),y^*,\alpha) \text{ for } \alpha \varepsilon [0,1]. \qquad (*)$$

2. A non-cooperative two-person game $(\{\alpha\}, \{p\}, I, B)$ with $\beta^*(\alpha) := \beta(x^*(\alpha),y^*,\alpha)$, where $x^* \varepsilon X_\alpha$ and $y^* \varepsilon Y_M$ are the solutions of the first game. The optimal strategies $\alpha^*$ and $p^*$ of this game are given by

$$\begin{aligned} (-a+(a-c)\cdot\beta^*(\alpha^*))\cdot p^* - e\cdot\alpha^*\cdot(1-p^*) &\geq (-a+(a-c)\cdot\beta^*(\alpha))\cdot p^* - e\cdot\alpha\cdot(1-p^*) \\ (-b+(b+d)\cdot\beta^*(\alpha^*))\cdot p^* - e\cdot\alpha^*\cdot(1-p^*) &\geq (-b+(b+d)\cdot\beta^*(\alpha^*))\cdot p - e\cdot\alpha^*\cdot(1-p). \end{aligned} \qquad (**)$$

*Proof*

From the left hand inequality of (*), and since $a < c$, we get

$$(-a+(a-c)\cdot\beta(x^*(\alpha),y^*,\alpha))\cdot p^*-e\cdot\alpha\cdot(1-p^*) \geq (-a+(a-c)\cdot\beta(x(\alpha),y^*,\alpha))\cdot p^*-e\cdot\alpha\cdot(1-p^*)$$

for arbitrary values of $p^*\varepsilon(0,1)$ and $\alpha\varepsilon(0,1)$. Therefore, we get with the upper inequality (**)

$$(-a+(a-c)\cdot\beta(x^*(\alpha^*),y^*,\alpha^*))\cdot p^*-e\cdot\alpha^*\cdot(1-p^*) \geq (-a+(a-c)\cdot\beta(x(\alpha),y^*,\alpha))\cdot p^*-e\cdot\alpha\cdot(1-p^*)$$

for arbitrary values of $\alpha$ and $p^*$. In the same way we have

$$(-b+(b+d)\cdot\beta(x^*(\alpha^*),y^*,\alpha^*))\cdot p^*-f\cdot\alpha^*\cdot(1-p^*) \geq (-b+(b+d)\cdot\beta^*(x^*(\alpha^*),y,\alpha^*))\cdot p-f\cdot\alpha^*\cdot(1-p)$$

for arbitrary values of $\alpha^*$ and p. This, however, means that $(x^*(\alpha^*),y^*,p^*)$ is in fact an equilibrium point of the game $(X,\{p\}\otimes Y_M,I,B)$. ⌑

*Theorem 2* (Avenhaus and Frick (1977))

The noncooperative two person game $(\{\alpha\},\{p\},I,B)$, defined in *Theorem 1*, has exactly one equilibrium point $(\alpha^*,p^*)$, if $\beta^*(\alpha)$ is convex and differentiable on $(0,1)$, and if the function $-b+(b+d)\cdot\beta(\alpha)+f\cdot\alpha$ has exactly one root in $[0,1]$. It is solution of the following two equations

$$f\cdot\alpha^*-b+(b+d)\cdot\beta(\alpha^*) = 0 \ , \quad (e+(a-c)\cdot\frac{d}{d\alpha^*}\beta(\alpha^*))\cdot p^*-1=0. \qquad ⌑$$

Besides the technical advantage this two-step procedure offers, it also has a very important substantial consequence: It has already been mentioned that the values of the payoff parameters a,...,b can hardly be estimated. However, we see that if we fix the value of the false alarm probability and consider only the illegal game, we must consider only a two-person zero sum game with the probability of detection as payoff to the inspector.

For the rest of this part, we will only consider the first step game.

So far, apparently only one further concrete game theoretical model for the problem of one inventory period has been published (Siri et al. (1978)). These authors formulate a two person zero sum game with the payoff

$$\beta + c \cdot y + x - b \cdot \min(x,y) + e \cdot |x - y|$$

to the operator, where

$\beta$ ≡ clean out inventory cost (fixed cost)

$x$ ≡ amount diverted by the operator

$y$ ≡ estimate of the inspector of amount diverted,

$c \cdot y$ ≡ recovery search cost (variable cost),

$b \cdot \min(y,x)$ ≡ value to inspector of assurance, or recovery of material diverted,

$e \circ |y - x|$ ≡ error penalty from wrong estimate of inspector.

According to the considerations given so far, this model is not suited for practical application, because

- in case of a false alarm, the objectives of the two players are at most partially conflicting, and
- the optimal strategy of the inspector, i.e., the significance threshold of the MUF-test, depends heavily on the values of the payoff parameters.

To enable the inspector to decide at the end of the reference time whether or not the operator has diverted material, he needs a *decision rule*. A decision rule $\delta$ is defined as a map of the space of observations Z into the interval $[0,1]$ and may be interpreted as the conditional probability of choosing the null hypothesis $H_o$:

If only the decisions $H_o$ or $H_1$ are permitted, we have

$$\delta(H_o|\hat{\underline{Z}}) + \delta(H_1|\hat{\underline{Z}}) = 1 \quad \text{for all } \hat{\underline{Z}} \varepsilon Z, \qquad (*)$$

which means that, if we decide with the probability $\delta$ for $H_o$ we decide with the probability $1-\delta$ for $H_1$.

In case of

$$\delta(H_o|\hat{\underline{Z}}) = \begin{cases} 1 & \text{if } \hat{\underline{Z}}\varepsilon Z_o \\ 0 & \text{if } \hat{\underline{Z}}\varepsilon Z_1, \end{cases} \qquad \delta(H_1|\hat{Z}_1) = \begin{cases} 0 & \text{if } \hat{\underline{Z}}\varepsilon Z_o \\ 1 & \text{if } \hat{Z}\varepsilon Z_1 \end{cases}$$

we call the decision rule a *non-randomized test;* it is simply characterized by the *acceptance region* $Z_o$ or, alternatively, by the critical region $Z_1 = Z\backslash Z_o$.

Let $f_o(\underline{z})$, defined by

$$f_o(\underline{z})d\underline{z} := \text{prob}\{\underline{z}<\underline{Z}\leq\underline{z}+d\underline{z}|H_o\},$$

be the probability density of $\underline{Z}$ under $H_o$, and correspondingly $f_1(\underline{z})$ the probability density under $H_1$. Then the false alarm probability $\alpha$ is given by

$$1-\alpha = \int_Z \delta(H_o|\underline{z})\cdot f_o(\underline{z})d\underline{z}.$$

Similarly, the probability of detection is given by

$$1-\beta = \int_Z \delta(H_1|\underline{z})\cdot f_1(\underline{z})d\underline{z}.$$

In case of non-randomized tests, these expressions reduce to

$$1-\alpha = \int_{Z_o} f_o(\underline{z})d\underline{z}\ , \qquad 1-\beta = \int_{Z_1} f_1(\underline{z})d\underline{z}\ .$$

In the following paragraphs we will characterize the non-randomized tests by their acceptance regions; only in Section 1.1, where we determine the Neyman-Pearson test, we have to make full use of the formalism developed so far.

Before discussing various test procedures in the framework of this static approach, it has to be emphasized that the use of a series of MUF-variables does *not* mean that a decision is made once a specific MUF-value is observed: The decision, whether or not any of the two hypotheses is accepted or rejected, is only made at the end of the reference time. Naturally, in case of the n-fold test procedure discussed in Section 1.4, this does not exclude the possibility to investigate, when the *first* significance occured, in order to locate the loss or diversion in time.

## 1.1 The Neyman-Pearson Test

If the distribution functions under the null and under the alternative hypotheses are specified, then the best test in the sense of the probability of the error second kind ( overall probability of no detection in one case) for a fixed probability of the error first kind (false alarm probability in our case) is the Neyman-Pearson test, the acceptance region of which is given by

*Theorem 3:* Lemma of Neyman and Pearson (1933)

Given the random vector $\underline{Z}$, the density function of which is $f_o(\underline{z})$ under $H_o$ and $f_1(\underline{z})$ under $H_1$. Then the acceptance region $Z_o^{NP}$ of that test, which minimizes the probability of the error second kind for a fixed probability of the error first kind, is given by

$$Z_o^{NP} = \left\{ \underline{z}_1 : \frac{f_1(\underline{z})}{f_o(\underline{z})} \leq \lambda \right\} ,$$

where the value of $\lambda$ is determined with the help of the probability of the error of the first kind.

*Proof* (after Shipley (1980))

According to the terminology introduced at the beginning of this part one has to determine the free minimum of the expression

$$\int_Z \delta(H_o|\underline{z}) \cdot f_1(\underline{z})d\underline{z} + \lambda \cdot \left[ \int_Z \delta(H_1|\underline{z}) \cdot f_o(\underline{z})d\underline{z} - \alpha \right],$$

where $\lambda$ is a Lagrange multiplier. With

$$\delta(H_1|\underline{z}) = 1-\delta(H_o|\underline{z})$$

this is equivalent to determining the free minimum of the expression

$$\lambda \cdot (1-\alpha) + \int_Z [f_1(\underline{z}) - \lambda \cdot f_o(\underline{z})] \delta(H_o \; \underline{z}) d\underline{z} \; .$$

As we have $0 \leq \delta(H_o|\underline{z}) \leq 1$, we minimize this expression, in case we choose $\delta(H_o|\underline{z}) = 1$ if the integrand is negative and $\delta(H_o|\underline{z}) = 0$ if it is positive. Therefore the Neyman-Pearson test has the acceptance region given above. ⌑

In the following we consider two different diversion scenarios. First, we analyze the possibility that during the reference time $[t_o, t_n]$ the diversion of the total amount M of material will be spread over the whole reference time, we call this *protracted diversion*. Second, the possibility that during the reference time $[t_o, t_n]$ the total amount M of material will be diverted in such a way that with probability $q_i$, $\sum_i q_i = 1$, it will be diverted in the i-th inventory period. We call this *abrupt diversion*. It is noted already here that only the first szenario leads to a simple solution.

*Protracted diversion*

The application of the Lemma of Neyman and Pearson leads in this case to a result which we formulate as

*Theorem 4* (Avenhaus and Jaech (1981), Zerrweck (1981))

Given the random vector $\underline{Z}$, defined in the introduction, with covariance Matrix $\underline{\underline{\Sigma}}$, and with expectation vector

$$E(\underline{Z}) = \underline{O} \text{ under } H_o \text{ and } E(\underline{Z}) = \underline{M},\ \underline{e}' \cdot \underline{M} = M, \text{ under } H_1 \; .$$

Let $\delta$ be a test for these two hypotheses with fixed fals alarm probability $\alpha$ and probability of detection $1 - \beta_\delta(\underline{M})$.

Then the power $1 - \beta_{\delta**}$ , defined by

$$\delta** = \begin{cases} 1 & \text{for } \underline{e}' \cdot \underline{X} > k_\alpha \\ 0 & \text{otherwise,} \end{cases}$$

where $k_\alpha$ is the significance threshold, fulfills the relations

$$1 - \beta_{\delta**} = \min_{\{\underline{M}\}} \sup_{\{\delta\}} (1 - \beta_\delta(\underline{M})) = \sup_{\{\delta\}} \min_{\{M\}} (1 - \beta_\delta(\underline{M})) = \Phi\left(\frac{M}{\sqrt{\underline{e}' \cdot \underline{\underline{\Sigma}} \cdot e}} - U_{1-\alpha}\right),$$

here, $\Phi(.)$ is the normal distribution function,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt ,$$

and U its inverse.

*Proof*

According to our assumptions the density functions $f_o$ and $f_1$ are explicitely given by

$$f_o(\underline{z}) = \frac{1}{(2\pi)^{\frac{n}{2}} \cdot |\underline{\underline{\Sigma}}|^{\frac{1}{2}}} \cdot \exp\left(-\frac{1}{2} \cdot \underline{z}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{z}\right),$$

$$f_1(\underline{z}) = \frac{1}{(2\pi)^{\frac{n}{2}} \cdot |\underline{\underline{\Sigma}}|^{\frac{1}{2}}} \cdot \exp\left(-\frac{1}{2}(\underline{z}-\underline{M})' \cdot \underline{\underline{\Sigma}}^{-1} \cdot (\underline{z}-\underline{M})\right) ,$$

where $\underline{\underline{\Sigma}}$ is the covariance matrix of $\underline{Z}$,

$$\underline{\underline{\Sigma}} := \operatorname{cov}(\underline{Z},\underline{Z}') = (\operatorname{cov}(Z_i,Z_j)) ,$$

the critical region of the Neyman-Pearson test is given by

$$Z_1^{NP} = \{\underline{z} : \underline{z}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{M} > k'\} ,$$

which means that the test statistic is given by

$$\underline{Z}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{M} .$$

As this linear form of multivariate normally distributed random variables is normally distributed with expectation values

$$E(\underline{Z} \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{M}) = 0 \quad \text{under } H_o \quad \text{and} \quad E(\underline{Z}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{M}) = \underline{M}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{M} \quad \text{under } H_1$$

and with the variance

$$\operatorname{var}(\underline{Z}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{M}) = \underline{M}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{M} ,$$

see e.g. Anderson (1958), the false alarm probability $\alpha$, defined by

$$\alpha = \text{prob}\{\underline{Z} \varepsilon Z_1^{NP} | H_o\} ,$$

is given by

$$\alpha = \Phi \left(-\frac{s}{\sqrt{\underline{M}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{M}}}\right) ,$$

where s is the *significance threshold* of the test, and where

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt$$

is the normal distribution function.

The probability of detection, defined by

$$1-\beta_{NP} = \text{prob}\{\underline{Z} \varepsilon Z^{NP} | H_1\} ,$$

is given by

$$1-\beta_{NP} = \Phi \left(\frac{\underline{M}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{M} - s}{\sqrt{\underline{M}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{M}}}\right) .$$

By elimination of the significance threshold with the help of its relation to the false alarm probability, one gets

$$1-\beta_{NP} = \Phi \left(\sqrt{\underline{M}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{M}} - U_{1-\alpha}\right),$$

where U is the inverse function of the normal distribution function $\Phi$.

As $\Phi(.)$ is a monotone function, the minimum of the probability of detection is given by the minimum of the scalar form

$$\underline{M}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{M} .$$

We determine the minimum of this form under the boundary condition

$$\sum_i M_i = \underline{e}' \cdot \underline{M} = M > 0, \quad \underline{e}' := (1 \ldots 1),$$

with the help of the Lagrange formalism: The free minimum of the form

$$\underline{M}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{M} + \lambda \cdot \underline{e}' \cdot \underline{M} ,$$

$\lambda$ being the Lagrange parameter, is given by the condition

$$2 \cdot \underline{M}^{*\prime} \cdot \underline{\underline{\Sigma}}^{-1} + \lambda \cdot \underline{e}' = 0.$$

Multiplication from right by $\underline{\underline{\Sigma}}$ gives

$$2 \cdot \underline{M}^{*\prime} + \lambda \cdot \underline{e}' \cdot \underline{\underline{\Sigma}} = 0.$$

Multiplication from right by $\underline{e}$ and use of the boundary condition gives

$$2 \cdot M + \lambda \cdot \underline{e}' \cdot \underline{\underline{\Sigma}} \cdot \underline{e} = 0,$$

therefore, elimination of $\lambda$ gives

$$\underline{M}^* = \frac{M}{\underline{e}' \cdot \underline{\underline{\Sigma}} \cdot \underline{e}} \cdot \underline{\underline{\Sigma}} \cdot \underline{e} ,$$

and the minimum of the probability of detection is

$$\min_{\underline{M}:\underline{e}'\cdot\underline{M}=M} (1-\beta_{NP}) =: 1-\beta^*_{NP} = \Phi \left(\frac{M}{\sqrt{\underline{e}' \cdot \underline{\underline{\Sigma}} \cdot \underline{e}}} - U_{1-\alpha}\right).$$

Now, as one immediately sees, we have

$$\underline{e}' \cdot \underline{\underline{\Sigma}} \cdot \underline{e} = \mathrm{var}(\underline{e}' \cdot \underline{Z}).$$

As we have $E(\underline{e}' \cdot \underline{Z}) = M$ under $H_1$, the minimum of $1-\beta_{NP}$ is in fact the probability of detection based on the test statistics

$$\underline{e}' \cdot \underline{Z} = \sum_i Z_i ,$$

i.e. the test based *only on the beginning and ending inventories and on the net transfers.*

Now, let $\delta_o$ be a test characterized by the test statistic $\underline{e}'\cdot\underline{X}$. As this test has the same power for all $\underline{\mu}$ satisfying the condition $\underline{e}'\cdot\underline{\mu} = M$, we have

$$\min_{\{\mu\}}(1-\beta_{\delta_o}(\underline{\mu})) = \min_{\{\mu\}}\sup_{\{\delta\}}(1-\beta_\delta(\underline{\mu})).$$

Now we have in general

$$\sup_{\{\delta\}}\min_{\{\mu\}}(1-\beta_\delta(\underline{\mu})) \geq \min_{\{\mu\}}(1-\beta_{\delta_o}(\underline{\mu})),$$

therefore,

$$\sup_{\{\delta\}}\min_{\{\mu\}}(1-\beta_\delta(\underline{\mu})) \geq \min_{\{\mu\}}\sup_{\{\delta\}}(1-\beta_\delta(\underline{\mu})). \qquad (*)$$

Furthermore, we have in general

$$\min_{\{\mu\}}\sup_{\{\delta\}}(1-\beta_\delta(\mu)) \geq \sup_{\{\delta\}}\min_{\{\mu\}}(1-\beta_\delta(\mu)). \qquad (**)$$

From (*) and (**), however, we get equity of both sides which completes the proof.

□

The following theorem shows that one obtains the same result if one considers instead of the original random vector $\underline{X}$ a linearly transformed vector:

*Theorem 5*

Let $\underline{X}$ be a normally distributed random vector with regular known covariance matrix $\underline{\underline{\Sigma}}$, and let $\underline{Y} = \underline{\underline{A}}\cdot\underline{X}$ be a linearly transformed vector with regular transformation matrix $\underline{\underline{A}}$. Then *Theorem 1* holds also if the test procedure is based on the transformed vector $\underline{Y}$ instead of the original vector $\underline{X}$.

*Proof*

As the expectation vector of $\underline{Y}$ is

$$E(\underline{Y}) = \underline{0} \quad \text{under } H_0 \text{ and } E(\underline{Y}) = \underline{\underline{A}}\cdot\underline{\mu} \text{ under } H_1 \ ,$$

and as the covariance matrix of $\underline{Y}$ is

$$\underline{\underline{\Lambda}} = \text{var}(\underline{Y},\underline{Y}') = \underline{\underline{A}}\cdot\underline{\underline{\Sigma}}\cdot\underline{\underline{A}}' \ ,$$

the Neyman Pearson test statistic of the test for fixed $\underline{\mu}$ is given by

$$\underline{Y}'\cdot\underline{\underline{\Lambda}}^{-1}\cdot\underline{\underline{A}}\cdot\underline{\mu} \ ,$$

and the power of this test is given by

$$\phi\ (\sqrt{(\underline{\underline{A}}\cdot\underline{\mu})'\cdot\underline{\underline{\Lambda}}^{-1}\cdot\underline{\underline{A}}\cdot\underline{\mu}} - U_{1-\alpha}).$$

As we see immediately, we have

$$\underline{\mu}'\cdot\underline{\underline{A}}'\cdot\underline{\underline{\Lambda}}^{-1}\cdot\underline{\underline{A}}\cdot\underline{\mu} = \underline{\mu}'\cdot\underline{\underline{\Sigma}}^{-1}\cdot\underline{\mu} \ ,$$

thus, we have the same expression as for the test using the original random vector $\underline{X}$.

□

*Theorem 4* and also *Theorem 5*, which is in fact a special case of the former one, show that the best test in the sense of Neyman and Pearson uses only the beginning and ending inventories and the net transfers, i.e., it ignores intermediate inventories. In other words, in the sense of the overall probability of detection criterion it is best to test only the book physical inventory difference $I_o+\sum_i T_1-I_n$ at the end point $t_n$ of the reference time interval $[t_o,t_n]$.

It should be emphasized that this result is based on the operator's best strategy $\underline{M}^*$ given above (even though the inspector does not take any notice of it, as he performs only one test at the end of the reference time!). If the operator chooses a different strategy, the probability of detection will become higher by definition.

On the other hand, the inspector could get a better result if he knew the strategy $\underline{M}$: $\underline{e}' \cdot \underline{M} = M$ of the operator: In this case his test statistics would be $\underline{Z}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{M}$, which would lead to a probability of detection as given above.

For equal diversion in each inventory period,

$$\underline{M} = \frac{M}{n} \cdot \underline{e} \,,$$

we would get

$$1-\beta_{NP} = \Phi\left(\frac{M}{n} \cdot \sqrt{\underline{e}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{e}} - U_{1-\alpha}\right).$$

Let us consider the case n=2. With

$$\underline{\underline{\Sigma}} = \begin{pmatrix} \sigma_1^2 & \rho \cdot \sigma_1 \cdot \sigma_2 \\ \rho \cdot \sigma_1 \cdot \sigma_2 & \sigma_2^2 \end{pmatrix} ; \qquad \underline{\underline{\Sigma}}^{-1} = \frac{1}{1-\rho^2} \cdot \begin{pmatrix} \frac{1}{\sigma_1^2} & -\frac{\rho}{\sigma_1 \cdot \sigma_2} \\ -\frac{\rho}{\sigma_1 \cdot \sigma_2} & \frac{1}{\sigma_2^2} \end{pmatrix}$$

we get

$$\underline{M}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{M} = \frac{1}{1-\rho^2} \cdot \left(\frac{M_1^2}{\sigma_1^2} - \frac{2 \cdot \rho - M_1 \cdot M_2}{\sigma_1 \cdot \sigma_2} + \frac{M_2^2}{\sigma_2^2}\right)$$

$$\frac{M^2}{\underline{e}' \cdot \underline{\underline{\Sigma}} \cdot \underline{e}} = \frac{M^2}{\sigma_1^2 + 2 \cdot \rho \cdot \sigma_1 \cdot \sigma_2 + \sigma_2^2}$$

$$\underline{M}^* = \begin{pmatrix} M_1^* \\ M_2^* \end{pmatrix} = \frac{M}{\sigma_1^2 + 2 \cdot \rho \cdot \sigma_1 \cdot \sigma_2 + \sigma_2^2} \cdot \begin{pmatrix} \sigma_1^2 + \rho \cdot \sigma_1 \cdot \sigma_2 \\ \sigma_2^2 + \rho \cdot \sigma_1 \cdot \sigma_2 \end{pmatrix}$$

Thus, for $M_1^* \neq M_2^*$ which is true for $\sigma_1^2 \neq \sigma_2^2$, and $M_1 = M_2 = \frac{M}{2}$ the test statistic $\underline{Z}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{Z}$ - which takes into account $I_1$ - leads to a higher probability of detection for all values of $\sigma_1^2$, $\sigma_2^2$, $\rho$ as we always have

$$\frac{M^2}{4} \cdot \underline{e}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{e} > \frac{M^2}{\underline{e}' \cdot \underline{\underline{\Sigma}} \cdot \underline{e}}$$

which is equivalent to

$$\frac{1}{4} \cdot \frac{1}{1-\rho^2} \cdot \frac{1}{\sigma_1^2 \cdot \sigma_2^2} \cdot (\sigma_1^2 - 2 \cdot \rho \cdot \sigma_1 \cdot \sigma_2 + \sigma_2^2) > \frac{1}{\sigma_1^2 + 2 \cdot \rho \cdot \sigma_1 \cdot \sigma_2 + \sigma_2^2}$$

which is equivalent to

$$(\sigma_1^2 + \sigma_2^2)^2 > 4 \cdot \sigma_1^2 \cdot \sigma_2^2$$

which is always true.

Finally, it should be noted that Shipley (1980) has treated the problem of the optimal number of inventory periods in a static approach as follows:
He defines the random vector $\underline{Z}$ by

$$\underline{Z}' = (I_o I_1 \ldots I_n \; T_1 \ldots T_n)$$

and the 2n+1 x n matrix $\underline{\underline{A}}$ by

$$\underline{\underline{A}}' = \left.\begin{pmatrix} \underbrace{\begin{matrix} 1 & -1 & 0 \ldots 0 \\ 0 & 1 & -1 \ldots 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 \ldots 1 & -1 \end{matrix}}_{n+1} & \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ 0 & 0 \ldots 1 & \end{matrix}}_{n} \end{pmatrix}\right\} n$$

and writes the two hypotheses $H_o$ and $H_1$ as

$$H_o: \underline{\underline{A}}' \cdot E_o(\underline{Z}) = 0$$

$$H_1: \underline{\underline{A}}' \cdot E_1(\underline{Z}) = \underline{\underline{H}} \cdot \underline{D} ,$$

where $\underline{\underline{H}}$ is a constant n x 2n+1 matrix and $\underline{D}$ is an unknown vector with n elements. Especially $\underline{D}$ may correspond to our $\underline{M}$, and $\underline{\underline{H}}$ to

$$\underline{\underline{H}} = \begin{pmatrix} 1 & 0 \ldots 0 & 0 \ldots 0 \\ 0 & 1 \ldots 0 & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot & \cdot \quad \cdot \\ 0 & 0 \ldots 1 & 0 \ldots 0 \end{pmatrix}$$

It should be noted that Stewart et al. (1979, p. 5ff) call this model, which contains the material balance principle, the *fundamental model* because it is the starting point of their estimation theory.

Shipley now constructs the *maximum likelihood ratio test* the critical region of which is given by

$$\left\{ \underline{Z}: \quad \frac{\max\limits_{E_1(\underline{Z}),\underline{D}} \exp(-\frac{1}{2} \cdot (\underline{z}-E_1(\underline{z}))' \cdot \underline{\underline{\Sigma}}^{-1} \cdot (\underline{z}-E_1(\underline{z}))}{\max\limits_{E_o(\underline{Z})} \exp(-\frac{1}{2} \cdot (\underline{z}-E_o(\underline{z}))' \cdot \underline{\underline{\Sigma}}^{-1} \cdot (\underline{z}-E_o(\underline{z}))} > k \right\},$$

where $\underline{\underline{\Sigma}}$ is the covariance matrix of the random vector $\underline{Z}$. This test leads to a test statistics which is quadratic in $\underline{Z}$. For the case n=2 and independent variables $\underline{Z}_i$, i=1,...,5, he shows numerically that it is better to neglect the inventory $I_1$ and to base the test on the global balance $I_o+T_1+T_2-I_1$; in a private discussion he supposed that it will be also true for n>2 and for dependent $Z_i$, i=1,...,2n+1.

Translating Shipley's approach into ours, the critical region of the maximum likelihood ratio test would be defined by

$$\left\{ \underline{x}: \quad \frac{\max\limits_{\underline{M}:\underline{e}'\cdot\underline{M}=M} \exp(-\frac{1}{2} \cdot (\underline{x}-\underline{M})' \cdot \underline{\underline{\Sigma}}^{-1} \cdot (\underline{x}-\underline{M}))}{\exp(-\frac{1}{2} \cdot \underline{x}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{x})} > k \right\}$$

which is equivalent to

$$\{\underline{x}: \max_{\underline{M}:\underline{e}'\cdot\underline{M}=M} (\underline{M}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{M} - 2 \cdot \underline{M}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{x}) > k'\}$$

which, using again the Lagrange formalism, is equivalent to

$$\{ \underline{x}:\underline{x}'\cdot\underline{\underline{\Sigma}}^{-1}\cdot\underline{x} + \frac{(M-\underline{e}'\cdot\underline{x})^2}{\underline{e}'\cdot\underline{\underline{\Sigma}}\cdot\underline{e}} < k''\} .$$

Thus, one is led to a test statistics which is quadratic in $\underline{x}$ and whose distribution function can therefore hardly be written down explicitly in the general case. This means that the probability of detection can be determined only for special cases. As, in addition, the maximum likelihood ratio test is *not* the best test in general, it does not seem reasonable to further investigate this approach; the Neyman-Pearson test as the most powerful tool has to be preferred in any case.

*Abrupt diversion*

In this case the density functions $f_o(\underline{Z})$ of the random vector $\underline{Z}$ under $H_o$ is the same as before. The density function of $\underline{Z}$ under $H_1$ is determined as follows:

Under the condition, that during the i-th inventory period the amount M is diverted, the distribution function $F_i^{(1)}(\underline{Z})$, i = 1...N, is given by the expression

$$F_1^{(i)}(\underline{Z}) = \frac{1}{(2\pi)^{\frac{n}{2}} \cdot |\underline{\underline{\Sigma}}|^{\frac{1}{2}}} \cdot \int_{-\infty}^{z_1} \cdots \int_{-\infty}^{z_N} d\underline{z} \cdot \exp\left(-\frac{1}{2}\cdot\left(\underline{z}-\underline{\mu}^{(i)'}\right)\cdot\underline{\underline{\Sigma}}^{-1}\cdot\left(\underline{z}-\underline{\mu}^{(i)}\right)\right),$$

$$i = 1...N ,$$

where $\underline{\mu}^{(i)}$ is a row vector which has only zero's except for the i-th component the value of which is M.

The unconditioned distribution function $F_1(\underline{z})$ is given by

$$F_1(\underline{z}) = \sum_{i=1}^{N} F_1^{(i)}(\underline{z}) \cdot q_i \quad \text{with} \quad \sum_{i=1}^{N} q_i = 1 ,$$

therefore, the density function under $H_1$ is given by

$$f_1(\underline{z}) = \frac{1}{(2\pi)^{\frac{n}{2}} \cdot |\underline{\underline{\Sigma}}|^{\frac{1}{2}}} \cdot \sum_{i=1}^{N} \exp\left(-\frac{1}{2}\cdot\left(\underline{z}-\underline{\mu}^{(i)}\right)' \cdot\underline{\underline{\Sigma}}^{-1}\cdot\left(\underline{z}-\mu^{(i)}\right)\right)\cdot q_i .$$

The critical region of the Neyman-Pearson test for the two hypotheses

$f(\underline{z}) = f_o(\underline{z})$ under $H_o$ and $f(\underline{z}) = f_1(\underline{z})$ under $H_1$

is according to Theorem 3 given by

$$K = \left\{ \underline{z} : \sum_{i=1}^{N} \exp\left(\underline{z}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \mu^{(i)}\right) \cdot q_i > k_\alpha \right\} .$$

For illustrative purposes we consider the case N = 2. With

$$\underline{z}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{\mu}^{(1)} = \frac{1}{1-\rho^2} \cdot (z_1, z_2) \cdot \begin{pmatrix} \frac{1}{\sigma_1^2} & -\frac{\rho}{\sigma_1\sigma_2} \\ -\frac{\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{pmatrix} \cdot \begin{pmatrix} M \\ 0 \end{pmatrix} =$$

$$= \frac{M}{1-\rho^2} \cdot (z_1, z_2) \cdot \begin{pmatrix} \frac{1}{\sigma^2} \\ -\frac{\rho}{\sigma_1\sigma_2} \end{pmatrix} = \frac{M}{1-\rho^2} \cdot \left( \frac{z_1}{\sigma_1^2} - \frac{\rho z_2}{\sigma_1\sigma_2} \right)$$

$$\underline{z}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{\mu}^{(2)} = \frac{M}{1-\rho^2} \cdot \left( - \frac{\rho z_1}{\sigma_1\sigma_2} + \frac{z_2}{\sigma_2^2} \right)$$

the critical region is given by

$$K = \left\{ (z_1, z_2) : \exp\left(\frac{M}{1-\rho^2} \cdot \left(- \frac{z_1}{\sigma_1^2} - \frac{\rho z_2}{\sigma_1\sigma_2}\right)\right) \cdot q + \exp\left(\frac{M}{1-\rho^2} \cdot \left(- \frac{\rho z_1}{\sigma_1\sigma_2} + \frac{z_1}{\sigma_2^2}\right)\right) \cdot (1-q) > k_2 \right\} .$$

The problem is that even in this special case the distribution of the test statistic cannot be written down explicitely, therefore, the optimal alternative hypothesis $q^*$ cannot be determined analytically. With the help of simulation studies (Horsch 1982) it has been shown, that the supposition

$$\underline{q}^* = \frac{\underline{\underline{\Sigma}} \cdot \underline{e}}{\underline{e}^1 \cdot \underline{\underline{\Sigma}} \cdot \underline{e}} = \frac{1}{\sigma_1^2 + 2\rho\, \sigma_1\sigma_2 + \sigma_2^2} \cdot \begin{pmatrix} \sigma_1 \cdot (\sigma_1 + \rho\sigma_2) \\ \sigma_2 \cdot (\sigma_2 + \rho\sigma_1) \end{pmatrix} ,$$

which seems to be intuitive because of the results of the next section, *cannot* be confirmed; in general $q^*$ depends on the false alarm probability $\alpha$.

## 1.2 A Special Test Studied by Jaech et al.

The idea of the test procedure to be discussed now is to establish a linear combination

$$S_n := \sum_{i=1}^{n} a_i \cdot Z_i = \underline{a}' \cdot \underline{Z}$$

with help of the n different MUF variables and to use this linear combination as the test statistic at the end of the reference time $[t_o, t_n]$. The acceptance region of this test is therefore simply given by

$$Z_o^{\mathfrak{J}} := \left\{ \hat{S}_n : \hat{S}_n \leq s \right\} .$$

Even though this test does not provide any information about the time of diversion at all, and therefore, in case of protracted diversion it would be better to use the test statistic $\sum_i Z_i$ - a special case of the statistic to be discussed here - we will analyze this test statistic as it has played a role in the literature and since abrupt diversion can be analyzed here.

*Protracted Diversion*

It is clear that Theorem 3 covers this test procedure, too, nevertheless, we present the main result separately as

*Theorem 6* (Frick 1979 a,b)

Given the random vector $\underline{Z}$, defined in the introduction, with known regular covariance matrix $\underline{\underline{\Sigma}}$. Let $\delta$ be a test for the two hypotheses $H_o$ and $H_1$,

$$H_o : E(\underline{Z}) = 0$$

$$H_1 : E(\underline{Z}) = \underline{\mu} : \underline{e}' \cdot \underline{\mu} = M > 0$$

with a fixed significance level and power $1 - \beta_\delta(\mu)$, defined by

$$\delta = \begin{cases} 1 & \text{for } \underline{a}' \cdot \underline{X}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\underline{a}' = (a_1 \ldots a_N)$ is an arbitrary real vector. Then the power $1 - \beta_{\delta**}$ of the test $\delta**$, defined by

$$\delta** = \begin{cases} 1 & \text{for } c \cdot \underline{e}' \cdot \underline{X},\ c > 0 \\ 0 & \text{otherwise,} \end{cases}$$

fulfills the relations

$$1-\beta_{\delta^{**}} = \min_{\{\mu\}} \max_{\{\delta\}}(1-\beta_\delta(\mu)) = \max_{\{\delta\}} \min_{\{\mu\}}(1-\beta_\delta(\mu)).$$

The minimizing alternative hypothesis is

$$\underline{\mu}^* = \frac{M}{\underline{e}' \cdot \underline{\underline{\Sigma}} \cdot \underline{e}} \cdot \underline{\underline{\Sigma}} \cdot \underline{e} \ .$$

*Proof*

As the linear form $\underline{a}'\cdot\underline{X}$ of multivariate normally distributed random variables is normally distributed with expectation values

$$E(\underline{a}'\cdot Z) = 0 \text{ under } H_0 \text{ and } E(\underline{a}'\cdot Z) = \underline{a}'\cdot\underline{\mu} \text{ under } H_1$$

and with the variance

$$\mathrm{var}(\underline{a}'\cdot Z) = \underline{a}'\cdot\underline{\underline{\Sigma}}\cdot\underline{a},$$

the power of the test is

$$1-\beta_\delta(\mu) = \phi\left(\frac{\underline{a}'\cdot\underline{\mu}}{\sqrt{\underline{a}'\cdot\underline{\underline{\Sigma}}\cdot\underline{a}}} - U_{1-\alpha}\right) .$$

Because of the monotony of $\phi$ we only have to prove

$$\frac{\underline{a}^{*\prime}\cdot\underline{\mu}^*}{\sqrt{\underline{a}^{*\prime}\cdot\underline{\underline{\Sigma}}\cdot\underline{a}^*}} = \min_{\underline{\mu}} \max_{\underline{a}} \frac{\underline{a}'\cdot\underline{\mu}}{\sqrt{\underline{a}'\cdot\underline{\underline{\Sigma}}\cdot\underline{a}}} = \max_{\underline{a}} \min_{\underline{\mu}} \frac{\underline{a}'\cdot\underline{\mu}}{\sqrt{\underline{a}'\cdot\underline{\underline{\Sigma}}\cdot\underline{a}}} \quad .$$

We perform this by showing that the *saddle point criterion*

$$\frac{\underline{a}^{*\prime}\cdot\underline{\mu}}{\sqrt{\underline{a}^{*\prime}\cdot\underline{\underline{\Sigma}}\cdot\underline{a}^*}} \geq \frac{\underline{a}^{*\prime}\cdot\underline{\mu}^*}{\sqrt{\underline{a}^{*\prime}\cdot\underline{\underline{\Sigma}}\cdot\underline{a}^*}} \geq \frac{\underline{a}'\cdot\underline{u}^*}{\sqrt{\underline{a}'\cdot\underline{\underline{\Sigma}}\cdot\underline{a}}}$$

is fulfilled. Now, these two inequalities are equivalent to

$$\frac{M}{\sqrt{\underline{e}'\cdot\underline{\underline{\Sigma}}\cdot\underline{e}}} \geq \frac{M}{\sqrt{\underline{e}'\cdot\underline{\underline{\Sigma}}\cdot\underline{e}}} \geq \frac{M}{\sqrt{\underline{e}'\cdot\underline{\underline{\Sigma}}\cdot\underline{e}}} \cdot \frac{\underline{a}'\cdot\underline{\underline{\Sigma}}\cdot\underline{e}}{\sqrt{\underline{a}'\cdot\underline{\underline{\Sigma}}\cdot\underline{a}}} \quad ,$$

which means that it suffices to show

$$(\underline{e}'\cdot\underline{\underline{\Sigma}}\cdot\underline{e})\cdot(\underline{a}'\cdot\underline{\underline{\Sigma}}\cdot\underline{a})\geq(\underline{a}'\cdot\underline{\underline{\Sigma}}\cdot\underline{e})^2 .$$

As the symmetric and regular matrix $\underline{\underline{\Sigma}}$ can be represented as the product of a regular matrix $\underline{\underline{D}}$ and its transposed matrix $\underline{\underline{D}}'$,

$$\underline{\underline{\Sigma}} = \underline{\underline{D}}'\cdot\underline{\underline{D}} ,$$

this inequality is equivalent to

$$(\tilde{\underline{a}}'\cdot\tilde{\underline{a}})\cdot(\tilde{\underline{e}}'\cdot\tilde{\underline{e}})\geq(\tilde{\underline{a}}\cdot\tilde{\underline{e}})^2 ,$$

where $\tilde{\underline{a}}$ and $\tilde{\underline{e}}$ are defined as

$$\tilde{\underline{a}} = \underline{\underline{D}}\cdot\underline{a}, \quad \tilde{\underline{e}} = \underline{\underline{D}}\cdot\underline{e} \quad ;$$

this, however, is nothing else than the *Schwartz' inequality* which completes the proof.

$\square$

Jaech (1977) has determined the best weighting coefficients under the assumption

$$M_i = \frac{M}{n} \text{ for } i=1,\ldots,n,$$

and, furthermore, by postulating that $S_n$ shall represent an unbiased estimate of the total diversion M. Because of

$$E_1(S_n) = \sum_i a_i\cdot\frac{M}{n} = \frac{M}{n}\cdot\underline{e}'\cdot\underline{a} = M$$

the probability of detection is given by

$$1-\beta_J = \Phi\left(\frac{M}{\sqrt{\underline{a}'\cdot\underline{\underline{\Sigma}}\cdot\underline{a}}} - U_{1-\alpha}\right)$$

and the problem is to minimize the quadratic form

$$\underline{a}' \cdot \underline{\underline{\Sigma}} \cdot \underline{a}$$

under the boundary condition

$$\underline{e}' \cdot \underline{a} = n ,$$

which leads, using again the Lagrange formalism, to the optimal weighting coefficients

$$\underline{a}^* = \frac{n}{\underline{e}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{e}} - \underline{\underline{\Sigma}}^{-1} \cdot e$$

and consequently, to the optimal acceptance region

$$Z_o^{*J} = \{ \underline{z} : \underline{e}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{z} \leq \sqrt{\underline{e}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{e}} \cdot U_{1-\alpha} \}$$

and to the optimal probability of detection

$$1-\beta_J^* = \Phi\left(\frac{M}{n} \cdot \sqrt{\underline{e}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{e}} - U_{1-\alpha}\right) ,$$

which, by the way, is the same as that of the Neyman-Pearson test under $H_1 : \underline{M} = \frac{M}{n} \cdot \underline{e}$.

As already pointed out by Stewart and Wincek (1980), this procedure aims at the optimal detection of *constant losses*, its value for the detection of *purposeful diversion* of material has to be doubted. It is also in this line of detecting constant losses that the postulate of the unbiasedness of $S_n$ has been formulated: Whereas an operator is interested to *estimate* his losses, an inspector must primarily be interested in the detection of diversion; therefore, from his point of view the unbiasedness of $S_n$ has no meaning at all.

*Abrupt Diversion*

Contrary to the situation in the foregoing section, here we can solve the problem completely:

*Theorem 7*

Given the random vector $\underline{Z}$, defined in the introduction, with known regular covariance matrix $\underline{\underline{\Sigma}}$. Let $\delta$ be a test for the two hypotheses $H_o$ and $H_1$,

$$H_o \; : \; E(\underline{Z}) = 0$$

$$H_1 \; : \; E(Z_i) = \begin{cases} M \text{ with prob } q_i, \; i = 1,..n, \\ 0 \text{ otherwise,} \end{cases}$$

with fixed significance level and power $1 - \beta_\delta(\mu)$, defined by

$$\delta = \begin{cases} 1 & \text{for } \underline{a}' \cdot \underline{X}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\underline{a}' = (a_1 ... a_n)$ is an arbitrary real vector. Then the power $1 - \beta_{\delta**}$ of the test $\delta**$, defined by

$$\delta** = \begin{cases} 1 & \text{for } c \cdot \underline{e}' \cdot \underline{X} \; , \; c > 0 \\ 0 & \text{otherwise,} \end{cases}$$

fulfills the relations

$$1 - \beta_{\delta**} = \min_{\{q_i\}} \max_{\{\underline{a}\}} \left(1 - \beta_\delta(\underline{a})\right) = \max_{\{\underline{a}\}} \min_{\{\delta\}} \left(1 - \beta_\delta(\underline{a})\right).$$

The minimizing alternative hypothesis is given by

$$q^* = \frac{\underline{e}' \cdot \underline{\underline{\Sigma}}}{\underline{e}' \cdot \underline{\underline{\Sigma}} \cdot \underline{e}} \; .$$

*Proof*

The probability of no detection $\beta_\delta \equiv \beta(\underline{a},\underline{q})$ is given by

$$\beta_\delta = \sum_i \Phi \left( U_{1-\alpha} - \frac{a_i \cdot M}{\sqrt{\underline{a} \cdot \underline{\underline{\Sigma}} \cdot \underline{a}}} \right) \cdot q_i \; ,$$

therefore, we have to show

$$\beta(\underline{a}^*,\underline{q}) \le \beta(\underline{a}^*,\underline{q}^*) \le \beta(\underline{a},\underline{q}^*) \; .$$

The left hand inequality is trivial. In order to prove the right hand inequality, we determine the minimum of $\beta(\underline{a},\underline{q}^*)$ with respect to $\underline{a}$. We get

$$\frac{\partial}{\partial a_j} \sum_i \Phi\left(U_{1-\alpha} - \frac{a_i \cdot M}{\sqrt{\underline{a}' \cdot \underline{\underline{\Sigma}} \cdot \underline{a}}}\right) \cdot q_i^* =$$

$$= \sum_{i \neq j} \Phi'\left(U_{1-\alpha} - \frac{a_i \cdot M}{\sqrt{\underline{a}' \cdot \underline{\underline{\Sigma}} \cdot \underline{a}}}\right) (a_i \cdot M) \cdot \frac{\partial}{\partial a_j}\left(\frac{1}{\sqrt{\underline{a}' \cdot \underline{\underline{\Sigma}} \cdot \underline{a}}}\right) \cdot q_i^* +$$

$$+ \Phi'\left(U_{1-\alpha} - \frac{a_j \cdot M}{\sqrt{\underline{a}' \cdot \underline{\underline{\Sigma}} \cdot \underline{a}}}\right)\left(\frac{M}{\sqrt{\underline{a}' \cdot \underline{\underline{\Sigma}} \cdot \underline{a}}} + a_i \cdot M \cdot \frac{\partial}{\partial a_i}\left(\frac{1}{\sqrt{\underline{a}' \cdot \underline{\underline{\Sigma}} \cdot \underline{a}}}\right)\right) \cdot q_j^* =$$

$$= M\left[\sum_i \Phi'\left(U_{1-\alpha} - \frac{a_i \cdot M}{\sqrt{\underline{a}' \cdot \underline{\underline{\Sigma}} \cdot \underline{a}}}\right) \cdot a_i \cdot \frac{\partial}{\partial a_j}\left(\frac{1}{\sqrt{\underline{a}' \cdot \underline{\underline{\Sigma}} \cdot \underline{a}}}\right) \cdot q_i^* + \Phi'\left(U_{1-\alpha} - \frac{a_j \cdot M}{\sqrt{\underline{a}' \cdot \underline{\underline{\Sigma}} \cdot \underline{a}}}\right) \cdot \frac{a_j^*}{\sqrt{\underline{a}' \cdot \underline{\underline{\Sigma}} \cdot \underline{a}}}\right].$$

Putting these first derivatives at $\underline{a} = \underline{a}^*$ equal to zero, we get with

$$\frac{\partial}{\partial \underline{a}} \frac{1}{\sqrt{\underline{a}' \cdot \underline{\underline{\Sigma}} \cdot a}} = - \frac{\underline{a}' \cdot \underline{\underline{\Sigma}}}{(\underline{a}' \cdot \underline{\underline{\Sigma}} \cdot \underline{a})^{3/2}}$$

the following determinants for $\underline{q}^*$

$$- \underline{e}' \cdot \underline{\underline{\Sigma}} + \underline{q}^* \cdot (\underline{e}' \cdot \underline{\underline{\Sigma}} \cdot \underline{e}) = 0$$

which completes the proof. ⌑

One very interesting observation can be made here: The *expected* amount of material to be diverted in the i-th inventory period, is

$$M \cdot q_i^* + 0 \cdot (1 - q_i^*) = M \cdot q_i^* .$$

This means, that the vector of expected diversions in the n inventory periods is

$$M \cdot \underline{q}^* = \frac{M}{\underline{e}' \cdot \underline{\underline{\Sigma}} \cdot \underline{e}} \cdot \underline{e}' \cdot \underline{\underline{\Sigma}} ,$$

which is *exactly* the same as that of the real diversions in the protracted diversion case.

Let us repeat what has already been said at the end of section 1.1, p. 28: The results for Jaech's test lead to the supposition that also for the Neyman-Pearson test the optimal abrupt diversion strategy is given by Theorem 7; this, however, cannot be confirmed.

### 1.3 The n-fold Test Procedure by Avenhaus and Frick

As two successive MUF variables $Z_i$ and $Z_{i+1}$, $i=1,\dots,n-1$, are not independent because of the common inventory $I_i$, the test procedure to be outlined now does not use the $Z_i$, $i=1,\dots,n$, directly, but uses instead linear combinations of them which are independent.

Stewart (1958) developed these linear combinations by estimating the starting inventory $S_{i-1}$ for the i-th inventory period with the help of a linear combination of the foregoing - appropriately defined - book and ending physical inventories,

$$S_{i-1} := a_{i-1} \cdot \tilde{B}_{i-1} + (1-a_{i-1}) \cdot I_{i-1}, \quad i=2,\dots,n$$
$$S_o = I_o ,$$

in such a way that this estimate is an *unbiased* estimate of the starting inventory under the null hypothesis and has a minimum variance. The resulting modified MUF variables $\tilde{Z}_i$,

$$\tilde{Z}_i := S_{i-1} + T_i - I_i = a_{i-1} \cdot (\tilde{B}_{i-1} - I_{i-1}) + T_{i-1} + T_i - I_i =$$
$$= a_{i-1} \cdot \tilde{Z}_{i-1} + Z_i, \quad i=2,\dots,n,$$
$$\tilde{Z}_1 = Z_1 = I_o + T_1 - I_1 ,$$

where $a_{i-1}$ is given by

$$a_{i-1} = \frac{\text{var}(I_{i-1})}{\text{var}(B_{i-1}) + \text{var}(I_{i-1})}, \quad i=2,\dots,n$$

$$a_o = 0,$$

in fact represent linear combinations of the $Z_i$ which are uncorrelated,

$$\text{cov}(\tilde{Z}_i, \tilde{Z}_j) = 0 \text{ for } i \neq j,$$

and they are *independent* because we assumed normal distribution for $Z_i$.

Thus, the covariance matrix of the random variables $\tilde{Z}_i$ is a diagonal matrix, the i-th diagonal element of which we call $\sigma_i^2$, and the probability density function of the random vector $\underline{\tilde{Z}}' = (\tilde{Z}_1,\ldots,\tilde{Z}_n)$ is the product of the density functions of the random variables $\tilde{Z}_i$, $i=1,\ldots,n$.

*Protracted Diversion*

Under the null hypothesis $H_o$ that no material is diverted at all, we have

$$H_o\colon\ E_o(\tilde{Z}_i) = E_o(Z_i) = 0,\ i=1,\ldots,n.$$

Under the alternative hypothesis $H_1$ that in the i-th inventory period the amount $M_i$, $i=1,\ldots,n$, is diverted, we have

$$E_1(Z_i) = M_i \text{ for } i=1,\ldots,n,$$

and therefore,

$$H_1\colon\ E_1(\tilde{Z}_i) = a_{i-1}\cdot E(\tilde{Z}_{i-1})+M_i,\ i=2,\ldots,n,$$
$$E_1(\tilde{Z}_1) = E_1(Z_1) = M_1.$$

Avenhaus and Frick (1974a) analyzed the following procedure for testing the hypotheses $H_o$ and $H_1$: For each of the n observations $\hat{\tilde{Z}}_i$ of the random variables $\tilde{Z}_i$, $i=1,\ldots,n$, a significance threshold $s_i$ is defined in such a way that the overall false alarm probability $\alpha$ does not exceed a given value. This means that the inspector decides

$$H_o \text{ is accepted if } \hat{\tilde{Z}}_i \leq s_i \text{ for all } i=1,\ldots,n$$
$$H_1 \text{ is accepted if } \hat{\tilde{Z}}_i > s_i \text{ for at least one i out of } 1,\ldots,n,$$

i.e. the acceptance region is given by

$$Z_o^{AF} := \{\hat{\tilde{Z}}_i \leq s_i,\ i=1,\ldots,n\}\ ,$$

where the $s_i$-values are subject to the boundary condition

$$1-\alpha = \text{prob}\{\tilde{Z}_1 \leq s_1 \wedge\ldots\wedge \tilde{Z}_n \leq s_n | H_o\}\ ,$$

or, because of the independence of the $\tilde{Z}_i$,

$$1-\alpha = \prod_{i=1}^{n} \text{prob}\{\tilde{Z}_i \leq s_i | H_o\} \quad ,$$

which, with the help of the single false alarm probability given by

$$1-\alpha_i = \text{prob}\{\tilde{Z}_i \leq s_i | H_o\}, \; i=1,\ldots,n \; ,$$

also can be written as

$$1-\alpha = \prod_{i=1}^{n} (1-\alpha_i).$$

The overall probability of detection defined by

$$1-\beta_{AF} = 1-\text{prob}\{\tilde{Z}_1 \leq s_1 \wedge \ldots \wedge \tilde{Z}_n \leq s_n | H_1\}$$

likewise because of the independence of the $\tilde{Z}_i$ and with the single probabilities of detection

$$1-\beta_i = \text{prob}\{\tilde{Z}_i > s_i | H_1\}, \; i=1,\ldots,n,$$

can be written as

$$1-\beta_{AF} = 1-\prod_{i=1}^{n} \beta_i \; .$$

Explicitly, the relation between the significance thresholds $s_i$ and the single false alarm probabilities $\alpha_i$ is given by

$$1-\alpha_i = \Phi\left(\frac{s_i}{\sigma_i}\right), \; i=1,\ldots,n \; .$$

Therefore, the single probabilities of detection $1-\beta_i$, given by

$$1-\beta_i = \Phi\left(\frac{a_{i-1} \cdot E_1(\tilde{Z}_{i-1})+M_i-s_i}{\sigma_i}\right), \; i=1,\ldots,n,$$

being functions of the single false alarm probabilities $\alpha_i$ are given by

$$1-\beta_i = \Phi\left(\frac{a_{i-1}\cdot E_1(\tilde{Z}_{i-1})+M_i}{\sigma_i} - U_{1-\alpha_i}\right), \quad i=1,\ldots,n,$$

and the total probability of detection $1-\beta_{AF}$ is given by

$$1-\beta_{AF} = 1 - \prod_{i=1}^{n} \Phi\left(U_{1-\alpha_i} - \frac{a_{i-1}\cdot E_1(\tilde{Z}_{i-1})+M_i}{\sigma_i}\right) .$$

As the strategies $X_\alpha$ and $Y_M$ of the inspector and of the operator are given by

$$X_\alpha := \{\underline{\alpha}': (\alpha_1,\ldots,\alpha_n): 0\leq\alpha_i\leq 1, \prod_i(1-\alpha_i) = 1-\alpha\}$$

$$Y_M := \{\underline{M}': (M_1,\ldots,M_n): 0\leq M_i, \sum_i M_i = M\} ,$$

the optimal significance thresholds $s_i$ (or equivalently, the optimal false alarm probabilities $\alpha_i$) are determined by the following optimization problem:

$$\underset{\underline{\alpha}\in X_\alpha}{\text{maximize}} \quad \min_{\underline{M}\in Y_M} \quad 1-\beta(\underline{\alpha},\underline{M}) .$$

The solution of this problem is

*Theorem 8* (Avenhaus and Frick 1974 a)

Given the random vector $\underline{\tilde{Z}}$, defined above, with known diagonal covariance matrix $\underline{\underline{\Sigma}}$. Let $\delta$ be a test for the two hypotheses $H_o$ and $H_1$,

$$H_o : E(\underline{\tilde{Z}}) = 0$$

$$H_1 : E(\underline{\tilde{Z}}_i) = a_{i-1} \cdot E(\tilde{Z}_{i-1}) + \mu_i , \quad i = 2\ldots n ,$$

$$E(\tilde{Z}_1) = \mu_1,$$

$$\underline{e}' \cdot \underline{\mu} = M > 0,$$

with fixed significance level and power $1 - \beta_\delta(\underline{\mu})$, defined by

$$\delta = \begin{cases} 0 & \text{for } \tilde{Z}_i \leq \sigma_i \cdot U_{1-\alpha_i} , \quad i = 1\ldots n , \\ 1 & \text{otherwise.} \end{cases}$$

Then the power $1 - \beta_{\delta**}$ of the test $\delta**$, defined by

$$\delta** = \begin{cases} 0 & \text{for } \tilde{Z}_i \leq \sigma_i \cdot U_{1-\alpha_i^*} , \quad i = 1\ldots n , \\ 1 & \text{otherwise ,} \end{cases}$$

where $\alpha_i^*$ is given by

$$\frac{\exp\left(-x_i - \frac{1}{2}\cdot U^2\left(\exp(x_i)\right)\right)}{\sigma_i \cdot (1-a_i)} - \frac{\exp\left(-x_{i-1} - \frac{1}{2}\cdot U^2\left(\exp(x_{i-1})\right)\right)}{\sigma_{i-1} \cdot (1-a_{i-1})} = 0 \ , \quad i = 2,\dots,n-1,$$

$$\frac{\exp\left(-x_n - \frac{1}{2}\cdot U^2\left(\exp(x_n)\right)\right)}{\sigma_n} - \frac{\exp\left(-x_{n-1} - \frac{1}{2}\cdot U^2\left(\exp(x_{n-1})\right)\right)}{\sigma_{n-1} \cdot (1-a_{n-1})} = 0$$

$$\sum_{j=1}^{n} x_j = \ln(1-\alpha) \ ; \ x_j = \ln(1-\alpha_j) \ ,$$

fulfills the relations

$$1 - \beta_{\delta**} = \min_{\{\mu\}} \max_{\{\delta\}} \left(1 - \beta_\delta(\mu)\right) = \max_{\{\delta\}} \min_{\{\mu\}} \left(1 - \beta_\delta(\mu)\right) .$$

The minimizing alternative hypothesis is given by

$$\exp\left(x_i + \frac{1}{2}\cdot U^2\left(e^{x_i}\right)\right)\cdot Q\left(U\left(e^{x_i}\right) - \frac{\mu_i}{\sigma_i}\right) - \exp\left(x_{i-1} + \frac{1}{2}\cdot U^2\left(e^{x_{i-1}}\right)\right)\cdot Q\left(U\left(e^{x_{i-1}} - \frac{\mu_{i-1}}{\sigma_{i-1}}\right) = 0 \ ,$$

$$i = 1,\dots,n \ ,$$

$$\mu_n + \sum_{j=1}^{n-1} (1-a_j)\cdot\mu_j = M \ ,$$

where $Q(x)$ is defined by

$$Q(x) := \frac{d}{dx} \ln \Phi(x) \ . \qquad \Pi$$

For illustrative and later purpose, we give the solution for n = 2:

$$(\alpha_1^*,\alpha_2^*) : 1-\alpha = (1-\alpha_1^*)\cdot(1-\alpha_2^*) \ \text{and} \ (M_1^*,M_2^*) : M_1^* + M_2^* = M$$

are solutions of the following system of equations

$$(1-\alpha_1)\cdot\sigma_1\cdot(1-a)\cdot\exp\left(\frac{1}{2}\cdot U^2_{1-\alpha_1}\right) - (1-\alpha_2)\cdot\sigma_2\cdot\exp\left(\frac{1}{2}\cdot U^2_{1-\alpha_2}\right) = 0$$

$$(1-\alpha_1)\cdot(1-\alpha_2) = 1-\alpha$$

and furthermore,

$$- \frac{1}{\sigma_1} \cdot \Phi'(U_{1-\alpha_1} - \frac{M_1}{\sigma_1}) \cdot \Phi(U_{1-\alpha_2} - \frac{M}{\sigma_2} + \frac{(1-a)\cdot M_1}{\sigma_2}) +$$

$$+ \frac{1-a}{\sigma_2} \cdot \Phi(U_{1-\alpha_1} - \frac{M_1}{\sigma_1})\cdot\Phi'(U_{1-\alpha_2} - \frac{M}{\sigma_2} + \frac{(1-a)\cdot M_1}{\sigma_2}) = 0$$

$$M_1 + M_2 = M,$$

We see that the optimal values of $\alpha_1$ and $\alpha_2$ *do not depend on the value of M.* (This is also true for n>2). They only depend on the single parameter $(1-a) \cdot \frac{\sigma_1}{\sigma_2}$ which means that they lend themselves to a convenient graphical representation. This is shown in Fig. 1a: the optimal values of $\alpha_1$ and $\alpha_2$ for given values of $(a-1)\cdot \frac{\sigma_1}{\sigma_2}$ and $\alpha$ are the intersections of the two curves with the appropriate values. In Fig. 1b the more interesting area $(0 < \alpha \leq 0.1)$ is shown on a larger scale.

According to the definitions of $\sigma_1$, $\sigma_2$ and a, we have

$$(1 - a)^2 \cdot \frac{\sigma_1^2}{\sigma_2^2} = \frac{(\mathrm{var}(D_1) + \mathrm{var}(I_1))^2}{\mathrm{var}(MUF_1) \cdot \mathrm{var}(MUF_2) - (\mathrm{var}(I_1))^2} .$$

If we assume stable plant conditions,

$$\mathrm{var}(I_o) = \mathrm{var}(I_1) = \mathrm{var}(I_2), \quad \mathrm{var}(D_1) = \mathrm{var}(D_2),$$

then we get

$$(1 - a)^2 \cdot \frac{\sigma_1^2}{\sigma_2^2} = \frac{\mathrm{var}(I) + \mathrm{var}(D)}{3 \cdot \mathrm{var}(I) + \mathrm{var}(D)} ,$$

and we can show immediately

$$\frac{1}{\sqrt{3}} \leq (1 - a) \cdot \frac{\sigma_1}{\sigma_2} \leq 1 .$$

The Figure shows us that in this latter case the optimal value of $\alpha_2$ is always larger than that of $\alpha_1$.

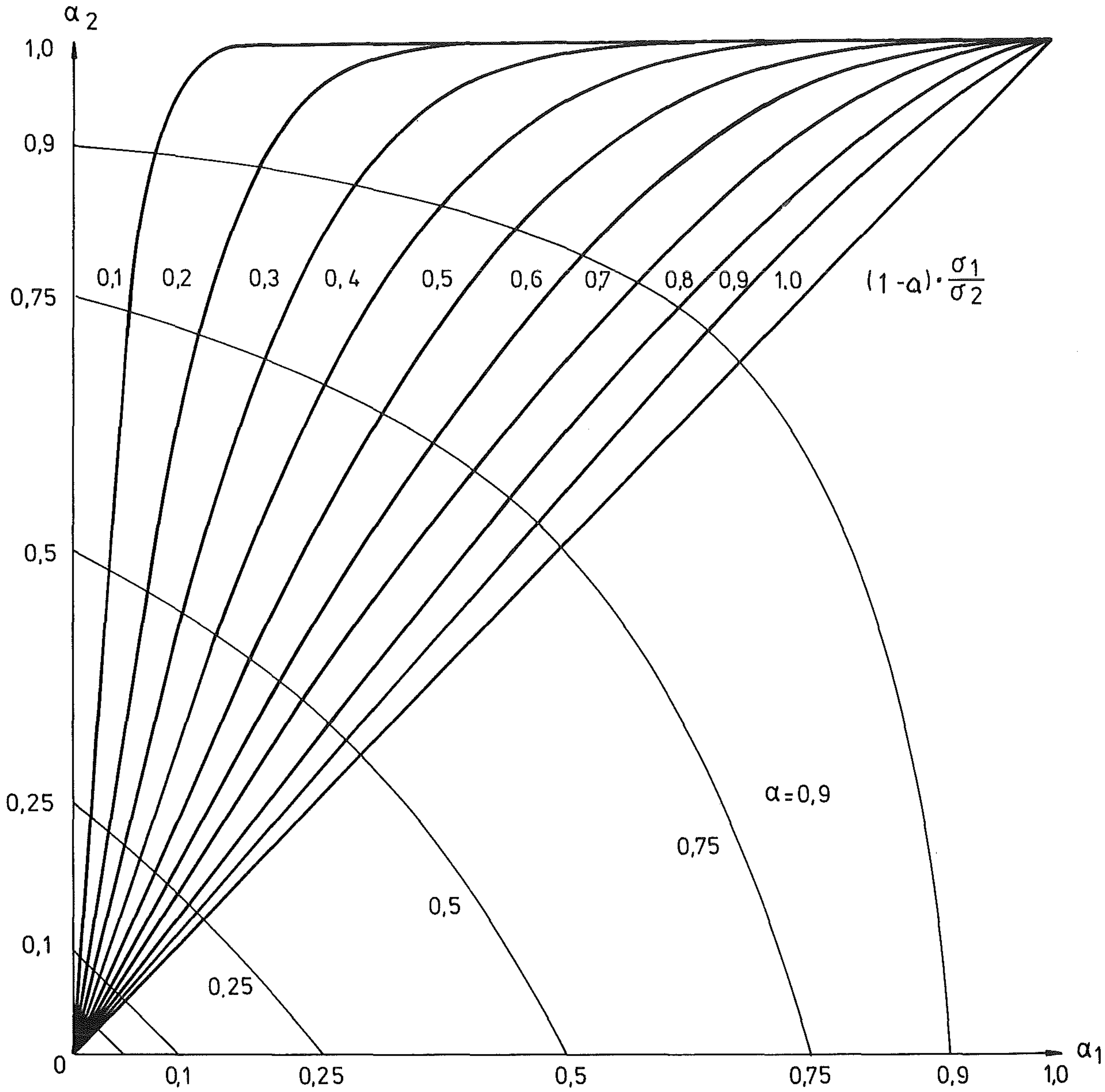


Figure 1a: Graphical representation of the equation

$$(1-\alpha_1)\cdot\sigma_1\cdot(1-a)\cdot\exp(\tfrac{1}{2}\cdot U^2_{1-\alpha_1})-(1-\alpha_2)\cdot\sigma_2\cdot\exp(\tfrac{1}{2}\cdot U^2_{1-\alpha_2}) = 0$$

for various values of $(1-\alpha)\cdot\frac{\sigma_1}{\sigma_2}$ and of the equation

$$(1-\alpha_1)\cdot(1-\alpha_2) = 1-\alpha$$

for various values of $\alpha$ (after Heidl and Schmidt (1980)).

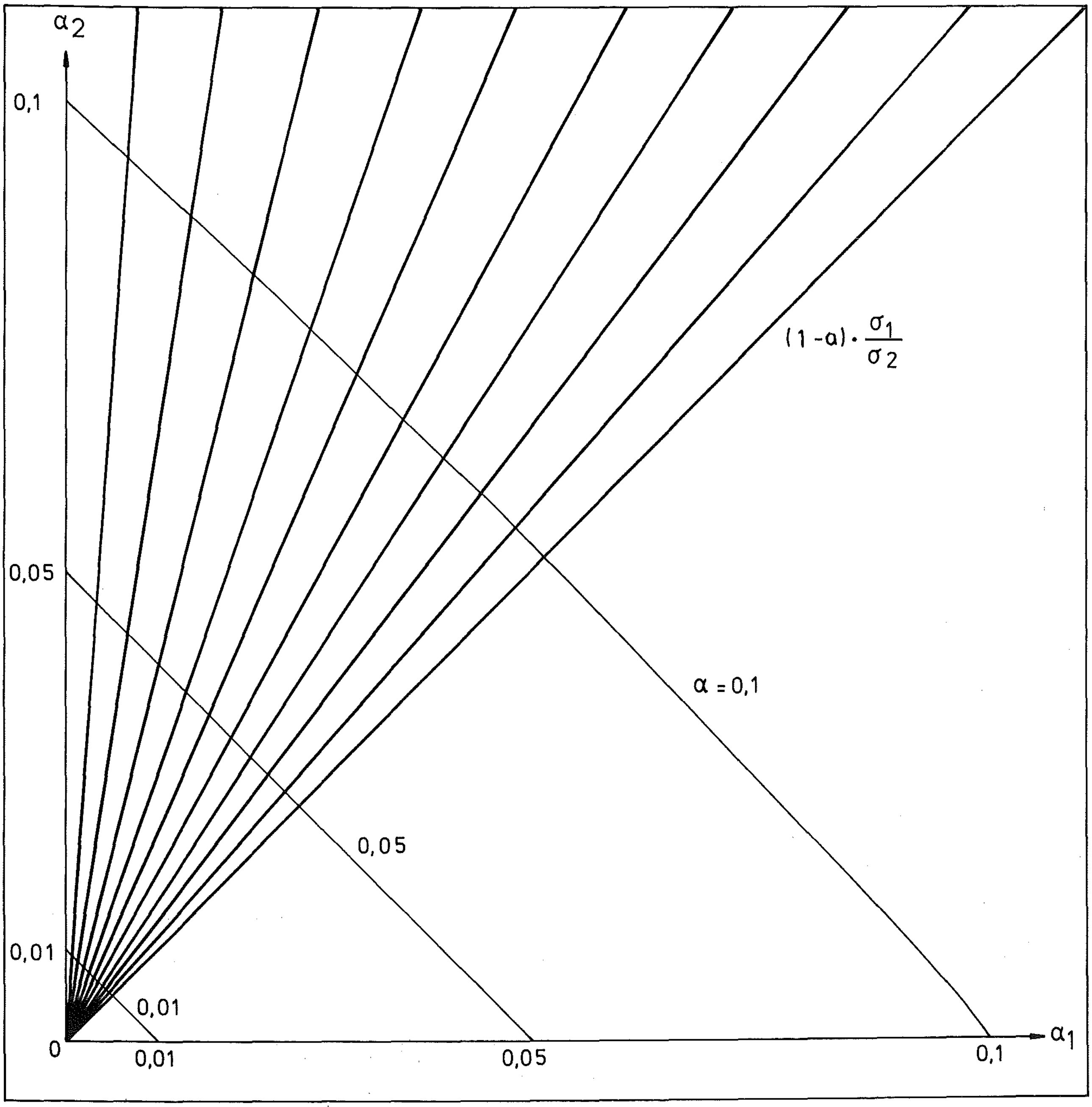


Figure 1b: Graphical representation of the same equation as represented in Fig. 1a, larger scale (after Heidl and Schmidt (1980)).

At least for $a_i = 0$ - i.e. for the case where the physical inventories are measured precisely and therefore are taken as the starting inventories for the next inventory period - one can show that the smaller the variances $\sigma_i^2$ of $\tilde{Z}_i$ are, the smaller are the corresponding optimal significance thresholds

$$s_i^* = \sigma_i \cdot U_{1-\alpha_i^*}$$

and the smaller are the amounts of material $M_i^*$ to be diverted in the i-th inventory period. This can be interpreted in such a way that in inventory periods during which the technical possibility (expressed by the variance of the measurement errors) of detecting diversion is good, the operator will divert only small amounts of material, if any, and the inspection team will check the material balance only in a rather loose way.

*Abrupt Diversion*

As in this case simple and intuitive solutions cannot be obtained, we will limit our discussion to the case $n = 2$.

Under the null hypothesis $H_o$, that no material is diverted at all, we have again

$$H_o : E(Z_1) = E(Z_2) = 0 .$$

Under the alternative hypothesis $H_1$ that in the i-th inventory period the amount M is diverted with probability $q_i$, $i = 1,2$, where $q_1 + q_2 = 1$, we have

$$H_1 : E(Z_i) = \begin{cases} M & \text{with probability } q_i,\ i = 1,2 \\ 0 & \text{otherwise.} \end{cases}$$

Under the condition that the amount M is diverted in the first inventory period, the overall probability of no detection is

$$\Phi\left(U_{1-\alpha_1} - \frac{M}{\sigma_1}\right) \cdot \Phi\left(U_{1-\alpha_2} - \frac{a \cdot M}{\sigma_2}\right) =: \beta_1(\alpha_1)$$

(we assume $\alpha_2$ to be determinated by $(1-\alpha_1) \cdot (1-\alpha_2) = 1-\alpha)$. Under the condition that the amount M is diverted in the second inventory period, the overall probability of no detection is

$$(1-\alpha_1) \cdot \Phi\left(U_{1-\alpha_2} - \frac{M}{\sigma_2}\right) =: \beta_2(\alpha_2) \quad .$$

The unconditioned probability of no detection therefore is

$$\Phi\left(U_{1-\alpha_1}-\frac{M_1}{\sigma_1}\right)\cdot\Phi\left(U_{1-\alpha_2}-\frac{a\cdot M}{\sigma_2}\right)\cdot q+(1-\alpha_1)\cdot\Phi\left(U_{1-\alpha_2}-\frac{M}{\sigma_2}\right)\cdot(1-q)=:\beta(\alpha_1,q)\,,$$

here we have defined

$$q_1 =: q\;,\; q_2 =: 1-q\;.$$

The optimal single false alarm probabilities $\alpha_1{}^*$ and $\alpha_2{}^*$ and the optimal diversion strategy $q^*$, i.e., those strategies which fulfill the saddlepoint criterion

$$\beta(\alpha_1{}^*,q)\;\le\;\beta(\alpha_1{}^*,q^*)\;\le\;\beta(\alpha_1,q^*)\;\;,$$

are given by the conditions

$$\beta_1(\alpha_1{}^*)\;=\;\beta_2(\alpha_2{}^*)$$

$$\frac{d}{d\alpha_1}\left(\beta_1(\alpha_1)\cdot q^*+\beta_2(\alpha_2)\cdot(1-q^*)\right)\Big|_{\alpha_1=\alpha_1{}^*}=0\;,$$

which leads to the following determinants for $\alpha_1{}^*$ and $q^*$:

$$\Phi\left(U_{1-\alpha_1{}^*}-\frac{M}{\sigma_1}\right)\cdot\Phi\left(U_{1-\alpha_2{}^*}-\frac{a\cdot M}{\sigma_2}\right)=(1-\alpha_1{}^*)\cdot\Phi\left(U_{1-\alpha_2{}^*}-\frac{M}{\sigma_2}\right)$$

$$\Bigg[\Phi'\left(U_{1-\alpha_1{}^*}-\frac{M}{\sigma_1}\right)\cdot\exp\left(\frac{1}{2}\cdot U^2_{1-\alpha_1{}^*}\right)\cdot\Phi\left(U_{1-\alpha_2{}^*}-\frac{a\cdot M}{\sigma_2}\right)+\Phi\left(U_{1-\alpha_1{}^*}-\frac{M}{\sigma_1}\right)\cdot\Phi'\left(U_{1-\alpha_2{}^*}-\frac{a\cdot M}{\sigma_2}\right)\otimes$$

$$\otimes\exp\left(\frac{1}{2}\cdot U^2_{1-\alpha_2{}^*}\right)\cdot\frac{d(1-\alpha_2)}{d\alpha_1}\Bigg|_{\alpha_1{}^*}\Bigg]\cdot q^*+$$

$$+\Bigg[-\frac{1}{\sqrt{2\pi}}\cdot\Phi\left(U_{1-\alpha_2{}^*}-\frac{M}{\sigma_2}\right)+(1-\alpha_1{}^*)\cdot\Phi'\left(U_{1-\alpha_2{}^*}-\frac{M}{\sigma_2}\right)\cdot\exp\left(\frac{1}{2}\cdot U^2_{1-\alpha_2{}^*}\right)\otimes$$

$$\otimes\frac{d(1-\alpha_2)}{d\alpha_1}\Bigg|_{\alpha_1{}^*}\Bigg]\cdot(1-q^*)=0\;.$$

Contrary to the situation in section 1.2, where we were able to establish an equivalence between the true optimal diversion fractions in the single inventory periods for the protracted diversion case on one hand, and the expected optimal diversion fractions for the abrupt diversion case on the other, we see here no relation at all. Furthermore, the optimal inspection strategy $(\alpha_1{}^*,\alpha_2{}^*)$ is *not* independent of the total diversion M.

*Detection Time*

The surprising result which we have derived in the two foregoing sections, namely that it is optimal in the sense of the overall probability of detection not to use the in-between inventories, highlights the question for the optimum number of inventory periods for one material balance area in a given reference time (O,T). In order to tackle this problem we have to formulate the appropriate optimization criterion. We will proceed in two steps.

First, let us assume that the criterion is again the *overall probability* of detection. In the sense of this criterion it is best, as we have seen in the foregoing sections, to have only one beginning and one ending inventory, i.e., to establish only one balance for the whole reference time.

One additional point has to be mentioned in this connection: If one considers n inventory periods in the reference time (O,T), and if one uses Stewart's starting inventory estimate for all inventory periods and determines the guaranteed overall probability of detection according to *Theorem 8*, then one can prove directly that only one inventory period is better than more periods in case of protracted diversion - which is not surprising according to *Theorem 4*. What one cannot prove is that $n_2$ periods are worse than $n_1$ periods for $n_2 > n_1$; however, numerical calculations indicate this. An example taken from Avenhaus and Frick (1974 b) is given in Figure 2a. This is important for the following reason:

Let us consider as second criterion the *detection time*. From what was been said up to now, one could make two assumptions about the detection time. First, one would assume that the shorter an inventory period is, the shorter the detection time is. Second, according to the result stated above, with an increasing number of inventory periods per reference time, the probability of detection decreases. Therefore, detection may depend on the values of the parameters of the stronger of the two effects. From these considerations one concludes that the expected detection time T is the appropriate optimization criterion from the detection point of view, because it takes into account both aspects, the actual time at which detection may occur and the probability for detection at that time.

Before entering into this subject, we repeat once more what has already been said at the end of the introduction to this part: Even if the observation of a specific MUF variable exceeds its significance threshold, the final decision is made only at the end of the reference time. Thus, one may speak here of a detection time in retrospect in the sense that at the end of the reference time a statement is made when the loss or diversion was observed the first time.

Let us work for the following with the MUF statistics, which are independent, as we saw. The expected detection time $E(T)$, which shall be measured in units of inventory periods, is the sum of the products of detection times, $i$, $i = 1 \ldots n$, and probabilities for first detection at time $i$.

In order to formalize these ideas, we introduce the concept of the *run length* RL. The run length is defined as the time from the beginning until the first "detection" (which may be a false alarm):

$$\{RL = i\} = \{\text{first detection at } t_i\} .$$

It should be noted that in the monograph of van Dobben (1968) the run length is defined as the time until the rejection of the null hypothesis $H_o$. Here, we will use it in the general sense.

The probabilities of these events are in our case given by the following expressions:

i) In case the null hypothesis $H_o$ is true

$$\text{prob}\{RL = i\} = \begin{cases} \alpha_i \cdot \prod_{j=1}^{i-1} (1 - \alpha_j) & \text{for } i = 2 \ldots n \\ \alpha_1 \quad \text{for } i = 1 . \end{cases}$$

ii) In case the alternative hypothesis $H_1$ is true

$$\text{prob}\{RL = i\} = \begin{cases} (1 - \beta_i) \cdot \prod_{j=1}^{i-1} \beta_j & \text{for } i = 2 \ldots n \\ 1 - \beta_1 & \text{for } i = 1 , \end{cases}$$

where $\beta_i$ is given by

$$\beta_i = \text{prob}\{\widetilde{MUF}_i \le s_i | H_1\} = \text{prob}\{\widetilde{MUF}_1 \le \sigma_i \cdot U_{1-\alpha_i} | H_1\} =$$

$$= \Phi\left(U_{1-\alpha_i} - \frac{a_{i-1} \cdot E_1(\widetilde{Z}_{i-1}) + M_i}{\sigma_i}\right) , \quad i = 1 \ldots n .$$

One is now faced with the difficulty of taking into account the probability that no detection at all accurs during the reference time, the probability of this event is given by

$$\prod_{i=1}^{n} \beta_i \; .$$

If we call a the detection time for the case in which detection occurs only after the end of the reference time, then the expected detection time is given by

$$E(T_2) = 1 - \beta_1 + \sum_{i=2}^{n} i \cdot (1 - \beta_i) \cdot \prod_{j=1}^{i-1} \beta_j + a \cdot \prod_{i=1}^{n} \beta_i \; .$$

The difficulty with this formula is that there exists no natural numerical value for a.

A more satisfying optimization criterion is the expected detection time $T_1$ under the condition that detection will take place during the reference time. This criterion again takes into account both aspects of detection time - actual time and probability of detection at that time. It is given by

$$E(T_1) = \frac{1 - \beta_1 + \sum_{i=2}^{n} i \cdot (1 - \beta_i) \cdot \prod_{j=1}^{i-1} \beta_j}{1 - \prod_{i=1}^{n} \beta_i} \; .$$

This relation can be understood as follows: By definition, we have

$$E(T_1) = \sum_i i \cdot \text{prob} \left\{ RL = i \,|\, \text{detection in } (0,T) \right\} = \frac{\sum_i i \cdot \text{prob} \left\{ RL = i \wedge \text{detection in } (0,T) \right\}}{\text{prob} \left\{ \text{detection in } (0,T) \right\}} \; .$$

Now, the event 'detection in (O,T)' may be described as the union of the events $\{RL = i\}$ for $i = 1 \ldots n$:

$$\left\{ \text{detection in } (0,T) \right\} = \bigcup_{i=1}^{n} \left\{ RL = i \right\} ,$$

therefore we have

$$\text{prob} \left\{ RL = i \wedge \text{detection in } (0,T) \right\} = \text{prob} \left\{ RL = i \right\} .$$

Furthermore, because of the independence of the events $\{RL = i\}$ - first detection at $t_i$ means that at $t_j, j < i$, no detection took place - and because of the independence of the $\widetilde{MUF}_i$, we have

$$\text{prob}\left\{\text{detection in } (0,T)\right\} = \sum_{i=1}^{n} \text{prob}\left\{RL = i\right\} =$$

$$= 1 - \beta_1 + \sum_{i=2}^{n} (1 - \beta_i) \cdot \prod_{j=1}^{i-1} \beta_j = 1 - \prod_{i=1}^{n} \beta_i \ ;$$

the latter relation is intuitive, but can also easily be proven by complete induction. Inserting these relations into the definitions completes the derivation.

Numerical calculations show that this conditioned expected detection is *not* a monotonically decreasing function of the number of inventories. A numerical illustration is given in Figure 2b.

The fact that we have difficulties with the appropriate definition of the expected detection time is a consequence of the fact that we consider a *fixed reference time interval* $(0,T)$. This we do in order to be able to control the overall false alarm probability $\alpha$. An alternative way, which leads to a more natural definition of the expected detection time and still allows to control the number of false alarms, goes as follows, if we work again with the $\widetilde{MUF}$ statistics.

If we consider an *infinite time interval* $(0,\infty)$ and, accordingly, an infinite number of inventory periods, we get

$$\sum_{i=1}^{\infty} \text{prob}\left\{RL = i\right\} = 1 \ ,$$

thus, detection is a certain event, and we can define the expected run length under the alternative hypothesis

$$E(RL) = \sum_{i=1}^{\infty} i \cdot \text{prob}\ \{RL = i\} = 1 - \beta_1 + \sum_{i=2}^{\infty} i \cdot (1 - \beta_i) \cdot \prod_{j=1}^{i-1} \beta_j \text{ under } H_1 \ .$$

Now, if we take for the moment for each inventory period the same value of $\alpha$ for the false alarm probability, then we get under the null hypothesis $H_0$ the expected run length

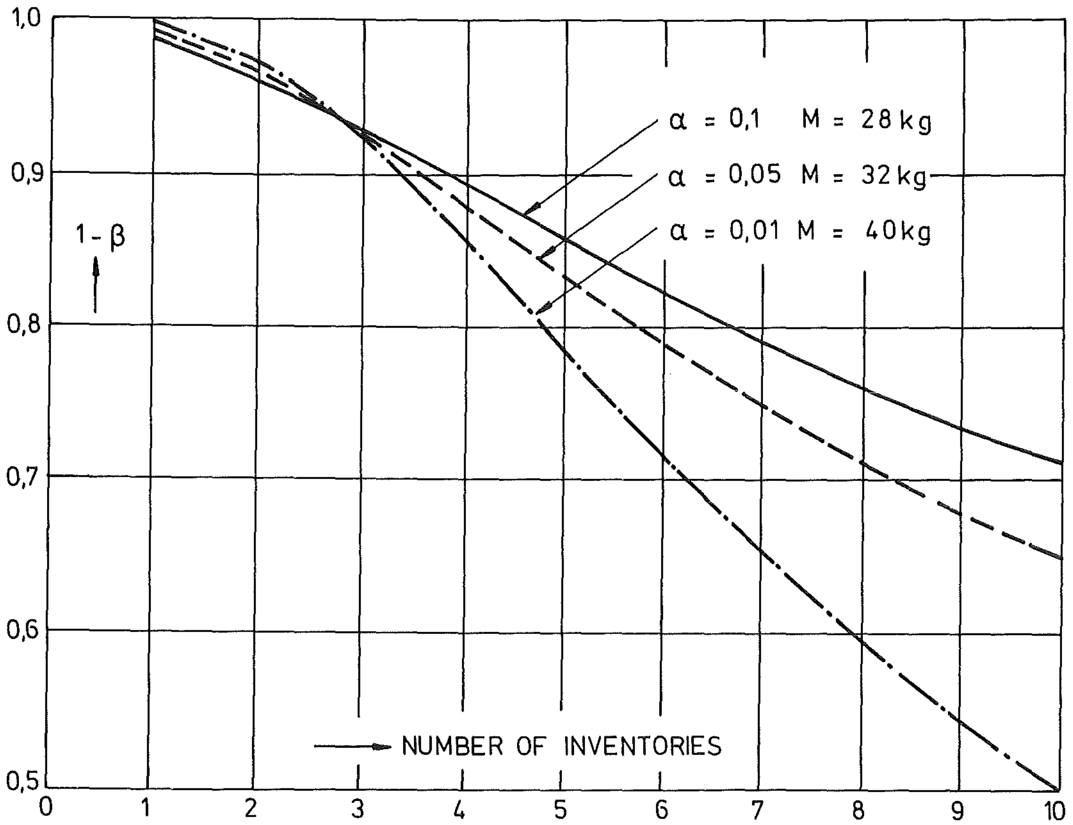


Figure 2a: Probability of detection as a function of the number of inventories per year with α and M as parameter.

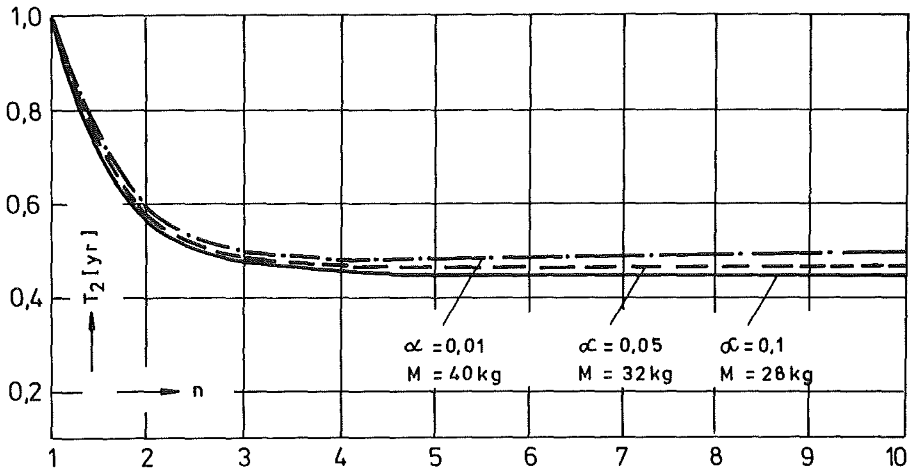


Figure 2b: Expected conditional detection time as a function of the number of inventories per year with α and M as parameters.

$$E(RL) = \alpha + \sum_{i=2}^{\infty} i \cdot \alpha \cdot \prod_{j=1}^{i-1} (1-\alpha) = \alpha + \prod_{i=2}^{\infty} i \cdot \alpha \cdot (1-\alpha)^{i-1} = \quad (i-1=j)$$

$$= \alpha \cdot \left(1 + \sum_{j=1} (j+1) \cdot (1-\alpha)^{j}\right) = \alpha \cdot \left(1 + \sum_{j=1}^{\infty} (1-\alpha)^{j} + \sum_{j=1}^{\infty} j \cdot (1-\alpha)^{j}\right) =$$

$$= \alpha \cdot \left(1 + \frac{1}{\alpha} - 1 + \frac{1-\alpha}{\alpha^2}\right) = \frac{1}{\alpha} \qquad \text{under } H_o \ .$$

This gives us the following idea: Instead of postulating a value for the overall false alarm probability for a fixed interval of time, we postulate a value for the expected run length under the null hypothesis $H_o$.

To take the detection time as an optimization criterion then would mean to minimize the expected run length under an appropriately defined alternative hypothesis $H_1$, subject to a fixed expected run length under the null hypothesis $H_o$. It is clear that such a program can only be tackled with the help of simulation studies.

There is one objection against this procedure from a political point of view: From the standpoint of the non-weapons NPT signatory states it is important that at regular fixed points of time (e.g., at the end of the year) the safeguards authority makes a statement that from the nuclear fuel cycle of that state no material has been diverted, if no significant differences occured. It is questionable if a procedure, where in principle all foregoing MUF's are taken into account for making a statement on a specific MUF at a given time, is compatible with such a postulate from the side of the non-weapons states. This postulate in fact, was the basic reason for working with a finite reference time.

Let us conclude this section with the observation that so far nobody has made an attempt to balance the two criteria "overall probability of detection" and "expected detection time" - one exception being the general sequential game theoretical approach to be discussed in the second part of this paper which, however, does not directly answer this question. From a methodological point of view, this is a difficult multiobjective decision theoretical problem. From a safeguards point of view, the question is far from being settled. There is only one recent official statement (Shea et al. 1981) which says

> "In this regard, in IAEA safeguards, the timeliness objective is never given equal importance in comparison to detection sensitivity."

Even though it is not explicitely stated, that detection sensitivity has to be understood as probability of detection, nor have the boundary conditions (e.g., false alarm probability) formulated quantitatively, one sees that different weights are given to different objective functions.

## 1.4 CUMUF Tests

The results of the foregoing sections showed that from the overall probability of detection point of view it is optimal to consider only the material balance over the total reference time. Nevertheless, there are reasons to consider the in-between inventories in order to localize any loss or diversion in time. The combination of these two aspects leads us to the conjecture that the so-called CUMUF - test, based on the sums of MUF-variables,

$$Y_j := \mathrm{CUMUF}_j := \sum_{i=1}^{j} Z_i ,$$

might be a useful test for our purposes.

This test represents a special application of the so-called CUSUM - test (see, e.g., van Dobben de Bruyn (1968)) to the nuclear material safeguards case and has, e.g., been studied by Stewart et al. (1979). The idea - same as in the foregoing section - is to test the hypotheses

$$H_o : E(Y_j) = 0$$

$$H_1 : E(Y_j) = \begin{cases} \sum\limits_{i=1}^{j} E(Z_i) = \sum\limits_{i=1}^{j} M_i , \quad \sum\limits_{i=1}^{n} M_i = M \\ \qquad\qquad \text{for protracted diversion} \\ \\ M \text{ with probability } \sum\limits_{i=1}^{j} q_j \text{ and zero otherwise} \\ \qquad\qquad \text{for abrupt diversion} \end{cases}$$

separately in such a way that the overall false alarm probability $\alpha$ does not exceed a given value. The acceptance region of this test has the form

$$Z_o^{CM} := \left\{ \hat{Y}_i \leq s_i , \; i = 1 \ldots n \right\} ,$$

where the $s_i$ values are subject to the boundary condition of a fixed overall probability of detection,

$$1 - \alpha = \text{prob}\left\{ \hat{Y}_1 \leq s_1 \wedge \ldots \wedge \hat{Y}_n \leq s_n | H_o \right\} .$$

Contrary to the situation in the foregoing section the random variables $\hat{Y}_j$ are *not* independent, therefore, it is impossible in general to give an explicit expression for the overall probability of detection $1 - \beta_{CM}$ defined by

$$1 - \beta_{CM} = 1 - \text{prob}\left\{ Y_1 \leq s_1 \wedge \ldots \wedge Y_n \leq s_n | H_1 \right\} .$$

Consequently, it is impossible as a rule to give explicit expressions for the optimal significance thresholds. For the purpose of illistration, we consider the special case $n = 2$.

*Protracted Diversion*

In this case we have

$$Y_1 = Z_1 = I_o + T_1 - I_1$$
$$Y_2 = Z_1 + Z_2 = I_o + T_1 + T_2 - I_2,$$

and therefore,

$$H_o: E(Y_1) = E(Y_2) = 0$$
$$H_1: E(Y_1) = M_1, \quad E(Y_2) = M.$$

The second moments are given by

$$\text{var}(Y_1) = \text{var}(I_o) + \text{var}(T_1) + \text{var}(T_2) =: \sigma_1^2$$
$$\text{var}(Y_2) = \text{var}(I_o) + \text{var}(T_1) + \text{var}(T_2) + \text{var}(I_2) =: \sigma_2^2$$
$$\text{cov}(Y_1, Y_2) = \text{cov}(I_o + T_1, I_o + T_1) = \text{var}(I_o) + \text{var}(T_1) =: \rho \cdot \sigma_1 \cdot \sigma_2 > 0.$$

The single false alarm probabilities $\alpha_i$, defined by

$$1 - \alpha_i = \text{prob}\{Y_i \leq s_i | H_o\},$$

are explicitly given by

$$1-\alpha_i = \Phi\left(\frac{s_i}{\sigma_i}\right).$$

They are related to the overall false alarm probability $\alpha$ by the following equation

$$1-\alpha = \text{prob}\{Y_1 \leq \sigma_1 \cdot U_{1-\alpha_1} \cap Y_2 \leq \sigma_2 \cdot U_{1-\alpha_2} | H_o\}.$$

As the common probability density of the random variables $Y_1$ and $Y_2$ under $H_o$ is given by

$$\frac{1}{2\pi \cdot \sqrt{1-\rho^2} \cdot \sigma_1 \cdot \sigma_2} \cdot \exp\left(-\frac{1}{2} \cdot \frac{1}{1-\rho^2} \cdot \left(\frac{t_1^2}{\sigma_1^2} - \frac{2\rho t_1 t_2}{\sigma_1 \sigma_2} + \frac{t_2^2}{\sigma_2^2}\right)\right),$$

the relation between $\alpha$, $\alpha_1$ and $\alpha_2$ is explicitly given by

$$1-\alpha = \frac{1}{2\pi} \cdot \frac{1}{\sqrt{1-\rho^2}} \cdot \int_{-\infty}^{U_{1-\alpha_1}} dt_1 \int_{-\infty}^{U_{1-\alpha_2}} dt_2 \exp\left(-\frac{1}{2} \cdot \frac{t_1^2 - 2\rho t_1 t_2 + t_2^2}{1-\rho^2}\right).$$

This relation has been studied extensively by Avenhaus (1979); a graphical representation of this relation for $\alpha = 0.05$ has been given in Fig. 3.

As the common probability density of the random variables $Y_1$ and $Y_2$ under $H_1$ is given by

$$\frac{1}{2\pi \cdot \sqrt{1-\rho^2} \cdot \sigma_1 \cdot \sigma_2} \cdot \exp\left(-\frac{1}{2} \cdot \frac{1}{1-\rho^2} \cdot \left(\frac{(t_1-M_1)^2}{\sigma_1^2} - \frac{2\rho(t_1-M_1)(t_2-M)}{\sigma_1 \sigma_2} + \frac{(t_2-M)^2}{\sigma_2^2}\right)\right),$$

the probability of detection as a function of the single false alarm probabilities defined by

$$1-\beta_{CM}^{P} = 1-\text{prob}\{Y_1 \leq \sigma_1 \cdot U_{1-\alpha_1} \cap Y_2 \leq \sigma_2 \cdot U_{1-\alpha_2} | H_1\}$$

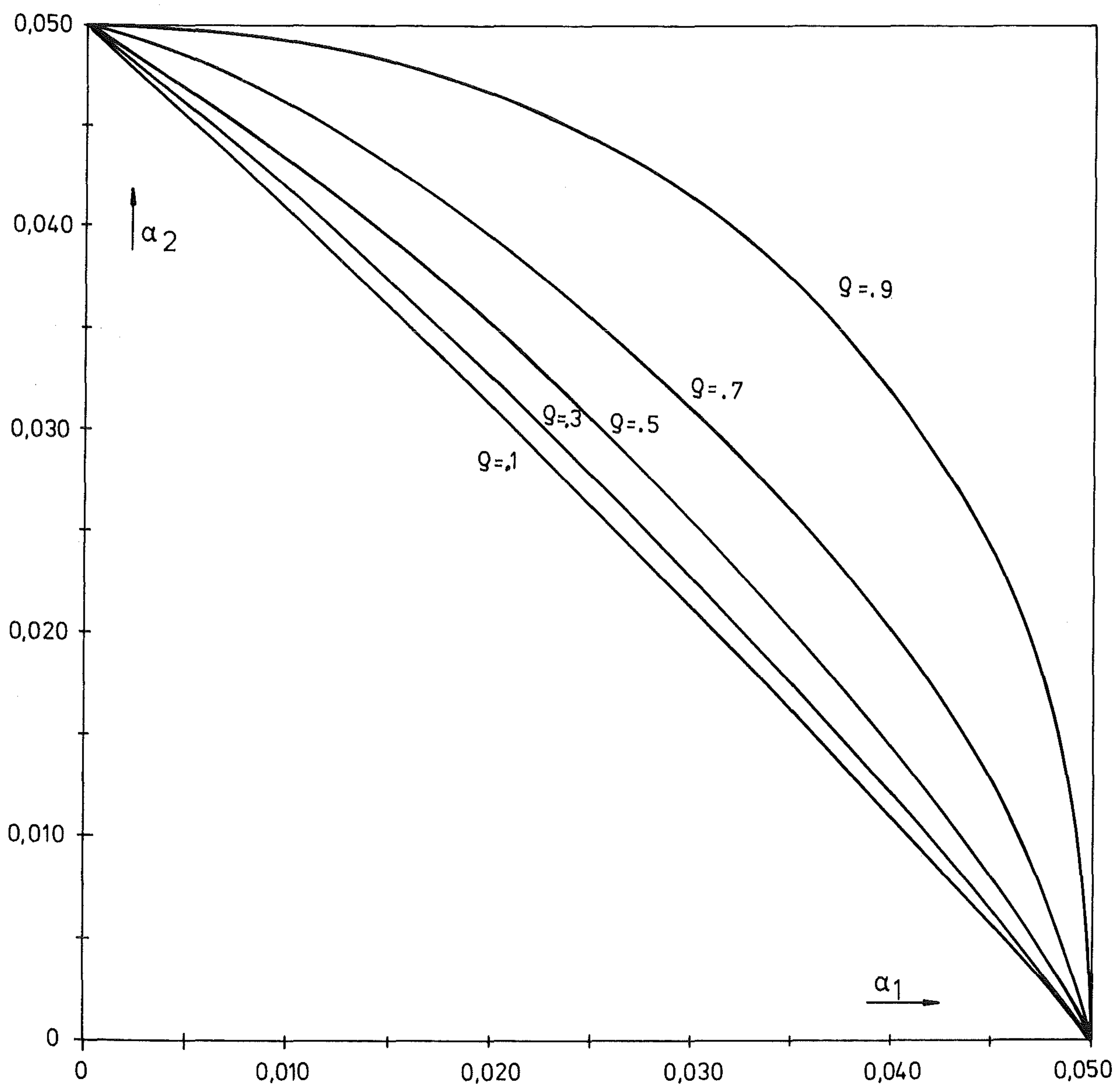


Figure 3: Graphical representation of the relation

$$1-\alpha = \frac{1}{2\pi\cdot\sqrt{1-\rho^2}}\int_{-\infty}^{U_{1-\alpha_1}} dt_1 \int_{-\infty}^{U_{1-\alpha_2}} dt_2 \exp\left(-\frac{t_1^2-2\rho t_1+t_2^2}{2(1-\rho^2)}\right)$$

between $\alpha_1$ and $\alpha_2$ for $\alpha$=0.05 and $\rho$ as parameter; after Avenhaus (1979).

is given explicitly by the following expression:

$$1-\beta_{CM}^{P} = 1- \frac{1}{2\pi\cdot\sqrt{1-\rho^2}} \cdot \int_{-\infty}^{U_{1-\alpha_1}-\frac{M_1}{\sigma_1}} dt_1 \int_{-\infty}^{U_{1-\alpha_2}-\frac{M}{\sigma_2}} dt_2 \; \exp\left(-\frac{1}{2}\cdot\frac{t_1^2-2\rho t_1 t_2+t_2^2}{1-\rho^2}\right).$$

To determine the optimal strategy $(\alpha_1^*,\alpha_2^*)$ which fulfills the total fals alarm boundary condition, and which counteracts any strategy of the operator, we first have to determine

$$\frac{\partial\beta}{\partial M_1} = \frac{1}{2\pi}\cdot\frac{1}{\sqrt{1-\rho^2}}\cdot \int_{-\infty}^{U_{1-\alpha_2}-\frac{M}{\sigma^2}} dt_2 \cdot \exp(-\frac{1}{2}\cdot\frac{1}{1-\rho^2}\cdot((U_{1-\alpha_1}-\frac{M_1}{\sigma_1})^2 -$$

$$- 2\rho\cdot(U_{1-\alpha_1}-\frac{M_1}{\sigma_1})\cdot t_2 + t_2^2))\cdot(-\frac{1}{\sigma_1}),$$

which is smaller than zero. This means that $\beta$ is monotonely decreasing with increasing $M_1$ meaning that for any values of $\alpha_1$ and $\alpha_2$, $\beta$ has its maximum with respect to $M_1$ for $M_1 = 0$. Therefore, we get by use of the derivation of a function of the type

$$F(x) = \int_{-\infty}^{g(x)} dt\cdot f(t,x)$$

the well-known formula

$$\frac{d}{dx}F(x) = f(g(x),x)\cdot\frac{d}{dx}g(x) + \int_{-\infty}^{g(x)} dt\cdot\frac{d}{dx}f(t,x)$$

the following expression for $\frac{d}{d\alpha_1}\beta$:

$$\frac{1}{2\pi}\cdot\frac{1}{\sqrt{1-\rho^2}} \int_{-\infty}^{U_{1-\alpha_2}-\frac{M}{\sigma_2}} dt_2 \; \exp(-\frac{1}{2}\cdot\frac{1}{1-\rho^2}\cdot(U_{1-\alpha_1}^2 - 2\rho\cdot U_{1-\alpha_1}\cdot t_2 + t_2^2)\cdot\frac{d}{d\alpha_1}U_{1-\alpha_1} +$$

$$+\frac{1}{2\pi}\cdot\frac{1}{\sqrt{1-\rho^2}}\cdot \int_{-\infty}^{U_{1-\alpha_1}} dt_1 \; \exp(-\frac{1}{2}\cdot\frac{1}{1-\rho^2}\cdot(t_1^2-2\rho t_1\cdot(U_{1-\alpha_2}-\frac{M}{\sigma_2})+(U_{1-\alpha_2}-\frac{M}{\sigma_2})^2)\;\frac{d}{d\alpha_1}U_{1-\alpha_2}.$$

Since

$$\frac{d}{d\alpha_1} U_{1-\alpha_1} = -\frac{d}{d\alpha_1} U_{\alpha_1} = -\sqrt{2\pi}\cdot\exp(\frac{1}{2}\cdot U_{\alpha_1}^2) < 0 \ , \qquad (*)$$

and because we get by implicit differentiation of the relation between $\alpha$, $\alpha_1$ and $\alpha_2$

$$0 = \Phi(\frac{1}{\sqrt{1-\rho^2}}\cdot(U_{1-\alpha_2} - \rho\cdot U_{1-\alpha_1})) + \Phi(\frac{1}{\sqrt{1-\rho^2}}\cdot(U_{1-\alpha_1} - \rho\cdot U_{1-\alpha_2}))\cdot\frac{d\alpha_2}{d\alpha_1} \ , \qquad (**)$$

which leads to

$$\frac{d}{d\alpha_1} U_{1-\alpha_2} = -\frac{d}{d\alpha_1} U_{\alpha_2} = \sqrt{2\pi}\cdot\exp(\frac{1}{2}\cdot U_{\alpha_2}^2)\cdot\frac{d\alpha_2}{d\alpha_1} > 0 \ ,$$

we finally get

$$\frac{d}{d\alpha_1}\beta = \frac{1}{2\pi}\cdot\frac{1}{\sqrt{1-\rho^2}}\int_{-\infty}^{U_{1-\alpha_2}-\frac{M}{\sigma_2}} dt_2 \ \exp(-\frac{1}{2}\cdot\frac{1}{1-\rho^2}\cdot(t_2-\rho\cdot U_{1-\alpha_1})^2)\cdot\exp(-\frac{1}{2}\cdot U_{1-\alpha_1}^2)\frac{d}{d\alpha_1} U_{1-\alpha_1} +$$

$$+\frac{1}{2\pi}\cdot\frac{1}{\sqrt{1-\rho^2}}\int_{-\infty}^{U_{1-\alpha_1}} dt_1 \ \exp(-\frac{1}{2}\cdot\frac{1}{1-\rho^2}\cdot(t_1-\rho\cdot(U_{1-\alpha_2}-\frac{M}{\sigma_2})^2)\cdot\exp(-\frac{1}{2}\cdot(U_{1-\alpha_2}-\frac{M}{\sigma_2})^2)\frac{d}{d\alpha_1} U_{1-\alpha_2} =$$

$$= \Phi(\frac{1}{\sqrt{1-\rho^2}}\cdot(U_{1-\alpha_2}-\frac{M}{\sigma_2}-\rho\cdot U_{1-\alpha_1})) +$$

$$+\Phi(\frac{1}{\sqrt{1-\rho^2}}\cdot(U_{1-\alpha_1}-\rho\cdot(U_{1-\alpha_2}-\frac{M}{\sigma_2})))\cdot\exp(\frac{M}{2}\cdot(U_{1-\alpha_2}-\frac{M}{\sigma_2}))\cdot\frac{\Phi(\frac{1}{\sqrt{1-\rho^2}}\cdot(U_{1-\alpha_2}-\rho\cdot U_{1-\alpha_1}))}{\Phi(\frac{1}{\sqrt{1-\rho^2}}\cdot(U_{1-\alpha_1}-\rho\cdot U_{1-\alpha_2}))} \ .$$

Whereas it does not seem possible to show for all parameter values of M, $\alpha$, $\sigma_2$, $\rho$ that

$$\frac{d}{d\alpha_1}\beta > 0 \ ,$$

i.e., $\alpha_1^*=0$, i.e. $s_1=\infty$, it proved to be like that for all special cases which were considered and, furthermore, it is intuitive: If in the first inventory period no material is diverted, the inspector does not need to check the first balance at all but has to concentrate his false alarm possibility on the second period, which means that only the global balance test is performed.

The fact that $\beta_{CM}$ has its maximum for $M_2 = 0$, yields an interesting interpretation which throws some light on the meaning of the CUMUF test: One can interpret the first step of this test as an attempt to *estimate* the diversion $M_1$ in the first period in order to specify the diversion $M-M_1$ in the second period (M given); in fact, for appropriately chosen $M_1$ and $\alpha_1$ and $\alpha_2$ ($\alpha$ given), one has

$$1-\beta_{CM}^{P} > 1-\beta_{NP} = \Phi\left(\frac{M}{\sigma_2} - U_{1-\alpha}\right).$$

Therefore, the best counterstrategy of the operator obviously is to choose $M_1 = 0$ in order not to permit such an estimate. This means again that this test procedure may be useful in cases where diversion strategies can be anticipated.

*Abrupt Diversion*

In this case the variables $Y_1$ and $Y_2$ are defined as before and therefore, now

$$\begin{aligned} H_o &: E(Y_1) = E(Y_2) = 0 \\ &\quad E(Y_1) = \begin{cases} M & \\ & \text{with probability} \\ 0 & \end{cases} \begin{matrix} q \\ \\ 1-q \end{matrix} \\ H_1 &: E(Y_2) = M \, . \end{aligned}$$

The second moments are again given as before, the same holds for the single and the overall false alarm probabilities.

The probability of detection $1-\beta_{CM}^{A}$ as a function of the single false alarm probabilities is given by

$$1-\beta_{CM}^{A} = 1 - \frac{1}{2\pi \cdot \sqrt{1-\rho^2}} \cdot \int\limits_{-\infty}^{U_{1-\alpha_1}-\frac{M}{\sigma_1}} dt_1 \int\limits_{-\infty}^{U_{1-\alpha_2}-\frac{M}{\sigma_2}} dt_2 \exp\left(-\frac{1}{2}\cdot\frac{1}{1-\rho^2}\cdot\left(t_1^2-2\rho t_1 t_2+t_2^2\right)\right)\cdot q +$$

$$- \frac{1}{2\pi \cdot \sqrt{1-\rho^2}} \cdot \int_{-\infty}^{U_{1-\alpha_1}} dt_1 \int_{-\infty}^{U_{1-\alpha_2} - \frac{M}{\sigma_2}} dt_2 \exp\left(- \frac{1}{2} \cdot \frac{1}{1-\rho^2} \cdot \left(t_1^2 - 2\rho t_1 t_2 + t_2^2\right)\right) \cdot (1-q).$$

$$=: 1 - \beta_{CM}^{A}(\alpha_1, q),$$

if we remember that $\alpha_2$ is fixed by $\alpha_1$ for given overall false alarm probability.

The optimal strategies $\alpha_1^*$ and $q^*$ have to fulfill the saddlepoint criterion

$$\beta_{CM}^{A}(\alpha_1^*, q) \leq \beta_{CM}^{A}(\alpha_1^*, q^*) \leq \beta_{CM}^{A}(\alpha_1, q^*) \quad .$$

This criterion is fulfilled if the two relations hold

$$\frac{1}{2\pi \cdot \sqrt{1-\rho^2}} \cdot \int_{-\infty}^{U_{1-\alpha_1^*} - \frac{M}{\sigma_1}} dt_1 \cdot \int_{-\infty}^{U_{1-\alpha_2^*} - \frac{M}{\sigma_2}} dt_2 \exp\left(- \frac{1}{2} \cdot \frac{1}{1-\rho^2} \cdot \left(t_1^2 - 2\rho t_1 t_2 + t_2^2\right)\right) =$$

$$= \frac{1}{2\pi \cdot \sqrt{1-\rho^2}} \cdot \int_{-\infty}^{U_{1-\alpha_1^*}} dt_1 \int_{-\infty}^{U_{1-\alpha_2^*} - \frac{M}{\sigma_2}} dt_2 \exp\left(- \frac{1}{2} \cdot \frac{1}{1-\rho^2} \left(t_1^2 - 2\rho t_1 t_2 + t_2^2\right)\right) \tag{1}$$

$$\frac{d}{d\alpha_1} \beta_{CM}^{A}(\alpha_1, q^*) = 0 \Big|_{\alpha_1 = \alpha_1^*} \quad . \tag{2}$$

For $\alpha_1^* = 0$ relation (1) is satisfied; this means that again only CUMUF is tested. Furthermore, we have

$$\frac{d}{d\alpha_1} \beta_{CM}^{A} = \frac{1}{2\pi \cdot \sqrt{1-\rho^2}} \cdot \Bigg\{ \Bigg[ \int_{-\infty}^{U_{1-\alpha_2} - \frac{M}{\sigma_2}} dt_2 \exp\left(-\frac{1}{2} \cdot \frac{1}{1-\rho^2} \cdot \left(\left(U_{1-\alpha_1} - \frac{M}{\sigma_1}\right)^2 - 2\rho\left(U_{1-\alpha_1} - \frac{M}{\sigma_1}\right) \cdot t_2 + t_2^2\right) \cdot \frac{dU_{1-\alpha_1}}{d\alpha_1} + \right.$$

$$+ \int_{-\infty}^{U_{1-\alpha_1} - \frac{M}{\sigma_1}} dt_1 \cdot \exp\left(-\frac{1}{2} \frac{1}{1-\rho^2} \cdot \left(t_1^2 - 2\rho t_1 \cdot \left(U_{1-\alpha_2} - \frac{M}{\sigma_2}\right) + \left(U_{1-\alpha_2} - \frac{M}{\sigma_2}\right)^2\right) \cdot \frac{dU_{1-\alpha_2}}{d\alpha_1}\right] \cdot q +$$

$$+ \left[ \int\limits_{-\infty}^{U_{1-\alpha_2}-\frac{M}{\sigma_2}} dt_2 \exp\left(-\frac{1}{2}\cdot\frac{1}{1-\rho^2}\cdot\left(U^2_{1-\alpha_1}-2\rho\cdot U_{1-\alpha_1}\cdot t_2+t_2^2\right)\right)\cdot\frac{dU_{1-\alpha_1}}{d\alpha_1}\right.$$

$$\left. + \int\limits_{-\infty}^{U_{1-\alpha_1}-\frac{M}{\sigma_1}} dt_1 \exp\left(-\frac{1}{2}\cdot\frac{1}{1-\rho^2}\cdot\left(t_1^2-2\rho t_1\cdot\left(U_{1-\alpha_2}-\frac{M}{\sigma_2}\right)+\left(U_{1-\alpha_2}-\frac{M}{\sigma_2}\right)^2\right)\right)\cdot\frac{dU_{1-\alpha_2}}{d\alpha_1}\right]\cdot(1-q)\Bigg\} =$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\Bigg\{\left[\exp\left(-\frac{1}{2}\cdot\left(U_{1-\alpha_1}-\frac{M}{\sigma_1}\right)^2\right)\cdot\Phi\left(\frac{U_{1-\alpha_2}-\frac{M}{\sigma_2}-\rho\cdot\left(U_{1-\alpha_1}-\frac{M}{\sigma_1}\right)}{\sqrt{1-\rho^2}}\right)\cdot\frac{dU_{1-\alpha_1}}{d\alpha_1}+\right.$$

$$\left.+\exp\left(-\frac{1}{2}\cdot\left(U_{1-\alpha_2}-\frac{M}{\sigma_2}\right)^2\right)\cdot\Phi\left(\frac{U_{1-\alpha_1}-\frac{M}{\sigma_1}-\rho\cdot\left(U_{1-\alpha_2}-\frac{M}{\sigma_2}\right)}{\sqrt{1-\rho^2}}\right)\cdot\frac{dU_{1-\alpha_2}}{d\alpha_1}\right]\cdot q +$$

$$+\left[\exp\left(-\frac{U^2_{1-\alpha_1}}{2}\right)\cdot\Phi\left(\frac{U_{1-\alpha_2}-\frac{M}{\sigma_2}-\rho\cdot U_{1-\alpha_1}}{\sqrt{1-\rho^2}}\right)\cdot\frac{dU_{1-\alpha_1}}{d\alpha_1}+\right.$$

$$\left.+\exp\left(-\frac{1}{2}\cdot\left(U_{1-\alpha_2}-\frac{M}{\sigma_2}\right)^2\right)\cdot\Phi\left(\frac{U_{1-\alpha_1}-\frac{M}{\sigma_1}-\rho\cdot\left(U_{1-\alpha_2}-\frac{M}{\sigma_2}\right)}{\sqrt{1-\rho^2}}\right)\cdot\frac{dU_{1-\alpha_2}}{d\alpha_1}\right]\cdot(1-q)\Bigg\}.$$

Because of (*) and (**) we therefore get

$$\frac{d}{d\alpha_1}\beta^A_{CM} = -\left[\exp\left(U_{1-\alpha_1}\cdot\frac{M}{\sigma_1}\right)\cdot\exp\left(-\frac{1}{2}\cdot\left(\frac{M}{\sigma_1}\right)^2\right)\cdot\Phi\left(\frac{U_{1-\alpha_1}-\frac{M}{\sigma_2}-\rho\cdot\left(U_{1-\alpha_1}-\frac{M}{\sigma_1}\right)}{\sqrt{1-\rho^2}}\right)+\right.$$

$$\left.\exp\left(U_{1-\alpha_2}\cdot\frac{M}{\sigma_2}\right)\cdot\exp\left(-\frac{1}{2}\cdot\left(\frac{M}{\sigma_2}\right)^2\right)\cdot\Phi\left(\frac{U_{1-\alpha_1}-\frac{M}{\sigma_1}-\rho\cdot\left(U_{1-\alpha_2}-\frac{M}{\sigma_2}\right)}{\sqrt{1-\rho^2}}\right)\cdot\frac{d\alpha_2}{d\alpha_1}\right]\cdot q +$$

$$-\left[\Phi\left(\frac{U_{1-\alpha_2}-\frac{M}{\sigma_2}-\rho\cdot U_{1-\alpha_1}}{\sqrt{1-\rho_2}}\right)+\right.$$

$$\left.+\exp\left(U_{1-\alpha_2}\cdot\frac{M}{\sigma_2}\right)\cdot\exp\left(-\frac{1}{2}\left(\frac{M}{\sigma_2}\right)^2\right)\cdot\Phi\left(\frac{U_{1-\alpha_1}-\frac{M}{\sigma_1}-\rho\cdot\left(U_{1-\alpha_2}-\frac{M}{\sigma_2}\right)}{\sqrt{1-\rho^2}}\right)\cdot\frac{d\alpha_2}{d\alpha_1}\right]\cdot(1-q) .$$

From (**) we get

$$\frac{d\alpha_2}{d\alpha_1} = 0 \quad \text{for } \alpha_1 = 0 \; ,$$

furthermore we get with L'Hospital's rule

$$\lim_{x\to\infty} \exp\left(x \cdot \frac{M}{\sigma_1}\right) \cdot \Phi\left(-\frac{\rho}{\sqrt{1-\rho^2}} \cdot x\right) = \lim_{x\to\infty} \frac{\bar{\Phi}\left(-\frac{\rho}{\sqrt{1-\rho^2}} \cdot x\right)}{\exp\left(-x \cdot \frac{M}{\sigma_1}\right)^2}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\,\frac{\rho^2}{1-\rho^2} \cdot x^2\right)\left(-\frac{\rho}{\sqrt{1-\rho^2}}\right)}{\exp\left(-x \cdot \frac{M}{\sigma_1}\right)} = 0 \; ,$$

therefore relation (2) is fulfilled for *any* $q^*$ .

In section 2.4 we will discuss some numerical results for CUMUF - tests, obtained by Pike, Wood and coworkers (1980, 1981).

## 1.5 Summary of Part I

Having in mind above all the objective of optimizing the overall probability of detecting the diversion of nuclear material during the reference time $[t_o, t_n]$, we have basically studied

- the Neyman-Pearson test the statistics of which is simply $\sum_i Z_i$,
- a special test studied first by Jaech,
- an n-fold test procedure with fixed the overall false alarm probability, and
- a CUMUF test.

We have seen in Section 1.1 that the best test in the sense of the overall probability of detection is the test based on the global balance $\sum_i Z_i = I_o + \sum_i T_i - I_n$; this has been supported by the results given in later sections,

To take into account in-between inventories $I_i$, $i=1,\ldots,n-1$, makes only sense if one is interested in having information about the time of loss or diversion. As the test with the statistics $\sum_i a_i \cdot Z_i$, described in Section 1.2, does not provide any relevant information, it is ruled out. Furthermore, the CUMUF test cannot be handled analytically and probably also not numerically for more than two inventory periods. In addition, to determine optimal significance thresholds, one has to make arbitrary assumptions on the diversion strategy; thus, its use cannot be recommended either.

So far only the Neyman-Pearson test and the n-fold test procedure of Section 1.3 are left. As the only purpose of additional inventories is to shorten the detection time, one has to study the appropriately defined, expected detection time - this, however, caused major difficulties, as we have seen. At the end of Section 1.3 it was illustrated by means of a numerical example that too many inventories shorten the conditional, expected detection time; thus, in general, there will be an "optimal" number of in-between inventories per reference time.

It cannot be recommended, however, at this stage of the discussion to use this "optimal" number of in-between inventories per reference time, as the tradeoff between the two objectives "global probability of detection" and "conditional expected detection time" has not yet been resolved.

# PART II

## The Sequential Approach

In this part it will be assumed that the operator decides at the beginning of the reference time whether or not he will divert material, and if yes, which amount he will divert in each inventory period. Furthermore, it will be assumed that the inspector decides at the beginning of the reference time which test procedure he will choose in order to make a decision *after each inventory taking* whether the book physical inventory difference is considered to be significant and, therefore, plant operations have to be stopped, or whether it is not considered to be significant and, therefore, plant operations can be continued. In the second section of this part we look into a model with the additional possibility that the inspector is indifferent in his statement.

The assumption that plant operations will be stopped in case a significant book physical inventory difference occurs, is identical with the supposition that the primary safeguards objective of the inspector is to detect a diversion as soon as possible and that the inspector considers a single significant book physical inventory difference to be so serious that it has to be clarified, if caused by an error of the first kind or, if justified, that immediate sanctions have to be applied.

Whereas for the static approach discussed in Part 1 we were able to formulate one general game in such a way that the various test procedures influenced the expected payoffs to the inspector and to the operator only via the probability of detection, and whereas we were able to show that the two-step procedure allowed us to neglect the payoff parameters as long as we fixed the value of the overall false alarm probability, this is no longer the case for the procedures to be considered now. The fact that the payoff parameters cannot be eliminated completely can be understood intuitively: It must be expressed in some way that it is more interesting for the inspector to detect a diversion already at time $t_i$ and not only at time $t_{i+1}$, $i=1,\ldots,n-1$.

As in this part either game theoretical models are analyzed which do not lead to explicit solutions, or - in section 2.3 - statistical procedures are considered which emphasize unilaterally the aspect of timely detection, we will not discuss the question of the appropriate inventory taking frequency, but take its value as fixed a priori.

In the following, a general frame will be presented for sequentialmodels with *infinite* reference time intervals. One specific model, again based on Stewart's estimate for the starting inventory, will be analized in some detail both for protracted and abrupt diversion. Finally, the present state of the art of sequential approaches for infinite reference time intervals will be discussed.

First, let us again consider a reference time interval $[t_o, t_n]$ and let us assume that at most n physical inventories $I_i$, $i = 1 \ldots n$ are performed during this reference time at time points $t_1, \ldots, t_n$. Let us assume furthermore that n book physical inventory differences $Z_i = I_{i-1} + T_i - I_i$, $i = 1 \ldots n$, are formed, and that after each inventory period the inspector performs a test in order to decide whether or not during this inventory period or in a foregoing period material has been lost or diverted.

In case of *protracted diversion* the set of strategies of the operator is

$$\left\{ p : p \in [0,1] \right\} \otimes \left\{ (M_1, \ldots, M_n) : \sum_i M_i = M \right\} =: \{p\} \cdot Y_M^P ,$$

where p is the probability, that he will divert material, and $M_i$ is the amount of material to be diverted in the i-th inventory period. In this case the pair of payoffs to the inspector as player 1 and to the operator as player 2 is

| | |
|---|---|
| $(-a_i, -b_i)$ | in case of diversion and detection after the i-th inventory taking, |
| $(-c, d)$ | in case of diversion and no detection, |
| $(-e_i, -f_i)$ | in case of no diversion and detection after the i-th inventory taking, and |
| $(0, 0)$ | in case of no diversion and no detection, |

where $(a_i, b_i, c, d, e_i, f_i) > (0, \ldots, 0)$ for $i=1, \ldots, n$, where $a_i < c$, i.e., the inspector's loss in case of a detected diversion is smaller than his loss in case of a not detected diversion, and where $b_i > f_i$, i.e., the operator's loss in case of a detected diversion is greater than his loss in case of a false alarm.

Let $1-\beta^{(i)}$ be the probability that in case of diversion the inspector decides after the i-th inventory period for the first time that material has been lost or diverted. Let furthermore $\alpha^{(i)}$ be the probability that in case of no diversion the inspector decides after the i-th inventory period that material has been lost or diverted whereas after the foregoing inventories he had decided that nomaterial had been lost or diverted. Then the conditioned expected payoffs to the inspector and to the operator are

$$(-c\cdot\beta^{(n)}-\sum_{i=1}^{n} a_i\cdot(1-\beta^{(i)}),\quad d\cdot\beta^{n}-\sum_{i=1}^{n} b_i\cdot(1-\beta^{(i)}))\text{ in case of diversion}$$

$$(-\sum_{i=1}^{n} e_i\cdot\alpha^{(i)},\ -\sum_{i=1}^{n} f_i\cdot\alpha^{(i)})\text{ in case of no diversion,}$$

and the unconditioned expected payoffs are

$$I:=\ (-c\cdot\beta^{(n)}-\sum_i \cdot_i(1-\beta^{(i)}))\cdot p+(-\sum_i e_i\cdot\alpha^{(i)})\cdot(1-p)\text{ for the inspector}$$

$$B:=\ (d\cdot\beta^{(n)}-\sum_i b_i\cdot(1-\beta^{(i)}))\cdot p+(-\sum_i f_i\cdot\alpha^{(i)})\cdot(1-p)\quad\text{for the operator.}$$

In case of *abrupt diversion* the set of strategies of the operator is

$$\left\{\ p\ :\ p\in[0,1]\ \right\}\otimes\left\{\ (q_1\ldots q_n)\ :\sum_i q_i\ =\ 1\ \right\}=:\ \{p\}\otimes\ Y_M^A\ ,$$

where $q_i$ is the probability that he will divert the amount M in the i-th inventory period. In this case the pair of payoffs to the inspector and to the operator in case of illegal behavior of the operator is much more complicated than in the case of protracted diversion, because one has to take into account the additional possibilities that

- first a 'false' alarm is raised and later material is diverted (which means that this 'false' alarm is not really false from the global point of view, and
- that a diversion, which takes place in the i-th inventory period, $i=1\ldots n-1$, will be detected after the j-th period, $j=2,\ldots,n$ .

In section 2.2 we will write down this explicitely for the case of two inventory periods.

The set of strategies of the inspector is the set of significance thresholds which we call X. The determination of the optimal significance thresholds means according to the general discussion in Part 1 the solution of the non-cooperative two person game $(X,\{p\}\otimes Y_M,I,B)$ which in turn means the determination of the equilibrium points $(x^*;\ p^*,y^*)$ defined by

$$I(x^*;p^*,y^*) \geq I(x;p^*,y^*)$$

$$B(x^*;p^*,y^*) \geq I(x^*;p,y),$$

where I and B are the payoffs to the two players. Again, for both cases, abrupt and protracted diversion, these equilibrium points can be found by a two step precedure

*Theorem 9* (Abel and Avenhaus 1980, 1981)

The noncooperative two person games defined above can be solved by the following two auxiliary games:

1) A noncooperative two person game $(X_\alpha,Y_M,I,B)$, where

$$X_\alpha := \left\{ x \in X \ : \ \alpha := \sum_i \alpha^{(i)} \text{ fixed} \right\},$$

and where p has a fixed value.

In case $e_i = e$ for $i=1...n$ it is sufficient to consider the conditioned expected payoffs in case of diversion, which for protracted diversion are

$$\tilde{I} := -c \cdot \beta^n - \sum_i a_i \cdot \left(1-\beta^{(i)}\right), \quad \tilde{B} := d \cdot \beta^n - \sum_i b_i \cdot \left(1-\beta^{(i)}\right);$$

the equilibrium strategies of this game, $x^*(\alpha)$ and $y^*$, are defined by

$$\tilde{I}(x^*(\alpha),y^*,\alpha) \geq \tilde{I}(x(\alpha),y^*,\alpha) \text{ for } \alpha\varepsilon[0,1].$$

2. A non-cooperative two-person game $(\{\alpha\},\{p\},\ \tilde{I}^*\ \tilde{B}^*)$ with $\tilde{I}^*(\alpha) := \tilde{I}(x^*(\alpha),y^*,\alpha)$ and $\tilde{B}^*(\alpha) := \tilde{B}(x^*(\alpha),y^*,\alpha)$.
   The equilibrium strategies of this game, $\alpha^*$ and $p^*$, are defined by

$$\tilde{I}^*(\alpha^*,p^*) \geq \tilde{I}^*(\alpha,p^*),\ \tilde{B}^*(\alpha^*,p^*) \geq \tilde{B}^*(\alpha^*,p).$$

If we compare this two step procedure with the one presented in Part 1, we see that the first step game is no longer a zero sum game and furthermore, contains the original payoff parameters.

Again, for the rest of this part we will consider only the first step game; before, however, we will come back once more to the problem of the two safeguards systems design criteria "overall probability of detection" and "short detection time."

In the framework of the game theoretical model just described, there is only one criterion or "objective" of the inspector, namely his expected payoff. His interest in a short detection time is expressed by the different parameters $a_i$, $i=1,2,\ldots,n$, his interest in a high overall probability of detection by the parameter c. The time points $t_i$, $i=1,\ldots,n$, for inventory taking are fixed a priori and not subject to optimization: This does not mean anything else than that between the given inventory taking times $t_1$ and $t_{i+1}$ of possible detection there is a difference in interest to the inspector; there is no difference in interest for time points closer to each other.

Contrary to the situation in Part 1, where we needed only one decision criterion at the end of the reference time, in this part we have to define a decision criterion for each inventory period. Let $\underline{\hat{Z}}^{(i)}$ be the observed vector of MUF values until the end of the i-th inventory period,

$$\underline{\hat{Z}}^{(i)} = (\hat{Z}_1,\ldots,\hat{Z}_i),\ i=1,\ldots,n.$$

Then $\delta^{(i)}$ is defined as a map of the space of observations $Z^{(i)}{}_i = \{\underline{\hat{Z}}^{(i)}\}$ into the interval $[0,1]$ and may again be interpreted as the conditional probability of choosing either the null hypothesis $H_o$ or the alternative hypothesis $H_1$, given the observation $\underline{\hat{Z}}^{(i)}$ *and, furthermore,* given that all foregoing i-1 decisions have led to the acceptance of $H_o$ (otherwise, plant operations would have been stopped as already mentioned):

$$\delta^{(i)} = \begin{cases} \delta(H_o|\underline{\hat{Z}}^{(i)}) & \\ & \text{for} \\ \delta(H_1|\underline{\hat{Z}}^{(i)}) & \end{cases} \begin{matrix} \underline{\hat{Z}}^{(i)}\varepsilon Z_o^{(i)} \\ \\ \underline{\hat{Z}}^{(i)}\varepsilon Z_1^{(i)} \end{matrix} .$$

Again, we have

$$Z_o^{(i)} \cap Z_1^{(i)} = \emptyset, \quad Z_o^{(i)} \cup Z_1^{(i)} = Z^{(i)},$$

and therefore

$$\delta(H_o|\hat{\underline{Z}}^{(i)}) + \delta(H_1|\hat{\underline{Z}}^{(i)}) = 1.$$

It should be noted that in Section 2.3 we consider a decision rule which permits the decision "no decision" (N) implying that we have in this case

$$\delta^{(i)} = \begin{cases} \delta(H_o|\hat{\underline{Z}}^{(i)}) & \hat{\underline{Z}}^{(i)} \varepsilon Z_o^{(i)} \\ \delta(N|\hat{\underline{Z}}^{(i)}) & \text{for} \quad \hat{\underline{Z}}^{(i)} \varepsilon Z_N^{(i)} \\ \delta(H_1|\hat{\underline{Z}}^{(i)}) & \hat{\underline{Z}}^{(i)} \varepsilon Z_1^{(i)} \end{cases}$$

and accordingly

$$Z_o^{(i)} \cap N^{(i)} = Z_o^{(i)} \cap Z_1^{(i)} = Z_N^{(i)} \cap Z_1^{(i)} = \emptyset, \; Z_o^{(i)} \cup Z_N^{(i)} \cup Z_1^{(i)} = Z^{(i)},$$

$$\delta(H_o|\hat{\underline{Z}}^{(i)}) + \delta(N|\hat{\underline{Z}}^{(i)}) + \delta(H_1|\hat{\underline{Z}}^{(i)}) = 1.$$

If $f_o(\underline{z}^{(i)})$ and $f_1(\underline{z}^{(i)})$ are the probability density functions of the random vector $\underline{Z}^{(i)}$ under the hypotheses $H_o$ and $H_1$, then in case of terminal decisions the false alarm probability $\alpha^{(i)}$ and the probability of detection $1-\beta^{(i)}$ after the i-th inventory period are given by the expressions

$$\alpha^{(i)} = \int_{Z^{(i)}} \delta(H_1|\underline{Z}^{(i)}) \cdot f_o(\underline{z}^{(i)}) \, d\underline{z}^{(i)}$$

$$1-\beta^{(i)} = \int_{Z^{(i)}} \delta(H_1|\underline{Z}^{(i)}) \cdot f_1(\underline{z}^{(i)}) \, d\underline{z}^{(i)}.$$

As the events "false alarm after the i-th period" and "false alarm after the j-th period" are exclusive events for $i \neq j$ (if after the i-th period an alarm is actuated, then plant operations are stopped therefore, an alarm after the j-th period requires that after the i-th period $i<j$, *no* alarm had been actuated), the overall false alarm probability $\alpha$ is given by the expression

$$\alpha = \sum_{i=1}^{n} \int_{Z^{(i)}} \delta(H_1|\underline{Z}^{(i)}) \cdot f_o(\underline{z}^{(i)})\, d\underline{z}^{(i)} .$$

With the same argument we obtain the following expression for the overall probability of detection

$$1-\beta = \sum_{i=1}^{n} \int_{Z^{(i)}} \delta(H_1|\underline{Z}^{(i)}) \cdot f_1(\underline{z}^{(i)})\, d\underline{z}^{(i)} .$$

As in Part 1, we will describe all non-randomized tests defined by

$$\delta(H_o|\hat{\underline{Z}}^{(i)}) = \begin{cases} 1 & \text{if } \hat{\underline{Z}}^{(i)} \varepsilon Z_o^{(i)} \\ 0 & \text{if } \hat{\underline{Z}}^{(i)} \varepsilon Z_1^{(i)} \end{cases} ,$$

by their acceptance regions $Z_o^{(i)}$, $i=1,\dots,n$. Only in those sections where we determine Neyman-Pearson tests, we will have to make use of the full formalism developed so far.

Second, we consider in this part an infinite reference time interval $[0,\infty]$, as we have seen in section 1.3 that for fixed reference time intervals it is difficult to formulate an appropriate criterion for the detection time, and as sequential methods for infinite reference time intervals have been discussed extensively in the last years in connection with the so-called *near real time accountancy*. We repeat , however, our objection from a political point of view, namely that for non-weapons NPT-signatory states it is important that at regular fixed points of time the safeguards authority should make a statement about the compliance of the state operations with the NPT.

## 2.1 Modified Neyman - Pearson Test

In section 1.1 it has been pointed out that, if the distribution functions under the null and under the alternative hypotheses are specified, the best test in the sense of overall probability of detection for a fixed false alarm probability is the Neyman - Pearson test. In the case of a sequential procedure for a finite number n of inventory periods its acceptance region is given by

*Theorem 10*

Given the random vector $\underline{Z}$, generated sequentially, the density function of which is $f_o(\underline{z})$ under $H_o$ and $f_1(\underline{z})$ under $H_1$. Then the acceptance region $Z_o^{(i)NP}$ of that test, which minimizes the probability of the error second kind for the whole sequence for a fixed overall error first kind probability $\alpha$, is given by

$$Z_o^{(i)NP} = \left\{ \underline{Z}^{(i)} : \frac{f_1(\underline{z}^{(i)})}{f_o(\underline{z}^{(i)})} \leq \frac{-\lambda}{c - a_i} \right\} , \quad i = 1 \ldots n ,$$

where $\lambda$ has to be determined with the help of the fixed value of the error first kind probability according to

$$1 - \alpha = \sum_{i=1}^{n} \int_{Z_o^{(i)NP}} f_o(\underline{z}^{(i)}) \, d\underline{z}^{(i)} ,$$

and where c and $a_i$, $i = 1 \ldots n$, are the payoff parameters, introduced in the introduction of this part.

*Proof*

According to the introduction of this part the decision rule after the i - th inventory at time $t_i$ is

$$\delta^{(i)} = \begin{cases} \delta^{(i)}(H_o|\underline{Z}^{(i)}) & \text{if } \underline{Z}^{(i)} \epsilon Z_o^{(i)} \\ \delta^{(i)}(H_1|\underline{Z}^{(i)}) & \text{if } \underline{Z}^{(i)} \epsilon Z_1^{(i)} \end{cases}$$

Again we restrict ourselves to terminal decisions so that either $H_o$ or $H_1$ must be accepted which means

$$Z_o^{(i)} \cup Z_1^{(i)} = Z^{(i)}, \; Z_o^{(i)} \cap Z_1^{(i)} = \emptyset,$$

and furthermore,

$$\delta(H_o|\underline{Z}^{(i)}) + \delta(H_1|\underline{Z}^{(i)}) = 1 \quad \text{for } \underline{Z}^{(i)} \varepsilon Z^{(i)}.$$

According to the introduction to this part, the problem is to maximize the inspector's payoff $\tilde{I}$ given by

$$\tilde{I} = -c\cdot\beta^{(n)} - \sum_i a_i \cdot (1-\beta^{(i)})$$

within the boundary of a fixed overall false alarm probability. This means explicitly that - if we use again the Lagrange formalism - we have to determine the free maximum of the form

$$-c\cdot(1-\sum_{i=1}^{n} \int_{Z^{(i)}} \delta(H_1|\underline{Z}^{(i)})\cdot f_1(\underline{z}^{(i)})d\underline{z}^{(i)}) - \sum_{i=1}^{n} a_i \cdot \int_{Z^{(i)}} \delta(H_1|Z^{(i)}) f_1(\underline{z}^{(i)})d\underline{z}^{i} +$$

$$+ \lambda(\sum_{i=1}^{n} \int_{Z^{(i)}} \delta(H_1|\underline{Z}^{(i)})\cdot f_o(\underline{z}^{(i)})d\underline{z}^{(i)} - \alpha),$$

where $\lambda$ is the Lagrange multiplier. If we neglect the constant terms, the problem is reduced to the determination of the free maximum of the form

$$\sum_{i=1}^{n} \int_{Z^{(i)}} [(c-a_i)\cdot f_1(\underline{z}^{(i)} + \lambda\cdot f_o(\underline{z}^{(i)})]\cdot\delta(H_1|\underline{Z}^{(i)})d\underline{z}^{(i)}.$$

As we have $0 \leq \delta(H_1|\underline{Z}^{(i)}) \leq 1$, we maximize this expression by choice of $\delta(H_1|\underline{Z}^{(i)}) = 1$, if the integrand is positive and zero if it is negative. Therefore, the modified Neyman-Pearson test has the rejection region for the decision at the i-th step

$$Z_1^{(i)NP} = \{\underline{Z}^{(i)}:\ (c-a_i)\cdot f_1(\underline{Z}^{(i)}) + \lambda\cdot f_o(\underline{Z}^{(i)}) > 0\},$$

and the acceptance region has the form given above, if we remember the assumption $a_i < c$ for $i = 1\ldots n$, which completes the proof. ⌑

*Protracted Diversion*

By going through the same formalism as used in Section 1.1, we see that the acceptance region $Z_o^{(i)NP}$ for the i-th decision is given by

$$Z_o^{(i)NP} = \{\underline{Z}^{(i)}: \underline{Z}^{(i)'} \cdot \underline{\underline{\Sigma}}^{(i)^{-1}} \cdot \underline{M}^{(i)} \leq \ln \frac{-\lambda}{c-a_i} + \frac{1}{2} \cdot \underline{M}^{(i)'} \cdot \underline{\underline{\Sigma}}^{(i)^{-1}} \cdot \underline{M}^{(i)}\},$$

where $\underline{\underline{\Sigma}}^{(i)}$ is the covariance matrix of the first i MUF variables $\underline{Z}^{(i)}$, and where $\underline{M}^{(i)'} = (M_1,\ldots,M_i)$, $i=1,\ldots,n$. Therefore, the relation between the false alarm probability $\alpha$ and the Lagrange multiplier $\lambda$ is explicitly given by

$$1-\alpha = \sum_{i=1}^{n} \Phi\left(\frac{\ln \frac{-\lambda}{c-a_i}}{\sqrt{\underline{M}^{(i)'} \cdot \underline{\underline{\Sigma}}^{(i)^{-1}} \cdot \underline{M}^{(i)}}} + \frac{1}{2} \cdot \sqrt{\underline{M}^{(i)'} \cdot \underline{\underline{\Sigma}}^{(i)^{-1}} \cdot \underline{M}^{(i)}}\right).$$

The payoff to the operator in case of illegal behavior is given by

$$\tilde{B} = d \cdot \beta^{(n)} - \sum_i b_i \cdot (1-\beta^{(i)}),$$

where $\beta^{(i)}$ is given by

$$\beta^{(i)} = \Phi\left(\frac{\ln \frac{-\lambda}{c-a_i}}{\sqrt{\underline{M}^{(i)'} \cdot \underline{\underline{\Sigma}}^{(i)^{-1}} \cdot \underline{M}^{(i)}}} - \frac{1}{2} \cdot \sqrt{\underline{M}^{(i)'} \cdot \underline{\underline{\Sigma}}^{(i)^{-1}} \cdot \underline{M}^{(i)}}\right).$$

Therefore, the optimal diversion strategy $\underline{M}^*$ with $\underline{e}' \cdot \underline{M}^* = M$ is given - if we neglect constant terms - by the free maximum of the form

$$\sum_i \left[(-d-b_i) \cdot \Phi\left(\frac{1}{2} \cdot \sqrt{\underline{M}^{(i)'} \cdot \underline{\underline{\Sigma}}^{(i)^{-1}} \cdot \underline{M}^{(i)}} - \frac{\ln \frac{-\lambda}{c-a_i}}{\sqrt{\underline{M}^{(i)'} \cdot \underline{\underline{\Sigma}}^{(i)^{-1}} \cdot \underline{M}^{(i)}}}\right) - \kappa \cdot M_i\right],$$

where $\kappa$ is the Lagrange parameter. Here, it has to be noted that $\lambda$ also depends on the diversion strategy $\underline{M}:\lambda = \lambda(\underline{M})$ via the false alarm relation. This means in turn that the equilibrium strategy of the inspector is to choose the acceptance region $Z_o^{(i)NP}$ as follows:

$$Z_o^{(i)NP} = \{\underline{Z}^{(i)}:\ \underline{Z}^{(i)'}\cdot\underline{\underline{\Sigma}}^{(i)^{-1}}\cdot\underline{M}^{*(i)} \leq \ln\frac{-\lambda(\underline{M}^*)}{c-a_i} + \frac{1}{2}\cdot\underline{M}^{*(i)}\cdot\underline{\underline{\Sigma}}^{(i)^{-1}}\cdot\underline{M}^{*(i)}\}.$$

Naturally, this program cannot be carried through analytically. Nevertheless, some interesting features of this approach can be observed:

- In general the test procedure after the i-th inventory period makes use of *all* foregoing MUF variables $Z_j$, $j=1\ldots i$. This should be kept in mind if we discuss another procedure in the next section.

- The acceptance regions after different inventories are different not only because of the different values of $a_i$ which express the operator's interest in *early* detection, but they are also different for fixed diversion strategies.

- In the case n=2 the relation between the Lagrange multiplier $\lambda$ and the false alarm probability $\alpha$ is given by

$$1-\alpha = \Phi\left(\frac{\sigma_1}{M_1}\cdot\ln\frac{-\lambda}{c-a_1}+\frac{1}{2}\cdot\frac{M_1}{\sigma_1}\right)+$$

$$+\ \Phi\left(\sqrt{\frac{1-\rho^2}{\frac{M_1^2}{\sigma_1^2}-\frac{2\rho M_1M_2}{\sigma_1\sigma_2}+\frac{M_2^2}{\sigma_2^2}}}\cdot\ln\frac{-\lambda}{c-a_2}+\frac{1}{2}\sqrt{\frac{\frac{M_1^2}{\sigma_1^2}-\frac{2\rho M_1M_2}{\sigma_1\sigma_2}+\frac{M_2^2}{\sigma_2^2}}{1-\rho^2}}\right)$$

and the form to be minimized with respect to $(M_1,M_2:\ M_1+M_2 = M)$ is given as

$$(-d-b_1)\cdot\Phi\left(\frac{1}{2}\cdot\frac{M_1}{\sigma_1}-\frac{\sigma_1}{M_1}\cdot\ln\frac{-\lambda}{c-a_1}\right)+(-d-b_2)\cdot\Phi\left(\sqrt{\frac{1-\rho^2}{\frac{M_1^2}{\sigma_1^2}-\frac{2\rho M_1M_2}{\sigma_1\sigma_2}+\frac{M_2^2}{\sigma_2^2}}}\cdot\ln\frac{-\lambda}{c-a_2}+\right.$$

$$\left.+\ \frac{1}{2}\cdot\sqrt{\frac{\frac{M_1^2}{\sigma_1^2}-\frac{2\rho M_1M_2}{\sigma_1\sigma_2}+\frac{M_2^2}{\sigma_2^2}}{1-\rho^2}}\right)+\kappa\cdot(M_1+M_2),$$

which means that in this very simple case it is not possible to obtain an explicit solution for general parameter values.

*Abrupt Diversion*

Let us first consider the very special abrupt diversion

$$\underline{M}' = (M,0,\ldots,0)$$

and assume that the inspector knows that the operator will either behave legally, or use this strategy. Then the acceptance regions $Z_o^{(i)NP}$ are given by

$$Z_o^{(i)} = \{\underline{Z}^{(i)}: Z_1 < \frac{\sigma_1^2}{M} \cdot |\underline{\underline{\Sigma}}^{(i)}| \cdot \ln \frac{-\lambda}{c-a_i} + \frac{1}{2} \cdot M\}$$

which means that only $Z_1$ is used as decision variable, but that *all* n tests are performed if no detection takes place. In the same way it can be shown that in case the diversion strategy consists in diverting the amount M of material during the i-th inventory period, n-i+1 tests, beginning after the i-th inventory and using only $Z_i$ as decision variable, are performed.

For a general abrupt diversion strategy, namely that in the i-th inventory period the amount M is diverted with probability $q_i$, $i=1\ldots n$, with $\sum_i q_i = 1$, the density function under the alternative hypothesis is given in section 1.1 . However, as already in the nonsequential case it was impossible to determine analytically optimal strategies - even for the case of only two inventory periods - it is clear that here, in the sequential case it is even more so, which means that only numerical studies can help to find appropriate solutions.

## 2.2 A Special Sequential Approach for a Fixed Reference Time

In this section we will consider again the test procedure based on Stewart's estimate for the starting inventory which we have used already in Section 1.1.

Let $\tilde{Z}_i$ be the modified MUF variable for the i-th inventory period as defined in Section 1.3:

$$\tilde{Z}_i := a_{i-1} \cdot \tilde{Z}_{i-1} + Z_i \qquad \text{for } i=2,\ldots n,$$

$$\tilde{Z}_1 = Z_1 .$$

Then the acceptance region $Z^{(i)}$ for the decision after the i-th inventory period to be discussed here is given by

$$Z_o^{(i)} = \{\hat{\tilde{Z}}_i : \hat{\tilde{Z}}_i \le s_i\}.$$

It should be noted that for the i-th decision only the modified MUF variable for the i-th inventory period is used and not the foregoing ones.

Let $\alpha_i$ be the probability that a false alarm is given after the i-th inventory period, expressed by

$$\alpha_i = 1 - \int_{Z_o^{(i)}} f_o(z_i) dz_i .$$

where $f_o(z_i)$ is the density of $\tilde{Z}_i$ under the null hypothesis $H_o$.

Because of the independence of the modified MUF variables, the probability of *first* giving a false alarm after the i-th inventory period $\alpha^{(i)}$ is expressed by

$$\alpha^{(i)} = \alpha_i \cdot \prod_{j=1}^{i-1} (1-\alpha_j), \quad i=2,\ldots,n,$$

$$\alpha^{(1)} = \alpha_1 .$$

The probability of giving a false alarm $\alpha$ is then expressed by

$$\alpha = \sum_{i=1}^{n} \alpha^{(i)} = 1 - \prod_{i=1}^{n} (1-\alpha_i),$$

as can be shown by complete induction (an analogous formula had been used in Section 1.3 in connection with the conditional, expected detection time).

*Protracted Diversion*

Furthermore, let $1-\beta_i$ be the probability that a diversion is detected after the i-th inventory period, given by

$$1-\beta_i = 1-\int_{Z_o^{(i)}} f_1(z_i)\, dz_i ,$$

where $f_1(z_i)$ is the density of $\tilde{Z}_i$ under the alternative hypothesis $H_1$.

The probability of *first* detecting a diversion after the i-th inventory period $1-\beta^{(i)}$ is given by

$$1-\beta^{(i)} = (1-\beta_i)\cdot \prod_{j=1}^{i-1} \beta_j, \quad j=2,\ldots,n,$$

$$\beta^{(1)} = \beta_1 .$$

The overall probability of detection $1-\beta$ is then given by

$$1-\beta = \sum_i (1-\beta^{(i)}) = 1-\prod_i \beta_i ,$$

as can be shown again by complete induction.

Explicitely $\alpha_i$ and $1-\beta_i$ are given by

$$\alpha_i = \Phi\left(\frac{s_i}{\sigma_i}\right) \quad \text{and} \quad 1-\beta_i = \Phi\left(\frac{E_1(\tilde{Z}_i)}{\sigma_i} - U_{1-\alpha_i}\right) ,$$

where $s_i$ is the significance threshold of the i-th test, and where $E_1(\tilde{Z}_i)$ is defined as in Section 1.3.

The expected payoffs to the inspector and to the operator are given by

$$I = \left(-c \prod_{i=1}^{n} \beta_i - \sum_{i=1}^{n} a_i\cdot(1-\beta_i)\cdot \prod_{j=1}^{i-1} \beta_j\right)\cdot p + \left(-\sum_{i=1}^{n} e_i\cdot\alpha_i\cdot \prod_{j=1}^{i-1} (1-\alpha_j)\right)\cdot(1-p),$$

$$B = \left(d\cdot \prod_{i=1}^{n} \beta_i - \sum_{i=1}^{n} b_i\cdot(1-\beta_i)\cdot \prod_{j=1}^{i-1} \beta_j\right)\cdot p + \left(-\sum_{i=1}^{n} f_i\cdot\alpha_i\cdot \prod_{j=1}^{i-1} (1-\alpha_j)\right)\cdot(1-p).$$

In the following paragraphs we will adapt the analysis to the case n=2 and consider only the first step game introduced in the introduction to this part, namely the "illegal" game with a fixed value of the overall false alarm probability; furthermore we assume $e_1 = e_2$ .

The payoff parameters and the expected payoffs are given explicitly in Fig. 4. The conditional expected payoffs $\tilde{I}$ and $\tilde{B}$, introduced at the beginning of this part, are given by

$$\tilde{I} = (a_2-c)\cdot\beta_1\cdot\beta_2+(a_1-a_2)\cdot\beta_1-a_1$$

$$\tilde{B} = (b_2+d)\cdot\beta_1\cdot\beta_2+(b_1-b_2)\cdot\beta_1-b_1.$$

As we are interested primarily in the optimal strategies, we consider only a linear transformation $\tilde{\tilde{I}}$ and $\tilde{\tilde{B}}$ of $\tilde{I}$ and $\tilde{B}$:

$$\tilde{\tilde{I}} = \beta_1\cdot\beta_2+\lambda_I\cdot\beta_1, \quad \lambda_I = \frac{a_1-a_2}{a_2-c}$$

$$\tilde{\tilde{B}} = \beta_i\cdot\beta_2+\lambda_B\cdot\beta_1, \quad \lambda_B = \frac{b_1-b_2}{b_2+d}$$

Explicitly these transformed payoffs are given by

$$\tilde{\tilde{I}}(\underline{\alpha},\underline{M}) = \Phi(U_{1-\alpha_1} - \frac{M_1}{\sigma_1})\cdot(\lambda_I+\Phi(U_{1-\alpha_2} - \frac{a\cdot M_1+M_2}{\sigma_2}))$$

$$\tilde{\tilde{B}}(\underline{\alpha},\underline{M}) = \Phi(U_{1-\alpha_1} - \frac{M_1}{\sigma_1})\cdot(\lambda_B+\Phi(U_{1-\alpha_2} - \frac{a\cdot M_1+M_2}{\sigma_2})),$$

where $\alpha_1$ and $\alpha_2$ and $M_1$ and $M_2$ have to fulfil the boundary conditions

$$(1-\alpha_1)\cdot(1-\alpha_2) = 1-\alpha, \quad M_1+M_2 = M.$$

The equilibrium points $\underline{\alpha}^*$ and $\underline{M}^*$ of the game $(\{\underline{\alpha}\}, \{\underline{M}\}, \tilde{\tilde{I}}, \tilde{\tilde{B}})$ are determined by the relations

$$\tilde{\tilde{I}}(\underline{\alpha}^*,\underline{M}^*) \geq \tilde{\tilde{I}}(\underline{\alpha},\underline{M}^*); \quad \tilde{\tilde{B}}(\underline{\alpha}^*,\underline{M}^*) \geq \tilde{\tilde{B}}(\underline{\alpha}^*,\underline{M});$$

they are fulfilled if the following two equations are satisfied:

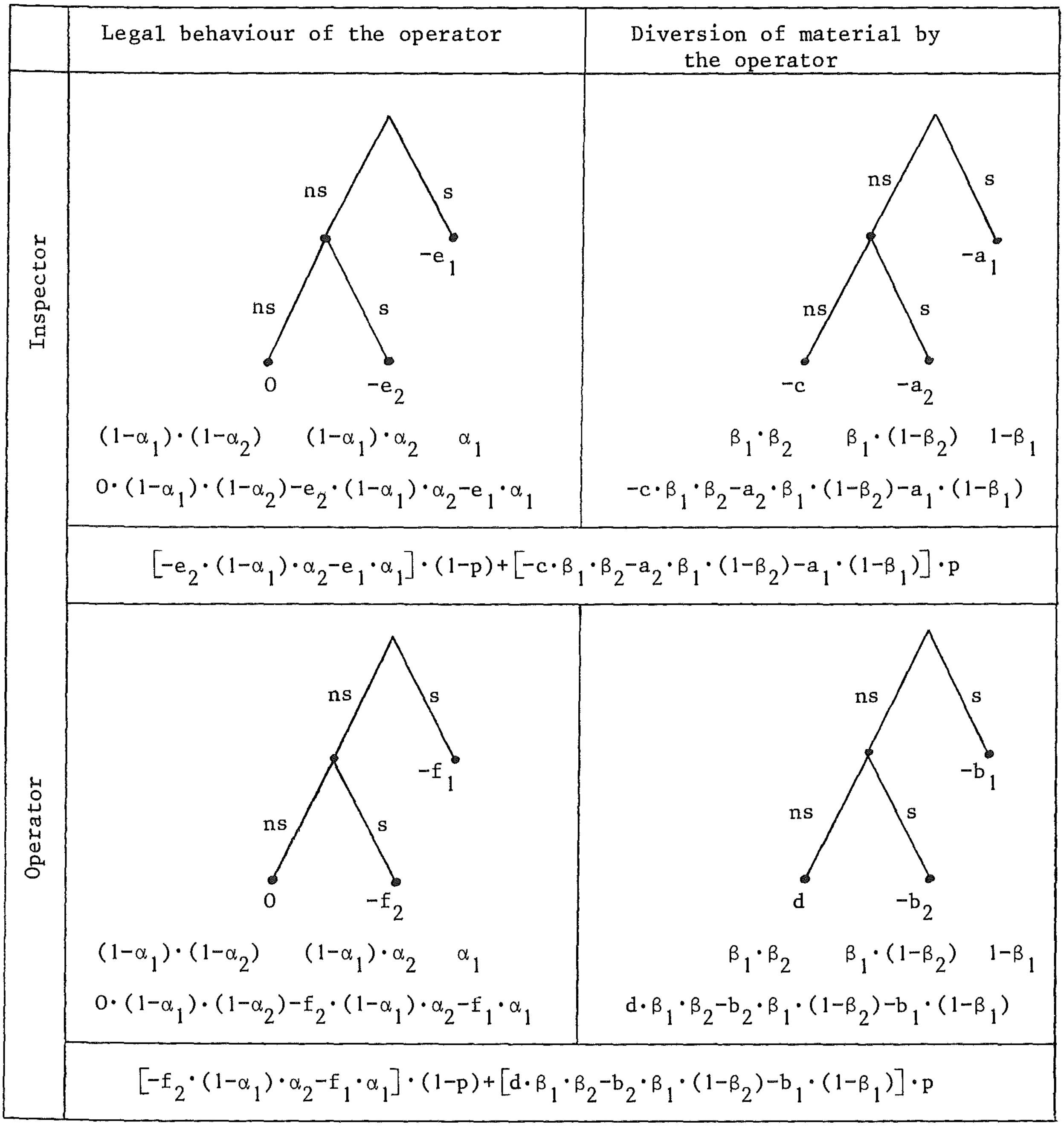


Figure 4: Extensive form of the sequential safeguards procedure for independent test statistics, Section 2.2: payoffs, probabilities, conditional expected payoffs and expected payoffs (ns: non-significant, s: significant, p: probability of diversion).

$$\Phi'(U_{1-\alpha_1} - \frac{M_1}{\sigma_1})\cdot(\lambda_I+\Phi(U_{1-\alpha_2} - \frac{a\cdot M_1+M_2}{\sigma_2}))\cdot\frac{d}{d\alpha_1}U_{1-\alpha_1}+\Phi(U_{1-\alpha_1} - \frac{M_1}{\sigma_1})\cdot\Phi'(U_{1-\alpha_2} -$$

$$- \frac{a\cdot M_1+M_2}{\sigma_2})\cdot\frac{d}{d\alpha_1}U_{1-\alpha_2} = 0$$

$$-\Phi'(U_{1-\alpha_1} - \frac{M_1}{\sigma_1})\cdot(\lambda_B+\Phi(U_{1-\alpha_2} - \frac{a\cdot M_1+M_2}{\sigma_2}))\cdot\frac{1}{\sigma_1} + \Phi(U_{1-\alpha_1} - \frac{M_1}{\sigma_1})\cdot\Phi'(U_{1-\alpha_2} - \frac{a\cdot M_1+M_2}{\sigma_2})\cdot$$

$$\cdot\frac{1-a}{\sigma_2} = 0.$$

Here $\Phi'(.)$ is again the first derivative of $\Phi(.)$.

If we use instead of these two equations the ratio of these and again the second equation, we get the following determinants for $\underline{\alpha}^*$ and $\underline{M}^*$:

$$\frac{\lambda_I+\Phi(U_{1-\alpha_2} - \frac{a\cdot M_1+M_2}{\sigma_2})}{\lambda_B+\Phi(U_{1-\alpha_2} - \frac{a\cdot M_1+M_2}{\sigma_2}}\cdot\sigma_1\cdot\frac{dU_{1-\alpha_1}}{d\alpha_1} + \frac{\sigma_2}{1-a}\cdot\frac{dU_{1-\alpha_2}}{d\alpha_1} = 0$$

$$-\Phi(U_{1-\alpha_1} - \frac{M_1}{\sigma_1})\cdot(\lambda_B+\Phi(U_{1-\alpha_2} - \frac{a\cdot M_1+M_2}{\sigma_2})+\Phi(U_{1-\alpha_1} - \frac{M_1}{\sigma_1})\cdot\Phi'(U_{1-\alpha_2} - \frac{a\cdot M_1+M_2}{\sigma_2}.\frac{1-\rho}{\sigma_2}) = 0,$$

together with the boundary conditions for $\underline{\alpha}$ and $\underline{M}$.

For $\lambda_I=\lambda_B$ the first equation is simplified to become

$$\sigma_1\cdot\frac{dU_{1-\alpha_1}}{d\alpha_1} + \frac{\sigma_2}{1-a}\cdot\frac{dU_{1-\alpha_2}}{d\alpha_1} = 0$$

which with

$$\frac{dU_{1-\alpha_1}}{d\alpha_1} = - \sqrt{2\pi} \cdot \exp(\frac{1}{2} \cdot U^2_{1-\alpha_1})$$

$$\frac{dU_{1-\alpha_2}}{d\alpha_1} = \frac{dU_{1-\alpha_2}}{d\alpha_2} \cdot \frac{d\alpha_1}{d\alpha_1} = \sqrt{2\pi} \cdot \exp(\frac{1}{2} \cdot U^2_{1-\alpha_2}) \cdot \frac{1-\alpha}{(1-\alpha_1)^2}$$

is equivalent to

$$-\sigma_1 \cdot \exp(\frac{1}{2} \cdot U^2_{1-\alpha_1}) + \frac{\sigma_2}{1-a} \cdot \exp(\frac{1}{2} \cdot U^2_{1-\alpha_2}) \cdot \frac{1-\alpha_2}{1-\alpha_1} = 0.$$

Thus, we see that in this case the optimal strategy of the inspector is the *same as that in the non-sequential case,* treated in Section 1.3 and represented graphically in Figs. 1a and 1b.

It can be understood intuitively that under certain circumstances, i.e. for some parameter value combinations, the optimal strategies in the sequential model are the same as in our non-sequential game, as both players choose their strategies already at the beginning of the game; therefore, the game is not really sequential. It is, however, not easy to understand the condition for this equality of optimal strategies: If we assume

$$\kappa_I := \frac{a_2}{a_1} > 1, \quad \kappa_B := \frac{b_1}{b_2} > 1,$$

the condition $\lambda_I = \lambda_B$ is equivalent to the condition

$$\frac{1 - \frac{1}{\kappa_I}}{\frac{c}{a_2} - 1} = \frac{\kappa_B - 1}{\frac{d}{b_2} + 1}$$

which still represents a relation between four parameters which, obviously, does not lend itself to an intuitive interpretation. Furthermore, it is a relation between parameters which, on the one hand, express the payoff to the two parties in case of *early* detection ($\kappa_I$ and $\kappa_B$), and parameters which, on the other hand, express the payoff to the strategies in case of *secure* detection (c, $a_2$, d and $b_2$).

As, however, these two criteria are somehow independent, this relation can only hold in very special cases.

The optimal strategy $\underline{M}^*$ of the operator in any case depends on the derived payoff parameter $\lambda_B$; therefore, also the overall probability of detection depends on $\lambda_B$. First numerical calculations (Heidl and Schmidt (1980)) indicate that in fact $\underline{M}^*$ strongly depends on the value of $\lambda_B$, but that the overall probability of detection practically does not depend on the value of $\lambda_B$.

*Abrupt Diversion*

Under the assumption that the operator diverts the total amount M of material with probability $q_i$, $i=1...n$, in the i-th inventory period, $\sum_i q_i = 1$, it is even difficult to formulate the expected payoffs to the inspector and to the operator in an understandable way, as because of the construction of the transformed MUF variables such a diversion still can be detected after later inventory periods. Therefore, we will consider only the case of two inventory periods.

The payoff parameters, the probabilities, the conditioned expected payoffs and the expected payoffs are given in Figure 5; according to Section 1.3 and according to the assumption that the total amount M of material is diverted *either* in the first *or* in the second inventory period, the probability to detect a diversion after the i-th inventory period is, if in fact in that inventory period the material is diverted

$$1 - \beta_i = \Phi\left(\frac{M}{\sigma_i} - U_{1-\alpha_i}\right), \quad i = 1,2,\dots ,$$

where $\sigma_i^2$ is the variance of the transformed MUF variables $\tilde{Z}_i$, $i=1,2$ :

$$\mathrm{var}(\tilde{Z}_1) = \mathrm{var}(Z_1) = \mathrm{var}(I_o + T_1 - I_1) =: \sigma_1^2 ,$$

$$\mathrm{var}(\tilde{Z}_2) = \mathrm{var}(a \cdot Z_1 + Z_2) = \mathrm{var}(a \cdot (I_o + T_1 - I_1) + I_1 + T_2 - I_2) =: \sigma_2^2 .$$

Furthermore, because of

$$E_1(\tilde{Z}_1) = a \cdot E_1(\tilde{Z}_1) + E_1(Z_2) \quad \text{under } H_1$$

in case of diversion in the *first* inventory period we have the probability

$$1-\beta^2 := \Phi\left(\frac{a \cdot M}{\sigma_2} - U_{1-\alpha_2}\right)$$

that this diversion will be detected after the *second* inventory period.

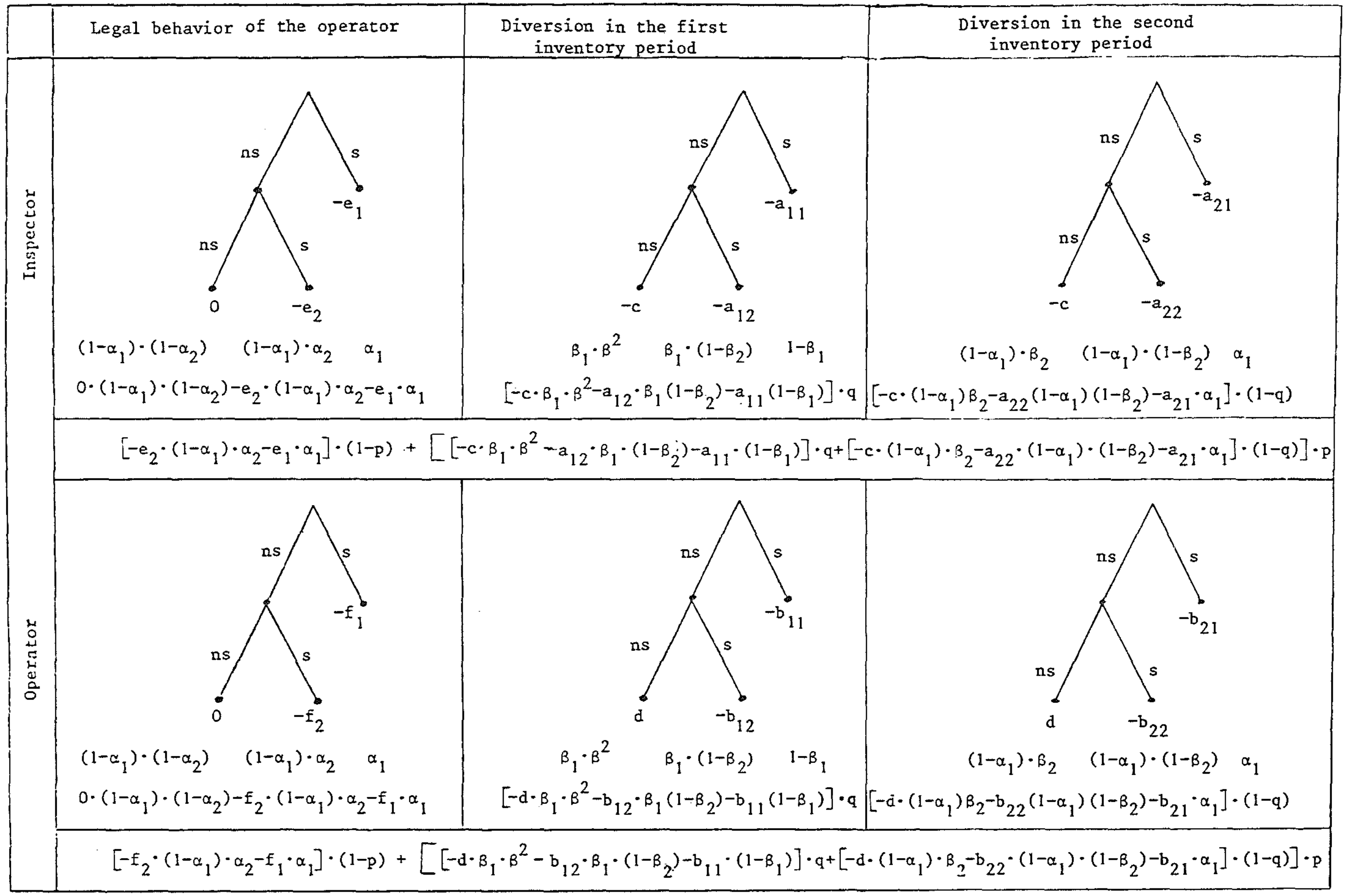


Figure 5: Extensive form of the sequential safeguards procedure for independent test statistics, Section 2.5: payoffs, probabilities, conditional expected payoffs and expected payoffs (ns: non-significant, s: significant, q: probability of diversion in the first period, p: probability of diversion at all).

Let q be the probability that the operator will divert the amount M of material in the first inventory period. Then according to our assumptions and as already denoted in Figure 5, the payoff to the operator is

$$B := \left[-f_2\cdot(1-\alpha_1)\cdot\alpha_2-f_1\cdot\alpha_1\right]\cdot(1-p) +$$

$$+\left[-d\cdot\beta_1\cdot\beta^2-b_{12}\cdot\beta_1\cdot(1-\beta^2)-b_{11}\cdot(1-\beta_1)\right]\cdot q+\left[-d\cdot(1-\alpha_1)\cdot\beta_2-b_{22}\cdot(1-\alpha_1)\cdot(1-\beta_2)-\right.$$

$$\left.-b_{21}\cdot\alpha_1\right]\cdot(1-q)]\cdot p$$

and his set of strategies is

$$\{(p,q):\ 0 \le p,q \le 1\};$$

the payoff to the inspector is

$$I := \left[-e_2\cdot(1-\alpha_1)\cdot\alpha_2-e_1\cdot\alpha_1\right]\cdot(1-p)+$$

$$+\ \left[\left[-c\cdot\beta_1\cdot\beta^2-a_{12}\cdot\beta_1\cdot(1-\beta^2)-a_{11}\cdot(1-\beta_1)\right]\cdot q+\left[-c\cdot(1-\alpha_1)\cdot\beta_2-a_{22}\cdot(1-\alpha_1)\cdot(1-\beta_2)-\right.\right.$$

$$\left.\left.-a_{21}\cdot\alpha_1\right](1-q)\right]\cdot p,$$

and his set of strategies is the same as before:

$$\{(\alpha_1,\alpha_2):\ 0 \le \alpha_1,\alpha_2 \le 1\}.$$

Now, it can be shown again that this game can be solved by solving two auxiliary games, the second one of which being characterized by a fixed value of p and a fixed overall false alarm probability $\alpha$ (Abel and Avenhaus 1981). In case of $e_1 = e_2$ it is sufficient to consider the conditioned expected payoffs in case of diversion,

$$\tilde{I} := \left[-c\cdot\beta_1\cdot\beta^2-a_{12}\cdot\beta_1\cdot(1-\beta^2)-a_{11}\cdot(1-\beta_1)\right]\cdot q+\left[-c\cdot(1-\alpha_1)\cdot\beta_2-a_{22}\cdot(1-\alpha_1)\cdot(1-\beta_2)\right.$$
$$\left.-a_{21}\cdot\alpha_1\right]\cdot(1-q)\ ,$$

$$\tilde{B} := \left[-d\cdot\beta_1\cdot\beta^2-b_{12}\cdot\beta_1\cdot(1-\beta^2)-b_{11}\cdot(1-\beta_1)\right]\cdot q+\left[-d\cdot(1-\alpha_1)\cdot\beta_2-\beta_{22}\cdot(1-\alpha_1)\cdot(1-\beta_2)\right.$$
$$\left.-b_{21}\cdot\alpha_1\right]\cdot(1-q)\ ,$$

which means that the second game to be considered is

$$\left(\left\{(\alpha_1,\alpha_2):(1-\alpha_1)\cdot(1-\alpha_2)=1-\alpha\right\},\left\{q:0\leq q\leq 1\right\},\tilde{I},\tilde{B}\right).$$

Only this game we will consider in the following.

Let us put furthermore

$$a_{11}=a_{22}\ ,\quad b_{11}=b_{22}\ ,$$

i.e., let us assume that neither for the inspector nor for the operator there is a difference if the abrupt diversion is detected immediately. Then the euqilibrium conditions are

$$\tilde{\tilde{I}}(\alpha_1{}^*,\alpha_2{}^*,q^*)\geq\tilde{\tilde{I}}(\alpha_1,\alpha_2,q^*)\qquad\text{for}\quad(\alpha_1,\alpha_2)\in X_\alpha$$

$$\tilde{\tilde{B}}(\alpha_1{}^*,\alpha_2{}^*,q^*)\geq\tilde{\tilde{B}}(\alpha_1{}^*,\alpha_2{}^*,q)\qquad\text{for}\quad q\in Y\ ,$$

where

$$\tilde{\tilde{I}}(\alpha_1,\alpha_2;q)=\left[\lambda_1\cdot\beta_1(\alpha_1)+\beta_1(\alpha_1)\cdot\beta^2(\alpha_2)\right]\cdot q+\left[\lambda_2\cdot\alpha_1+\lambda_3\cdot(1-\alpha_1)\cdot\beta_2(\alpha_2)\right]\cdot(1-q)\ ,$$

$$\lambda_1=\frac{a_{11}-a_{12}}{a_{12}-c}\ ,\qquad\lambda_2=\frac{a_{11}-a_{21}}{a_{12}-c}\ ,\qquad\lambda_3=\frac{a_{11}-c}{a_{12}-c}$$

$$\tilde{\tilde{B}}(\alpha_1,\alpha_2,q)=\left[\mu_1\cdot\beta_1(\alpha_1)+\beta_1(\alpha_1)\cdot\beta^2(\alpha_2)\right]\cdot q+\left[\mu_2\cdot\alpha_1+\mu_3\cdot(1-\alpha_1)\cdot\beta_2(\alpha_2)\right]\cdot(1-q)\ ,$$

$$\mu_1=\frac{b_{11}-b_{12}}{b_{12}+d}\ ,\qquad\mu_2=\frac{b_{11}-b_{21}}{b_{12}+d}\ ,\qquad\mu_3=\frac{b_{11}+d}{b_{12}+d}\ .$$

One realizes immediately

$$\lambda_3-\lambda_1=1\quad\text{and}\quad\mu_3-\mu_1=1\ ,$$

which means that, e.g., $\lambda_1$ and $\mu_1$ can be eliminated. If one assumes, furthermore

$$a_{11}<a_{12}<c,\ a_{11}<a_{21},\ b_{21}<b_{11}\ ,\ b_{12}<b_{11}\ ,$$

one gets

$$0<\lambda_2<\lambda_3\ ,\ 0<\lambda_1\ ,\ 0<\mu_1\ ,\ 0<\mu_2$$

which leads to

$$1 < \lambda_3 \quad , \quad 1 < \mu_3 \ .$$

If one introduces the parameters

$$\lambda := \lambda_3 \ , \ \kappa := \frac{\lambda_2}{\lambda_3} \ , \ \mu := \mu_3 \ , \ \varepsilon := \frac{\mu_2}{\mu_3} \quad ,$$

than one sees that the payoff functions $\approx{I}$ and $\approx{B}$ and therefore the equilibrium strategies depend only on the four parameters $\lambda,\kappa,\mu$ and $\varepsilon$ instead of the original ten parameters, with

$$1 < \lambda \ , \ 0 < \kappa < 1 \ , \ 1 < \mu \ , \ 0 < \varepsilon < 1 \ .$$

(In the case of protracted diversion we arrived at two parameters instead of the original six.)

Explicitely the equilibrium strategies are determined by the relations

$$\beta_1(\alpha_1{}^*) \cdot \left(\mu - 1 - \beta^2(\alpha_2{}^*)\right) - \varepsilon \cdot \mu \cdot \alpha_1{}^* - \mu \cdot (1 - \alpha_1{}^*) \cdot \beta_2(\alpha_2{}^*) = 0$$

$$(1 - \alpha_1{}^*) \cdot (1 - \alpha_2{}^*) = 1 - \alpha$$

$$\frac{\partial}{\partial \alpha_1} \tilde{\tilde{I}} (\alpha_1, \alpha_2; q^*) \Big|_{\alpha_1 = \alpha_1{}^*} = 0 \ .$$

Numerical calculations (Abel and Avenhaus 1981b) indicate that the optimal single false alarm probabilities $\alpha_1{}^*$ and $\alpha_2{}^*$ may be very different in case of protracted and of abrupt diversion, however, that the resulting overall probabilities of detection are nearly identical. This means that the inspector must not worry too much about the strategy chosen by the operator. Nevertheless it should be kept in mind that the optimal total false alarm probabilities $\alpha^*$ resulting from the general game, may have different values for the two diversion scenarios.

## 2.3 Sequential Approaches for Infinite Time Horizons

In general it is impossible to construct auxiliary games which are independent of the payoff parameters as we have seen. Therefore, in this section we discuss safeguards procedures from a purely statistical point of view which, however, means that we are no longer able to determine 'optimal' strategies. It is also much more difficult in case of sequential procedures to determine the probability of detection for any diversion strategy as we have seen in the foregoing two sections. Therefore, we will not discuss explicitely the cases of protracted and abrupt diversion even though many simulation studies have been performed recently (Pike 1980, Sellinschegg 1981). Even the false alarm probability for fixed intervals of time cannot be related to the significance thresholds in general, instead, the concept of the *average run length* is the central one , as we also saw already in section 1.3.

*Application of Wald's test*

Shipley (1980)[1)] has approached the problem of the determination of the optimal test statistics for a sequential test procedure at fixed points of time $t_i$, i=1,2,..., in the following way:

After the i-th inventory taking at time $t_i$, i=1,2,..., a test will be performed which has three different outcomes:

> Accept $H_o$ and continue, or
> accept $H_1$ and stop plant operations, or
> make no decision.

> It should be noted that Shipley writes in case of the third decision that another data point should be taken without saying what this means. In addition, he does not say explicitly "stop plant operations" in case of the second decision. We will come back to this point.

According to this test procedure, the space of observations $Z^{(i)}$ for the i-th test must be partitioned into three sets $Z_o^{(i)}$, $Z_N^{(i)}$, and $Z_1^{(i)}$ with

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1) In Shipley's document a large list of papers of the Los Alamos Group is given to which explicit reference is made at this point.

$$Z_o^{(i)} \cup Z_N^{(i)} \cup Z_1^{(i)} = Z^{(i)}, \quad Z_o^{(i)} \cap Z_N^{(i)} \cap Z_1^{(i)} = \emptyset$$

and the following decision rule

$$\delta = \begin{cases} \delta(H_o|\hat{\underline{Z}}^{(i)}) \\ \delta(N|\hat{\underline{Z}}^{(i)}) \\ \delta(H_1|\hat{\underline{Z}}^{(i)}), \end{cases}$$

where "N" indicates "no decision".

Let $P^i_{ND_j}$, $P^i_{D_j}$ and $P^i_{N_j}$, j=0,1, be the probabilities of no detection, detection and no decision under the hypothesis $H_j$ after the i-th inventory taking. Obviously, one has for these probabilities for i=1,2,...

$$P^i_{ND_o} + P^i_{N_o} + P^i_{D_o} = 1 \quad \text{for } H_o$$

$$P^i_{ND_1} + P^i_{N_1} + P^i_{D_1} = 1 \quad \text{for } H_1.$$

Furthermore, let the corresponding overall probabilities be $P_{ND_j}$, $P_{N_j}$ and $P_{D_j}$, j=0,1, given by

$$P_{ND_j} = \sum_i \int_{Z^{(i)}} \delta(H_o|\underline{z}^{(i)})\cdot f_j(z^{(i)})d\underline{z}^{(i)}$$

$$P_{D_j} = \sum_i \int_{Z^{(i)}} \delta(H_1|\underline{z}^{(i)})\cdot f_j(z^{(i)})d\underline{z}^{(i)}$$

where $f_j(\underline{z}^{(i)})$ are the densities of the first i MUF variables under the hypotheses j.

Now Shipley proceeds as follows: He is interested in determining those tests which have the properties

$$P_{ND_o} + P_{D_o} \to 1, \quad P_{ND_1} + P_{D_1} \to 1$$

implying that the test terminates eventually. To achieve this, he uses a modified Neyman-Pearson criterion in which both the values of $P_{D_o}$ and $P_{D_1}$ are fixed (which is possible because the terminal point is *not* fixed) and minimizes the sum of the probabilities of not making the right decision (including no decision) under the two hypotheses $H_o$ and $H_1$. This means that he uses the objective function

$$1-P_{ND_o} + 1-P_{ND_1} + \lambda_1 \cdot (P_{D_o} - \alpha) + \lambda_2 \cdot (P_{ND_1} - \beta),$$

where $\alpha$ and $\beta$ are the desired values of $P_{D_o}$ and $P_{D_1}$, respectively, and $\lambda_1$ and $\lambda_2$ are Lagrange multipliers.

The solution of this optimization problem, obtained in exactly the same manner as that in Section 2.1, leads Shipley to the following test: The three sets $Z_o^{(i)}$, $Z_N^{(i)}$ and $Z_1^{(i)}$ are given by

$$Z_o^{(i)} = \{\underline{Z}^{(i)}: \frac{f(\underline{Z}^{(i)}|H_1)}{f(\underline{Z}^{(i)}|H_o)} < \frac{1}{\lambda_2}\}$$

$$Z_N^{(i)} = \{\underline{Z}^{(i)}: \frac{1}{\lambda_2} < \frac{f(\underline{Z}^{(i)}|H_1)}{f(\underline{Z}^{(i)}|H_o)} < \lambda_1\}$$

$$Z_1^{(i)} = \underline{Z}^{(i)}: \lambda_1 < \frac{f(\underline{Z}^{(i)}|H_1)}{f(\underline{Z}^{(i)}|H_o)}\} \ , \ i = 1,2,\ldots,$$

which means that one has again a likelihood ratio test, this time with two thresholds.

Without going into further details of the analysis (e.g. the thresholds $\lambda_1$ and $\lambda_2$ in general cannot be expressed explicitly by $\alpha$ and $\beta$; therefore, approximations have to be used which lead to the thresholds $T_o$ and $T_1$, given by

$$T_o = \frac{\beta}{1-\alpha} \geq \frac{1}{\lambda_2} \text{ and } T_1 = \frac{1-\beta}{\alpha} \leq \lambda_1),$$

we will give some general comments on this procedure:

i) The criteria of overall probability of detection and false alarm probability do not take into account the aspect of timeliness which according to our discussion at the beginning of this part is the only justification for sequential procedures with their very strong implications for eventually stopping plant operations.
It should be noted that also in the procedure described in the foregoing sections the "overall" payoffs were optimized; these overall payoffs, however, contained parameters expressing the gains and losses in the cases of early and late detection, respectively.
Furthermore, the overall probabilities of detection and of false alarm refer to an interval of time the end point of which is a random variable and may be infinite. This is not in accordance with the concept of a fixed reference time interval, after which it shall be decided definitely whether or not the operator behaved legally.

ii) The concept of "no decision" can hardly be realized in practice: What action will be taken which is different from the action taken in case of the decision "accept $H_1$?"
It is clear that Wald's sequential test (1977) has its merits, e.g., in the area of quality control: Let the problem be given that a production lot has to be accepted if the percentage of bad items does not exceed a given value (null hypothesis); otherwise it has to be rejected. One starts with a random sample of given size and draws additonal items if no decision is taken. This way the expected sample size for given values of $\alpha$ and $\beta$ is *smaller* than in the case in which one would work with a fixed sample size and no region of indifference.
A sequence of inventory periods, however, represents a *stochastic* process, the different realizations of which require different actions; thus, Wald's theory simply does not seem to represent the appropriate analytical tool for this problem.

*Page's Test*

We will introduce Page's test (Page 1954) quite generally before discussing its application to the sequence of inventory periods problem.

The classical statistical theory for testing a hypotheses $H_o$ against a specific alternative $H_1$, based on a fixed sample of observations, was developed by considering the likelihood functions of the data under each of the hypotheses; examples were given in Section 1.1 and 1.2. This situation was later modified to take into account situations where the observations arrive one at a time (Wetherill 1975), and this theory was used by Page (1954) to develop his CUSUM test for the detection of a *shift in the mean value* of a random variable observed at regular intervals and goes as follows.

Given a sequence of random variables $X_1, X_2, \ldots$ with known variances $\sigma^2$ and expectation values ('target values') zero under the null hypothesis $H_o$. In case of a *onesided test* the null hypotheses $H_o$ is rejected after the i-th observation, if

$$S_i := \max\,(0, S_{i-1} + X_i - k) > h$$

where $S_o := 0$, and where k and h are called *reference* value and *decision* value. Explicitely it means that $H_o$ is rejected after the first step if

$$\max(0, X_1 - k) > h$$

it is rejected after the second step if

$$\max(0, S_1 + X_2 - k) = \left\{ \begin{array}{lcl} \max\,(0, X_1 + X_2 - 2k) & & X_1 - k > 0 \\ & \text{for} & \\ \max\,(0, X_2 - k) & & X_1 - k \leq 0 \end{array} \right\} \geq h \,,$$

and so on. In words, this procedure means that all variables, which are normalized to zero expectation values, are added unless their sum is smaller than the reference value, and that the null hypothesis is rejected if the decision variable h is passed. It should be noted that, contrary to the test discussed in section 1.4, the observations are *not* simply added and tested.

Before discussing the properties of this test procedure, it should be mentioned that the so-called V - *masque-technique* (see, e.g., van Dobben 1968) represents a graphical technique for the application of this test. However, as this technique is not easy to be understood, i.e., it is not easy to relate the parameters of the V - masque to statistical quantities, and as the test can easily be performed with the help of small computers, we will not go into further details.

The test is usually characterized by three quantities, namely by P(z), N(z) and L(z). Here, P(z) is the probability that a test, which starts at the value z of the test statistic, goes below zero; N(z) is the expected sample size of a single test which starts at z and ends either below zero ore above h, and L(z) is the expected sample size, i.e., the expected number of observations until h is passed. L(z) is also called *average run length.*

> Note: It should be observed that we consider single tests, i.e., series of observations which either end at zero or h, and series of tests, which end at h.

In general **explizite** expressions cannot be given for these quantities, however, integral equations can be derived under the assumption that all observations are from the same sample space characterized by the distribution F(x) and density f(x) (van Dobben 1968, Wetherill 1977). In order to achieve this, we start with the value z of the test statistic and assume that the observed value of the next sample is x. Then according to the definition we get the new 'score' for the test statistic

$$\begin{array}{rll} 0 & \text{for} & z+x \leq k \\ z+x-k & \text{for} & k < z+x \leq h+k \\ h & \text{for} & z+x \geq h+k. \end{array}$$

From the definition of P(z), N(z) and L(z) we therefore get immediately the equations

$$P(z) = F(k-z) + \int_{k-z}^{k-z-h} f(x) \cdot P(z+x-k)\ dx = F(k-z) + \int_{0}^{h} f(y+k-z) P(y)\ dy \ ,$$

$$N(z) = 1 + \int_{k-z}^{h-z-k} f(x) \cdot N(z+x-h)\ dx = 1 + \int_{0}^{h} f(y-k-z) \cdot N(y)\ dy \ ,$$

$$L(z) = 1 + L(0) \cdot F(k-z) + \int_{0}^{h} L(y) \cdot f(y+k-z)\ dy \ .$$

Furthermore, there exists a relation between L(z), N(z) and P(z) which heuristically can be understood as follows: The probability that - starting from zero - exactly s tests are performed until the decision is taken is

$$P(0)^{s-1} \cdot (1 - P(0)), \quad s = 1,2,\ldots,$$

therefore, the expected number of tests is

$$E(S) = \sum_{s=1}^{\infty} s \cdot P(0)^{s-1} \cdot (1 - P(0)) = \frac{1}{1 - P(0)} .$$

Thus, one expects that during the whole test procedure the statistic goes $\frac{1}{1 - P(0)} - 1$ times below zero and thereafter above h which means

$$L(0) = \frac{1}{1 - P(0)} \cdot \left(N^{l}(0) \cdot P(0) + N^{u}(0) \cdot (1 - P(0)\right) .$$

As, however, the average sample size N(0) of a single test starting at zero, is the average sample size under the condition that it ends at zero, $N^{l}(0)$, times the probability that it ends at zero, P(0), plus the average sample size under the condition that it ends at h, $N^{u}(0)$, times the probability that it ends at h, $1 - P(0)$, we finally get

$$L(0) = \frac{N(0)}{1 - P(0)} .$$

Finally, we get a relation between L(z), N(z) and P(z) as follows. The average run length L(z), starting from z, is the average run length of a single test, L(0), times the probability that the single test considered before ends at zero, P(z) - if it ended at h the whole procedure would have come to an end. Thus

$$L(z) = N(z) + L(0) \cdot P(z).$$

Exact solutions of the integral equations for special distributions have been given by van Dobben (1968). Numerical procedures for their solutions have been proposed by Kemp (1958); approximations are given by Kemp (1967 a,b).

An appropriate relation between h and P(0) can be established by comparing Page's test with the sequential probability ratio test by Wald (1947) in case the random variables $X_i$, $i = 1,2,\ldots$ are normally distributed with variances $var(X_i) = \sigma^2$ for $i = 1,2,\ldots$ and

$$E(X_i) = \begin{cases} 0 & \text{for } H_0 \\ \mu & \phantom{for } H_1, \end{cases} \quad i = 1,2,\dots .$$

Wald's test requires to

continue the procedure if $\ln B < \frac{\mu}{\sigma^2} \cdot \sum_i x_i - \frac{n}{2} \cdot \frac{\mu^2}{\sigma^2} < \ln A$ ,

reject $H_0$ if $\quad \frac{\mu}{\sigma^2} \cdot \sum_i x_i - \frac{n}{2} \cdot \frac{\mu^2}{\sigma^2} \geq \ln A$

reject $H_1$ if $\quad \frac{\mu}{\sigma^2} \cdot \sum_i x_i - \frac{n}{2} \cdot \frac{\mu^2}{\sigma^2} \leq \ln B$ .

It has been shown by Wald, that the following relations hold between the error first and second kind probabilities, $\alpha$ and $\beta$, and A and B :

$$A \leq \frac{1-\beta}{\alpha} \quad \text{and} \quad B \geq \frac{\beta}{1-\alpha} \quad .$$

Thus, for

$$A \sim A' := \frac{1-\beta}{\alpha} \quad \text{and } B \sim B' := \frac{\beta}{1-\alpha}$$

and furthermore, $\alpha,\beta \ll 1,1$, the procedure has to be continued if

$$\frac{\sigma^2}{\mu} \cdot \ln \beta < \sum_i x_i - n \cdot \frac{\mu}{2} < - \frac{\sigma^2}{\mu} \cdot \ln \alpha \ .$$

If we compare this procedure to that of Page's test, we can identify one series of Page's test with Wald's test, which means under $H_0$, that $\alpha = 1 - P(0)$, and therefore

$$k = \frac{\mu}{2}\ , \quad h \approx - \frac{\sigma^2}{\mu} \cdot \ln\ (1 - P(0))\ .$$

It should be noted, however that the probability P(0) does not give any information about the frequency of false alarms thus, in the sense of what has been said in section 1.3, the significance threshold h should be related to L(0).

Page's test was derived under the assumption that the observations were independent of one another, which might suggest that it is only relevant to the independently transformed MUF variables, considered in sections 1.3 and 2.2. However, it has been shown by Bagshaw and Johnson (1975) that, if Page's test is applied to a stream of negatively correlated observations - in our case the original MUF variables - the false alarm rate is not greatly affected. This suggests that the test will perform better for negatively correlated data than it does for independent observations.

Pike, Woods and Rose (1980, 1980 a,b) commented on the basis of their extensive numerical studies the properties of this procedure as follows: The alternative hypothesis is accepted only when sufficient evidence has been accumulated (sequentially) and each time the alternative hypothesis is accepted, the values used in the decision are discarded and the process begins anew. Hence, if a constant loss suddenly begins after a long time without any loss, the test is likely to signal quickly, since it does not wish to be confused by an early sequence of observations when no loss was occuring. On the other hand, the CUMUF test and many of those based on minimum variance unbiased estimates, continues to use all the MUF values until a positive signal is given so that a loss which begins late in the sequence may be masked by a long early run of acceptable values.

*Power One Test*

Recently it has been proposed by Cobb (1981) and by Shipley (1981) to use the so-called *power one test* which has been developed by Robbins and coworkers (1969, 1970), and which goes as follows:
Let $X_1, X_2, \ldots$ be a sequence of independent normally distributed random variables with variances

$$\mathrm{var}\ (X_i) = \sigma_i^2\ ,\ i = 1,2,\ldots$$

and with zero expectation values under the null hypothesis, The null hypothesis is rejected if

$$\sum_{i=1}^{k} \frac{x_i}{\sigma_i} > \left((k+m)\cdot\left(a^2 + \ln\ (\frac{k}{m}+1)\right)\right)^{\frac{1}{2}}\ ,$$

where $a > 0$, $m > 0$. It has been shown, now, that the overall false alarm probability of this test is bounded according to

$$\alpha \leq \frac{1}{2\Phi(a)}\cdot\exp\left(-\frac{a^2}{a}\right)\ ,$$

and furthermore, that for the alternative hypothesis is $H_1\ :\ E(X_i) = \mu > 0$ for $i = 1,2,\ldots$ the power of this test tends towards one for an infinite series of variables.

As these properties of the test hold only for independent random variables, it can only be applied to independently transformed MUF variables, e.g., to those used in section 1.3. The advantages of this test compared to those sequential tests discussed so far are obvious; it remains to be shown, how this test works in case of different alternative hypotheses, i.e., diversion scenarios. Preliminary calculations by Sellinschegg and Bicking (1981), based on with the help of Kalman Filter techniques transformed MUF-variables indicate good performance characteristics for various diversion scenarios.

## 2.4 Summary of Part II

Having in mind not only the objective of optimizing the overall probability of detecting a diversion of nuclear material during a finite reference time, but also the objective of early detecting any diversion, we proceeded in this part along two different lines: First, we tried to model the interest of the inspector to detect a diversion already at time $t_i$, and not only at time $t_{i+1}$ - this means to face the problem of estimating the values of the payoff parameters which describe this interest of the inspector (and also that of the operator). Second, because of this problem we tried to avoid the use of payoff parameters and instead to take the expected detection time as optimization criterion. We realized, by the way, that such a criterion cannot be derived from a game theoretical model. As this approach posed conceptual problems in the case of a finite reference time, we considered infinite reference time, and we were led to the new concept of average run length.

Whereas in the static approach it was possible to find explicite and intuitive solutions of the optimization problems, this was not possible for any of the sequential models. Partly, this is so because sequential models are more difficult than static ones from an analytical point of view - which naturally reflects inherent difficulties. More important, however, is the fact that it is much more difficult to formulate quantitatively diversion scenarios: How do we describe protracted and abrupt diversion of a finite goal quantity in case of an infinite time horizon?

In the last chapter we will try to draw some conclusions from the experience gained so far.

# CONCLUSION

## Where do we stand?

Let us repeat once more the nuclear material safeguards objectives as they are agreed today: First, any diversion of nuclear material should be detected as securely as possible - high probability of detection - and second, it should be detected as early as possible - short expected detection time.

In addition, there are ideas about which diversion strategies have to be met by a safeguards system: they range from *protracted* diversion of small amounts of material until *abrupt* diversion of large amounts. If possible, *all* strategies lying between these extremes should be met as well.

In order to meet these widespread and partially conflicting requirements on the safeguards system, a variety of test procedures have been developed by various research groups around the world, the most important variants of which have been presented in this paper.

Furthermore, there was a complementary effort to develop estimation procedures; their primary purpose was to obtain in this way best test statistics (from the plant operator's point of view they also should serve the purpose of estimating losses). With the exception of Stewart's estimate of the starting inventory, they were not presented in this paper. Just this example, however, showed that best estimates do not necessarily lead to best tests: Stewart's estimate did *not* lead to the Neyman-Pearson test.

Where do we stand now? We have seen that the problem may be considered as solved if the only objective is the probability to detect the diversion of a given amount of material in a given reference time. More than that, this objective can be derived from a non-cooperative two person game, if we consider the false alarm probability as a parameter of the problem.

> Sanborn (1980) has formulated a different criterion for the optimization of inspector's and operator's strategies: He defined the *information* $I(H_1,H_0;Z)$ as the expected value of the natural logarithm

> of the ratio of the density functions $L_1$ (Z) of the observations under the two hypotheses $H_1$ and $H_0$. The question to be answered is whether or not this criterion can be derived from first principles such that one can argue that it will be used by both opponents, inspector and operator, for the optimization of their strategies.

Things get difficult, if we introduce the objective of timeliness of detection. As this concept requires by its very nature immediate action after an alarm, we have to consider sequential procedures which makes the game theoretical model much more complicated; it has become clear that no simple timeliness criterion can be derived from these models. Furthermore, we have difficulties to define any expected detection time, as long as we consider finite reference times - this, however, we did in order to specify the probability of detection and the false alarm probability for a given reference time. If we consider infinite reference times, we can define the expected detection time in a satisfying way and furthermore, the average run length under the null and under the alternative hypotheses may take the role of the probability of detection and of the false alarm probability, however, we run into new problems:

First, we have no final decisions at fixed points of time which is an important political requirement, as we pointed out in section 1.3. Second, we can no longer specify the strategies of protracted and abrupt diversion for given total goal quantities as easily as in the static or in the sequential, fixed reference time approach.

> It should be mentioned that in view of these latter problems some time ago it has been proposed to use "batteries of tests" where every test is taylored for a specific diversion scenario. Because of the difficulty, that the overall false alarm probability of such a battery can no longer be kept under control, experts agreed that this proposal was not practical.

Now, what conclusions can be drawn at this stage of development, and which further actions should be taken?

First, politicians, responsible safeguards authority representatives and plant operators have to be asked to more carefully specify the objectives and boundaries of the safeguards system: How important is early detection compared to detection with high probability after some time? What are the diversion scenarios (protracted, abrupt, finite or infinite time horizon)? Which ones have to be considered? How many inventories, and what kind of inventories can economic plant operation tolerate? Will plant operations be stopped if the alternative hypothesis is accepted? What are the second and third action levels? How important is it for non-weapons states to get official statements from the side of the safeguards authority at fixed points of time (e.g., once a year)? And so on. This does not mean that one hopes to arrive in this way at parameter value estimates; it will rather permit to better structure the procedures and, consequently, the models. For example, the answer to questions like those raised above, should lead to a decision whether or not sequential test procedures will be applied.

Second, it is clear that in practice only *single* and simple test procedures (see, e.g., Woods and Pike 1981) will be applied which may differ from plant type to plant type. Therefore, it is necessary to perform concrete case studies in order to evaluate the sensitivity of the procedures: This analysis of sampling procedures in connection with data verification problems (Avenhaus and Beedgen (1980), Beedgen and Neu (1980)) showed that "second best solutions" frequently are not far from best solutions. In our case this would mean that a specific procedure which is best with respect to the primary objective, is expected to be not so much worse with respect to the secondary goal, for which a different procedure would be best.

## Terminology

### a) Logical and Set Theoretical Symbols

| | |
|---|---|
| $a = b$ | a equals b |
| $a := b$ | a is defined to be equal to b |
| $a < b$ | a is smaller than b |
| $a << b$ | a is much smaller than b |
| $\sum_{i=1}^{n} a_i = a_1+a_2+\ldots+a_n$ | sum of $a_1 \ldots a_n$ |
| $\prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdots a_n$ | product of $a_1 \ldots a_n$ |
| $A := \{a_1, \ldots, a_n\}$ | set A consists of the elements $a_1 \ldots a_n$ |
| $a \varepsilon A$ | a is element of set A |
| $A \cup B$ | union of sets A and B |
| $A \cap B$ | intersection of sets A and B |
| $A \setminus B$ | difference of A and B |
| $\emptyset$ | empty set |
| $A \otimes B := \{(a_i, b_j) : a_i \varepsilon A, b_j \cdot \varepsilon B\}$ | Cartesian product of the two sets A and B |
| $\underline{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$ | row vector with n components |
| $\underline{X}' = (X_1 \ldots X_n)$ | column vector with n components (transposed vector of $\underline{X}$) |
| $\underline{\underline{Z}} = (Z_{ij})$ | matrix with elements $Z_{ij}$ |
| $\lvert\underline{\underline{Z}}\rvert$ | determinant of matrix $\underline{\underline{Z}}$ |

### b) Statistical and Decision Theoretical Symbols

| | |
|---|---|
| $F(x) := \text{prob}\{X \leq x\}$ | distribution function of random variable X |
| $f(x) := F'(x) =: \frac{d}{dx} F(x)$ | density function of random variable X |
| $F(x) = \int_{\infty}^{x} f(t)dt$ | distribution function of random variable X, expressed by the density function |
| $E(X)$ | expected value of random variable X |
| $var(X)$ | variance of random variable X |

| | |
|---|---|
| $cov(X,Y)$ | covariance of random variables X and Y |
| $\underline{\underline{\Sigma}} := cov(X_i, X_j)$ | covariance matrix of random vector $\underline{X}$ |
| $\Phi(x)$ | normal distribution function |
| $U_\gamma$ | inverse of the normal distribution function |
| $H_o$ | null hypothesis |
| $H_1$ | alternative hypothesis |
| $\alpha$ | probability of error of the first kind (false alarm probability) |
| $\beta$ | probability of error of the second kind (probability of no detection) |
| $\delta$ | decision rule |

c) Model Symbols

| | |
|---|---|
| $[t_o, t_n]$ | reference time |
| $t_o, t_1, \ldots, t_{n-1}, t_n$ | inventory taking points of time |
| $I_i$, $i=1\ldots n$ | physical inventory at $t_i$ |
| $T_i$, $i=1\ldots n$ | transfers between $t_{i-1}$ and $t_i$ |
| $B_i = I_{i-1}+T_i$, $i=1\ldots n$, | book inventory at $t_i$ |
| $Z_i := I_{i-1}+T_i-I_i$, $i=1\ldots n$ | |
| $\underline{Z}' := (Z_1, \ldots, Z_n)$ | |
| $E_o(Z_i) = 0$, $i=1\ldots n$ | expected value of $Z_i$ under $H_o$ |
| $E_1(Z_i) = M_i$, $i=1\ldots n$ | expected value of $Z_i$ under $H_1$ |
| $M = \sum_i M_i$ | |
| $\sigma_i^2 = var(Z_i)$ | (note: for simplicity in Sections 1.3 and 2.2 $\sigma_i^2 = var(\tilde{Z}_i)$, in Section 1.5 $\sigma_i^2 = var(Y_i)$) |
| $\tilde{Z}_i = a_{i-1} \cdot \tilde{Z}_{i-1} + Z_i$, $i=1\ldots n$ | transformed $Z_i$, using Stewart's estimate of the starting inventory |
| $\hat{\underline{Z}} = \underline{z}$ | observed vector of $\underline{Z}$ |
| $Z := \{\hat{\underline{Z}}\} = \{\underline{z}\}$ | set of observations of $\underline{Z}$ |
| $Z_o \subset Z$ | acceptance region |
| $Z_1 \subset Z$ | rejection region |

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