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ON CALCULATING MATRIX ELEMENTS OF SLATER DETERMINANT WAVE FUNCTIONS IN THE CLUSTER MODEL. II. CLUSTERS WITH INTRINSIC ORBITAL ANGULAR MOMENTUM

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ABSTRACT

A method for projecting angular momentum in two-cluster systems with intrinsic orbital angular momentum is presented. The method is an analytical one making use of Racah algebra and exploiting tensor properties of two-cluster shell model wave functions. As an application, reduced matrix elements of spin-isospin independent scalar operators and of the electric charge multipole operator are calculated in the case where one of the clusters in the two-cluster wave function may carry an intrinsic orbital angular momentum.

ZUR BERECHNUNG VON MATRIXELEMENTEN VON VIELTEILCHENWELLENFUNKTIONEN IM CLUSTER MODELL.
II. CLUSTER MIT INNEREM BAHNDREHIMPULS

ZUSAMMENFASSUNG

Die vorliegende Arbeit befaßt sich mit der Drehimpulsprojektion in Zwei-Cluster-Systemen, wobei die einzelnen Cluster einen nicht-verschwindenden inneren Bahndrehimpuls besitzen können. Die Methode beruht auf einer Analyse der Tensorstruktur von Vielteilchenwellenfunktionen und deren Matrixelementen und macht ausführlichen Gebrauch von der Racah Algebra.
Als Anwendung werden reduzierte Matrixelemente von skalaren Operatoren, sowie dem elektrischen Ladungsmultipoloperator zwischen Zwei-Cluster Wellenfunktionen berechnet.

CONTENTS

1. INTRODUCTION

The Generator Coordinate Method (GCM) has been used as the basis for approximate calculations both of bound and scattering states [1] in light nuclei. In most of these studies it is assumed that the dominant cluster structure of the nuclear system consists of (i) a few "simple" clusters, as in the alpha-particle model[2] or in three-cluster models of ^{6}Li[3], ^{7}Li[4] and ^{8}Be[5], (ii) two "complicated" clusters with nucleons in 1s and 1p orbits[3,6].

One of the major problems in treating three-cluster systems or clusters with p-waves is the handling of angular momentum projection[7]. In a previous report[8], an analytical method of angular momentum projection for two- and three-cluster systems when the clusters carry no intrinsic orbital angular momentum and the operators involved are spin-isospin independent scalars was developed.

In continuation of this work, we extend this projection method to cases of two-cluster systems when one of the clusters carries an intrinsic orbital angular momentum and include also spherical tensor operators such as the electric charge multipole operator. As an application of this method, we study[9] a two-cluster model[3] of ^{6}Li which includes nucleons in 1p-wave orbits. The aim of this report is to provide the necessary formulas concerning the projection of angular momentum.

In sec. 2 we study two-center SM functions and their matrix elements in the context of irreducible spherical tensors. Section 3 deals with the calculation of reduced matrix elements of a scalar spin-isospin independent operator between many body SM functions with intrinsic orbital angular momentum.
This analysis is extended in sec. 4 to the electric charge multipole operator. Section 5 contains the application of these results to a cluster model of ^{6}Li and provides formulas for the elastic and inelastic charge form factors, the rms radius and the ground state radiative width $B(E2,\ 1^{+} \rightarrow 3^{+})$. In

sec. 6 we discuss very briefly some practical aspects of the projection of linear momentum of the center-of-mass motion and of angular momentum.

2. EXPANSION OF TWO-CENTER SHELL MODEL(SM) WAVE FUNCTIONS INTO SPHERICAL TENSORS

2.1 Tensor Properties of Two-Center SM Functions

In the two-center shell model, the N-particle wave function (N=A+B) is of the form[8)]

$$\Phi_{I\nu}(\{\underset{\sim}{x}\} \ell m \underset{\sim}{s})$$

$$= \sum_{m_A m_B} (\ell_A m_A, \ell_B m_B \mid \ell m)$$

$$* \left[\mathcal{A}\, \phi_A(\{\underset{\sim}{x}\}_A \ell_A m_A \underset{\sim}{S}_A)\, \phi_B(\{\underset{\sim}{x}\}_B \ell_B m_B \underset{\sim}{S}_B) \right]_{I\nu} \qquad (2.1)$$

$$\underset{\sim}{s} = \underset{\sim}{S}_A - \underset{\sim}{S}_B$$

$$A \underset{\sim}{S}_A + B \underset{\sim}{S}_B = 0$$

$$\{\underset{\sim}{x}\} = (\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_N)$$

$$\{\underset{\sim}{x}\}_A = (\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_A)$$

$$\{\underset{\sim}{x}\}_B = (\underset{\sim}{x}_{A+1}, \ldots \underset{\sim}{x}_N)$$

The functions $\phi_A(\phi_B)$ are A(B)-particle Slater determinant wave functions centered at the position $\underset{\sim}{S}_A(\underset{\sim}{S}_B)$. The symbol $[\ ]_{I\nu}$ means that the spins of all N particles are coupled to the total spin I and its projection ν along the Z axis.

The quantum numbers $\ell_A(\ell_B)$ and $m_A(m_B)$ refer to the intrinsic orbital angular momentum of the cluster A(B). Upon coupling $\ell_A m_A$ and $\ell_B m_B$ into ℓm, we obtain the total intrinsic orbital angular momentum of the system. The operator $\mathcal{A}$ antisymmetrizes all particle coordinates.

In order to study the tensor properties of the functions (2.1) in coordinate space, we use an expansion in terms of spherical harmonics

$$\phi(\{\underset{\sim}{x}\}\ell m \underset{\sim}{s}) = \frac{1}{s} \sum_{LM} \phi(\{\underset{\sim}{x}\} LM\ell m s)\, Y^{*}_{LM}(\hat{s}) \tag{2.2}$$

and investigate the behaviour of $\phi(\{\underset{\sim}{x}\}LM\ell ms)$ under transformations. Because various operations (for instance, reflection) cannot be carried out directly on a physical system but only on a coordinate frame, we limit ourselves to the "passive" interpretation of transformations. We mean the change in description which the system undergoes when the coordinate frame is changed while the system remains fixed (redescription). In the case of rotation, we interprete this to mean a redescription which changes the functions (2.2) in the following way[10)]

$$\begin{aligned} &R_{\underset{\sim}{x}}(\Omega)\, R_{\underset{\sim}{s}_A}(\Omega)\, R_{\underset{\sim}{s}_B}(\Omega)\, \phi(\{\underset{\sim}{x}\}\ell m \underset{\sim}{s}) \\ &= R_{\underset{\sim}{x}}(\Omega)\, R_{\underset{\sim}{s}}(\Omega)\, \phi(\{\underset{\sim}{x}\}\ell m \underset{\sim}{s}) \\ &= \sum_{m'} D^{\ell}_{m'm}(\Omega)\, \phi(\{\underset{\sim}{x}\}\ell m' \underset{\sim}{s}) \end{aligned} \tag{2.3}$$

where $R_{\underset{\sim}{x}}(\Omega)$ is the unitary rotation operator which rotates the basis of the vectors $\{\underset{\sim}{x}\}$ through Euler angles Ω and

$$D^{\ell}_{m'm}(\Omega) = \langle \ell m' | R(\Omega) | \ell m \rangle \tag{2.4}$$

is the usual[10] rotation matrix. Upon inserting the expansion (2.2) into eq. (2.3) we find (see Appendix A)

$$\begin{aligned} & R_{\underset{\sim}{x}}(\Omega)\, \phi(\{\underset{\sim}{x}\}\, L\ell\mathcal{L}\mathcal{M}s) \\ & = \sum_{\mathcal{M}'} D^{\mathcal{L}}_{\mathcal{M}'\mathcal{M}}(\Omega)\, \phi(\{\underset{\sim}{x}\}\, L\ell\mathcal{L}\mathcal{M}'s) \end{aligned} \tag{2.5}$$

where we introduced the wave function in the coupled representation

$$\begin{aligned} & \phi(\{\underset{\sim}{x}\}\, L\ell\mathcal{L}\mathcal{M}s) \\ & = \sum_{mM} (LM, \ell m | \mathcal{L}\mathcal{M})\, \phi(\{\underset{\sim}{x}\}\, LM\, \ell m s) \end{aligned} \tag{2.6}$$

We conclude that the set of many body functions (2.6) with $\mathcal{M} = -\mathcal{L}, \ldots \mathcal{L}$ form a spherical tensor of rank $\mathcal{L}$ in the space of the spatial coordinates $\{\underset{\sim}{x}\}$.

The properties of this tensor under the parity transformation $\{\underset{\sim}{x}\} \to -\{\underset{\sim}{x}\}$ may be derived from the relation

$$\phi(\{-\underset{\sim}{x}\}\, \ell m \underset{\sim}{s}) = (-)^{\ell}\, \phi(\{\underset{\sim}{x}\}\, \ell m\, -\underset{\sim}{s}) \tag{2.7}$$

which follows from the fact that all spatial coordinates occur in the combination[8] $(\underset{\sim}{x}_i - \underset{\sim}{S}_A)^2$ and $(\underset{\sim}{x}_i - \underset{\sim}{S}_B)^2$ except for those particles occupying a $\ell \neq 0$ orbit. The latter ones contribute the overall phase $(-)^{\ell}$ in eq. (2.7).

Inserting the expansion (2.2) into eq. (2.7), we obtain

$$\phi(\{\underset{\sim}{x}\}\, LM\ell m s) = (-)^{L+\ell}\, \phi(\{\underset{\sim}{x}\}\, LM\ell m s) \tag{2.8a}$$

and

$$\phi(\{-\underset{\sim}{x}\}\, L\ell\mathcal{L}\mathcal{M}s) = (-)^{L+\ell}\,\phi(\{\underset{\sim}{x}\}\, L\ell\mathcal{L}\mathcal{M}s) \tag{2.8b}$$

Upon coupling the spin and coordinate tensors in $\phi_{I\nu}(\{\underset{\sim}{x}\}$ $L\ell\mathcal{L}\mathcal{M}s)$, we may obtain wave functions which are eigenfunctions of the total angular momentum

$$\begin{aligned}\phi_I(\{\underset{\sim}{x}\}\, L\ell\mathcal{L}JMs) \\ = \sum_{\nu\mathcal{M}} (I\nu,\mathcal{L}\mathcal{M}|JM)\,\phi_{I\nu}(\{\underset{\sim}{x}\}\, L\ell\mathcal{L}\mathcal{M}s)\end{aligned} \tag{2.9}$$

where $M = \nu + \mathcal{M}$. The wave functions of the GC type with fixed total angular momentum and parity are constructed by superposing the functions (2.9),

$$\begin{aligned}\Psi(\{\underset{\sim}{x}\}\, J\mathcal{M}\pi) \\ = \sum_{IL\ell\mathcal{L}} \int_0^\infty ds\, f_I(L\ell\mathcal{L}JM)\,\phi_I(\{\underset{\sim}{x}\}\, L\ell\mathcal{L}JMs)\end{aligned} \tag{2.10}$$

where the parity relations (2.8) restrict the possible orbital angular momenta

$$\pi = (-)^{L+\ell}$$

2.2 Expansion of Matrix Elements of a Scalar Spin-Isospin Independent Operator O between Two-Center SM Functions into Spherical Tensors

In the sequel, we consider separately the cases where the wave functions entering the matrix elements possess a (non) vanishing intrinsic orbital angular momentum. In addition we assume that the operator O has even parity.

(i) Both wave functions carry no intrinsic orbital angular momentum:

In accordance with eq. (2.2), we expand the many body wave functions into spherical harmonics and obtain

$$\langle \phi(\underset{\sim}{s}) | O | \phi'(\underset{\sim}{s}') \rangle$$

$$= 1/(ss') \sum_{\mathcal{L}\mathcal{M}\mathcal{L}'\mathcal{M}'} \langle \phi(\mathcal{L}\mathcal{M}s) | O | \phi'(\mathcal{L}'\mathcal{M}'s') \rangle Y_{\mathcal{L}\mathcal{M}}(\hat{s}) Y^{*}_{\mathcal{L}'\mathcal{M}'}(\hat{s}')$$

$$= 1/(ss') \sum_{\mathcal{L}\mathcal{M}} (-)^{\mathcal{L}+\mathcal{M}} \begin{pmatrix} \mathcal{L} & 0 & \mathcal{L} \\ -\mathcal{M} & 0 & \mathcal{M} \end{pmatrix} Y_{\mathcal{L}\mathcal{M}}(\hat{s}) Y_{\mathcal{L}-\mathcal{M}}(\hat{s}') (-)^{\mathcal{M}}$$

$$* \langle \phi(\mathcal{L}s) \| O \| \phi'(\mathcal{L}s) \rangle \tag{2.11}$$

$$= 1/(ss') \sum_{\mathcal{L}} B(\mathcal{L}\mathcal{L}00\hat{s}\hat{s}') \langle \phi(\mathcal{L}s) \| O \| \phi'(\mathcal{L}s') \rangle$$

The functions

$$B(l_1 l_2 LM \hat{s}\hat{s}') = \sum_{m_1 m_2} (l_1 m_1, l_2 m_2 | LM) Y_{l_1 m_1}(\hat{s}) Y_{l_2 m_2}(\hat{s}')$$

are bipolar spherical harmonics[11]. The reduced matrix elements $\langle \phi(\mathcal{L}s) || O || \phi'(\mathcal{L}s') \rangle$ are defined in accordance with ref. 10).

The derivation of eq. (2.11) rests on the fact that $\phi(\mathcal{L}\mathcal{M}s)$ is a spherical tensor of rank $\mathcal{L}$ as expressed by eq. (2.5). We conclude that the matrix element $\langle \phi(\underset{\sim}{s}) | O | \phi'(\underset{\sim}{s}') \rangle$ is a scalar in the space of the generator coordinates $\underset{\sim}{s}$ and $\underset{\sim}{s}'$.

(ii) the ket vector carries an intrinsic orbital angular momentum:

$$\langle \Phi(\underset{\sim}{s}) | O | \Phi(\ell' m' \underset{\sim}{s}') \rangle$$

$$= 1/(ss') \sum_{\mathcal{L}\mathcal{M}L'M'} \langle \Phi(\mathcal{L}\mathcal{M}s) | O | \Phi(L'M'\ell'm's') \rangle Y_{\mathcal{L}\mathcal{M}}(\hat{s}) Y^{*}_{L'M'}(\hat{s}')$$

$$= 1/(ss') \sum_{\mathcal{L}\mathcal{M}L'M'\mathcal{L}'\mathcal{M}'} \langle \Phi(\mathcal{L}\mathcal{M}s) | O | \Phi(L'\ell'\mathcal{L}'\mathcal{M}'s') \rangle$$

$$* Y_{\mathcal{L}\mathcal{M}}(\hat{s}) Y^{*}_{L'M'}(\hat{s}') \begin{pmatrix} L' & \ell' & \mathcal{L}' \\ M' & m'-\mathcal{M}' \end{pmatrix} \hat{\mathcal{L}}' (-)^{L'+\ell'+\mathcal{M}'} \qquad (2.12)$$

From the discussion in conjunction with eq. (2.6) it is clear that only terms with $\mathcal{L} = \mathcal{L}'$ and $\mathcal{M} = \mathcal{M}'$ contribute to the sum in eq. (2.12). Thus

$$\langle \Phi(\underset{\sim}{s}) | O | \Phi(\ell' m' \underset{\sim}{s}') \rangle$$

$$= 1/(ss') \sum_{\mathcal{L}\mathcal{M}L'M'} (-)^{\mathcal{L}+\mathcal{M}} \begin{pmatrix} \mathcal{L} & 0 & \mathcal{L} \\ -\mathcal{M} & 0 & \mathcal{M} \end{pmatrix}$$

$$* \langle \Phi(\mathcal{L}s) \| O \| \Phi(L'\ell'\mathcal{L}s') \rangle Y_{\mathcal{L}\mathcal{M}}(\hat{s}) Y_{L'M'}(\hat{s}')$$

$$* \begin{pmatrix} \mathcal{L} & L' & \ell' \\ \mathcal{M} & M' & -m' \end{pmatrix} \hat{\mathcal{L}} (-)^{\mathcal{L}+\mathcal{M}+M'}$$

$$= 1/(ss') \sum_{\mathcal{L}L'} \langle \Phi(\mathcal{L}s) \| O \| \Phi(L'\ell'\mathcal{L}s) \rangle$$

$$* (-)^{L'} \hat{\ell}'^{-1} B(\mathcal{L}L'\ell'm'\hat{s}\hat{s}') \qquad (2.13)$$

where $\hat{\ell}' = \sqrt{2\ell'+1}$.

Equation (2.13) expresses the fact that the matrix elements $\langle \phi(\underset{\sim}{s}) | O | \phi(\ell' m' s') \rangle$ form a spherical tensor of rank ℓ' in the space of the generator coordinates $\underset{\sim}{s}$ and $\underset{\sim}{s}'$.

This implies that the dependence of the unprojected matrix elements on the magnetic quantum number m' is given by

$$\langle \phi(\underset{\sim}{s}) | O | \phi(l'm'\underset{\sim}{s}') \rangle$$

$$= \mathcal{Y}_{l'm'}(\quad) \times \text{(scalar function of } \underset{\sim}{s} \text{ and } \underset{\sim}{s}', \text{ independent of } m') \tag{2.14}$$

where the argument of the solid spherical harmonic function $\mathcal{Y}_{l'm'}$ is either $\underset{\sim}{s}$ or $\underset{\sim}{s}'$.

(iii) both the ket and the bra vector carry an intrinsic orbital angular momentum:

$$\langle \phi(lm\underset{\sim}{s}) | O | \phi'(l'm'\underset{\sim}{s}') \rangle$$

$$= 1/(ss') \sum_{LML'M'\mathcal{L}\mathcal{M}\mathcal{L}'\mathcal{M}'} \langle \phi(Ll\mathcal{L}\mathcal{M}s) | O | \phi'(L'l'\mathcal{L}'\mathcal{M}'s) \rangle$$

$$* (LM, lm | \mathcal{L}\mathcal{M})(L'M', l'm' | \mathcal{L}'\mathcal{M}') \, Y_{LM}(\hat{s}) \, Y^{*}_{L'M'}(\hat{s}')$$

$$= 1/(ss') \sum_{LML'M'\mathcal{L}\mathcal{M}\mathcal{L}'\mathcal{M}'} \langle \phi(Ll\mathcal{L}s) \| O \| \phi'(L'l'\mathcal{L}'s) \rangle \hat{\mathcal{L}}$$

$$* \begin{pmatrix} L & l & \mathcal{L} \\ M & m & -\mathcal{M} \end{pmatrix} \begin{pmatrix} l' & L' & \mathcal{L} \\ -m' & M' & \mathcal{M} \end{pmatrix} (-)^{M'} \, Y_{LM}(\hat{s}) \, Y_{L'M'}(\hat{s}') \tag{2.15}$$

Parity conservation (eq. (2.8 b)) restricts the summation over L and L' to $L + L' + l + l' =$ even.

The tensor property (2.5) of the many body wave functions restricts the summation to $\mathcal{L} = \mathcal{L}'$ and $\mathcal{M} = \mathcal{M}'$. Upon recoupling the angular momenta, this equation may be expressed in terms of 6-j symbols [11]

$$
\begin{aligned}
&\langle \phi(lm\underset{\sim}{s})|O|\phi'(l'm'\underset{\sim}{s}')\rangle \\
&= 1/(ss')\sum_{LML'M'\mathcal{L}\mathcal{M}\mathcal{L}'\mathcal{M}'} \langle \phi(Ll\mathcal{L}s)\|O\|\phi'(L'l'\mathcal{L}s')\rangle\,\hat{\mathcal{L}}\,\hat{\mathcal{L}}'^{2} \\
&\quad * \begin{Bmatrix} L & l & \mathcal{L} \\ l' & L' & \mathcal{L}' \end{Bmatrix} \begin{pmatrix} L & L' & \mathcal{L}' \\ M & M' & -\mathcal{M}' \end{pmatrix} (-)^{\mathcal{L}+L+L'+m} \\
&\quad * Y_{LM}(\hat{s})\,Y_{L'M'}(\hat{s}')\begin{pmatrix} l & l' & \mathcal{L}' \\ m & -m' & \mathcal{M}' \end{pmatrix} \\
&= 1/(ss')\sum_{LL'\mathcal{L}\mathcal{L}'\mathcal{M}'} \langle \phi(Ll\mathcal{L}s)\|O\|\phi'(L'l'\mathcal{L}s')\rangle\,\hat{\mathcal{L}} \\
&\quad * \begin{Bmatrix} L & l & \mathcal{L} \\ l' & L' & \mathcal{L}' \end{Bmatrix} (-)^{\mathcal{L}+\mathcal{L}'+m} \quad B(LL'\mathcal{L}'\mathcal{M}'\hat{s}\hat{s}') \\
&\quad * (l-m,\, l'm'|\mathcal{L}'\mathcal{M}')
\end{aligned}
$$

Using the orthogonality of the CG-coefficients, we obtain from this equation the result

$$
\begin{aligned}
&\sum_{mm'} \langle \phi(lm\underset{\sim}{s})|O|\phi'(l'm'\underset{\sim}{s}')\rangle (-)^{m} (l-m,\, l'm'|\mathcal{L}'\mathcal{M}') \\
&= 1/(ss')\sum_{LL'\mathcal{L}} \langle \phi(Ll\mathcal{L}s)\|O\|\phi'(L'l'\mathcal{L}s')\rangle\,\hat{\mathcal{L}} \\
&\quad * \begin{Bmatrix} L & l & \mathcal{L} \\ l' & L' & \mathcal{L}' \end{Bmatrix} (-)^{\mathcal{L}+\mathcal{L}'} \; B(LL'\mathcal{L}'\mathcal{M}'\hat{s}\hat{s}')
\end{aligned}
\qquad (2.16)
$$

where $\mathcal{M}' = -m+m'$.

This equation shows that the expression on the lefthand side is the $\mathcal{M}'$-th component of a spherical tensor of rank $\mathcal{L}'$ in the space of the generator coordinates $\underset{\sim}{s}$ and $\underset{\sim}{s}'$. Consequently, there are five different ways in which the unprojected matrix elements may depend on the magnetic quantum numbers m and m':

(a) $\langle \phi(\ell m \underset{\sim}{s}) | O | \phi'(\ell' m' \underset{\sim}{s}') \rangle$

$= \delta_{mm'} \delta_{\ell\ell'} \times$ (scalar function of $\underset{\sim}{s}$ and $\underset{\sim}{s}'$, independent of m and m') (2.17a)

(b) $\langle \phi(\ell m \underset{\sim}{s}) | O | \phi'(\ell' m' \underset{\sim}{s}') \rangle$

$= \mathcal{Y}^{*}_{\ell m}(\;) \mathcal{Y}_{\ell' m'}(\;) \times$ (scalar function of $\underset{\sim}{s}$ and $\underset{\sim}{s}'$, independent of m and m')

where the argument of the solid spherical harmonics is either $\underset{\sim}{s}$ or $\underset{\sim}{s}'$.

3. CALCULATION OF REDUCED MATRIX ELEMENTS OF A SCALAR SPIN-ISOSPIN INDEPENDENT OPERATOR O BETWEEN TWO-CENTER SM FUNCTIONS WITH INTRINSIC ORBITAL ANGULAR MOMENTUM

The development in the remaining sections of this paper is tailored to the treatment of a simplified version of a two-cluster model[3] of ^{6}Li. In our calculation[9,12], the wave function of ^{6}Li is taken as a linear superposition of the clusterizations ^{4}He-^{2}H and ^{5}He-p. The nuclei ^{4}He, ^{2}H and ^{5}He are described by their lowest shell model configurations $(1s)^4$, $(1s)^2$ and $(1s)^4(1p)$ centered at the positions of the respective potential wells. Consequently, the cluster (^{5}He) carries a nonvanishing intrinsic orbital angular momentum (ℓ=1).

The general idea of calculating reduced matrix elements or - alternatively - carry out angular momentum projection is to parametrize the unprojected matrix elements in accordance with their tensor properties studied in the last section and to expand them in terms of spherical tensors. In this paper we are concerned only with spin independent interactions. This leads to the simplification that we have to consider only spherical tensors in the spatial coordinates.

3.1 Reduced Matrix Elements $\langle \ell=0||O||\ell'=0\rangle$

This case has been treated earlier[8], and we include it for completeness only. The unprojected overlap may be parametrized in the form

$$\langle\phi(\underset{\sim}{s})|O|\phi'(\underset{\sim}{s}')\rangle = \sum_i v^{(i)} \exp\left[-\beta\left(u^{(i)}s^2+u'^{(i)}s'^2\right)+\beta w^{(i)}\underset{\sim}{s}\cdot\underset{\sim}{s}'\right] \tag{3.1a}$$

while the reduced matrix element is given by

$$\begin{aligned}&\langle\phi(Ls)||O||\phi'(L's')\rangle\\ &=\delta_{LL'}\hat{L}\sum_i v^{(i)}\exp\left[-\beta\left(u^{(i)}s^2+u'^{(i)}s'^2\right)\right]\\ &\quad * 4\pi ss'\, i_L(\beta w^{(i)}ss')\end{aligned} \tag{3.1b}$$

The operator O stands for the identity and the scalar spin-isospin independent part of the two-nucleon interaction. Of course, the corresponding coefficients $v^{(i)}$, $u^{(i)}$ etc. are different for the two operators. Matrix elements of the kinetic energy operator, however, may be expressed in terms of the coefficients belonging to the identity operator[8]

$$\begin{aligned}&\langle\phi(Ls)||T-T_{CM}||\phi'(L's')\rangle\\ &=\delta_{LL'}\hat{L}\left(\frac{\hbar^2\beta}{4m}\right)\sum_i v^{(i)}\exp\left[-\beta\left(u^{(i)}s^2+u'^{(i)}s'^2\right)\right]\\ &\quad * 4\pi ss'\left\{\left[3(N-1)+2L-2\beta\left(u^{(i)}s^2+u'^{(i)}s'^2\right)\right]\right.\\ &\quad \left. * i_L(\beta w^{(i)}ss')+2\beta w^{(i)}ss'\, i_{L+1}(\beta w^{(i)}ss')\right\}\end{aligned} \tag{3.2}$$

3.2 Reduced Matrix Elements $\langle \ell = 0 || O || \ell' = 1 \rangle$

In terms of our model of ^{6}Li, this means that we want to calculate matrix elements of the type $\langle \alpha\text{-d}, \underset{\sim}{s} | O | {}^5\text{He-p}, 1m'\underset{\sim}{s}' \rangle$ where $\underset{\sim}{s} = \underset{\sim}{S}_\alpha - \underset{\sim}{S}_d$ and $\underset{\sim}{s}' = \underset{\sim}{S}_{^5\text{He}} - \underset{\sim}{S}_p$. The magnetic quantum number m' refers to the p-wave of the neutron in ^{5}He. In ref. 12) it is shown that the unprojected overlap may be parametrized in the following way

$$\langle \phi(\underset{\sim}{s}) | O | \phi'(1m'\underset{\sim}{s}') \rangle$$
$$= \sum_i \{ v_1^{(i)} \mathcal{Y}_{1m'}(\sqrt{\beta}\underset{\sim}{s}') \exp[-\beta(u_1^{(i)} s^2 + u_1'^{(i)} s'^2) + \beta w_1^{(i)} \underset{\sim}{s}\cdot\underset{\sim}{s}']$$
$$+ v_2^{(i)} \mathcal{Y}_{1m'}(\sqrt{\beta}\underset{\sim}{s}) \exp[-\beta(u_2^{(i)} s^2 + u_2'^{(i)} s'^2) + \beta w_2^{(i)} \underset{\sim}{s}\cdot\underset{\sim}{s}'] \} \tag{3.3}$$

where again, O is the unit operator or the central part of the two-body interaction. We notice that the parametrization (3.3) is in agreement with eq. (2.14). Let us consider the two terms in eq. (3.3) separately

(i) $\langle \phi(\underset{\sim}{s}) | O | \phi(1m'\underset{\sim}{s}') \rangle_1$

$$= \sum_i v_1^{(i)} \mathcal{Y}_{1m'}(\sqrt{\beta}\underset{\sim}{s}') \exp[-\beta(u_1^{(i)} s^2 + u_1'^{(i)} s'^2) + \beta w_1^{(i)} \underset{\sim}{s}\cdot\underset{\sim}{s}']$$
$$= \sum_i v_1^{(i)} \sqrt{\beta}\, s' Y_{1m'}(\hat{s}') \exp[-\beta(u_1^{(i)} s^2 + u_1'^{(i)} s'^2)]$$
$$* 4\pi \sum_{\mathcal{L}\mathcal{M}} i_{\mathcal{L}}(\beta w_1^{(i)} s s') Y_{\mathcal{L}\mathcal{M}}(\hat{s}) Y^*_{\mathcal{L}\mathcal{M}}(\hat{s}')$$
$$= \sum_i v_1^{(i)} \sqrt{\beta}\, s' \exp[-\beta(u_1^{(i)} s^2 + u_1'^{(i)} s'^2)] \sqrt{4\pi} \sum_{\mathcal{L}\mathcal{M}L'M'} i_{\mathcal{L}}(\beta w_1^{(i)} s s')$$
$$* (-)^{m'} Y_{\mathcal{L}\mathcal{M}}(\hat{s}) Y_{L'M'}(\hat{s}')\, \hat{1}\hat{\mathcal{L}}\hat{L}' \begin{pmatrix} 1 & \mathcal{L} & L' \\ -m' & \mathcal{M} & M' \end{pmatrix} \begin{pmatrix} 1 & \mathcal{L} & L' \\ 0 & 0 & 0 \end{pmatrix}$$
$$= -\sum_i v_1^{(i)} \sqrt{\beta}\, s' \exp[-\beta(u_1^{(i)} s^2 + u_1'^{(i)} s'^2)] \sqrt{4\pi}$$
$$* \sum_{\mathcal{L}L'} i_{\mathcal{L}}(\beta w_1^{(i)} s s') B(\mathcal{L}L'1m'\hat{s}\hat{s}')\, \hat{\mathcal{L}}\hat{L}' \begin{pmatrix} \mathcal{L} & L' & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.4}$$

Upon comparing eqs. (2.13) and (3.4) we have

$$\langle \phi(\mathcal{L}s) \| O \| \phi(L'1\mathcal{L}s') \rangle_1$$
$$= (ss')\sqrt{\beta}s' \sum_i v_1^{(i)} \exp\left[-\beta\left(u_1^{(i)}s^2 + u_1'^{(i)}s'^2\right)\right]\sqrt{4\pi}$$
$$* i_{\mathcal{L}}(\beta w_1^{(i)}ss')(-)^{\mathcal{L}}\,\hat{1}\,\hat{L}'\hat{\mathcal{L}}\begin{pmatrix}\mathcal{L} & L' & 1\\ 0 & 0 & 0\end{pmatrix}$$

(3.5)

Parity conservation is contained in the 3-j symbol $\begin{pmatrix}L' & 1 & \mathcal{L}\\ 0 & 0 & 0\end{pmatrix}$ which is zero unless $\mathcal{L}$ + L' + 1 = even, or

$$(-)^{\mathcal{L}} = -(-)^{L'} \quad (3.6)$$

(ii) $\langle \phi(\underset{\sim}{s}) | O | \phi(1m'\underset{\sim}{s}') \rangle_2$

$$= \sum_i v_2^{(i)} \mathcal{Y}_{1m'}(\sqrt{\beta}\underset{\sim}{s})\exp\left[-\beta\left(u_2^{(i)}s^2 + u_2'^{(i)}s'^2\right) + \beta w_2^{(i)}\underset{\sim}{s}\cdot\underset{\sim}{s}'\right]$$
$$= \sqrt{\beta}s \sum_i v_2^{(i)} Y_{1m'}(\hat{s})\exp\left[-\beta\left(u_2^{(i)}s^2 + u_2'^{(i)}s'^2\right)\right]$$
$$* 4\pi \sum_{L'M'} i_{L'}(\beta w_2^{(i)}ss') Y^*_{L'M'}(\hat{s}) Y_{L'M'}(\hat{s}')$$
$$= \sqrt{\beta}s \sum_i v_2^{(i)} \exp\left[-\beta\left(u_2^{(i)}s^2 + u_2'^{(i)}s'^2\right)\right]\sqrt{4\pi}$$
$$* \sum_{L'M'\mathcal{L}\mathcal{M}} i_{L'}(\beta w_2^{(i)}ss')(-)^{m'} Y_{\mathcal{L}\mathcal{M}}(\hat{s}) Y_{L'M'}(s')$$
$$* \hat{1}\,\hat{L}'\hat{\mathcal{L}}\begin{pmatrix}1 & L' & \mathcal{L}\\ -m' & M' & \mathcal{M}\end{pmatrix}\begin{pmatrix}1 & L' & \mathcal{L}\\ 0 & 0 & 0\end{pmatrix}$$
$$= -\sqrt{\beta}s \sum_i v_2^{(i)} \exp\left[-\beta\left(u_2^{(i)}s^2 + u_2'^{(i)}s'^2\right)\right]\sqrt{4\pi}$$
$$* \sum_{L'\mathcal{L}} i_{L'}(\beta w_2^{(i)}ss')\,\hat{L}'\hat{\mathcal{L}}\;B(\mathcal{L}L'1m'ss')\begin{pmatrix}\mathcal{L} & L' & 1\\ 0 & 1 & 0\end{pmatrix}$$

(3.7)

Equating the coefficients of $B(\mathcal{L}L'1m'\hat{s}\hat{s}')$ in eqs. (2.13) and (3.7), we obtain

$$\langle\phi(\mathcal{L}s)\|O\|\phi(L'1\mathcal{L}s')\rangle_2$$
$$= (ss')\sqrt{\beta}\, s\sum_i v_2^{(i)}\exp[-\beta(u_2^{(i)}s^2+u_2^{\prime(i)}s'^2)]\sqrt{4\pi}$$
$$* i_{L'}(\beta w_2^{(i)}ss')(-)^{\mathcal{L}}\,\hat{1}\,\hat{L}'\hat{\mathcal{L}}\begin{pmatrix}\mathcal{L} & L' & 1\\ 0 & 0 & 0\end{pmatrix} \qquad (3.8)$$

Parity conservation is expressed by the same relation (3.6) as in the previous case. Adding eqs. (3.5) and (3.8), we obtain the final expression of the reduced matrix element of the operator O.

$$\langle\phi(\mathcal{L}s)\|O\|\phi(L'1\mathcal{L}s')\rangle$$
$$= \Big\{\sqrt{\beta}\, s'\sum_i v_1^{(i)}\exp[-\beta(u_1^{(i)}s^2+u_1^{\prime(i)}s'^2)]\, i_{\mathcal{L}}(\beta w_1^{(i)}ss')$$
$$+\sqrt{\beta}\, s\sum_i v_2^{(i)}\exp[-\beta(u_2^{(i)}s^2+u_2^{\prime(i)}s'^2)]\, i_{L'}(\beta w_2^{(i)}ss')\Big\}$$
$$*(ss')\sqrt{4\pi}\,(-)^{\mathcal{L}}\,\hat{1}\,\hat{L}'\hat{\mathcal{L}}\begin{pmatrix}\mathcal{L} & L' & 1\\ 0 & 0 & 0\end{pmatrix}$$
$$(3.9)$$

In order to calculate reduced matrix elements of the kinetic energy operator we use the formula [12])

$$\langle\phi|T-T_{CM}|\phi'\rangle$$
$$=\Big\{\left(\frac{\hbar^2\beta}{4m}\right)[3(N-1)+n]+\left(\frac{\hbar^2\beta^2}{2m}\right)\frac{d}{d\beta}\Big\}\langle\phi|\phi'\rangle \qquad (3.10)$$

where n is the total number of p-wave single particle orbits in ϕ and ϕ'. The Slater determinant wave functions ϕ and ϕ' are restricted to those which are composed of harmonic oscillator single particle wave functions with the same oscillator parameter β. Combining eqs. (3.9) and (3.10), we find

$$
\begin{aligned}
&\langle \Phi(\ell s) \,\|\, T - T_{CM} \,\|\, \Phi(L' 1 \ell s') \rangle \\
&= (\hbar^2 \beta / m) \Big(\sqrt{\beta}\, s' \sum_i v_1^{(i)} \exp\left[-\beta\left(u_1^{(i)} s^2 + u_1'^{(i)} s'^2\right)\right] \\
&\quad * \Big\{ \left[17/4 + \ell/2 - \beta/2 \left(u_1^{(i)} s^2 + u_1'^{(i)} s'^2\right)\right] i_\ell\left(\beta w_1^{(i)} s s'\right) \\
&\qquad + 1/2\, w_1^{(i)} \beta s s'\, i_{\ell+1}\left(\beta w_1^{(i)} s s'\right) \Big\} \\
&\qquad + \sqrt{\beta}\, s \sum_i v_2^{(i)} \exp\left[-\beta\left(u_2^{(i)} s^2 + u_2'^{(i)} s'^2\right)\right] \\
&\quad * \Big\{ \left[17/4 + L'/2 - \beta/2 \left(u_2^{(i)} s^2 + u_2'^{(i)} s'^2\right)\right] i_{L'}\left(\beta w_2^{(i)} s s'\right) \\
&\qquad + 1/2\, w_2^{(i)} \beta s s'\, i_{L'+1}\left(\beta w_2^{(i)} s s'\right) \Big\} \Big) \\
&\qquad * (s s') \sqrt{4\pi}\, (-)^\ell\, \hat{1}\, \hat{L}'\, \hat{\ell} \begin{pmatrix} \ell & L' & 1 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{3.11}
$$

where the coefficients $u^{(i)}$, $v^{(i)}$ etc. refer to the normlization overlap.

3.3 Reduced Matrix Elements $<\ell=1||O||\ell'=1>$

In this subsection, we shall carry out angular momentum projection in the matrix elements $<{}^5He\text{-}p,\ 1m\underset{\sim}{s}|O|{}^5He\text{-}p,\ 1m'\underset{\sim}{s}>$ where, again, m and m' refer to the p wave of the neutron in ^{5}He. We use the parametrization derived in ref. 12) which is, of course, in agreement with the tensor properties of these matrix elements expressed by eqs. (2.17).

$$\langle \phi(1m\underset{\sim}{s})|O|\phi'(1m'\underset{\sim}{s}')\rangle$$
$$= \delta_{mm'} \sum_i v_1^{(i)} \exp[-\beta(u_1^{(i)}s^2 + u_1'^{(i)}s'^2) + \beta w_1^{(i)} \underset{\sim}{s}\cdot\underset{\sim}{s}']$$
$$+ Y_{1m}^*(\sqrt{\beta}\underset{\sim}{s}) Y_{1m'}(\sqrt{\beta}\underset{\sim}{s}') \sum_i v_2^{(i)} \exp[-\beta(u_2^{(i)}s^2 + u_2'^{(i)}s'^2) + \beta w_2^{(i)} \underset{\sim}{s}\cdot\underset{\sim}{s}']$$
$$+ Y_{1m}^*(\sqrt{\beta}\underset{\sim}{s}') Y_{1m'}(\sqrt{\beta}\underset{\sim}{s}) \sum_i v_3^{(i)} \exp[-\beta(u_3^{(i)}s^2 + u_3'^{(i)}s'^2) + \beta w_3^{(i)} \underset{\sim}{s}\cdot\underset{\sim}{s}']$$
$$+ Y_{1m}^*(\sqrt{\beta}\underset{\sim}{s}) Y_{1m'}(\sqrt{\beta}\underset{\sim}{s}) \sum_i v_4^{(i)} \exp[-\beta(u_4^{(i)}s^2 + u_4'^{(i)}s'^2) + \beta w_4^{(i)} \underset{\sim}{s}\cdot\underset{\sim}{s}']$$
$$+ Y_{1m}^*(\sqrt{\beta}\underset{\sim}{s}') Y_{1m'}(\sqrt{\beta}\underset{\sim}{s}') \sum_i v_5^{(i)} \exp[-\beta(u_5^{(i)}s^2 + u_5'^{(i)}s'^2) + \beta w_5^{(i)} \underset{\sim}{s}\cdot\underset{\sim}{s}'] \qquad (3.12)$$

As in the previous case, we proceed by calculating the reduced matrix elements separately for the five different terms in eq. (3.12), starting from eq. (2.16).

(i) $$\sum_{mm'} \langle \phi(1m\underset{\sim}{s})|O|\phi'(1m'\underset{\sim}{s}')\rangle_1 (-)^m (1-m, 1m'|\mathcal{L}'\mathcal{M}')$$
$$= \sum_i v_1^{(i)} \exp[-\beta(u_1^{(i)}s^2 + u_1'^{(i)}s'^2) + \beta w_1^{(i)} \underset{\sim}{s}\cdot\underset{\sim}{s}']$$
$$* \sum_m (-)^m (1-m, 1m|\mathcal{L}'0)$$
$$= -\delta_{\mathcal{L}'0}\, \hat{1} \sum_i v_1^{(i)} \exp[-\beta(u_1^{(i)}s^2 + u_1'^{(i)}s'^2)]$$
$$* 4\pi \sum_{LM} i_L(\beta w_1^{(i)} s s') Y_{LM}(\hat{s}) Y_{L-M}(\hat{s})(-)^M$$
$$= \delta_{\mathcal{L}'0}\, \hat{1} \sum_i v_1^{(i)} \exp[-\beta(u_1^{(i)}s^2 + u_1'^{(i)}s'^2)]$$
$$* 4\pi \sum_L i_L(\beta w_1^{(i)} s s')\, B(LL00\hat{s}\hat{s}')(-)^{L+1}\hat{L}$$

(3.13)

A comparison with eq. (2.15) shows that the only nonvanishing reduced matrix elements are those with L=L' and $\mathcal{L}'$=0, i.e.

$$1/(ss') \sum_{\mathcal{L}} \langle \phi(L1\mathcal{L}s)\|O\|\phi'(L1\mathcal{L}s')\rangle_1 \hat{\mathcal{L}} \begin{Bmatrix} L & 1 & \mathcal{L} \\ 1 & L & 0 \end{Bmatrix} (-)^{\mathcal{L}}$$
$$= \hat{1} \sum_i v_1^{(i)} \exp[-\beta(u_1^{(i)}s^2 + u_1'^{(i)}s'^2)]\, 4\pi i_L(\beta w_1^{(i)} s s')(-)^{L+1}\hat{L}$$

From the orthogonality relation[10)] of the 6-j symbols

$$\sum_e (2e+1)(2f+1)\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}\begin{Bmatrix} a & b & e \\ d & c & g \end{Bmatrix} = \delta_{fg}$$

it follows that $<\phi(L1\mathcal{L}s)||O||\phi'(L1\mathcal{L}\,s')>(-)^{\mathcal{L}}/\hat{\mathcal{L}}$ is proportional to the 6-j symbol $\begin{Bmatrix} L1\mathcal{L} \\ 1L0 \end{Bmatrix} = (-)^{L+1+\mathcal{L}}/(\hat{1}\hat{L})$.

Therefore

$$\begin{aligned} &\langle\phi(L1\mathcal{L}s)\|O\|\phi'(L'1\mathcal{L}s')\rangle_1 \\ &= (ss')\sum_i v_1^{(i)}\exp\left[-\beta\left(u_1^{(i)}s^2+u_1'^{(i)}s'^2\right)\right] \\ &\quad * 4\pi\, i_L(\beta w_1^{(i)}ss')\;\delta_{LL'}\,\hat{\mathcal{L}} \end{aligned} \tag{3.14}$$

(ii) $$\sum_{mm'}\langle\phi(1m\underset{\sim}{s})|O|\phi'(1m'\underset{\sim}{s}')\rangle_2(-)^m(1-m,1m'|\mathcal{L}'\mathcal{M}')$$

$$\begin{aligned} &= \beta ss'\sum_i v_2^{(i)}\exp\left[-\beta\left(u_2^{(i)}s^2+u_2'^{(i)}s'^2\right)+\beta w_2^{(i)}\underset{\sim}{s}\cdot\underset{\sim}{s}'\right] \\ &\quad * \sum_{mm'} Y_{1m}^*(\hat{s})\,Y_{1m'}(\hat{s}')\,(-)^m(1-m,1m'|\mathcal{L}'\mathcal{M}') \\ &= \beta ss'\sum_i v_2^{(i)}\exp\left[-\beta\left(u_2^{(i)}s^2+u_2'^{(i)}s'^2\right)\right] \\ &\quad * 4\pi\sum_{mm'\mathcal{L}\mathcal{M}} i_{\mathcal{L}}(\beta w_2^{(i)}ss')\,Y_{1m}^*(\hat{s})\,Y_{\mathcal{L}\mathcal{M}}^*(\hat{s})\,Y_{1m'}(\hat{s}')\,Y_{\mathcal{L}\mathcal{M}}(\hat{s}') \\ &\quad * (-)^m(1-m,1m'|\mathcal{L}'\mathcal{M}') \\ &= \beta ss'\sum_i v_2^{(i)}\exp\left[-\beta\left(u_2^{(i)}s^2+u_2'^{(i)}s'^2\right)\right] \\ &\quad * \sum_{\mathcal{L}LL'} i_{\mathcal{L}}(\beta w_2^{(i)}ss')\,\hat{1}^2\hat{\mathcal{L}}^2\hat{L}\hat{L}'\begin{pmatrix} 1 & \mathcal{L} & L \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & \mathcal{L} & L' \\ 0 & 0 & 0 \end{pmatrix} \\ &\quad * B(LL'\mathcal{L}'\mathcal{M}'\hat{s}\hat{s}')\,(-)^{\mathcal{L}+\mathcal{L}'}\begin{Bmatrix} L & 1 & \mathcal{L} \\ 1 & L' & \mathcal{L}' \end{Bmatrix} \end{aligned} \tag{3.15}$$

where in the last step we used eq. (B.1).

If we compare this expression with eq. (2.16) we obtain

$$\langle \phi(L1\mathcal{L}s) \| O \| \phi'(L'1\mathcal{L}s') \rangle_2$$
$$= \beta (ss')^2 \sum_i v_2^{(i)} \exp\left[-\beta\left(u_2^{(i)} s^2 + u_2'^{(i)} s'^2\right)\right] i_{\mathcal{L}}\left(\beta w_2^{(i)} ss'\right)$$
$$* \hat{1}^2 \hat{L} \hat{L}' \hat{\mathcal{L}} \begin{pmatrix} L & 1 & \mathcal{L} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L' & 1 & \mathcal{L} \\ 0 & 0 & 0 \end{pmatrix} \qquad (3.16)$$

(iii) $\langle \phi(1m\underset{\sim}{s}) | O | \phi'(1m'\underset{\sim}{s}') \rangle_3$

$$= \beta ss' \sum_i v_3^{(i)} \exp\left[-\beta\left(u_3^{(i)} s^2 + u_3'^{(i)} s'^2\right) + \beta w_3^{(i)} \underset{\sim}{s} \cdot \underset{\sim}{s}'\right] Y_{1m}^*(\hat{s}') Y_{1m'}(\hat{s})$$

Upon inserting this expression into eq. (2.15) we have

$$\sum_{LML'M'\mathcal{L}\mathcal{M}} \langle \phi(L1\mathcal{L}s) \| O \| \phi'(L'1\mathcal{L}s') \rangle_3 \hat{\mathcal{L}}$$
$$* \begin{pmatrix} L & 1 & \mathcal{L} \\ M & m & -\mathcal{M} \end{pmatrix} \begin{pmatrix} L' & 1 & \mathcal{L} \\ M' & m' & -\mathcal{M} \end{pmatrix} Y_{LM}(\hat{s}) Y_{L'M'}^*(\hat{s}')$$
$$= \beta (ss')^2 \sum_i v_3^{(i)} \exp\left[-\beta\left(u_3^{(i)} s^2 + u_3'^{(i)} s'^2\right)\right] 4\pi$$
$$* \sum_{\lambda\mu} i_\lambda\left(\beta w_3^{(i)} ss'\right) Y_{1m'}(\hat{s}) Y_{\lambda\mu}(\hat{s}) Y_{1m}^*(\hat{s}') Y_{\lambda\mu}^*(\hat{s}')$$

$$= \sum_{LML'M'\mathcal{L}\mathcal{M}\lambda} \beta (ss')^2 \sum_i v_3^{(i)} \exp\left[-\beta\left(u_3^{(i)} s^2 + u_3'^{(i)} s'^2\right)\right]$$
$$* i_\lambda\left(\beta w_3^{(i)} ss'\right) \hat{1}^2 \hat{\lambda}^2 \hat{L} \hat{L}' \hat{\mathcal{L}}^2 \begin{Bmatrix} 1 & L' & \lambda \\ 1 & L & \mathcal{L} \end{Bmatrix} \begin{pmatrix} 1 & \lambda & L \\ 0 & 0 & 0 \end{pmatrix}$$
$$* \begin{pmatrix} 1 & \lambda & L' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & 1 & \mathcal{L} \\ M & m & -\mathcal{M} \end{pmatrix} \begin{pmatrix} L' & 1 & \mathcal{L} \\ M' & m' & -\mathcal{M} \end{pmatrix} Y_{LM}(\hat{s}) Y_{L'M'}^*(\hat{s}')$$

where in the last step we used eq. (B.2). Thus

$$\langle \Phi(L1\mathcal{L}s) \| O \| \Phi'(L'1\mathcal{L}s') \rangle_3$$
$$= \beta(ss')^2 \sum_i v_3^{(i)} \exp[-\beta(u_3^{(i)} s^2 + u_3'^{(i)} s'^2)] \sum_\lambda i_\lambda(\beta w_3^{(i)} ss')$$
$$* \hat{1}^2 \hat{\lambda}^2 \hat{L} \hat{L}' \hat{\mathcal{L}} \begin{Bmatrix} 1 & L' & \lambda \\ 1 & L & \mathcal{L} \end{Bmatrix} \begin{pmatrix} 1 & \lambda & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \lambda & L' \\ 0 & 0 & 0 \end{pmatrix} \qquad (3.17)$$

(iv) $$\sum_{mm'} \langle \Phi(1m\underset{\sim}{s}) | O | \Phi'(1m'\underset{\sim}{s}') \rangle_4 (-)^m (1-m, 1m' | \mathcal{L}'\mathcal{M}')$$

$$= \beta s^2 \sum_i v_4^{(i)} \exp[-\beta(u_4^{(i)} s^2 + u_4'^{(i)} s'^2)]$$
$$* 4\pi \overline{\sum_{L'M'mm'}} i_{L'}(\beta w_4^{(i)} ss')$$
$$* Y_{1m}^*(\hat{s})\, Y_{1m'}(\hat{s})\, Y_{L'M'}^*(\hat{s})\, Y_{L'M'}(\hat{s})$$
$$* \begin{pmatrix} 1 & 1 & \mathcal{L}' \\ -m & m' & -\mathcal{M}' \end{pmatrix} (-)^{\mathcal{M}'+m} \hat{\mathcal{L}}'$$

$$= \beta s^2 \sum_i v_4^{(i)} \exp[-\beta(u_4^{(i)} s^2 + u_4'^{(i)} s'^2)]$$
$$* \sum_{LL'} i_{L'}(\beta w_4^{(i)} ss') \hat{1}^2 \hat{L} \hat{L}' (-)^{\mathcal{L}'}$$
$$* \begin{pmatrix} 1 & 1 & \mathcal{L}' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L' & \mathcal{L}' & L \\ 0 & 0 & 0 \end{pmatrix} B(LL'\mathcal{L}'\mathcal{M}'\hat{s}\hat{s}') \qquad (3.18)$$

where eq. (B.3) was used. Equating the coefficients of $B(LL'\mathcal{L}'\mathcal{M}'\hat{s}\hat{s}')$ in eqs. (2.16) and (3.18), we obtain

$$\sum_{\mathcal{L}} \langle \phi(L1\mathcal{L}s) \| O \| \phi'(L'1\mathcal{L}s) \rangle_4 \hat{\mathcal{L}} (-)^{\mathcal{L}} \begin{Bmatrix} L & L' & \mathcal{L}' \\ 1 & 1 & \mathcal{L} \end{Bmatrix}$$
$$= (ss')\beta s^2 \sum_i v_4^{(i)} \exp[-\beta(u_4^{(i)}s^2 + u_4^{\prime(i)}s'^2)] \, i_{L'}(\beta w_4^{(i)}ss')$$
$$* \hat{1}^2 \hat{L} \hat{L}' \begin{pmatrix} 1 & 1 & \mathcal{L}' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L' & \mathcal{L}' & L \\ 0 & 0 & 0 \end{pmatrix}$$

Using the orthogonality of the 6-j symbols we find

$$\langle \phi(L1\mathcal{L}s) \| O \| \phi'(L'1\mathcal{L}s') \rangle_4$$
$$= (ss')\beta s^2 \sum_i v_4^{(i)} \exp[-\beta(u_4^{(i)}s^2 + u_4^{\prime(i)}s'^2)] \, i_{L'}(\beta w_4^{(i)}ss')$$
$$* (-)^{\mathcal{L}} \sum_{\mathcal{L}'} \hat{1}^2 \hat{L} \hat{L}' \hat{\mathcal{L}} \hat{\mathcal{L}}'^2 \begin{pmatrix} L' & \mathcal{L}' & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & \mathcal{L}' \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} L & L' & \mathcal{L}' \\ 1 & 1 & \mathcal{L} \end{Bmatrix}$$

$$= (ss')\beta s^2 \sum_i v_4^{(i)} \exp[-\beta(u_4^{(i)}s^2 + u_4^{\prime(i)}s'^2)]$$
$$* i_{L'}(\beta w_4^{(i)}ss') \, \hat{1}^2 \hat{L} \hat{L}' \hat{\mathcal{L}} \begin{pmatrix} L & 1 & \mathcal{L} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L' & 1 & \mathcal{L} \\ 0 & 0 & 0 \end{pmatrix} \quad (3.19)$$

(V) $$\sum_{mm'} \langle \phi(1m\underset{\sim}{s}) | O | \phi'(1m'\underset{\sim}{s}') \rangle_5 (-)^{m}$$
$$* (1-m, 1m' | \mathcal{L}'\mathcal{M}')$$
$$= \beta s'^2 \sum_i v_5^{(i)} \exp[-\beta(u_5^{(i)}s^2 + u_5^{\prime(i)}s'^2)] \, 4\pi$$
$$* \sum_{LMmm'} i_L(\beta w_5^{(i)}ss') \, Y_{LM}(\hat{s}) \, Y^*_{1m}(\hat{s}') \, Y_{1m'}(\hat{s}')$$
$$* Y^*_{LM}(\hat{s}') \begin{pmatrix} 1 & 1 & \mathcal{L}' \\ -m & m' & -\mathcal{M}' \end{pmatrix} (-)^{\mathcal{M}'+m} \hat{\mathcal{L}}'$$

$$= \beta s'^2 \sum_i v_5^{(i)} \exp[-\beta(u_5^{(i)} s^2 + u_5'^{(i)} s'^2)]\, i_L(\beta w_5^{(i)} s s')$$
$$* \hat{1}^2 \hat{L} \hat{L}' \begin{pmatrix} 1 & 1 & \mathcal{L}' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & \mathcal{L}' & L' \\ 0 & 0 & 0 \end{pmatrix} (-)^{\mathcal{L}'} B(LL'\mathcal{L}'\mathcal{M}'\hat{s}\hat{s}') \tag{3.20}$$

where eq. (B.4) was used.

Comparing this equation with eq. (2.16) gives

$$\sum_{\mathcal{L}} \langle \phi(L1\mathcal{L}s) \| O \| \phi'(L'1\mathcal{L}s') \rangle_5 \hat{\mathcal{L}} \begin{Bmatrix} L & L' & \mathcal{L}' \\ 1 & 1 & \mathcal{L} \end{Bmatrix} (-)^{\mathcal{L}}$$
$$= (ss')\beta s'^2 \sum_i v_5^{(i)} \exp[\quad]\, i_L(\quad)\, \hat{1}^2 \hat{L} \hat{L}' \begin{pmatrix} 1 & 1 & \mathcal{L}' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & \mathcal{L}' & L' \\ 0 & 0 & 0 \end{pmatrix}$$

and thus

$$\langle \phi(L1\mathcal{L}s) \| O \| \phi'(L'1\mathcal{L}s') \rangle_5$$
$$= (ss')\beta s'^2 \sum_i v_5^{(i)} \exp[-\beta(u_5^{(i)} s^2 + u_5'^{(i)} s'^2)]\, i_L(\beta w_5^{(i)} s s')$$
$$* \sum_{\mathcal{L}'} (-)^{\mathcal{L}} \begin{Bmatrix} L & L' & \mathcal{L}' \\ 1 & 1 & \mathcal{L} \end{Bmatrix} \hat{1}^2 \hat{L} \hat{L}' \hat{\mathcal{L}} \hat{\mathcal{L}}'^2 \begin{pmatrix} L' & L & \mathcal{L}' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & \mathcal{L}' \\ 0 & 0 & 0 \end{pmatrix}$$
$$= (ss')\beta s'^2 \sum_i v_5^{(i)} \exp[-\beta(u_5^{(i)} s^2 + u_5'^{(i)} s'^2)]\, i_L(\beta w_5^{(i)} s s')$$
$$* \hat{1}^2 \hat{L} \hat{L}' \hat{\mathcal{L}} \begin{pmatrix} L & 1 & \mathcal{L} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L' & 1 & \mathcal{L} \\ 0 & 0 & 0 \end{pmatrix} \tag{3.21}$$

If we add the eqs. (3.14), (3.16), (3.17), (3.19) and (3.21), we obtain the complete reduced matrix element

$$\langle \phi(L1\mathcal{L}s) \| O \| \phi'(L'1\mathcal{L}s') \rangle$$
$$= (ss') \{ \delta_{LL'} \sum_i v_1^{(i)} \exp[-\beta(u_1^{(i)} s^2 + u_1'^{(i)} s'^2)]\, 4\pi\, i_L(\beta w_1^{(i)} s s')$$

(continued next page)

$$+ \beta s s' \sum_i v_2^{(i)} \exp[-\beta(u_2^{(i)} s^2 + u_2^{\prime(i)} s'^2)]\, i_{\mathcal{L}}(\beta w_2^{(i)} s s')$$
$$* 3\sqrt{(2L+1)(2L'+1)} \begin{pmatrix} L & 1 & \mathcal{L} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} L' & 1 & \mathcal{L} \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ \beta s s' \sum_i v_3^{(i)} \exp[-\beta(u_3^{(i)} s^2 + u_3^{\prime(i)} s'^2)]$$
$$* \sum_{\lambda=\max(|L-1|,|L'-1|)}^{\min(L+1,L'+1)} i_\lambda(\beta w_3^{(i)} s s')\, 3(2\lambda+1)\sqrt{(2L+1)(2L'+1)}$$
$$* \begin{Bmatrix} 1 & L' & \lambda \\ 1 & L & \mathcal{L} \end{Bmatrix}\begin{pmatrix} L & 1 & \lambda \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} L' & 1 & \lambda \\ 0 & 0 & 0 \end{pmatrix}$$
$$+ \Big[(\beta s^2)\sum_i v_4^{(i)} \exp[-\beta(u_4^{(i)} s^2 + u_4^{\prime(i)} s'^2)]\, i_{L'}(\beta w_4^{(i)} s s')$$
$$+ (\beta s'^2)\sum_{i'} v_5^{(i)} \exp[-\beta(u_5^{(i)} s^2 + u_5^{\prime(i)} s'^2)]\, i_L(\beta w_5^{(i)} s s')\Big]$$
$$* 3\sqrt{(2L+1)(2L'+1)} \begin{pmatrix} L & 1 & \mathcal{L} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} L' & 1 & \mathcal{L} \\ 0 & 0 & 0 \end{pmatrix}\Big\}\sqrt{2\mathcal{L}+1} \qquad (3.21)$$

where O stands for the identity or the operator of the central two-body interaction.

Reduced matrix elements of the kinetic-energy operator are obtained from those of the normalization using eq. (3.10).

$$\langle \Phi(L1\mathcal{L}s) \| T - T_{CM} \| \Phi'(L'1\mathcal{L}s') \rangle$$
$$= (ss')\Big\{ \delta_{LL'} \sum_i v_1^{(i)} \exp[-\beta(u_1^{(i)} s^2 + u_1^{\prime(i)} s'^2)]\, 4\pi$$
$$* \big\{[17/4 + L/2 - \beta/2\,(u_1^{(i)} s^2 + u_1^{\prime(i)} s'^2)]\, i_L(\beta w_1^{(i)} s s')$$
$$+ 1/2\, \beta w_1^{(i)} s s'\, i_{L+1}(\beta w_1^{(i)} s s')\big\}$$

(continued next page)

$$+(\beta s s')\sum_i v_2^{(i)} \exp\left[-\beta\left(u_2^{(i)} s^2 + u_2^{\prime(i)} s'^2\right)\right]$$

$$* \left\{\left[19/4 + \ell/2 - \beta/2\left(u_2^{(i)} s^2 + u_2^{\prime(i)} s'^2\right)\right] i_\ell\left(\beta w_2^{(i)} s s'\right)\right.$$

$$\left. + 1/2\,\beta w_2^{(i)} s s'\, i_{\ell+1}\left(\beta w_2^{(i)} s s'\right)\right\}$$

$$* 3\sqrt{(2L+1)(2L'+1)} \begin{pmatrix} L & 1 & \ell \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} L' & 1 & \ell \\ 0 & 0 & 0 \end{pmatrix}$$

$$+(\beta s s')\sum_i v_3^{(i)} \exp\left[-\beta\left(u_3^{(i)} s^2 + u_3^{\prime(i)} s'^2\right)\right]$$

$$* \sum_{\lambda=\max(|L-1|,|L'-1|)}^{\min(L+1,\,L'+1)} \left\{\left[19/4 + \lambda/2 - \beta/2\left(u_3^{(i)} s^2 + u_3^{\prime(i)} s'^2\right)\right] i_\lambda\left(\beta w_3^{(i)} s s'\right)\right.$$

$$\left. + 1/2\,\beta w_3^{(i)} s s'\, i_{\lambda+1}\left(\beta w_3^{(i)} s s'\right)\right\} 3\,(2\lambda+1)\sqrt{(2L+1)(2L'+1)}$$

$$* \begin{Bmatrix} L & 1 & \ell \\ L' & 1 & \lambda \end{Bmatrix}\begin{pmatrix} L & 1 & \lambda \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} L' & 1 & \lambda \\ 0 & 0 & 0 \end{pmatrix}$$

$$+(\beta s^2)\sum_i v_4^{(i)} \exp\left[-\beta\left(u_4^{(i)} s^2 + u_4^{\prime(i)} s'^2\right)\right]$$

$$* \left\{\left[19/4 + L'/2 - \beta/2\left(u_4^{(i)} s^2 + u_4^{\prime(i)} s'^2\right)\right] i_{L'}\left(\beta w_4^{(i)} s s'\right)\right.$$

$$\left. + 1/2\,\beta w_4^{(i)} s s'\, i_{L'+1}\left(\beta w_4^{(i)} s s'\right)\right\}$$

$$+(\beta s'^2)\sum_i v_5^{(i)} \exp\left[-\beta\left(u_5^{(i)} s^2 + u_5^{\prime(i)} s'^2\right)\right]$$

$$* \left\{\left[19/4 + L/2 - \beta/2\left(u_5^{(i)} s^2 + u_5^{\prime(i)} s'^2\right)\right] i_L\left(\beta w_5^{(i)} s s'\right)\right.$$

$$\left. + 1/2\,\beta w_5^{(i)} s s'\, i_{L+1}\left(\beta w_4^{(i)} s s'\right)\right\}\Big]$$

$$* 3\sqrt{(2L+1)(2L'+1)} \begin{pmatrix} L & 1 & \ell \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} L' & 1 & \ell \\ 0 & 0 & 0 \end{pmatrix}\Bigg\}$$

$$* \sqrt{2\ell+1}\left(\frac{\hbar^2\beta}{4m}\right) \qquad (3.22)$$

4. CALCULATION OF REDUCED MATRIX ELEMENTS OF THE ELECTRIC CHARGE MULTIPOLE OPERATOR $M(C\lambda\mu q)$

4.1 Some General Considerations

We summarize some standard formulas [13] of the theory of electron nucleus scattering. In Born approximation the charge form factor is given by

$$|F_{ch}(q)|^2 = \frac{4\pi}{Z^2}(2J_i+1)^{-1}\sum_{\lambda=0}^{\infty} |\langle J_f \| M(C\lambda q) \| J_i \rangle|^2 \tag{4.1}$$

where

$$M(C\lambda\mu q) = \int d\underset{\sim}{x}\, j_\lambda(qx)\, Y_{\lambda\mu}(\hat{x})\, \rho(\underset{\sim}{x}) \tag{4.2}$$

is the electric charge multipole operator with

$$\rho(\underset{\sim}{x}) = \sum_{j=1}^{N} e_j\, \delta^3(\underset{\sim}{x} - \underset{\sim}{r}_j) \tag{4.3}$$

being the nuclear charge density operator. For later convenience, we rewrite eqs. (4.2) and (4.3) in the following form

$$M(C\lambda\mu q) = (4\pi i^\lambda)^{-1} \int d\hat{q}\, O(\underset{\sim}{q})\, Y_{\lambda\mu}(\hat{q}) \tag{4.4}$$

$$O(\underset{\sim}{q}) = \sum_{j=1}^{N} e_j \exp(i\underset{\sim}{q}\cdot\underset{\sim}{r}_j) \tag{4.5}$$

The sum over λ in eq. (4.1) is restricted by the parity conservation

$$\pi_f = \pi_i\, (-)^\lambda \tag{4.6}$$

We consider two corrections to the charge form factor (4.1):

(i) The finite size of the nucleons is taken into account by multiplying all multipole operators by the appropriate nucleon form factor. The charge form factor of the proton may be approximated by [14)]

$$f(q) = \exp\left(-\frac{q^2 a_p^2}{4}\right) \qquad a_p^2 = 0.43\ \mathrm{fm}^2$$

(ii) If shell model wave functions are used for the nucleon wave functions, their lack of translational invariance due to the assumption of fixed potential wells has to be remedied by projecting the total linear momentum of the system. If the oscillator parameters ß of the wells in a multi-cluster wave function are the same, the center-of-mass degree of freedom in the wave function can be factored out as a Gaussian function. Projection of linear momentum may then be accounted for by multiplying the matrix elements of each multipole operator by the factor [14)]

$$g(q) = \exp\left(\frac{q^2}{4\beta N}\right)$$

Thus the corrected charge form factor reads

$$F_{ch}(q)_{corr} = F_{ch}(q) \exp\left[\frac{q^2}{4}\left(\frac{1}{\beta N} - a_p^2\right)\right] \tag{4.7}$$

Studying the long wavelength limit ($q\to 0$), one may extract further information from the charge form factor (4.1).

(i) Elastic scattering ($J_f=J_i=J$):

The multipole expansion for the total charge form factor (4.1) is given by

$$|F_{ch}(q)|^2 = |F_{c0}(q)|^2 + \left(\frac{q^4}{180 Z^2}\right)\frac{(J+1)(2J+3)}{J(2J-1)} Q^2 \, |F_{c2}(q)|^2 + \ldots \tag{4.8}$$

where $F_{co}(q)$ is the electric monopole form factor and $F_{c2}(q)$ is the electric quadrupole form factor and Q is the electric quadrupole moment. For $q\to 0$, $|F_{co}(q)|^2$ may be expanded as[13)]

$$|F_{co}(q)|^2 = \left[1 - \frac{q^2\langle r^2\rangle}{6} + \frac{q^4\langle r^4\rangle}{120} + \ldots\right]^2 \quad (4.9)$$

where $\langle r^2\rangle$ is the mean-square radius. In ref. 12) it is shown how the q-dependence of the reduced matrix elements and hence the charge form factor may be obtained analytically for a two-cluster model. $\langle r^2\rangle$ may then be obtained from the monopole charge form factor

$$\langle r^2\rangle = -3 \frac{d^2}{dq^2}\left|F_{co}(q)\right|_{q=0} \quad (4.10)$$

(ii) Inelastic scattering ($J_i\to J_j$):

In the long wavelength limit ($q\to 0$), the multipole component λ of the inelastic charge form factor is identical[14)] to the reduced matrix element of an electric λ-pole transition

$$B(E\lambda, J_i\to J_f) \underset{q\to 0}{=} (2J_i+1)^{-1}\left[\frac{(2\lambda+1)!!}{q^\lambda}\right]^2 \left|\langle J_f \| M(C\lambda q) \| J_i\rangle\right|^2 \quad (4.11)$$

The charge multipole operators (4.2) do not act on the spins of the system. Thus, in the LS coupling scheme, we may express the reduced matrix elements by those of orbital angular momentum eigenstates only[10)]

$$\langle \mathcal{L}_f I J_f \| M(C\lambda q) \| \mathcal{L}_i I J_i\rangle$$
$$= (-)^{\mathcal{L}_f + I + J_i + \lambda} \hat{J}_f \hat{J}_i \begin{Bmatrix} \mathcal{L}_f & J_f & I \\ J_i & \mathcal{L}_i & \lambda \end{Bmatrix} \langle \mathcal{L}_f \| M(C\lambda q) \| \mathcal{L}_i\rangle \quad (4.12)$$

In sections 4.2 - 4.5 we shall calculate the reduced matrix elements $\langle \mathcal{L}_f || M(C\lambda q) || \mathcal{L}_i \rangle$ with wave functions of a two-cluster model of ^{6}Li starting from analytical expressions[12] for matrix elements of the operator O, eq. (4.5). For the sake of easy reference, we list them below

$$\langle \phi(\underset{\sim}{s}) | O(\underset{\sim}{q}) | \phi(\underset{\sim}{s}') \rangle = \sum_i v^{(i)} \exp\left[-\beta(u^{(i)} s^2 + u'^{(i)} s'^2) + \beta w^{(i)} \underset{\sim}{s} \cdot \underset{\sim}{s}' + i \underset{\sim}{q} \cdot (-p^{(i)} \underset{\sim}{s} + p'^{(i)} \underset{\sim}{s}') - q^2/(4\beta) \right] \tag{4.13a}$$

$$\langle \phi(\underset{\sim}{s}) | O(\underset{\sim}{q}) | \phi(1m'\underset{\sim}{s}') \rangle = \mathcal{Y}_{1m'}(\sqrt{\beta}\underset{\sim}{s}) \sum_i v^{(i)} \exp\left[-\beta(u^{(i)} s^2 + u'^{(i)} s'^2) + \beta w^{(i)} \underset{\sim}{s} \cdot \underset{\sim}{s}' + i \underset{\sim}{q} \cdot (-p^{(i)} \underset{\sim}{s} + p'^{(i)} \underset{\sim}{s}') - q^2/(4\beta) \right] \tag{4.13b}$$

$$\langle \phi(1m\underset{\sim}{s}) | O(\underset{\sim}{q}) | \phi(\underset{\sim}{s}') \rangle = \mathcal{Y}^*_{1m}(\sqrt{\beta}\underset{\sim}{s}') \sum_i v^{(i)} \exp\left[-\beta(u^{(i)} s^2 + u'^{(i)} s'^2) + \beta w^{(i)} \underset{\sim}{s} \cdot \underset{\sim}{s}' + i \underset{\sim}{q} \cdot (-p^{(i)} \underset{\sim}{s} + p'^{(i)} \underset{\sim}{s}') - q^2/(4\beta) \right] \tag{4.13c}$$

$$\langle \phi(1m\underset{\sim}{s}) | O(\underset{\sim}{q}) | \phi(1m'\underset{\sim}{s}') \rangle = \delta_{mm'} \sum_i v^{(i)} \exp\left[-\beta(u^{(i)} s^2 + u'^{(i)} s'^2) + \beta w^{(i)} \underset{\sim}{s} \cdot \underset{\sim}{s}' + i \underset{\sim}{q} \cdot (-p^{(i)} \underset{\sim}{s} + p'^{(i)} \underset{\sim}{s}') - q^2/(4\beta) \right] \tag{4.13d}$$

The notation used in eqs. (4.13) is the same as in eqs. (3.1a), (3.3) and (3.12). Of course, the coefficients $v^{(i)}$, $u^{(i)}$ etc. are different for each of the matrix elements.

4.2 Reduced Matrix Elements $<\ell=o||M(C\lambda q)||\ell'=0>$

We introduce the decomposition (2.2)

$$\phi(\{\underset{\sim}{x}\}\underset{\sim}{s}) = \frac{1}{s}\sum_{LM}\phi(\{\underset{\sim}{x}\}LMs)\,Y^*_{LM}(\hat{s})$$

where $\phi(\{\underset{\sim}{x}\}$ LMs) form a spherical tensor of rank L in the coordinate space. Thus we may use the Wigner-Eckart theorem

$$\langle\phi(\underset{\sim}{s})|\,M(C\lambda\mu q)\,|\phi(\underset{\sim}{s}')\rangle$$

$$= 1/(ss')\sum_{LML'M'} Y_{LM}(\hat{s})\,Y^*_{L'M'}(\hat{s}')\,\langle\phi(LMs)|M\,(C\lambda\mu q)|\phi(L'M's')\rangle$$

$$= 1/(ss')\sum_{LML'M'} Y_{LM}(\hat{s})\,Y^*_{L'M'}(\hat{s}')\,(-)^{L+M}\begin{pmatrix} L & \lambda & L' \\ -M & \mu & M'\end{pmatrix}$$

$$* \langle\phi(Ls)\|\,M\,(C\lambda q)\,\|\,\phi(L's')\rangle$$

$$= 1/(ss')\sum_{LL'}(-)^{L'}\hat{\lambda}^{-1}\,B(LL'\lambda\mu\hat{s}\hat{s}')$$

$$* \langle\phi(Ls)\|\,M\,(C\lambda q)\,\|\,\phi(L's')\rangle \qquad (4.14)$$

The lefthand side of this equation may be evaluated using eqs. (4.4), (4.5) and (4.13a)

$$\langle\phi(\underset{\sim}{s})|\,M(C\lambda\mu q)\,|\phi(\underset{\sim}{s}')\rangle\exp(q^2/4\beta)$$

$$= (4\pi i^\lambda)^{-1}\int d\hat{q}\,Y_{\lambda\mu}(\hat{q})\sum_i v^{(i)}\exp[-\beta(u^{(i)}s^2 + u'^{(i)}s'^2)$$

$$+\beta w^{(i)}\underset{\sim}{s}\cdot\underset{\sim}{s}' + i\underset{\sim}{q}\cdot(-p^{(i)}\underset{\sim}{s} + p'^{(i)}\underset{\sim}{s}')]$$

$$= (4\pi i^\lambda)^{-1}\sum_i v^{(i)}\exp[-\beta(u^{(i)}s^2 + u'^{(i)}s'^2)](4\pi)^3$$

$$* \sum_{abc} i^{a+b}\,j_a(-qp^{(i)}s)\,j_b(qp'^{(i)}s')\,i_c(\beta w^{(i)}ss')$$

(continued next page)

$$* \sum_{\alpha\beta\gamma} Y^*_{a\alpha}(\hat{s})\, Y_{c\gamma}(\hat{s})\, Y^*_{b\beta}(\hat{s}')\, Y^*_{c\gamma}(\hat{s}')$$

$$* \int d\hat{q}\, Y_{\lambda\mu}(\hat{q})\, Y_{a\alpha}(\hat{q})\, Y_{b\beta}(\hat{q})$$

The integral over the three $Y(\hat{q})$ in this equation may be expressed in terms of 3-j symbols[10] and is equal to

$$\frac{\hat{\lambda}\,\hat{a}\,\hat{b}}{\sqrt{4\pi}} \begin{pmatrix} \lambda & a & b \\ \mu & \alpha & \beta \end{pmatrix} \begin{pmatrix} \lambda & a & b \\ 0 & 0 & 0 \end{pmatrix}$$

The summation over the magnetic quantum numbers α, β and γ is then easily carried out (see eq. (B.1)) with the result that

$$\langle \phi(\underset{\sim}{s}) | M(C\lambda\mu q) | \phi(s') \rangle \exp(q^2/4\beta)$$
$$= \sqrt{4\pi} \sum_i v^{(i)} \exp\left[-\beta(u^{(i)} s^2 + u'^{(i)} s'^2)\right]$$
$$* \sum_{abcLL'} j_a(q p^{(i)} s)\, j_b(q p'^{(i)} s')\, i_c(\beta w^{(i)} s s')$$
$$* \hat{a}^2 \hat{b}^2 \hat{c}^2 \hat{L} \hat{L}' (-)^{L'} i^{a+b+\lambda} \begin{pmatrix} \lambda & a & b \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & a & c \\ 0 & 0 & 0 \end{pmatrix}$$
$$* \begin{pmatrix} L' & b & c \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} L & \lambda & L' \\ b & c & a \end{Bmatrix} B(LL'\lambda\mu\hat{s}\hat{s}') \qquad (4.15)$$

If we insert this equation into eq. (4.14) and equate the coefficients of $B(LL'\lambda\mu\hat{s}\hat{s}')$, we find

$$\langle \phi(Ls) \| M(C\lambda q) \| \phi(L's') \rangle$$
$$= \sqrt{4\pi}\,(ss') \sum_i v^{(i)} \exp\left[-\beta(u^{(i)} s^2 + u'^{(i)} s'^2) - q^2/4\beta\right]$$

(continued next page)

$$* \sum_{abc} j_a(qp^{(i)}s)\, j_b(qp'^{(i)}s')\, i_c(\beta w^{(i)}ss')\, \hat{a}^2 \hat{b}^2 \hat{c}^2 \hat{L} \hat{L}' \hat{\lambda}$$

$$* i^{a+b+\lambda} \begin{pmatrix} \lambda & a & b \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & a & c \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L' & b & c \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} L & \lambda & L' \\ b & c & a \end{Bmatrix} \tag{4.16}$$

The conservation of parity (4.6), is contained in the three 3-j symbols, i.e. $\lambda + L + L' =$ even.

4.3 Reduced Matrix Elements $<\ell=0||M(C\lambda\ q)||\ell'=1>$

We use the expansions (2.2) and (2.5)

$$\phi(\{\underset{\sim}{x}\}\, 1m\, \underset{\sim}{s})$$
$$= 1/s \sum_{LM} \phi(\{\underset{\sim}{x}\}\, LM\, 1m\, s)\, Y^*_{LM}(\hat{s})$$
$$= 1/s \sum_{LM\mathcal{L}\mathcal{M}} \hat{\mathcal{L}}\, (-)^{L+1+\mathcal{M}} \begin{pmatrix} L & 1 & \mathcal{L} \\ M & m & -\mathcal{M} \end{pmatrix} \phi(\{\underset{\sim}{x}\}\, L1\mathcal{L}\mathcal{M}s)\, Y^*_{LM}(\hat{s})$$

where the set of functions $\phi(\{\underset{\sim}{x}\}L1\,\mathcal{L}\,\mathcal{M}s)$ form a spherical tensor of rank $\mathcal{L}$ in the coordinate space.

Thus we may use again the Wigner-Eckart theorem in calculating matrix elements

$$\langle \phi(\underset{\sim}{s})|\, M(C\lambda\mu q)\, |\phi(1m'\underset{\sim}{s}')\rangle$$
$$= 1/(ss') \sum_{LML'M'\mathcal{L}\mathcal{M}} Y_{LM}(\hat{s})\, Y^*_{L'M'}(\hat{s}')\, \hat{\mathcal{L}}\, (-)^{L'+1+\mathcal{M}} \begin{pmatrix} L' & 1 & \mathcal{L} \\ M' & m' & -\mathcal{M} \end{pmatrix}$$
$$* \langle \phi(LMs)|\, M(C\lambda\mu q)\, |\phi(L'1\mathcal{L}\mathcal{M}s')\rangle$$
$$= 1/(ss') \sum_{LL'\mathcal{L}f\phi} \hat{\mathcal{L}}\hat{f}\, (-)^{L+\lambda+1+f+\phi} \begin{Bmatrix} 1 & L' & \mathcal{L} \\ L & \lambda & f \end{Bmatrix}$$
$$* \begin{pmatrix} \lambda & 1 & f \\ \mu & m' & -\phi \end{pmatrix} B(LL'f\phi\hat{s}\hat{s}')\langle\phi(Ls)\|\, M(C\lambda q)\, \|\phi(L'1\mathcal{L}s')\rangle$$

$$\tag{4.17}$$

where eq. (B.6) was used. On the left we use eqs. (4.4), (4.5) and (4.13b) and obtain

$$\langle \Phi(\underset{\sim}{s}) | M(c\lambda\mu q) | \Phi(l m' \underset{\sim}{s}') \rangle \exp(q^2/4\beta)$$

$$= (4\pi i^{\lambda})^{-1} \sqrt{\beta s} \int d\hat{q}\, Y_{\lambda\mu}(\hat{q})\, Y_{lm'}(\hat{s}) \sum_i v^{(i)}$$
$$* \exp[-\beta(u^{(i)}s^2 + u'^{(i)}s'^2) + \beta w^{(i)} \underset{\sim}{s}\cdot\underset{\sim}{s}' + i\underset{\sim}{q}\cdot(-p^{(i)}\underset{\sim}{s} + p'^{(i)}\underset{\sim}{s}')]$$

$$= (4\pi i^{\lambda})^{-1} \sqrt{\beta s} \sum_i v^{(i)} \exp[-\beta(u^{(i)}s^2 + u'^{(i)}s'^2)]$$
$$* (4\pi)^3 \sum_{abc} i^{a+b}\, j_a(-q p^{(i)} s)\, j_b(q p'^{(i)} s')\, i_c(\beta w^{(i)} s s')$$
$$* \sum_{\alpha\beta\gamma} Y_{lm'}(\hat{s})\, Y^*_{a\alpha}(\hat{s})\, Y_{c\gamma}(\hat{s})\, Y^*_{b\beta}(\hat{s}')\, Y^*_{c\gamma}(\hat{s}')$$
$$* \int d\hat{q}\, Y_{\lambda\mu}(\hat{q})\, Y_{a\alpha}(\hat{q})\, Y_{b\beta}(\hat{q})$$

Again, the sum over the magnetic quantum nubmers α, β and γ may be carried out (see eq. (B.5)).

$$\langle \Phi(\underset{\sim}{s}) | M(\lambda\mu q) | \Phi(l m' \underset{\sim}{s}') \rangle \exp(q^2/4\beta)$$

$$= \sqrt{\beta s} \sum_i v^{(i)} \exp[-\beta(u^{(i)}s^2 + u'^{(i)}s'^2)]$$
$$* \sum_{abcdLL'f\phi} j_a(q p^{(i)} s)\, j_b(q p'^{(i)} s')\, i_c(\beta w^{(i)} s s')$$

(continued next page)

$$* \; \hat{a}^2 \hat{b}^2 \hat{c}^2 \hat{d}^2 \hat{1} \hat{L} \hat{L}' \hat{f} \hat{\lambda} \, (-)^{L+L'+\lambda+1+\phi} \, i^{a+b+\lambda}$$
$$* \begin{pmatrix} \lambda & a & b \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & c & d \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} b & c & L' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & d & L \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} \lambda & L' & d \\ c & a & b \end{Bmatrix}$$
$$* \begin{Bmatrix} L' & \lambda & d \\ 1 & L & f \end{Bmatrix} \begin{pmatrix} \lambda & 1 & f \\ \mu & m' & -\phi \end{pmatrix} B(LL'f\phi\hat{s}\hat{s}') \qquad (4.18)$$

Comparing eqs. (4.17) and (4.18), we obtain for the reduced matrix elements the set of equations

$$\sum_{\mathcal{L}'} \langle \Phi(Ls) \| M(c\lambda q) \| \Phi(L'1\mathcal{L}'s') \rangle \hat{\mathcal{L}}' \hat{f} \begin{Bmatrix} 1 & L' & \mathcal{L}' \\ L & \lambda & f \end{Bmatrix}$$
$$= (ss') \sqrt{\beta}\, s \sum_i v^{(i)} \exp\left[-\beta(u^{(i)}s^2 + u'^{(i)}s'^2) - q^2/4\beta\right]$$
$$* \sum_{abcd} j_a(qp^{(i)}s)\, j_b(qp'^{(i)}s')\, i_c(\beta w^{(i)}ss')\, \hat{a}^2 \hat{b}^2 \hat{c}^2 \hat{d}^2 \hat{1} \hat{L} \hat{L}' \hat{f} \hat{\lambda}$$
$$* (-)^{L'+f} \, i^{a+b+\lambda} \begin{pmatrix} \lambda & a & b \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & c & d \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} b & c & L' \\ 0 & 0 & 0 \end{pmatrix}$$
$$* \begin{pmatrix} 1 & L & d \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} \lambda & L' & d \\ c & a & b \end{Bmatrix} \begin{Bmatrix} L' & \lambda & d \\ 1 & L & f \end{Bmatrix} \qquad (4.19)$$

Using the orthogonality relation (see Appendix C) we solve eq. (4.19) for the reduced matrix element with the result that

$$\langle \Phi(Ls) \| M(c\lambda q) \| \Phi(L'1\mathcal{L}s') \rangle$$
$$= (ss') \sqrt{\beta}\, s \sum_i v^{(i)} \exp\left[-\beta(u^{(i)}s^2 + u'^{(i)}s'^2) - q^2/4\beta\right]$$
$$* \sum_{abcd} j_a(qp^{(i)}s)\, j_b(qp'^{(i)}s')\, i_c(\beta w^{(i)}ss')$$
$$* \; \hat{a}^2 \hat{b}^2 \hat{c}^2 \hat{d}^2 \hat{1} \hat{L} \hat{L}' \hat{\mathcal{L}} \hat{\lambda} \, (-)^{L'+d+\mathcal{L}} \, i^{a+b+\lambda}$$
$$* \begin{pmatrix} \lambda & a & b \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & c & d \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} b & c & L' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & L & d \\ 0 & 0 & 0 \end{pmatrix}$$
$$* \begin{Bmatrix} \lambda & L' & d \\ c & a & b \end{Bmatrix} \begin{Bmatrix} L' & 1 & \mathcal{L} \\ L & \lambda & d \end{Bmatrix} \qquad (4.20)$$

Again, the conservation of parity is contained in the product of four 3-j symbols, i.e., L+L'+ λ +1 = even.

4.4 Reduced Matrix Elements $\langle \ell=1 || M(C\lambda q) || \ell'=0 \rangle$

Proceeding in a similar way as in the previous cases, we obtain

$$\langle \phi(1 m \underset{\sim}{s}) | M(C\lambda\mu q) | \phi(\underset{\sim}{s}') \rangle$$
$$= 1/(ss') \sum_{LML'M'\mathcal{L}\mathcal{M}} Y_{LM}(\hat{s}) Y^{*}_{L'M'}(\hat{s}') \hat{\mathcal{L}} (-)^{L+1+\mathcal{L}}$$
$$* \begin{pmatrix} L & 1 & \mathcal{L} \\ M & m & -\mathcal{M} \end{pmatrix} \begin{pmatrix} \mathcal{L} & \lambda & L' \\ -\mathcal{M} & \mu & M' \end{pmatrix} \langle \phi(1 L \mathcal{L} s) \| M(C\lambda q) \| \phi(L's') \rangle$$
$$= 1/(ss') \sum_{LL'\mathcal{L}f\phi} \hat{\mathcal{L}} \hat{f} (-)^{\mathcal{L}+\mu} \begin{Bmatrix} 1 & L & \mathcal{L} \\ L' & \lambda & f \end{Bmatrix} \begin{pmatrix} \lambda & 1 & f \\ \mu & -m & -\phi \end{pmatrix}$$
$$* B(LL'f\phi\hat{s}\hat{s}') \langle \phi(L1\mathcal{L}s) \| M(C\lambda q) \| \phi(L's') \rangle \tag{4.21}$$

where eq. (B.7) was used. The lefthand side is evaluated using eqs. (4.4), (4.5), (4.13c) and (B.8)

$$\langle \phi(1 m \underset{\sim}{s}) | M(C\lambda\mu q) | \phi(\underset{\sim}{s}') \rangle \exp(q^2/4\beta)$$
$$= \sqrt{\beta}\, s' \sum_i v^{(i)} \exp\left[-\beta\left(u^{(i)} s^2 + u'^{(i)} s'^2\right)\right] (4\pi)^{3/2}$$

(continued next page)

$$* \sum_{abc} i^{a+b-\lambda} j_a(-qp^{(i)}s)\, j_b(qp'^{(i)}s')\, i_c(\beta w^{(i)} ss')$$

$$* \sum_{\alpha\beta\gamma} Y^*_{a\alpha}(\hat{s})\, Y_{c\gamma}(\hat{s})\, Y^*_{b\beta}(\hat{s}')\, Y^*_{c\gamma}(\hat{s}')\, Y^*_{1m}(\hat{s}')$$

$$* \begin{pmatrix} \lambda & a & b \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda & a & b \\ \mu & \alpha & \beta \end{pmatrix} \hat{a}\,\hat{b}\,\hat{\lambda}$$

$$= \sqrt{3}\, s' \sum_i v^{(i)} \exp\left[-\beta\left(u^{(i)} s^2 + u'^{(i)} s'^2\right)\right]$$

$$* \sum_{abcd\,LL'f\phi} j_a(qp^{(i)}s)\, j_b(qp'^{(i)}s')\, i_c(\beta w^{(i)} ss')$$

$$* \hat{a}^2 \hat{b}^2 \hat{c}^2 \hat{d}^2\, \hat{1}\, \hat{L}\, \hat{L}'\, \hat{f}\, \hat{\lambda}\, (-)^{d+f+L'+\mu}\, i^{a+b+\lambda}$$

$$* \begin{pmatrix} \lambda & a & b \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & c & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} b & c & d \\ 0 & 0 & 0 \end{pmatrix}$$

$$* \begin{pmatrix} d & 1 & L' \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} \lambda & a & b \\ c & d & L \end{Bmatrix} \begin{Bmatrix} 1 & d & L' \\ L & f & \lambda \end{Bmatrix}$$

$$* \begin{pmatrix} \lambda & 1 & f \\ \mu & -m & -\phi \end{pmatrix} B(LL'f\phi\, \hat{s}\hat{s}') \qquad (4.22)$$

If compare the coefficients of $B(LL'f_\phi\hat{s}\hat{s}')$ in eqs. (4.21) and (4.22) we find that

$$\sum_{\mathcal{L}'} \langle \phi(L1\mathcal{L}'s) \| M(C\lambda q) \| \phi(L's') \rangle$$

$$* \hat{\mathcal{L}}' \hat{f} (-)^{\mathcal{L}'} \begin{Bmatrix} 1 & L & \mathcal{L}' \\ L' & \lambda & f \end{Bmatrix}$$

$$= (ss')\sqrt{\beta}\, s' \sum_{i} v^{(i)} \exp\left[-\beta\left(u^{(i)}s^2 + u'^{(i)}s'^2\right) - q^2/4\beta\right]$$

$$* \sum_{abcd} j_a(qp^{(i)}s)\, j_b(qp'^{(i)}s')\, i_c(\beta w^{(i)}ss')$$

$$* \hat{a}^2 \hat{b}^2 \hat{c}^2 \hat{d}^2 \hat{1} \hat{L} \hat{L}' \hat{f} \hat{\lambda} (-)^{d+f+L'} i^{a+b+\lambda} \begin{pmatrix} \lambda & a & b \\ 0 & 0 & 0 \end{pmatrix}$$

$$* \begin{pmatrix} a & c & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} b & c & d \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} d & 1 & L' \\ 0 & 0 & 0 \end{pmatrix}$$

$$* \begin{Bmatrix} \lambda & L & d \\ c & b & a \end{Bmatrix} \begin{Bmatrix} 1 & L' & d \\ L & \lambda & f \end{Bmatrix} \qquad (4.23)$$

It is easy to solve the set of equations (4.23) for the desired reduced matrix element (see Appendix C)

$$\langle \phi(L1\mathcal{L}s) \| M(C\lambda q) \| \phi(L's') \rangle$$

$$= (ss')\sqrt{\beta}\, s' \sum_{i} v^{(i)} \exp\left[-\beta\left(u^{(i)}s^2 + u'^{(i)}s'^2\right) - q^2/4\beta\right]$$

$$* \sum_{abcd} j_a(qp^{(i)}s)\, j_b(qp'^{(i)}s')\, i_c(\beta w^{(i)}ss')$$

$$* \hat{a}^2 \hat{b}^2 \hat{c}^2 \hat{d}^2 \hat{1} \hat{L} \hat{L}' \hat{\mathcal{L}} \hat{\lambda} (-)^{L'} i^{a+b+\lambda}$$

$$* \begin{pmatrix} \lambda & a & b \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & c & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} b & c & d \\ 0 & 0 & 0 \end{pmatrix}$$

$$* \begin{pmatrix} d & 1 & L' \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} L & 1 & \mathcal{L} \\ L' & \lambda & d \end{Bmatrix} \begin{Bmatrix} \lambda & L & d \\ c & b & a \end{Bmatrix}$$

(4.24)

4.5 Reduced Matrix Elements $\langle \ell=1 || M(C\lambda q) || \ell'=1 \rangle$

Upon expanding the unprojected matrix element, we find

$$\begin{aligned}
&\langle \phi(1m\underset{\sim}{s}) | M(C\lambda\mu q) | \phi(1m'\underset{\sim}{s}') \rangle \\
&= 1/(ss') \sum_{LML'M'\mathcal{L}\mathcal{M}\mathcal{L}'\mathcal{M}'} Y_{LM}(\hat{s}) Y^*_{L'M'}(\hat{s}') \hat{\mathcal{L}}\hat{\mathcal{L}}' (-)^{L+L'+\mathcal{L}+\mathcal{M}'} \\
&\quad * \begin{pmatrix} L & 1 & \mathcal{L} \\ M & m & -\mathcal{M} \end{pmatrix} \begin{pmatrix} L' & 1 & \mathcal{L}' \\ M' & m' & -\mathcal{M}' \end{pmatrix} \begin{pmatrix} \mathcal{L} & \lambda & \mathcal{L}' \\ -\mathcal{M} & \mu & \mathcal{M}' \end{pmatrix} \\
&\quad * \langle \phi(L1\mathcal{L}s) || M(C\lambda q) || \phi(L'1\mathcal{L}'s') \rangle
\end{aligned}$$

The sum over the magnetic quantum numbers in this equation may be carried out (see eq. (B.9)) with the result that

$$\begin{aligned}
&\langle \phi(1m\underset{\sim}{s}) | M(C\lambda\mu q) | \phi(1m'\underset{\sim}{s}') \rangle \\
&= 1/(ss') \sum_{LL'\mathcal{L}\mathcal{L}'b\beta c\gamma} \hat{\mathcal{L}}\hat{\mathcal{L}}'\hat{b}^2\hat{c}\, (-)^{L'+c+1+m'+\gamma} \\
&\quad * \begin{pmatrix} \lambda & b & c \\ \mu & \beta & -\gamma \end{pmatrix} \begin{pmatrix} b & 1 & 1 \\ \beta & -m' & m \end{pmatrix} \begin{Bmatrix} \lambda & b & c \\ \mathcal{L}' & 1 & L' \\ \mathcal{L} & 1 & L \end{Bmatrix} B(LL'c\gamma\hat{s}\hat{s}') \\
&\quad * \langle \phi(L1\mathcal{L}s) || M(C\lambda q) || \phi(L'1\mathcal{L}'s') \rangle
\end{aligned}$$

If we use the orthogonality relation of the 3-j symbols and the property that the 9-j symbol is invariant under reflection about either diagonal, we find

$$\begin{aligned}
&\sum_{mm'} (-)^{1+m'} \begin{pmatrix} d & 1 & 1 \\ \delta & -m' & m \end{pmatrix} \langle \phi(1m\underset{\sim}{s}) | M(C\lambda\mu q) | \phi(1m'\underset{\sim}{s}') \rangle \\
&= 1/(ss') \sum_{LL'\mathcal{L}\mathcal{L}'c\gamma} \hat{\mathcal{L}}\hat{\mathcal{L}}'\hat{c}\, (-)^{L'+c+\gamma} \begin{pmatrix} \lambda & d & c \\ \mu & \delta & -\gamma \end{pmatrix}
\end{aligned}$$

(continued next page)

$$* \; B(LL'c\gamma\hat{s}\hat{s}') \begin{Bmatrix} L & 1 & \ell \\ L' & 1 & \ell' \\ c & d & \lambda \end{Bmatrix} \qquad (4.25)$$

$$* \; \langle \phi(L1\ell s) \| M(c\lambda q) \| \phi(L'1\ell' s') \rangle$$

On the lefthand side of this equation we use the property that the matrix element vanishes unless m=m' and is independent of m(cf. eq. (4.13d)). The summation over m may then be carried out to give

$$\langle \phi(1ms) | M(c\lambda\mu q) | \phi(1m's') \rangle \; \hat{1} \; \delta_{d0}$$

$$= (4\pi)^{1/2} \; \hat{1} \; \delta_{d0} \sum_i v^{(i)} \exp\left[-\beta(u^{(i)}s^2 + u'^{(i)}s'^2) - q^2/4\beta\right]$$

$$* \sum_{abcLL'} j_a(qp^{(i)}s) \, j_b(qp'^{(i)}s') \, i_c(\beta w^{(i)}ss')$$

$$* \; \hat{a}^2 \hat{b}^2 \hat{c}^2 \hat{L} \hat{L}' (-)^{L'} i^{a+b+\lambda} \begin{pmatrix} \lambda & a & b \\ 0 & 0 & 0 \end{pmatrix}$$

$$* \begin{pmatrix} L & a & c \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L' & b & c \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} L & \lambda & L' \\ b & c & a \end{Bmatrix}$$

$$* \; B(LL'\lambda\mu\hat{s}\hat{s}') \qquad (4.26)$$

where we used eq. (4.15). The orthogonality relation of the 9-j symbols, eq. (6.4.6) of ref. 10) shows that the dependence on ℓ and ℓ' of the reduced matrix element in eq. (4.25) must be given by the 9-j symbol

$$\hat{\mathcal{L}}\hat{\mathcal{L}}' \begin{Bmatrix} L & 1 & \mathcal{L} \\ L' & 1 & \mathcal{L}' \\ \lambda & 0 & \lambda \end{Bmatrix} = (-)^{L+\lambda+\mathcal{L}'+1} \begin{Bmatrix} L' & L & \lambda \\ \mathcal{L} & \mathcal{L}' & 1 \end{Bmatrix} \frac{\hat{\mathcal{L}}\,\hat{\mathcal{L}}'}{\hat{\lambda}\,\hat{1}}$$

Indeed, it is easy to see that

$$\begin{aligned} &\langle \phi(L1\mathcal{L}s) \| M(C\lambda q) \| \phi(L'1\mathcal{L}'s') \rangle \\ &= (ss')\sqrt{4\pi}\sum_i v^{(i)} \exp\left[-\beta(u^{(i)}s^2 + u'^{(i)}s'^2) - q^2/4\beta\right] \\ &\quad * \sum_{abc} j_a(qp^{(i)}s)\, j_b(qp'^{(i)}s')\, i_c(\beta w^{(i)}ss') \\ &\quad * \hat{a}^2\hat{b}^2\hat{c}^2\,\hat{L}\hat{L}'\,\hat{\lambda}\,\hat{\mathcal{L}}\hat{\mathcal{L}}'(-)^{L+\mathcal{L}'+\lambda+1}\, i^{a+b+\lambda} \\ &\quad * \begin{pmatrix} \lambda & a & b \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & a & c \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} b & c & L' \\ 0 & 0 & 0 \end{pmatrix} \\ &\quad * \begin{Bmatrix} L' & L & \lambda \\ \mathcal{L} & \mathcal{L}' & 1 \end{Bmatrix} \begin{Bmatrix} L & \lambda & L' \\ b & c & a \end{Bmatrix} \end{aligned} \tag{4.27}$$

is the solution of eqs. (4.25) and (4.26).

5. APPLICATION TO A CLUSTER MODEL OF ^{6}Li

The model [12] is designed to describe the ground and first excited state of ^{6}Li. In a LS-coupling scheme, we assign the following quantum numbers to these two states:

$\mathcal{L} = 0,\ I = 1,\ J = 1,\ \pi = +$	ground state
$\mathcal{L} = 2,\ I = 1,\ J = 3,\ \pi = +$	first excited state

where $\mathcal{L}$, I and J are the orbital, spin and total angular momentum, respectively .

5.1 Elastic Charge Form Factor and RMS Radius

For $\mathcal{L}_i = \mathcal{L}_f = \mathcal{L} = 0$, the only term which contributes to the elastic charge form factor (4.1) is that with $\lambda = 0$. In this case, the reduced matrix element (4.12) is given by

$$\begin{aligned} &\left| \langle \mathcal{L} I J \| M(C0q) \| \mathcal{L} I J \rangle_{\mathcal{L}=0,\, I=J=1} \right| \\ &= \sqrt{3} \left| \langle \mathcal{L} \| M(C0q) \| \mathcal{L} \rangle_{\mathcal{L}=0} \right| \end{aligned}$$

and we obtain from eqs. (4.1) and (4.7)

$$\begin{aligned} &|F_{el}(q)| \\ &= \sqrt{4\pi}/3 \left| \langle \mathcal{L} \| M(C0q) \| \mathcal{L} \rangle_{\mathcal{L}=0} \right| \exp\left[\frac{q^2}{4} \left(1/\beta_N - a_p^2 \right) \right] \end{aligned} \tag{5.1}$$

Due to the fact that the Model Hamiltonian [12] is independent of the spin, the GC functions (2.10) to be used in the reduced matrix elements in eq. (5.1) simplify

$$\begin{aligned} &\psi(\{\underset{\sim}{x}\} \mathcal{L} \mathcal{M}) \\ &= \sum_{L\ell} \int_0^\infty ds\, f(L \ell \mathcal{L} s)\, \phi(\{\underset{\sim}{x}\} L \ell \mathcal{L} \mathcal{M} s) \end{aligned} \tag{5.2}$$

The calculation of the elastic charge form factor is thus reduced to the determination of the GC amplitude f(L$\ell\mathcal{L}$s) by solving the corresponding Hill-Wheeler equation [9] and the evaluation of the reduced matrix elements (4.16), (4.20), (4.24) and (4.27).

According to eq. (4.10), we may obtain the rms radius from the monopole charge form factor

$$\langle r^2 \rangle = -3 \frac{d^2}{dq^2} \left| F_{C0}(q) \right|_{q=0} \tag{5.3a}$$

where

$$\begin{aligned}
|F_{C0}(q)| &= \sqrt{4\pi}/3\,(2J+1)^{-1/2} \\
&\quad * |\langle J \| M(C0q) \| J \rangle| \exp\left[\frac{q^2}{4}(1/\beta N - a_p^2)\right] \\
&= \sqrt{4\pi}/3\,(2\mathcal{L}+1)^{-1/2} |\langle \mathcal{L} \| M(C0q) \| \mathcal{L} \rangle \exp\left[\frac{q^2}{4}(1/\beta N - a_p^2)\right] \\
&= \sqrt{4\pi}/3\,(2\mathcal{L}+1)^{-1/2} \Big[\sum_{LL'\ell\ell'} \int_0^\infty ds \int_0^\infty ds'\, f(L\ell\mathcal{L}s) f(L'\ell'\mathcal{L}s') \\
&\quad * \langle \phi(L\ell\mathcal{L}s) \| M(C0q) \| \phi(L'\ell'\mathcal{L}s') \rangle \Big] \\
&\quad * \Big[\sum_{LL'\ell\ell'} \int_0^\infty ds \int_0^\infty ds'\, f(L\ell\mathcal{L}s) f(L'\ell'\mathcal{L}s') \\
&\quad * \langle \phi(L\ell\mathcal{L}s) | \phi(L'\ell'\mathcal{L}s') \rangle \Big]^{-1} \\
&\quad * \exp\left[\frac{q^2}{4}(1/\beta N - a_p^2)\right]
\end{aligned} \tag{5.3b}$$

From the development in sections 4.2 - 4.5 it is clear that we have to evaluate the following expression

$$\begin{aligned}
&(-3) \frac{d^2}{dq^2} \left\{ j_a(q p^{(i)} s)\, j_a(q p'^{(i)} s') \exp\left[-\frac{q^2}{4}\left(\frac{N-1}{\beta N} + a_p^2\right)\right] \right\}_{q=0} \\
&= \delta_{a0} \left[\frac{3}{2}\left(\frac{N-1}{\beta N} + a_p^2 \right) + (p^{(i)} s)^2 + (p'^{(i)} s')^2 \right] - \delta_{a1} \frac{2}{3} p^{(i)} p'^{(i)} s s'
\end{aligned} \tag{5.4}$$

Substitution of eq. (5.4) into eqs. (4.16), (4.20), (4.24) and (4.27) gives the corresponding expressions which are needed in eq. (5.3) to calculate the rms radius.

5.2 Inelastic Charge Form Factor and Radiative Width

The inelastic charge form factor for the transition from the ground to the first excited state ($1^+ \rightarrow 3^+$) in ^{6}Li is given by the term $\lambda = 2$ in eq. (4.1)

$$
\begin{aligned}
&|F_{inel}(q)|^2 \\
&= \frac{4\pi}{27} \left| \langle \mathcal{L}_f = 2 \;\; I_f = 1 \;\; J_f = 3 \,\|\, M(C2q) \,\|\, \mathcal{L}_i = 0 \;\; I_i = 0 \;\; J_i = 1 \rangle \right. \\
&\quad * \exp\left[- \frac{q^2}{2} \left(1/\beta N - a_p^2 \right) \right]
\end{aligned}
\tag{5.5}
$$

with the reduced matrix element

$$
\begin{aligned}
&|\langle \mathcal{L}_f = 2 \;\; I_f = 1 \;\; J_f = 3 \,\|\, M(C2q) \,\|\, \mathcal{L}_i = 0 \;\; I_i = 1 \;\; J_f = 1 \rangle|^2 \\
&= 7/5 \; |\langle \mathcal{L}_f = 2 \,\|\, M(C2q) \,\|\, \mathcal{L}_i = 0 \rangle|^2 \\
&= 7/5 \Big[\sum_{LL'\ell\ell'} \int_0^\infty ds \int_0^\infty ds' \; f(L\ell\mathcal{L}_f s) \; f(L'\ell'\mathcal{L}_i s') \\
&\qquad * \langle \phi(L\ell\mathcal{L}_f s) \,\|\, M(C2q) \,\|\, \phi(L'\ell'\mathcal{L}_i s') \rangle \Big]^2 \\
&\qquad * \Big[\sum_{LL'\ell\ell'} \int_0^\infty ds \int_0^\infty ds' \; f(L\ell\mathcal{L}_f s) \; f(L'\ell'\mathcal{L}_f s') \\
&\qquad * \langle \phi(L\ell\mathcal{L}_f s) \,|\, \phi(L'\ell'\mathcal{L}_f s') \rangle \Big]^{-1} \\
&\qquad * \Big[\sum_{LL'\ell\ell'} \int_0^\infty ds \int_0^\infty ds' \; f(L\ell\mathcal{L}_i s) \; f(L'\ell'\mathcal{L}_i s') \\
&\qquad * \langle \phi(L\ell\mathcal{L}_i s) \,|\, \phi(L'\ell'\mathcal{L}_i s') \rangle \Big]^{-1}
\end{aligned}
\tag{5.6}
$$

In order to calculate the ground state radiative width $B(E2,\, 1^+ \rightarrow 3^+)$, we use eq. (4.11). According to the foregoing treatment, we have to evaluate

$$\lim_{q \to 0} q^{-\lambda} (2\lambda+1)!! \, j_a(qp^{(i)}s) \, j_b(qp'^{(i)}s') \begin{pmatrix} \lambda & a & b \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \delta_{a+b,\lambda} \frac{(2\lambda+1)!!}{(2a+1)!!\,(2b+1)!!}$$

$$* \, (p^{(i)}s)^a \, (p'^{(i)}s')^b \begin{pmatrix} \lambda & a & b \\ 0 & 0 & 0 \end{pmatrix}$$

(5.7)

If we combine eqs. (4.11), (4.16), (4.20), (4.24), (4.27) (5.10) and (5.11), we obtain the final expression for the radiative width B(E2, $1^+ \rightarrow 3^+$).

6. DISCUSSION

The basic input to the projection method developed in this report are the formulas (3.1 a), (3.3), (3.12) and (4.13) of the unprojected many body matrix elements. The general structure of these expresssions may be derived analytically starting from single particle matrix elements and using various cofactor expansions for matrix elements of Slater determinant wave functions. This is corroborated by an independent study of the tensor properties of these matrix elements.

The specific information on the physical system of interest is contained in the numerical coefficients $u_j^{(i)}$, $v_j^{(i)}$, $w_j^{(i)}$ etc. The determination of these coefficients, using the algebraic computer code Reduce II is treated in ref. 12).

In this report, we did not dwell on the projection of the total linear momentum of the system. If we assume that the oscillator parameters β of all the single particle orbits involved are the same, the spurious contribution to matrix elements of the kinetic energy operator and the electric charge multipole operator may easily be eliminated[8,13].

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APPENDIX A

In order to derive eq. (2.4), we need the following identity

$$R_{\underset{\sim}{s}}^{-1}(\Omega)\, Y_{LM}^{*}(\hat{s}) = \sum_{M'} D_{MM'}^{L}(\Omega)\, Y_{LM'}^{*}(\Omega) \qquad \text{(A.1)}$$

which may be proved by rewriting the lefthand side,

$$R_{\underset{\sim}{s}}(-\Omega)\, Y_{L-M}(\hat{s})\, (-)^{M}$$
$$= \sum_{M'} D_{M'-M}^{L}(-\Omega)\, Y_{LM'}(\hat{s})\, (-)^{M}$$
$$= \sum_{M'} D_{-MM'}^{L}(\Omega)^{*}\, Y_{LM'}(\hat{s})\, (-)^{M}$$
$$= \sum_{M'} D_{M-M'}^{L}(\Omega)\, Y_{LM'}(\hat{s})\, (-)^{M'}$$
$$= \sum_{M'} D_{MM'}^{L}(\Omega)\, Y_{LM'}^{*}(\hat{s})$$

If we insert the expansion (2.2) into eq. (2.3) and multiply by $R_{\underset{\sim}{s}}^{-1}(\Omega)$, we find for the lefthand side

$$l.h.s. = \sum_{LM} 1/s\, Y_{LM}^{*}(\Omega)\, R_{\underset{\sim}{x}}(\Omega)\, \phi(\{\underset{\sim}{x}\}LMlms)$$

and for the righthand side

$$r.h.s. = \sum_{m'M} 1/s\, D_{m'm}^{l}(\Omega)\, \phi(\{\underset{\sim}{x}\}LMlm's)\, R_{\underset{\sim}{s}}^{-1}(\Omega)\, Y_{LM}^{*}(\hat{s})$$
$$= \sum_{m'MM'} 1/s\, D_{MM'}^{L}(\Omega)\, D_{m'm}^{l}(\Omega)\, \phi(\{\underset{\sim}{x}\}LMlm's)\, Y_{LM'}^{*}(\hat{s})$$

$$= \sum_{m'MM'\mathcal{L}\mathcal{M}\mathcal{M}'} 1/s \, (LM, lm' | \mathcal{L}\mathcal{M}')(LM', lm | \mathcal{L}\mathcal{M})$$
$$* \, D^{\mathcal{L}}_{\mathcal{M}'\mathcal{M}}(\Omega) \, \phi(\{\underset{\sim}{x}\} \, LM \, lm's) \, Y^{*}_{LM'}(\hat{s})$$

Upon comparing the coefficients of $Y^{*}_{LM'}(\hat{s})$ on both sides of this equation we obtain eq. (2.4).

$$R_{\underset{\sim}{x}}(\Omega) \, \phi(\{\underset{\sim}{x}\} \, LM' \, lm \, s)$$
$$= \sum_{\mathcal{L}\mathcal{M}\mathcal{M}'} \phi(\{\underset{\sim}{x}\} \, Ll\mathcal{L}\mathcal{M}'s) \, (LM', lm | \mathcal{L}\mathcal{M}) \, D^{\mathcal{L}}_{\mathcal{M}'\mathcal{M}}(\Omega)$$

or

$$R_{\underset{\sim}{x}}(\Omega) \, \phi(\{\underset{\sim}{x}\} \, Ll\mathcal{L}\mathcal{M} s)$$
$$= \sum_{\mathcal{M}'} D^{\mathcal{L}}_{\mathcal{M}'\mathcal{M}}(\Omega) \, \phi(\{\underset{\sim}{x}\} \, Ll\mathcal{L}\mathcal{M}'s) \qquad \text{(A2)}$$

APPENDIX B

In this appendix we show how some products of spherical harmonics of different arguments and Clebsch-Gordan coefficients may be summed over their magnetic quantum numbers in a closed form in terms of 3-j and 6-j symbols[11]

$$(i)\quad (4\pi)\sum_{\alpha\beta\gamma} Y_{a\alpha}^{*}(\hat{s})\,Y_{c\gamma}(\hat{s})\,Y_{b\beta}^{*}(\hat{s}')\,Y_{c\gamma}^{*}(\hat{s}')\begin{pmatrix}\lambda & a & b\\ \mu & \alpha & \beta\end{pmatrix}$$

$$=\sum_{\alpha\beta\gamma LML'M'} \hat{a}\,\hat{b}\,\hat{c}^{2}\,\hat{L}\,\hat{L}'\begin{pmatrix}a & c & L\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}b & c & L'\\ 0 & 0 & 0\end{pmatrix}(-)^{b+c+L'+\gamma}$$

$$*\begin{pmatrix}\lambda & a & b\\ \mu & \alpha & \beta\end{pmatrix}\begin{pmatrix}c & L' & b\\ -\gamma & -M' & -\beta\end{pmatrix}\begin{pmatrix}a & c & L\\ \alpha & -\gamma & M\end{pmatrix} Y_{LM}(\hat{s})\,Y_{L'M'}(\hat{s}')$$

$$=\sum_{LML'M'} \hat{a}\,\hat{b}\,\hat{c}^{2}\,\hat{L}\,\hat{L}'\,\hat{\lambda}^{-1}\begin{pmatrix}a & c & L\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}b & c & L'\\ 0 & 0 & 0\end{pmatrix}$$

$$*(-)^{L+a}\begin{Bmatrix}\lambda & a & b\\ c & L' & L\end{Bmatrix}(LM,L'M'|\lambda\mu)\,Y_{LM}(\hat{s})\,Y_{L'M'}(\hat{s}')$$

$$=\sum_{LL'} \hat{a}\,\hat{b}\,\hat{c}^{2}\,\hat{L}\,\hat{L}'\,\hat{\lambda}^{-1}\begin{pmatrix}a & c & L\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}b & c & L'\\ 0 & 0 & 0\end{pmatrix}$$

$$*(-)^{L+a}\begin{Bmatrix}\lambda & a & b\\ c & L' & L\end{Bmatrix} B(LL'\lambda\mu\hat{s}\hat{s}')$$

(B.1)

(ii) $(4\pi) \sum_{\mu} Y_{1m'}(\hat{s})\, Y_{\lambda\mu}(\hat{s})\, Y^{*}_{1m}(\hat{s}')\, Y^{*}_{\lambda\mu}(\hat{s}')$

$$= \sum_{\mu L M L' M'} \hat{1}^2 \hat{\lambda}^2 \hat{L} \hat{L}' \begin{pmatrix} 1 & \lambda & L \\ m' & \mu & -M \end{pmatrix} \begin{pmatrix} 1 & \lambda & L \\ 0 & 0 & 0 \end{pmatrix}$$

$$* \begin{pmatrix} 1 & \lambda & L' \\ m & \mu & -M' \end{pmatrix} \begin{pmatrix} 1 & \lambda & L' \\ 0 & 0 & 0 \end{pmatrix} (-)^{M+M'} Y_{LM}(\hat{s})\, Y^{*}_{L'M'}(\hat{s}')$$

$$= \sum_{\mu L M L' M'} \hat{1}^2 \hat{\lambda}^2 \hat{L} \hat{L}' \begin{pmatrix} 1 & L' & \lambda \\ m & -M' & \mu \end{pmatrix} \begin{pmatrix} 1 & L & \lambda \\ -m' & M & -\mu \end{pmatrix} (-)^{1+\lambda+L'}$$

$$* \begin{pmatrix} 1 & \lambda & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \lambda & L' \\ 0 & 0 & 0 \end{pmatrix} (-)^{M+M'} Y_{LM}(\hat{s})\, Y^{*}_{L'M'}(\hat{s}')$$

$$= \sum_{L M L' M' \mathcal{L} \mathcal{M}} \hat{1}^2 \hat{\lambda}^2 \hat{L} \hat{L}' \hat{\mathcal{L}}^2 \begin{Bmatrix} 1 & L' & \lambda \\ 1 & L & \mathcal{L} \end{Bmatrix} \begin{pmatrix} 1 & \lambda & L \\ 0 & 0 & 0 \end{pmatrix}$$

$$* \begin{pmatrix} 1 & \lambda & L' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & 1 & \mathcal{L} \\ M & m & -\mathcal{M} \end{pmatrix} \begin{pmatrix} L' & 1 & \mathcal{L} \\ M' & m' & -\mathcal{M} \end{pmatrix} Y_{LM}(\hat{s})\, Y^{*}_{L'M'}(\hat{s}') \tag{B.2}$$

(iii) $(4\pi) \sum_{m m' M'} Y^{*}_{L'M'}(\hat{s})\, Y^{*}_{1m}(\hat{s})\, Y_{1m'}(\hat{s})\, Y_{L'M'}(\hat{s}')$

$$* \hat{\mathcal{L}}' (-)^{\mathcal{M}'+m} \begin{pmatrix} 1 & 1 & \mathcal{L}' \\ -m & m' & -\mathcal{M}' \end{pmatrix}$$

$$= \sqrt{4\pi} \sum_{m m' L M M'} \hat{1}^2 \hat{L} \hat{\mathcal{L}}' \begin{pmatrix} 1 & 1 & L \\ -m & m' & M \end{pmatrix} \begin{pmatrix} 1 & 1 & L \\ 0 & 0 & 0 \end{pmatrix}$$

(continued next page)

$$* \begin{pmatrix} 1 & 1 & \mathcal{L}' \\ -m & m' & -\mathcal{M}' \end{pmatrix} (-)^{\mathcal{M}'} Y^*_{L'M'}(\hat{s}) Y^*_{LM}(\hat{s}) Y_{L'M'}(\hat{s}')$$

$$= \sqrt{4\pi} \sum_{M'} \hat{1}^2 \begin{pmatrix} 1 & 1 & \mathcal{L}' \\ 0 & 0 & 0 \end{pmatrix} (-)^{\mathcal{M}'} Y^*_{L'M'}(\hat{s}) Y^*_{\mathcal{L}'-\mathcal{M}'}(\hat{s}) Y_{L'M'}(\hat{s}')$$

$$= \sum_{LMM'} \hat{1}^2 \hat{L} \hat{L}' \hat{\mathcal{L}}' \begin{pmatrix} 1 & 1 & \mathcal{L}' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & L' & \mathcal{L}' \\ M & \mu & -\mathcal{M}' \end{pmatrix} \begin{pmatrix} L' & \mathcal{L}' & L \\ 0 & 0 & 0 \end{pmatrix}$$
$$* (-)^{\mathcal{M}'} Y_{LM}(\hat{s}) Y_{L'M'}(\hat{s}')$$

$$= \sum_{L} \hat{1}^2 \hat{L} \hat{L}' (-)^{\mathcal{L}'} \begin{pmatrix} 1 & 1 & \mathcal{L}' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L' & \mathcal{L}' & L \\ 0 & 0 & 0 \end{pmatrix} B(L L' \mathcal{L}' \mathcal{M}' \hat{s} \hat{s}') \tag{B.3}$$

(iv) $$(4\pi) \sum_{mm'M} Y_{LM}(\hat{s}) Y^*_{1m}(\hat{s}') Y_{1m'}(\hat{s}') Y^*_{LM}(\hat{s}')$$
$$* \begin{pmatrix} 1 & 1 & \mathcal{L}' \\ -m & m' & -\mathcal{M}' \end{pmatrix} (-)^{\mathcal{M}'+m} \hat{\mathcal{L}}'$$

$$= \sum_{L'} \hat{1}^2 \hat{L} \hat{L}' \begin{pmatrix} 1 & 1 & \mathcal{L}' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & \mathcal{L}' & L' \\ 0 & 0 & 0 \end{pmatrix} (-)^{\mathcal{L}'} B(L' L \mathcal{L}' \mathcal{M}' \hat{s}' \hat{s})$$

$$= \sum_{L'} \hat{1}^2 \hat{L} \hat{L}' \begin{pmatrix} 1 & 1 & \mathcal{L}' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & \mathcal{L}' & L' \\ 0 & 0 & 0 \end{pmatrix} (-)^{\mathcal{L}'} B(L L' \mathcal{L}' \mathcal{M}' \hat{s} \hat{s}') \tag{B.4}$$

where we used eq. (B.3) and the symmetry relation

$$B(L' L \mathcal{L}' \mathcal{M}' \hat{s}' \hat{s}) = (-)^{L+\lambda+\mathcal{L}'} B(L L' \mathcal{L}' \mathcal{M}' \hat{s} \hat{s}')$$

(v) $$(4\pi)^{3/2} \sum_{\alpha\beta\gamma} Y_{1m'}(\hat{s})\, Y_{a\alpha}^{*}(\hat{s})\, Y_{c\gamma}(\hat{s})\, Y_{b\beta}^{*}(s')$$

$$* \, Y_{c\gamma}^{*}(\hat{s}') \begin{pmatrix} \lambda & a & b \\ \mu & \alpha & \beta \end{pmatrix}$$

$$= (4\pi)^{1/2} \sum_{d\delta L'M'} \hat{a}\,\hat{b}\,\hat{c}^2\,\hat{d}\,\hat{L}' \begin{pmatrix} a & c & d \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} b & c & L' \\ 0 & 0 & 0 \end{pmatrix}$$

$$* (-)^{L'+a+\mu} \begin{Bmatrix} \lambda & L' & d \\ c & a & b \end{Bmatrix} \begin{pmatrix} d & L' & \lambda \\ \delta & M' & -\mu \end{pmatrix}$$

$$* \, Y_{1m'}(\hat{s})\, Y_{d\delta}(\hat{s})\, Y_{L'M'}(\hat{s}')$$

$$= \sum_{d\delta LML'M'f\phi} \hat{a}\,\hat{b}\,\hat{c}^2\,\hat{d}^2\,\hat{1}\,\hat{L}\,\hat{L}'\,\hat{f} \begin{pmatrix} a & c & d \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} b & c & L' \\ 0 & 0 & 0 \end{pmatrix}$$

$$* \begin{pmatrix} 1 & d & L \\ 0 & 0 & 0 \end{pmatrix} (-)^{L+a+\mu+M+\phi} \begin{Bmatrix} \lambda & L' & d \\ c & a & b \end{Bmatrix} \begin{pmatrix} \lambda & d & L' \\ -\mu & \delta & M' \end{pmatrix}$$

$$* \begin{pmatrix} L & f & L' \\ -M & \phi & -M' \end{pmatrix} \begin{pmatrix} d & L & 1 \\ \delta & -M & m' \end{pmatrix} B(LL'f\phi\hat{s}\hat{s}')$$

$$= \sum_{dLL'f\phi} \hat{a}\,\hat{b}\,\hat{c}^2\,\hat{d}^2\,\hat{1}\,\hat{L}\,\hat{L}'\,\hat{f} \begin{pmatrix} a & c & d \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} b & c & L' \\ 0 & 0 & 0 \end{pmatrix}$$

$$* \begin{pmatrix} 1 & d & L \\ 0 & 0 & 0 \end{pmatrix} (-)^{L+L'+a+1+\phi} \begin{Bmatrix} \lambda & L' & d \\ c & a & b \end{Bmatrix}$$

$$* \begin{Bmatrix} \lambda & d & L' \\ L & f & 1 \end{Bmatrix} \begin{pmatrix} \lambda & 1 & f \\ \mu & m' & -\phi \end{pmatrix} B(LL'f\phi s s')$$

(B.5)

(vi) $\displaystyle\sum_{MM'\mathcal{M}} (-)^{\mathcal{M}+M} \begin{pmatrix} L' & 1 & \mathcal{L} \\ M' & m' & -\mathcal{M}\end{pmatrix}\begin{pmatrix} L & \lambda & \mathcal{L} \\ -M & \mu & \mathcal{M}\end{pmatrix} Y_{LM}(\hat{s})\, Y^{*}_{L'M'}(\hat{s}')$

$$= \sum_{MM'\mathcal{M}} \begin{pmatrix} L & \lambda & \mathcal{L} \\ M & -\mu & -\mathcal{M}\end{pmatrix}\begin{pmatrix} 1 & L' & \mathcal{L} \\ -m' & M' & \mathcal{M}\end{pmatrix} (-)^{M+M'+\mathcal{M}+L+\lambda+\mathcal{L}}$$

$$* \; Y_{LM}(\hat{s})\, Y_{L'M'}(\hat{s}')$$

$$= \sum_{MM'f\phi} \hat{f}^2 \begin{Bmatrix} L & \lambda & \mathcal{L} \\ 1 & L' & f\end{Bmatrix}\begin{pmatrix} L & L' & f \\ M & M' & -\phi\end{pmatrix}\begin{pmatrix} \lambda & 1 & f \\ -\mu & -m' & \phi\end{pmatrix}$$

$$* \; (-)^{L+1}\, Y_{LM}(\hat{s})\, Y_{L'M'}(\hat{s}')$$

$$= \sum_{f\phi} \hat{f} \begin{Bmatrix} 1 & L' & \mathcal{L} \\ L & \lambda & f\end{Bmatrix}\begin{pmatrix} \lambda & 1 & f \\ \mu & m' & -\phi\end{pmatrix} (-)^{L'+\lambda+f+\phi}$$

$$* \; B(LL'f\phi\,\hat{s}\hat{s}') \tag{B.6}$$

(vii) $\displaystyle\sum_{MM'\mathcal{M}} (-)^{L+1} \begin{pmatrix} L & 1 & \mathcal{L} \\ M & m & -\mathcal{M}\end{pmatrix}\begin{pmatrix} \mathcal{L} & \lambda & L' \\ -\mathcal{M} & \mu & M'\end{pmatrix} Y_{LM}(\hat{s})\, Y^{*}_{L'M'}(\hat{s}')$

$$= \sum_{MM'\mathcal{M}} \begin{pmatrix} L' & \lambda & \mathcal{L} \\ M' & -\mu & \mathcal{M}\end{pmatrix}\begin{pmatrix} 1 & L & \mathcal{L} \\ m & M & -\mathcal{M}\end{pmatrix} (-)^{\mathcal{L}+M'}\, Y_{LM}(\hat{s})\, Y_{L'M'}(\hat{s}')$$

$$= \sum_{MM'f\phi} \hat{f}^2 \begin{Bmatrix} L' & \lambda & \mathcal{L} \\ 1 & L & f\end{Bmatrix}\begin{pmatrix} L & L' & f \\ M & M' & -\phi\end{pmatrix}\begin{pmatrix} \lambda & 1 & f \\ -\mu & m & \phi\end{pmatrix}$$

$$* \; (-)^{f+L+L'+1+\lambda+m}\, Y_{LM}(\hat{s})\, Y_{L'M'}(\hat{s}')$$

$$= \sum_{f\phi} \hat{f} \begin{Bmatrix} L' & \lambda & \mathcal{L} \\ 1 & L & f\end{Bmatrix}\begin{pmatrix} \lambda & 1 & f \\ \mu & -m & -\phi\end{pmatrix} B(LL'f\phi\,\hat{s}\hat{s}')\, (-)^{\mu}$$

(B.7)

(viii')

$$(4\pi)^{3/2}\sum_{\alpha\beta\gamma} Y^*_{a\alpha}(\hat{s})\, Y_{c\gamma}(\hat{s})\, Y^*_{b\beta}(\hat{s}')\, Y^*_{c\gamma}(\hat{s}')\, Y^*_{1m}(\hat{s}') \begin{pmatrix} \lambda & a & b \\ \mu & \alpha & \beta \end{pmatrix}$$

$$= (4\pi)^{1/2} \sum_{LM d\delta} \hat{a}\,\hat{b}\,\hat{c}^2\,\hat{d}\,\hat{L} \begin{pmatrix} a & c & L \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} b & c & d \\ 0 & 0 & 0 \end{pmatrix} (-)^{a+d+\mu+\delta}$$

$$* \begin{pmatrix} L & d & \lambda \\ M & -\delta & -\mu \end{pmatrix} \begin{Bmatrix} \lambda & a & b \\ c & d & L \end{Bmatrix} Y_{LM}(\hat{s})\, Y^*_{d\delta}(\hat{s}')\, Y^*_{1m}(\hat{s}')$$

$$= \sum_{d\delta LML'M'f\phi} \hat{a}\,\hat{b}\,\hat{c}^2\,\hat{d}^2\,\hat{1}\,\hat{L}\,\hat{L}'\,\hat{f} \begin{pmatrix} a & c & L \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} b & c & d \\ 0 & 0 & 0 \end{pmatrix}$$

$$* \begin{pmatrix} d & 1 & L' \\ 0 & 0 & 0 \end{pmatrix} (-)^{a+L+f+\lambda+\mu+\delta+\phi} \begin{Bmatrix} \lambda & a & b \\ c & d & L \end{Bmatrix}$$

$$* \begin{pmatrix} 1 & d & L' \\ -m & -\delta & -M' \end{pmatrix}\begin{pmatrix} L & f & L' \\ M & -\phi & M' \end{pmatrix}$$

$$* \begin{pmatrix} d & L & \lambda \\ -\delta & M & -\mu \end{pmatrix} B(LL'f\phi\hat{s}\hat{s}')$$

$$= \sum_{dLL'f\phi} \hat{a}\,\hat{b}\,\hat{c}^2\,\hat{d}^2\,\hat{1}\,\hat{L}\,\hat{L}'\,\hat{f} \begin{pmatrix} a & c & L \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} b & c & d \\ 0 & 0 & 0 \end{pmatrix}$$

$$* \begin{pmatrix} d & 1 & L' \\ 0 & 0 & 0 \end{pmatrix} (-)^{a+d+\lambda+f+L'+\mu} \begin{Bmatrix} \lambda & a & b \\ c & d & L \end{Bmatrix}$$

$$* \begin{Bmatrix} 1 & d & L' \\ L & f & \lambda \end{Bmatrix} \begin{pmatrix} \lambda & 1 & f \\ \mu & -m & -\phi \end{pmatrix} B(LL'f\phi\hat{s}\hat{s}')$$

(B.8)

(ix)

$$\sum_{MM'\mathcal{M}\mathcal{M}'} (-)^{\mathcal{M}'} \begin{pmatrix} L & 1 & \mathcal{L} \\ M & m & -\mathcal{M} \end{pmatrix} \begin{pmatrix} L' & 1 & \mathcal{L}' \\ M' & m' & -\mathcal{M}' \end{pmatrix} \begin{pmatrix} \mathcal{L} & \lambda & \mathcal{L}' \\ -\mathcal{M} & \mu & \mathcal{M}' \end{pmatrix}$$

$$* \, Y_{LM}(\hat{s}) \; Y^{*}_{L'M'}(\hat{s}')$$

$$= \sum_{MM'\mathcal{M}\mathcal{M}'} (-)^{L+1+\mathcal{L}+m'} \begin{pmatrix} \lambda & \mathcal{L}' & \mathcal{L} \\ \mu & \mathcal{M}' & -\mathcal{M} \end{pmatrix} \begin{pmatrix} \mathcal{L}' & 1 & L' \\ \mathcal{M}' & -m' & M' \end{pmatrix}$$

$$* \begin{pmatrix} \mathcal{L} & 1 & L \\ -\mathcal{M} & m & M \end{pmatrix} Y_{LM}(\hat{s}) \, Y_{L'M'}(\hat{s}')$$

$$= \sum_{b\beta c\gamma MM'} \hat{b}^2 \hat{c}^2 (-)^{L'+c+1+\mathcal{L}+m'} \begin{Bmatrix} \lambda & b & c \\ \mathcal{L}' & 1 & L' \\ \mathcal{L} & 1 & L \end{Bmatrix}$$

$$* \begin{pmatrix} \lambda & b & c \\ \mu & \beta & -\gamma \end{pmatrix} \begin{pmatrix} b & 1 & 1 \\ \beta & -m' & m \end{pmatrix} \begin{pmatrix} L & L' & c \\ M & M' & -\gamma \end{pmatrix} Y_{LM}(\hat{s}) \, Y_{L'M'}(\hat{s}')$$

$$= \sum_{b\beta c\gamma} \hat{b}^2 \hat{c} \, (-)^{L+c+1+\mathcal{L}+m'+\gamma} \begin{pmatrix} \lambda & b & c \\ \mu & \beta & -\gamma \end{pmatrix}$$

$$* \begin{pmatrix} b & 1 & 1 \\ \beta & -m' & m \end{pmatrix} \begin{Bmatrix} \lambda & b & c \\ \mathcal{L}' & 1 & L' \\ \mathcal{L} & 1 & L \end{Bmatrix} B(LL'c\gamma \, \hat{s}\hat{s}') \qquad \text{(B. 9)}$$

APPENDIX C

We give a derivation of eq. (4.20): If we multiply eq. (4.19) by $\hat{\mathcal{L}}\ \hat{f}\ \begin{Bmatrix} 1 & L' & \mathcal{L} \\ L & \lambda & f \end{Bmatrix}$ and sum over f we may exploit on the lefthand side the orthogonality (eq. (6.2.10) of ref. 10)of the 6-j symbols and obtain the reduced matrix element $\langle\phi(Ls)||M(C\lambda q)||\phi'(L'1\mathcal{L}s)\rangle$. On the righthand side, the sum over f may be carried out with the help of the sum rule (eq. (6.2.11) of ref. 10)) of the 6-j symbols. Indeed, from

$$\sum_f (-)^{f+\mathcal{L}+d}\,\hat{f}^2 \begin{Bmatrix} 1 & L & d \\ L' & \lambda & f \end{Bmatrix} \begin{Bmatrix} 1 & L' & \mathcal{L} \\ L & \lambda & f \end{Bmatrix} = \begin{Bmatrix} L' & 1 & \mathcal{L} \\ L & \lambda & d \end{Bmatrix} \tag{C.1}$$

we find eq. (4.20).

The proof of eq. (4.24) is similar. Multiplying eq. (4.23) by $(-)^{\mathcal{L}}\ \hat{\mathcal{L}}\ \hat{f}\ \begin{Bmatrix} 1 & L & \mathcal{L} \\ L' & \lambda & f \end{Bmatrix}$ and summing over f gives on the lefthand side the reduced matrix element $\langle\phi(1L\mathcal{L}s)||M(C\lambda q)||\phi\ (L's)\rangle$. On the righthand side we use again the sum rule (C.1).