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A Complete and Certain Riemann Solver

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A Complete and Certain Riemann Solver

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Abstract:

An efficient algorithm is presented for solving the Riemann problem for polytropic gas. It enables the user to compute the solution for all physically reasonable data. The convergence of the algorithm is shown. The accuracy of the solution is limited only by the accuracy of the computing machine. There is an a-priori estimation of the required number of iterations. The rate of convergence turns out to be much higher than that of the usual fixed point iteration scheme.

Ein vollständiger und sicherer Riemannlöser

Kurzfassung:

Es wird ein effizienter Algorithmus zur Lösung des Riemannproblems für polytropes Gas vorgestellt. Er ermöglicht die Berechnung der Lösung für alle physikalisch sinnvollen Vorgaben. Die Konvergenz des Verfahrens wird gezeigt. Die Genauigkeit der Lösung wird nur durch die Maschinengenauigkeit begrenzt. Es gibt eine a-priori Abschätzung der Anzahl der erforderlichen Iterationen. Die Konvergenzgeschwindigkeit erweist sich als viel besser als die des üblichen Fixpunktiterationsverfahrens.

AMS Subject Classifications: 65M99, 76L05, 76N15

1. Introduction

The Riemann problem for gas dynamics is an important subject in applied science in two respects. Firstly, there is the possibility of getting exact solutions to the equations of gas dynamics and of comparing numerical results of any calculation scheme [13,14,20] with them. Secondly, Riemann solvers are a central element of several fluid dynamics codes like the random choice numerical methods [2,7,8,20] and others [4].

An initial value problem for a hyperbolic system of conservation laws

$$u_t + f(u)_x = 0 \quad , \quad -\infty < x < \infty \ , \ 0 \le t < \infty \ , \tag{1}$$

with the initial condition

$$u(x,0) = \begin{cases} u_l & , \quad x < 0 \ , \\ u_r & , \quad x > 0 \ , \end{cases} \tag{2}$$

where u_l, u_r are given constant vectors, is called a Riemann problem. The question of existence and uniqueness of its solution is nontrivial in general. The conservation laws of compressible inviscid fluid dynamics are of primary interest in practical applications. Results for the general case have been published in [9,15] and more recently in [18,19]. The computation of the solution is described in this paper. For simplicity the equation of state is assumed to be that of a polytropic gas. But there are no basic difficulties in handling other equations of state [4,7]. The conservation quantity u and the flux f(u) in (1) are:

$$u = \begin{pmatrix} \rho \\ m \\ e \end{pmatrix} \quad , \begin{cases} \rho \ : \ \text{mass density} \ , \\ m \ : \ \text{momentum density} \ , \\ e \ : \ \text{total energy density} \ , \end{cases} \tag{3}$$

$$f(u) = \begin{pmatrix} m \\ (\gamma-1)e + ((3-\gamma)/2)(m^2/\rho) \\ (\gamma e - ((\gamma-1)/2)(m^2/\rho))(m/\rho) \end{pmatrix} , \quad \gamma : \text{adiabatic exp.} \tag{4}$$

The substitutions

$$u := m/\rho \quad \text{and} \quad p := (\gamma-1)(e - \tfrac{1}{2}(m^2/\rho)) \quad , \tag{5}$$

where u and p denote the velocity and the pressure of the fluid, respectively, lead to the system of equations in the non conservation form

$$\begin{aligned} \rho_t + \rho u_x + u\rho_x &= 0 \\ u_t + uu_x + p_x/\rho &= 0 \\ p_t + up_x + \gamma p u_x &= 0 \end{aligned} \quad . \tag{6}$$

The initial conditions are

$$(\rho,u,p)^T = \begin{cases} (\rho_l,u_l,p_l)^T \quad , & x < 0 \quad , \\ (\rho_r,u_r,p_r)^T \quad , & x > 0 \quad , \end{cases} \tag{7}$$

with given constant values $\rho_l,\rho_r,u_l,u_r,p_l,p_r$. The following restrictions apply to the inital data

$$\begin{aligned} &\rho_l \geq 0 \quad , \quad p_l \geq 0 \quad , \quad \text{if} \ \rho_l = 0 \ \text{then} \ p_l = 0 \ , \\ &\rho_r \geq 0 \quad , \quad p_r \geq 0 \quad , \quad \text{if} \ \rho_r = 0 \ \text{then} \ p_r = 0 \ . \end{aligned} \tag{8}$$

The inital value problem described has no classical solution in general. The weak solution however is unique [18] and may contain discontinuities and constant states, i.e. a region in the x-t-plane where $(\rho,u,p)^T$ is constant. Since perturbations propagate at a finite speed there are two constant states S_l and S_r resulting from the constant initial values $(\rho_l,u_l,p_l)^T$ and $(\rho_r,u_r,p_r)^T$. In the general case there is a third state S_* being divided by a slip line into two constant states I and II and separated by centered waves from the constant states S_l and S_r. The velocity and the pressure in I and II are the same and therefore denoted by u_* and p_*, respectively. The centered waves are shocks or rarefaction waves. The solution to the Riemann problem depends on the initial values. The constant states S_l and S_r exist in any case but S_* or one of the constant states I and II may not appear. The slip line and the centered waves need not occur either. If there are two centered waves they need not be of different types, i.e. both of them can be shocks or rarefaction waves. The solution is constant on the lines x/t = const hence the boundaries of the centered waves and the constant states are straight lines originating in x = 0 , t = 0. A qualitative example of a solution is given in Fig. 1. The left centered wave is a rarefaction wave while the right one is a shock.

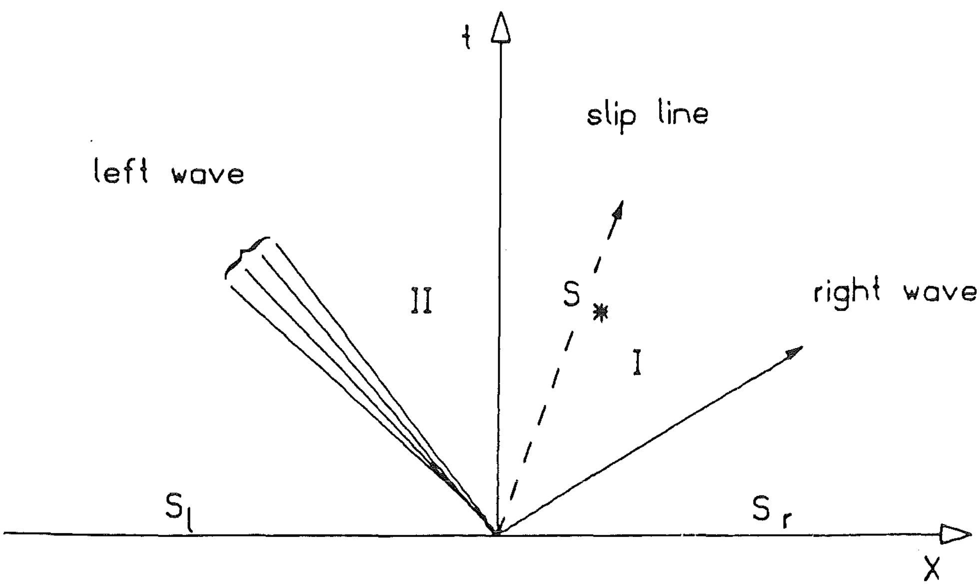

Fig. 1. The solution to the Riemann problem

2. The fixed point problem

Solving the Riemann problem means determining the values $p_*, u_*, \rho_I, \rho_{II}$, the type and location of the centered waves and the quantities inside a rarefaction fan if there is one. The central task has been shown to be the calculation of p_* [2,3,16,20]. When p_* has been determined, all other work is accomplished easily (for details see [20]).

Defining the quantities (misprint in [20])

$$M_l := -(p_l - p_*)/(u_l - u_*) \quad , \quad M_r := (p_r - p_*)/(u_r - u_*) \quad , \tag{9}$$

one can show that

$$M_l = (p_l \rho_l)^{\frac{1}{2}} \Phi(p_*/p_l) \quad , \quad M_r = (p_r \rho_r)^{\frac{1}{2}} \Phi(p_*/p_r) \tag{10}$$

with

$$\Phi(w) := \begin{cases} (((\gamma+1)w+\gamma-1)/2)^{\frac{1}{2}} & , \quad 1 \le w \quad , \\ ((\gamma-1)/(2\sqrt{\gamma}))((1-w)/(1-w^{\kappa})) & , \quad 0 \le w < 1 \quad , \\ \kappa := (\gamma-1)/(2\gamma) \quad . & \end{cases} \tag{10a}$$

Eliminating u_* in (9) one obtains

$$p_* = ((u_l - u_r)M_l M_r + p_l M_r + p_r M_l)/(M_l + M_r) \tag{11}$$

and the substitution of M_l and M_r given in (10) leads to

$$p_* = F(p_*) \tag{12}$$

where the function F contains the initial values as parameters. Looking for a fixed point of F is equivalent to the search of a zero of the function $G(p_*) := F(p_*) - p_*$ in $[0,\infty)$.

Theorem: The fixed point problem (12) is solvable iff

$$u_l - u_r + 2(c_l + c_r)/(\gamma - 1) \geq 0 \tag{13}$$

where c_l, c_r denote the values of the local sound velocity

$$c = (\gamma p/\rho)^{\frac{1}{2}} \tag{14}$$

in the states S_l, S_r.

The idea underlying the proof (for details see [11]) is the following. If $G(0) < 0$ then $G(p_*) < 0$ for all p_* in $(0,\infty)$. G is continuous and $G(p_*) \to -\infty$ if $p_* \to \infty$. Since (13) is equivalent to $G(0) \geq 0$ the proof is complete.

3. The solution to the fixed point problem

The criterion (13) contains only the initial data and is easy to handle. It permits one to decide whether the fixed point problem is solvable or not.

Let us assume (13) to be fulfilled. Now the question is how to calculate the solution. A modified form of the so-called Godunov iteration [10] is recommended in some papers [2,20]. Starting from the value

$$p_*^0 := \tfrac{1}{2}(p_l + p_r) \tag{15}$$

an iteration

$$p_*^{n+1} := \alpha F(p_*^n) + (1-\alpha)p_*^n \quad , \quad n=0,1,2,\ldots, \tag{16}$$

follows. The parameter α is initially chosen to be 1. The iteration is finished after having reached a certain accuracy (see [2]), otherwise it is continued. After L iterations (proposal in [2]: $L = 20$) α is taken to be one half of the previous value of α. The convergence of this procedure has not been proved and not all of our practical computations confirm the good experience reported in [2].

In [4] a secant iteration is presented. The first two guesses used to start the iteration are obtained from Godunov's iteration scheme. As with the above method, the convergence of this method is also not certain.

The Brent algorithm [1] is a combination of the bisection method and the secant iteration which retains the advantages and avoids the disadvantages of both procedures. The calculation effort is not significantly greater, i.e. there is only one function evaluation per iteration. The convergence is certain and an a-priori estimation of the required number of iterations can be made as for the bisection method. The rate of convergence is that of the secant method. The only requirement for the applicability of the Brent algorithm is to find an interval where the function changes its sign, i.e. an interval (p_{*min}, p_{*max}) has to be found in which G has its zero. Since (13) is now assumed to be valid one knows that $G(0) > 0$ and p_{*min} can be taken to be equal to zero. An appropriate value p_{*max} with $G(p_{*max}) \leq 0$ is required. Some estimations [11] lead to

$$p_{*max} = \begin{cases} \max\{p_l, p_r\} & \text{if } u_l - u_r \leq 0 \,, \\ \max\{p_l, p_r, p_+\} & \text{if } u_l - u_r > 0 \,, \end{cases} \tag{17}$$

where

$$p_+ := \frac{2\gamma^2(u_l-u_r)^2\rho_l\rho_r}{(\gamma+1)(\sqrt{\rho_l}+\sqrt{\rho_r})^2} + 4\,\frac{p_r\sqrt{\rho_l}+p_l\sqrt{\rho_r}}{\sqrt{\rho_l}+\sqrt{\rho_r}} \quad . \tag{17a}$$

Now the Brent algorithm can be applied. The desired accuracy of the result has to be specified and is only limited by the accuracy of the computing machine. In order to compare the Godunov iteration with the Brent algorithm one hundred data sets have been chosen arbitrarily for the Riemann problem all of them satisfying (13). Both methods have been successfully applied and furnished the same results within the accuracy required. The number of iterations needed is shown in Fig. 2. For a few cases, the Godunov scheme converged very quickly and necessitated only few iterations. However, the average number of iterations was 24.32 for the Godunov method and only 6.97 for the Brent algorithm. The superiority of the Brent algorithm is quite obvious.

4. The complete Riemann solver

If the solvability condition (13) of the fixed point problem is not fulfilled the solution to the Riemann problem can be written explicitly. It contains two rarefaction waves and a vacuum zone in between. With

$$\begin{aligned} u_{Vl} &= ((\gamma-1)u_l+2(x/t+c_l))/(\gamma+1) \\ \rho_{Vl} &= ((u_{Vl}-x/t)^2\rho_l^{\gamma}/(\gamma p_l))^{(1/(\gamma-1))} \\ p_{Vl} &= (\rho_{Vl}/\rho_l)^{\gamma} p_l \end{aligned} \tag{18}$$

and

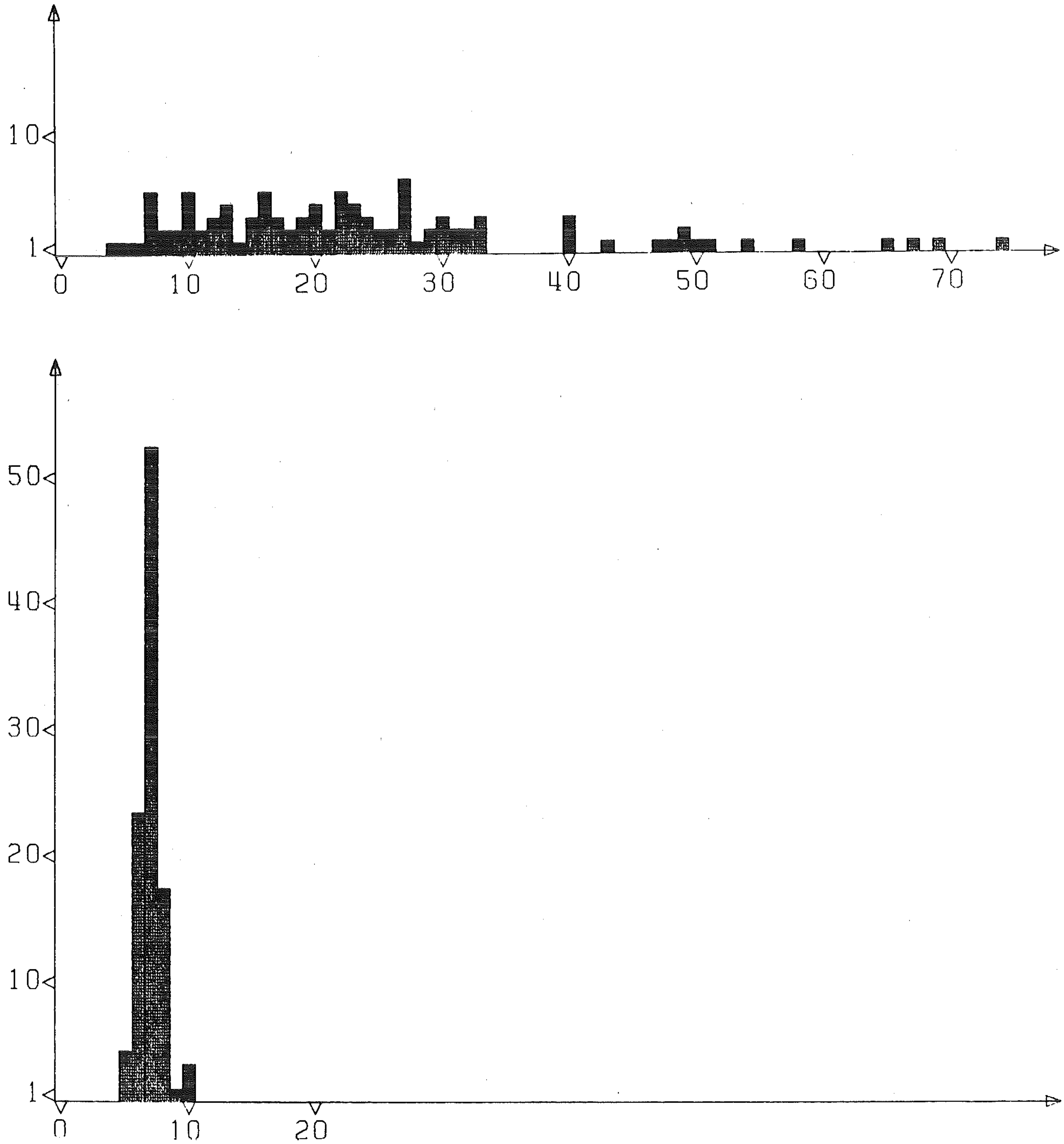

Fig. 2. The required number of iterations of the Godunov method (upper figure) and of the Brent algorithm (lower figure)

$$\begin{aligned}
u_{Vr} &= ((\gamma-1)u_r+2(x/t-c_r))/(\gamma+1) \\
\rho_{Vr} &= ((x/t-u_{Vr})^2\rho_r^{\gamma}/(\gamma p_r))^{(1/(\gamma-1))} \\
p_{Vr} &= (\rho_{Vr}/\rho_r)^{\gamma} p_r
\end{aligned} \tag{19}$$

one can define

$$(\rho,u,p)^T = \begin{cases} (\rho_1,u_1,p_1)^T & , \quad x/t < u_1-c_1 \\ (\rho_{V1},u_{V1},p_{V1})^T & , \quad u_1-c_1 < x/t < u_1+2c_1/(\gamma-1) \\ \text{vacuum} & , \quad u_1+2c_1/(\gamma-1) < x/t < u_r-2c_r/(\gamma-1) \\ (\rho_{Vr},u_{Vr},p_{Vr})^T & , \quad u_r-2c_r/(\gamma-1) < x/t < u_r+c_r \\ (\rho_r,u_r,p_r)^T & , \quad u_r+c_r < x/t\ . \end{cases} \tag{20}$$

The solution to the Riemann problem cannot be given by the quantities ρ,u,p. But looking at (5) the transformation

$$m = \rho u \quad , \quad e = p/(\gamma-1)+\tfrac{1}{2}\rho u^2 \tag{21}$$

leads to a (weak) solution of the conservation laws after substituting u and p from (20) and defining $\rho = m = e = 0$ in the vacuum zone. From the point of view of physics the natural variables in fluid mechanics are the conservation quantities ρ, m, e. There are also mathematical reasons to use these coordinates, especially if vacuum regions occur. But in the computations one uses ρ,u,p and refers to (21).

The Riemann solver described above is not yet complete. Some special data sets for the Riemann problem (like in [14]) to which the previous formulas do not apply need to be considered. If for example $p_1 = 0$ ($p_r = 0$) and $\rho_1 > 0$ ($\rho_r > 0$) the quantitiy M_1 (M_r) cannot be computed from (10). Some investigations [11] lead to

$$\begin{aligned}
M_1 &= ((\gamma+1)\rho_1 p_*/2)^{\frac{1}{2}} \quad \text{if} \quad p_1 = 0 \quad , \quad \rho_1 > 0 \quad , \\
(M_r &= ((\gamma+1)\rho_r p_*/2)^{\frac{1}{2}} \quad \text{if} \quad p_r = 0 \quad , \quad \rho_r > 0\ ).
\end{aligned} \tag{22}$$

The rest of the procedure can be accomplished as above, with only one exception. The function G now has two zeros if

$$u_l - u_r + 2(c_l+c_r)/(\gamma-1) > 0 \tag{23}$$

one of which is $p_* = 0$, and does not provide a solution to the Riemann problem. In order to compute the other zero $p_* > 0$ by means of the Brent algorithm, one should give p_{*min} any appropriate small positive value. If (23) is not satisfied the solution is obtained from (20), with the left (right) rarefaction wave vanishing, since $c_l = 0$ ($c_r = 0$). If the initial data contain a vacuum region, i.e. $\rho_l = 0$, $p_l = 0$ ($\rho_r = 0$, $p_r = 0$), the velocity specified for it has no meaning. From the jump conditions it is obvious that the transition of fluid flow to vacuum cannot be a shock. If $\rho_r > 0$, $p_r > 0$ ($\rho_l > 0$, $p_l > 0$) the solution is given in (20) with the central vacuum zone extending to the left (right) hand side. Moreover, if $\rho_r > 0$, $p_r = 0$ ($\rho_l > 0$, $p_l = 0$) the remaining rarefaction wave vanishes.

5. Conclusions

We have presented a procedure to solve the Riemann problem for polytropic gas. The initial data are first examined for their physical relevance (8). An easy to handle criterion (23) leads to either a solvable fixed point problem or to the direct solution (20). A fast algorithm [1] which converges with certainty has been made applicable to the fixed point problem. Some special cases for the initial data have been treated in order to complete the method. This Riemann solver is able to operate reliably and is recommended for use in practical applications.

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